8-2021

A Combinatorial Formula for Kazhdan-Lusztig Polynomials of Sparse Paving Matroids

George Nasr
University of Nebraska-Lincoln, george.nasr@huskers.unl.edu

A COMBINATORIAL FORMULA FOR KAZHDAN-LUSZTIG POLYNOMIALS OF SPARSE PAVING MATROIDS

by

George David Nasr

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professors Kyungyong Lee and Jamie Radcliffe

Lincoln, Nebraska

August, 2021

A COMBINATORIAL FORMULA FOR KAZHDAN-LUSZTIG POLYNOMIALS OF SPARSE PAVING MATROIDS

George David Nasr, Ph.D.

University of Nebraska, 2021

Advisers: Kyungyong Lee and Jamie Radcliffe

We present a combinatorial formula using skew Young tableaux for the coefficients of Kazhdan-Lusztig polynomials for sparse paving matroids. These matroids are known to be logarithmically almost all matroids, but are conjectured to be almost all matroids. We also show the positivity of these coefficients using our formula. In special cases, such as for uniform matroids, our formula has a nice combinatorial interpretation.

DEDICATION

Dedicated to my first math teacher, who helped a younger version of me who was struggling with math, and never gave up. Though I may have not appreciated it at the time, I can see now how much patience, selflessness, and compassion you must have had to help your son with math, especially after coming home from a full day of work. Thank you for everything, Mom.

ACKNOWLEDGMENTS

I first must thank my advisors, Kyungyong Lee and Jamie Radcliffe. Doing mathematics with you both was always fun. I can't thank you enough for your patience, support, and all you've taught me over the last few years.

Next, I'd like to thank some of my closest friends at UNL, namely Vince, Su Ji, Andrew, Kelsey, Stephen, Emily, Adam, and Dana, for all they've done for me. The emotional support, help with homework, random office conversations, baking, climbing, gaming, and binge-watching made all the difference during my time in Lincoln, especially in the first couple years of grad school. You're all incredible.

Next, I'd like to thank my Mom, Dad, and my sister for their love and support with grad school. Though it was often met with a delayed response, your regular texts to check-in on me were appreciated.

Next, I'd to thank Aaron, Matt, and Kirk for their long-time continued friendship, support, and understanding with my very slow replies to our long-standing group messages.

Finally, I thank the math faculty at Cal Lutheran University. If not for you, I definitely would have never considered pursuing mathematics. You helped expose me to the beauty of mathematics and helped foster my excitement for it. Your advice and support regarding grad school really helped me have faith early on that my dedication and hard work would make me successful in grad school, even if things weren't looking great.

Table of Contents

List of Figures

Chapter 1

Introduction

In 1979, Kazhdan and Lusztig found a polynomial that corresponds to a pair of elements in a Coxeter group [10]. The definition of this polynomial is recursive and uses the Bruhat order to induce a poset structure on the elements of a given Coxeter group. This polynomial, later gets called the *Kazhdan-Lusztig polynomial* (which we sometimes abbreviate as "KL polynomials"). Since then, the definition of these polynomials have been generalized—for instance, see Stanley's work in [29] and Brenti's continuation of Stanley's work in [6, 7]—so that one may define these polynomials using different combinatorial structures.[1]

In 2016, Elias, Proudfoot, and Wakefield introduced the Kazhdan-Lusztig polynomial of combinatorial objects called matroids. The definition of these polynomials and matroids can be found in the next chapter. After their introduction, these polynomials quickly drew active research interest due to their conjectured properties such as the non-negativity of coefficients, and real-rootedness (see [9, 17, 13, 32]). These are natural properties to be curious about in light of what we know for the original Kazhdan-Lusztig polynomials. For instance, Elias and Williamson in 2014 [11] prove the non-negativity for

[1] The generalized theory is sometimes referred to as the *Kazhdan-Lusztig-Stanley* theory.

all Kazhdan-Lustig polynomials. However, they are not real-rooted, as Polo proved that any polynomial with constant term 1 and non-negative integer coefficients can be realized as the Kazhdan-Lusztig polynomial for some pair of elements in some Coxeter group [25].

