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Numerical Methods 101 - Convergence of Numerical Models

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Abstract

A numerical model is convergent if and only if a sequence of model solutions with increasingly refined solution domains approaches a fixed value. Furthermore, a numerical model is consistent only if this sequence converges to the solution of the continuous equations which govern the physical phenomenon being modeled. Given that a model is consistent, it is insufficient to apply it to a problem without testing for sensitivity to the size of the time and distance steps which form the discrete approximation of the solution domain. That is, convergence testing is a required component of any modeling study.

Two models were examined for this paper. The first model is a four-point implicit method applied to the unsteady one-dimensional open-channel flow equations. The second model is an unsteady one-dimensional or depth-averaged two-dimensional explicit diffusion-wave approximation of the shallow-water flow equations. Both models were applied to a one-dimensional channel problem. The two-dimensional model was applied to a simple two-dimensional flow field. Convergence testing is demonstrated in this paper by applying these models and examining the impact of increased spatial and temporal resolution on the results. It is demonstrated that both models are sensitive to changes in the spatial resolution and that erroneous solutions may result if this sensitivity is not understood during application of these models.

Introduction

Few analytical solutions of practical importance to the hydrodynamic equations exist. Therefore, discrete approximations of the continuous partial differential and integral equations are made, often using finite difference or finite element methods. The resulting numerical models have the common property that they rely on discretization of the solution domain into a grid of points for which discrete approximations are made. Solution of a problem hinges on solution of the resulting approximate equations.

Because of these discrete approximations, solutions computed using these models exhibit dependence on the time and distance steps comprising the discrete solution domain. Such dependence should be determined and minimized prior to

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calibration of model parameters. It is the responsibility of the model user to determine appropriate time and distance steps for a given application of a model. This procedure is called convergence testing.

Definition of Convergence

Given a physical problem described by continuous partial differential or integral equations which have an exact solution, $\boldsymbol{A}$, define A_n to be a solution based on a particular discrete scheme using a distance step, Δx_n, and time step, Δt_n. Let $\{A_n\}$ be a sequence of solutions A_n such that increasing values of the subscript denote increasing refinement of the underlying solution grid, that is,

$$\Delta x_i < \Delta x_k \text{ and } \Delta t_i < \Delta t_k \text{ for i > k}$$

A model is said to be convergent if and only if $\{A_n\}$ asymptotically approaches some fixed value (Burnett, 1987). Of course, there is a potential problem in that $\{A_n\}$ could converge to the incorrect solution! Therefore, it is required that a model be consistent in addition to convergent. That is,

$$|\boldsymbol{A} - A_n| \to 0 \text{ as n} \to \text{infinity}$$

Therefore, a single application of a model is insufficient to ensure a converged solution of a problem. Multiple solutions with different values of the time and distance steps are required to ensure that dependence of the solution on domain discretization is removed from the model. This if often approached by repeatedly running the model with successively halved time and distance steps and examining model output at several points common to all solutions (Roache, 1982). The model is converged if further refinement of time and distance steps produces insignificant change in model output.

Problem 1

The first problem is a one-dimensional channel. The channel is 70,000 feet long and has a rectangular section 100 feet wide with bed slope of 0.001 and with Mannings n of 0.045. Initial conditions are specified by a discharge of 250 cubic feet per second (cfs) and a depth of 1.71 feet (uniform flow). The upstream boundary condition is specified with an inflow hydrograph defined by

$$Q \quad \begin{aligned} &= 250 + 238.7\,(1 - \cos(\pi t/4500)), && t \,\varepsilon\, [0, 9000] \\ &= 250, && \text{otherwise} \end{aligned}$$

where Q is the discharge in cfs, and t is time in seconds. The downstream boundary condition is specified by a constant depth of 1.71 feet. Time series of stage and discharge taken at 50,000 feet downstream from the upstream boundary are used for examination of model results.

Two unsteady flow models were applied to this channel. The first model is FourPt, a four-point implicit solution of the one-dimensional dynamic flow equations developed and used by the U.S. Geological Survey for training at its National Training Center in Denver, Colorado. The second model is the Diffusion Hydrodynamic Model, (DHM; Hromadka, 1987). DHM solves a simplified flow equation derived by dropping the local and convective acceleration terms from the momentum equations. Furthermore, DHM solves unsteady one-dimensional

problems using a simplified flow equation derived by dropping the local and convective acceleration terms from the one-dimensional momemtum equation.

FourPt was applied using a time step of 6 minutes for all runs. From previous studies, this value was determined to be sufficiently small to give convergence. Convergence in the distance step was determined by successively halving the distance step, beginning with a maximum of 10,000 feet, running the model, and plotting hydrographs from successive runs. These results are shown on Figure 1.