Recently, using algebro-geometric methods, Braden, Huh, Matherne, Proudfoot, and Wang [4] proved the non-negativity of the coefficients for these polynomials. There has also been much effort put into finding relations between these polynomials or generalizations thereof (see [5, 26, 31]). These polynomials have been explicitly calculated only for very special classes of matroids (for instance, see [19, 13, 16, 20, 27]), and yet many of the known formulas have left much room for improvement. In particular, as of now, there is no enlightening interpretation for such coefficients. This is also a relevant part of the history for the original Kahzdan-Lusztig polynomials. For instance, in [2], the authors discuss some combinatorics that led to a non-recursive definition for the original Kazhdan-Lusztig polynomials.

In this paper, we provide a combinatorial formula for the Kazhdan-Lusztig polynomials of *sparse paving* matroids. We will also provide a proof for the positivity of our formula. While this may not seem necessary in light of [4], we still share our proof as it only utilizes elementary methods.

Sparse paving matroids have attracted much research interest due to a conjecture given by Mayhew, Newman, Welsh, and Whittle [22] based on a prediction by Crapo and Rota [8]. The conjecture is that sparse paving matroids will eventually predominate in any asymptotic enumeration of matroids. There is a concise way of writing this mathematically. First, note that every matroid is associated to a finite set. If s_n is the number of sparse paving matroids on n elements and m_n is the number of matroids on n elements, that the conjecture is that

$$\lim_{n\to\infty} \frac{s_n}{m_n} = 1.$$

In pursuit of this conjecture, Pendavingh and van der Pol [24] have shown that

$$\lim_{n\to\infty} \frac{\log s_n}{\log m_n} = 1.$$

That is, so far, what we know is that logarithmically almost all matroids are sparse paving matroids.

We will wait to define matroids, and namely sparse paving matroids, until the next chapter, but it will be useful to have some more notation for the remainder of the introduction. In addition to being associated to a finite groundset, every matroid also has a parameter called the *rank*. A *sparse paving* matroid can be uniquely constructed given the size of the ground set, the rank, and a collection of subsets $\mathcal{CH}$, so that each member have cardinality equal to the rank and satisfies the property that for distinct $C, C' \in \mathcal{CH}$, we have $|C \triangle C'| \geq 4$, where $C \triangle C' := (C \setminus C') \cup (C' \setminus C)$ is the *symmetric difference.* To this end, we let $S_{m,d}(\mathcal{CH})$ be the sparse paving matroid of rank d with ground set $[m+d]$ so that $\mathcal{CH}$ satisfies the conditions described above.

The last thing we need to define before stating our main result is the object that will allow us to write our combinatorial formula for the coefficients of the polynomials. Define $\mathrm{Skyt}(a,i,b)$ to be the set of fillings of the shape in Figure 1.1 below so that the rows and columns strictly increase with entries in $[a+b+2i-2]$, where for an integer n we have $[n] := \{1, 2, 3, \dots, n\}$.

We define a related object which we denote $\overline{\mathrm{Skyt}}(i,b)$, the subset of $\mathrm{Skyt}(2,i,b)$ where the value 1 appears at the top of the left-most column. We set $\mathrm{skyt}(a,i,b) := \#\,\mathrm{Skyt}(a,i,b)$ and $\overline{\mathrm{skyt}}(i,b) := \#\,\overline{\mathrm{Skyt}}(i,b)$. There are some conventions for

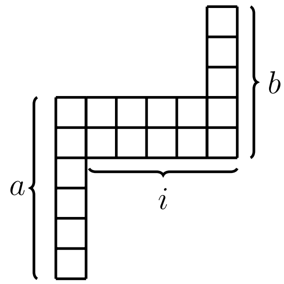


Figure 1.1: The left-most column has height a, followed by $i-1$ columns of height 2, followed by the right-most column of height b.

special values of a, i and b, but we leave these for Chapter 3.

We are now ready to state our main result.

Theorem 1. *Let $c^i_{m,d}(\mathcal{CH})$ be the i-th coefficient for the Kazhdan-Lusztig polynomial for the sparse paving matroid $S_{m,d}(\mathcal{CH})$. Then*

$$c^i_{m,d}(\mathcal{CH}) = \text{skyt}(m+1, i, d-2i+1) - |\mathcal{CH}| \cdot \overline{\text{skyt}}(i, d-2i+1).$$

Moreover, this formula is always non-negative.

What is truly remarkable about this formula is that it is not effected by how the elements of $\mathcal{CH}$ relate to one-another. Keep in mind that $\mathcal{CH}$ could be *any* set of elements so that their pairwise symmetric difference is at least 4. Given a fixed m, d, and i, the value of the coefficient is invariant of selection of $\mathcal{CH}$ so long as $|\mathcal{CH}|$ remains the same. It is also worth noting that the above formula has already inspired other mathematical results [12, 14].

When $\mathcal{CH}$ is a disjoint collection, we also have the formula in Theorem 1 has a manifestly positive interpretation. Consider the subset of $\text{Skyt}(m+1, i, d-2i+1)$ satisfying at least one of the following three conditions:

- the top entry of the right-most column is 1; or

- the bottom entry of the right-most column is greater than $d + |\mathcal{CH}|$; or
- the third entry (from the top) of the left-most column is less than $d+1$.

Then the size of this subset agrees with the formula we give in Theorem 1. In the special case where $\mathcal{CH} = \emptyset$, the second condition becomes tautological as the bottom of the right-most column is guaranteed to be at least $d+1$ for any tableaux. So when $\mathcal{CH} = \emptyset$, we get the entire size of $\mathrm{Skyt}(m+1, i, d-2i+1)$ as our coefficient, as Theorem 1 indicates. Also in this case we have $S_{m,d}(\mathcal{CH}) = U_{m,d}$, the uniform matroid of rank d on $m + d$ elements. [2]

In light of this, we have proven the following conjecture in the case of sparse paving matroids.

Conjecture 1. *Let M be a matroid of rank d on $m + d$ elements, and let c^i be the i-th coefficient for $P_M(t)$. Then*

$$c^i \leq c^i_{m,d}(\emptyset).$$

That is, among all matroids with rank d and ground set size $m + d$, the Kazhdan-Lusztig polynomial for $U_{m,d}$ has the largest coefficients.

This conjecture was posed by Katie Gedeon. It has no written source, but was communicated to us by Nicholas Proudfoot.

One final thing to note that is interesting about our formula is that if $m + 1 = 2$ or $d - 2i + 1 = 2$, then $\mathrm{skyt}(m + 1, i, d - 2i + 1)$ becomes equal to a well-known number, namely the number of polygon dissections, that is,

[2] The first (and only known) manifestly positive integral interpretation for uniform matroids was given in [15, Remark 3.4], which requires possibly many Young diagrams.

ways of drawing non-intersecting lines in a regular polygon [30]. Hence, when $m+1 = d-2i+1 = 2$, it becomes a Catalan number.[3]

[3]This connection to polygon dissections was already mentioned in several places, namely in Remark 1.3 in [28] and Remark 5.3 of [16], but with the discovery of our combinatorial object, this fact follows directly from [30].

Chapter 2

Background

The following sections provide the background information necessary to understand the rest of this document.

2.1 Poset Theory

Let P be a set with a partial ordering $\leq$. For $x, y \in P$, we set

$$[x, y] = \{z \in P : x \leq z \leq y\}.$$

We may replace a bracket in $[x, y]$ with a parenthesis to change the corresponding inequality into a strict inequality. We say y *covers* x, denoted $x \lessdot y$, if $x < y$ and $[x, y] = \{x, y\}$.

A *chain* in P is a collection of elements $x_1, x_2, \ldots, x_n$ in P so that $x_i \leq x_{i+1}$ for every i. We say a chain is *saturated* if it is maximal with respect to the number of elements it contains. That is, for each i, $x_i \lessdot x_{i+1}$.

A poset is *ranked* if every saturated chain in every interval $[x, y]$ has the same number of elements. A poset is *graded* if it is ranked and has a unique maximum and minimum element. In this case, we can define a *rank* function, $\mathrm{rk} : P \to \mathbb{Z}$, satisfying the following two condition.

1. The minimum element has rank 0.

2. For $x, y \in P$, if $x \lessdot y$ then $\operatorname{rk} y = \operatorname{rk} x + 1$.

Remark 1. *In general, one can replace condition 1 with "defining the rank arbitrarily of some element", but the condition we provide will be the most natural for our purposes.*

Let $\mu : P \times P \to \mathbb{Z}$ be given by

$$\mu(x,y) = \begin{cases} 1 & x = y \\ -\sum_{z \in [x,y)} \mu(x,z) & x < y \\ 0 & \text{otherwise.} \end{cases}$$

μ is often called the *Möbius function* on P.

Remark 2. *One can define μ instead as the inverse to another function $\zeta : P \times P \to \mathbb{Z}$ under an appropriate definition for multiplication in the incidence algebra (the set of function from $P \times P$ to $\mathbb{Z}$), but we will omit the necessary details for that here.*

With μ, one can define a the characteristic polynomial for a graded poset P, as follows. Let $\hat{0}$ be the minimal element in P, and $\hat{1}$ be the maximal element in P.