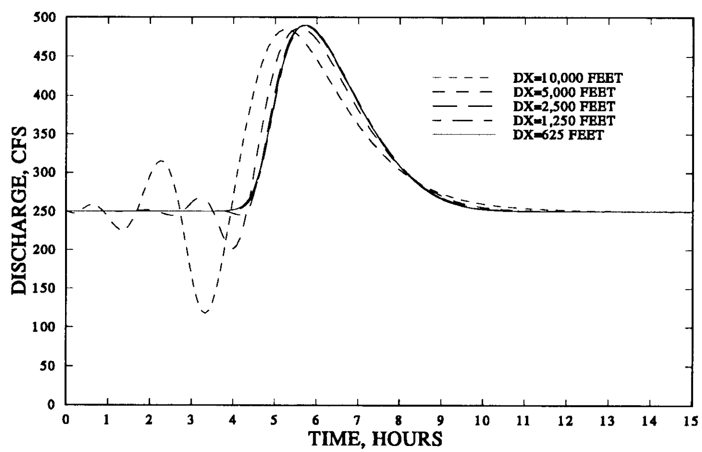

Figure 1. FourPt simulated discharge hydrographs at 50,000 feet for problem 1.

Clearly, FourPt is convergent when operated with a distance step of 1250 feet (and a time step of 6 minutes), because little change in the solution results from further reduction of the distance step. Similarly, it is also clear that use of the model with large distance steps (greater than 2500 feet) results in significant leading phase error resulting from the four-point implicit formulation. Therefore, application of the model to this example problem requires a distance step of less than 2500 feet. If greater values of time and distance steps are used, then the results produced by the model are questionable at best, and under some circumstances the model may produce no output, because leading phase error could cause the model to fail catastrophically.

DHM was applied in one-dimensional mode to this problem using the same initial and boundary conditions as those used in the FourPt application. Convergence was determined by successively halving the distance step, beginning

with a maximum of 10,000 feet, running the model, and plotting hydrographs from successive runs. DHM changes the time step dynamically during its operation over a range provided by the user as input. The time step was allowed to vary over a range of 0.1 seconds to 10 seconds. For distance steps greater than 156.25 feet, the model consistently used 10 seconds for the time step. For a distance step of 156.25 feet, the model used a range of values between 3 seconds and 6 seconds, with an average time step of 5 seconds. Results from theese runs are shown on Figure 2.

DHM is convergent with a distance step of 312.5 feet. There is little change in the discharge hydrograph between the simulation using a distance step of 312.5 feet and that using a distance step of 156.25 feet. Furthermore, the hydrograph is similar in shape and peak discharge to that produced by FourPt.

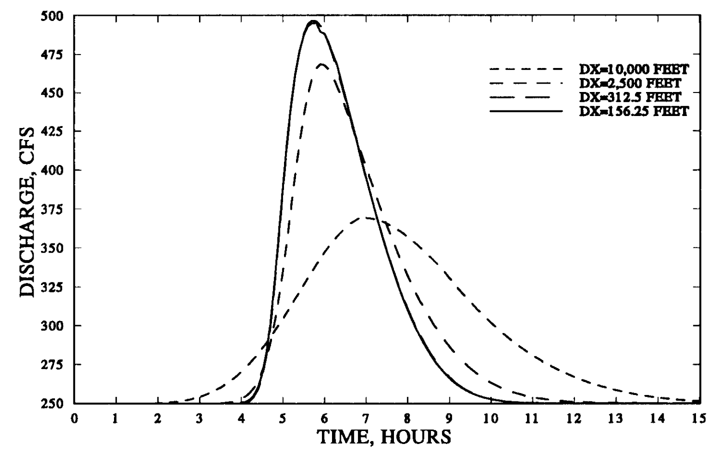

Figure 2. DHM simulated discharge hydrographs at 50,000 feet for problem 1.

Problem 2

The second problem is flow through a constriction, such as that caused by a bridge crossing over a river. A plan view of the problem is shown on Figure 3. The bottom elevation of the reach is 0.0 feet and Mannings n is 0.035. Initial conditions are specified as a constant depth of 4.0 feet throughout the solution domain (no flow). The downstream boundary condition is specified as a constant stage of 4.0 feet, and the upstream boundary condition is specified as a constant inflow of 4000 cfs distributed uniformly across the width of the flow field.