Definition 1. The *characteristic polynomial* for a poset P is

$$\chi_P = \sum_{x \in P} \mu(\hat{0}, x) t^{\operatorname{rk}(\hat{1}) - \operatorname{rk}(x)}.$$

2.2 Matroids

Let E be a finite set of elements. A *matroid* is the pair $M = (E, \mathcal{I})$, where I satisfies the following three axioms.

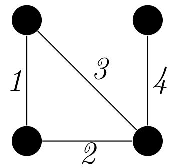
1. $\emptyset \in \mathcal{I}$
2. If $I \in \mathcal{I}$ and $J \subseteq I$ then $J \in \mathcal{I}$.
3. For all pairs of elements $I, J \in \mathcal{I}$, if $|I| < |J|$, then there exists an element $j \in J \setminus I$ such that $I \cup \{j\} \in \mathcal{I}$.

Conditions 1 and 2 imply that M is a *simplicial complex.* The sets $\mathcal{I}$ are called independent sets for M. E is called the ground set for M. This does not coincidentally overlap with terminology from Linear Algebra. Taking E to be the collection of edges of a matrix over a field k, and defining $\mathcal{I}$ to be the subsets of E that are linearly independent over a field k, $M = (E, \mathcal{I})$ is a matroid, and is called *representable.*

Example 1. *Consider the matrix* $\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$ *over the field* $\mathbb{R}$*, where* $c_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$*,* $c_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$*, and* $c_3 = 2c_1$*. Then* $M = (E, \mathcal{I})$ *is a representable matroid where* $E = \{c_1, c_2, c_3\}$ *and* $\mathcal{I} = \{\emptyset, \{c_1\}, \{c_2\}, \{c_3\}, \{c_1, c_2\}, \{c_1, c_3\}\}$.

The definition of a matroid was motivated by the fact that this combinatorial structure arising from Linear Algebra also appears in Graph Theory. Indeed, if E is the edge set of a graph, and $\mathcal{I}$ is the acyclic edge sets of the graph (that is, edge sets containing no cycle), then $M = (E, \mathcal{I})$ is a matroid, and is called *graphic.*

Example 2. *Consider the following edge-labeled graph.*



Then $M = (E, \mathcal{I})$ *is a graphic matroid, where* $E = \{1, 2, 3, 4\}$ *and*

$$\mathcal{I} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\},$$
$$\{2,3\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}\}.$$

Note that in the previous two examples, the maximal independent sets always had the same size. This is no coincidence, as given two independent sets of different sizes, axiom 3 for a matroid implies that the smaller set can not be maximal. Such a maximal independent set is called a *basis*. Bases enjoy the following property.

Proposition 1. *Let* $\mathcal{B}$ *be the bases for a matroid. Consider any two sets* $B, B' \in \mathcal{B}$*. Then for all* $b \in B$*, there exists a* $b' \in B'$ *so that* $(B \setminus \{b\}) \cup \{b'\} \in \mathcal{B}$*. Consequently,* $(B' \setminus \{b'\}) \cup \{b\} \in \mathcal{B}$*.*

Bases alone are in fact enough to define a matroid.

Proposition 2. *Let* $\mathcal{B}$ *be a collection of* d*-subsets of a finite set* E*, for some non-negative integer* d*, satisfying the condition in proposition 1. Let* $\mathcal{I} = \{I \subseteq B : B \in \mathcal{B}\}$*. Then* $M = (E, I)$ *is a matroid.*

We provide a few more useful definitions from Matroid Theory that we will need for future chapters. Let $M = (E, \mathcal{I})$ be a matroid with ground set E and independent sets $\mathcal{I}$. Given a subset $F \subseteq E$, the *rank* of F, denote $\operatorname{rk} F$, is given by

$$\operatorname{rk} F = \max\{|I| : I \subseteq F \text{ and } I \in \mathcal{I}\}.$$

That is, the rank of the set is the size of the largest independent set contained inside of it. If a set is maximal (with respect to set containment) for its rank it is said to be a *flat*.

Example 3. *Consider the matrix* $\begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}$ *over the field* $\mathbb{R}$*, where* $c_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$*,* $c_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$*, and* $c_3 = 2c_1$*. Recall that* $M = (E, \mathcal{I})$ *is a representable matroid. The flats of rank* i *are given below.*

$i = 0 : \emptyset$

$i = 1 : \{c_2\}, \{c_1, c_3\}$

$i = 2 : \{c_1, c_2, c_3\}.$

Note that the groundset of the matroid will always be the unique flat whose rank is the size of any basis. The subset of the groundset containing all elements not in any independent set will always be the unique flat of rank 0. For our purposes, this set will always be the empty set. That is, there will be no element of the groundset missing from every indepedent set.

One can induce a partial order on the set of flats for a matroid M with respect to set containment, which is often called the *lattice of flats*, which we denote $L(M)$.

Now, we discuss two related operations to a matroid, localizations and contractions. A *localization* of a matroid $M = (E, \mathcal{I})$ at a flat F, denoted M^F, is the matroid with ground set F, and whose indepdents sets are the subsets of F that are in $\mathcal{I}$. A *contraction* of $M = (E, \mathcal{I})$ at a flat F is the matroid M_F whose ground set if $E \setminus F$, and whose independent sets are the subsets of $E \setminus F$ whose union with a basis for F is in $\mathcal{I}$.

It is worth knowing that one can interpret the lattice of flats of localiztions and contractions as intervals in $L(M)$. For instance, $L(M^F)$ is isomorphic (as a poset) to the interval $[\emptyset, F]$ in $L(M)$. Also, $L(M_F)$ is isomorphic (as a poset) to the interval $[E \setminus F, E]$ in $L(M)$.

Finally, the *characteristic polynomial for a matroid* $M = (E, \mathcal{I})$, denoted χ_M, is simply the characteristic polynomial for its lattice of flats. That is,

$$\chi_M = \sum_{F \in L(M)} \mu(\emptyset, F) t^{\mathrm{rk}(E) - \mathrm{rk}(F)}.$$

2.3 Sparse Paving Matroids

Now that we have explicitly defined matroids, we will go into more detail defining and discussing sparse paving matroids. There are several known characterizations of sparse paving matroids. To understand these, we will need to define a few more terms. *Circuits* are minimal dependent sets, *hyperplanes* are flats whose rank is one less than that of the matroid, and *ciruit-hyperplanes* are sets that are both circuits and hyperplanes. If M is matroid, its dual M^* is the matroid with the same ground set and whose bases are the complement of the bases in M. We now define sparse paving matroids.

Definition 2. Let M be a matroid of rank d so that the ground set has $m+d$ elements. Let $\mathcal{B}$ be the set of bases for M, so in particular $\mathcal{B} \subseteq \binom{[m+d]}{d}$. Set $\mathcal{CH} := \binom{[m+d]}{d} \setminus \mathcal{B}$. Then M is *sparse paving* if any (and hence all) of the following hold.

1. $\mathcal{CH}$ is the set of circuit-hyperplanes for M.

2. For distinct $C, C' \in \mathcal{CH}$, we have $|C \triangle C'| \geq 4$, where $C \triangle C' := (C \setminus C') \cup (C' \setminus C)$ is the *symmetric difference.*

3. Every nonspanning circuit is a hyperplane.

4. M and its dual M^* are both paving; that is, their circuits have cardinality at least d.

Recall that in this case, M is denoted $S_{m,d}(\mathcal{CH})$. It is worth noting that when $\mathcal{CH}$ is a disjoint collection, $S_{m,d}(\mathcal{CH})$ can be seen to be representable. This in turn gives a combinatorial formula for the intersection cohomology Poincaré polynomial of the corresponding reciprocal plane over a finite field, thanks to [9]. In general, though, almost all sparse paving matroids are not representable. This is due in large part to Nelson [23] who showed that asymptotically almost all matroids are not representable. In particular, his work implies that the logarithmic growth of representable matroids is bounded by a polynomial. Meanwhile, the logarithmic growth of matroids in general are known to have at least exponential growth, and so the same must be true for sparse paving matroids.

We mention some other attributes of sparse paving matroids relating to topics from the last section.

Proposition 3. *The flats of* $S_{m,d}(\mathcal{CH})$ *are*

1. *the sets of cardinality at most* $d-2$*;*

2. *the sets of cardinality* $d-1$ *not contained in any element of* $\mathcal{CH}$*;*

3. *the elements of* $\mathcal{CH}$*;*

4. $[m+d]$.

Proof. We address each of the statements in the proposition, numbering our arguments accordingly. First, though, it will be useful for us to note that the members of $\mathcal{CH}$ are circuits and all have size d. Hence every set of size at most $d-1$ is independent.