DHM was applied to this problem to demonstrate convergence of a two-dimensional model. DHM uses squares to discretize the solution domain. Distance steps of 100 feet, 50 feet, and 25 feet were used to examine the convergence properties of the model. Because the problem is steady-state and DHM simulates unsteady flow, the model was allowed to advance in time until no further changes were evident in stage or velocity throughout the study reach. For distance steps of 100 feet and 50 feet, the stage changed only 0.01 feet between 720 seconds and 1800 seconds of model time. Because of excessive run time distance steps of 50 feet and less, and because the closure criterion for the model was 0.01 feet, convergence comparisons were made at 720 seconds of model simulation time.

Three points were selected from the flow field for comparison. The first point is located approximately midway between the upstream terminus of the model and the constriction in the region where streamlines would not have significant horizontal curvature. The second point is a short distance upstream from the constriction in a region where streamlines were expected to have significant horizontal curvature. The third point is in the constriction where the fewest relative number of computational points exist, and therefore where it is expected that a model would have difficulty converging. The location of these points is shown on Figure 3.

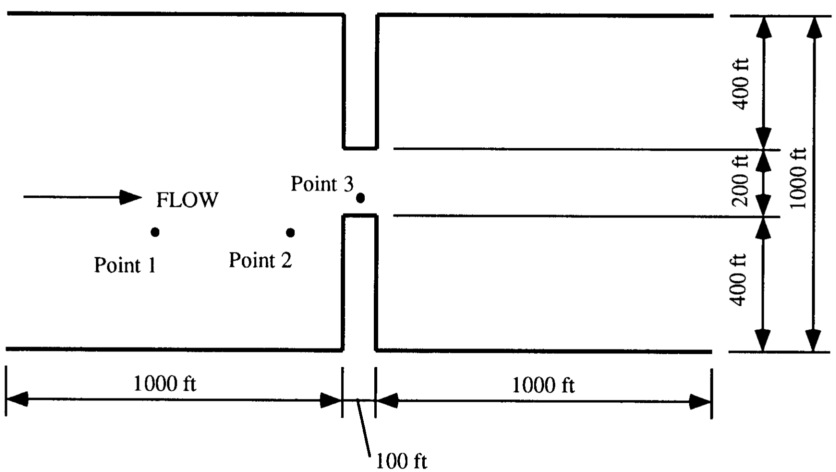

Figure 3. Plan view of the channel studied in problem 2.

According to the data in Table 1, DHM has not converged with a distance step of 25 feet. That is, the solution continues to be dependent on the distance step used to solve the problem. The lack of convergence exemplifies the need for convergence testing of a numerical model before the model is used to answer specific questions about the physical system being modeled.

Discussion

For the one-dimensional problem, each model required different time and

distance steps in order to converge. This occurs because each model approaches the solution of the governing equations differently. In fact, it was possible to achieve non-convergent solutions to the problem, some of which appeared to be reasonable solutions. In the case of the FourPt, had the initial conditions been slightly different, solutions computed with large distance steps may have failed if the oscillations leading the hydrograph had intersected the bed of the stream. With DHM, a reasonable looking hydrograph was produced for all runs. With both models, the only way to determine that the solution was dependent on the grid size was to make multiple runs and compare the results. This indicates the necessity for conducting a convergence test before relying on the output of a model.

Table 1. Convergence of DHM

Computed Steady-State Stage in Feet

Point	Distance Step: 100 Feet	50 Feet	25 Feet
1	4.53	4.49	4.45
2	4.50	4.46	4.42
3	4.30	4.28	4.26

For the two-dimensional problem, convergence was not achieved even with a distance step of 25 feet. When the distance step was halved from 50 feet to 25 feet, the solution changed at the same rate as it did when the distance step was reduced from 100 feet to 50 feet. Therefore, another run with a distance step of 12.5 feet should be executed (at least) before convergence can be ascertained.

Conclusions

Because of the discrete nature of numerical hydrodynamic models, solutions computed using these models exhibit dependence on the time and distance steps used to discretize the solution domain. In both the one-dimensional and two-dimensional problems, it was possible to attain a solution without convergence testing, and indeed without a convergent model. On completion of several runs of each model, it was evident that model output was dependent on the distance step in particular, especially for DHM. Therefore, such dependence should be determined and removed through convergence testing through appropriate discretization as a part of the application of these models. It is the responsibility of the model user to determine the correct grid spacing and time increment for a given application.

References

Burnett, D.S., 1987. Finite Element Analysis: From Concepts to Applications, Addison-Wesley.

Hromadka, T.V. II, and Yen, C.C., 1987. "A Diffusion Hydrodynamic Model," Water Resources Investigation Report 87-4137, U.S. Geological Survey.

Roache, P.J., 1982. Computational Fluid Dynamics, Hermosa Publishers, Albuquerque, NM.