1. Any set of size at most $d-2$ is maximal with respect to its rank, since every set of size $d-1$ is independent.

2. If every set of size d containing a set of size $d-1$ is independent, this $(d-1)$-set is maximal with respect to its rank.

3. The elements of $\mathcal{CH}$ have rank $d-1$ since they are sets of size d and are not independent. If there exists an element $C \in \mathcal{CH}$ that is not maximal, then there is an element of the groundset, say x, so that $D := C \cup \{x\}$ has rank $d-1$. Then this implies every subset of D of size d is not independent. If C' is such a subset, distinct from C, then $|C \triangle C'| = 2$. But $C' \in \mathcal{CH}$ since it is not independent, which contradicts the consrtaint of $\mathcal{CH}$ for sparse paving matroids.

4. The groundset of a matroid is always a flat.

□

With this, we can now discuss the localizations and contractions of $S_{m,d}(\mathcal{CH})$. First, note that the localizations and contractions of $U_{m,d}$, the uniform matroid of rank d with groundset $[m+d]$.

$$(U_{m,d})^F = \begin{cases} U_{m,d} & F = [m+d] \\ U_{0,|F|} & F \neq [m+d] \end{cases},$$

and

$$(U_{m,d})_F = \begin{cases} U_{0,0} & F = [m+d] \\ U_{m,d-|F|} & F \neq [m+d] \end{cases}.$$

The corresponding equations for $S_{m,d}(\mathcal{CH})$ can also be described in a similar manner. In what follows, if F is a flat, then we define $\mathcal{CH}(F) := \{C \setminus F : C \in \mathcal{CH}$ such that $F \subseteq C\}$. It is worth noting that if $\mathcal{CH}$ is the set of circuit-hyperplanes for a sparse paving matroid, then so is $\mathcal{CH}(F)$, so long as F is strictly contained in some circuit-hyperplane. One way to check this is by verifying $\mathcal{CH}(F)$ satisfies the condition that any pair has symmetric difference at least 4.

Proposition 4.

$$S_{m,d}(\mathcal{CH})^F = \begin{cases} S_{m,d}(\mathcal{CH}) & F = [m+d] \\ U_{1,d-1} & F \in \mathcal{CH} \\ U_{0,|F|} & \text{otherwise} \end{cases}$$

and

$$S_{m,d}(\mathcal{CH})_F = \begin{cases} S_{m,d}(\mathcal{CH}) & F = \emptyset \\ U_{m-1,1} & F \in \mathcal{CH} \\ S_{m,d-|F|}(\mathcal{CH}(F)) & \emptyset \subsetneq F \subsetneq C, \text{ for some } C \in \mathcal{CH} \\ (U_{m,d})_F & \text{otherwise.} \end{cases}$$

Proof. For the localization, the only new case necessary to mention in comparison to the uniform case is for $F \in \mathcal{CH}$; the other cases follow from the uniform case. The localization of this matroid at F treats F as the ground set, with independent sets being those that are independent in $S_{m,d}(\mathcal{CH})$. We know every *proper* subset of F is independent, giving $U_{1,d-1}$.

Now for the contraction. If we have $F \nsubseteq C$ for all $C \in \mathcal{CH}$, then the structure of $S_{m,d}(\mathcal{CH})_F$ is exactly that of $(U_{m,d})_F$. For the case where $F \in \mathcal{CH}$, we want the subsets of $S := [m+d] \setminus F$ such that their union with a basis for F is independent in $S_{m,d}(\mathcal{CH})$. The bases for F are the elements of $\binom{F}{d-1}$. Note if $B \in \binom{[m+d]}{d}$ satisfies $|B \triangle F| = 2$, then B is independent in $S_{m,d}(\mathcal{CH})$. This means the desired subsets of S are the empty set and every singleton of S. This gives a matroid isomorphic to $U_{m-1,1}$. Finally, when $\emptyset \subsetneq F \subsetneq C$, for some $C \in \mathcal{CH}$, note that F is independent, and hence a basis for itself. Thus, the independent sets for $S_{m,d}(\mathcal{CH})_F$ are the subsets X of $[m+d] \setminus F$ so that $X \cup F$ is independent in $S_{m,d}(\mathcal{CH})$. That is, $|X| \leq d - |F|$. When $|X| < d - |F|$, $|X \cup F| < d$ and every subset of $[m+d]$ of size smaller than d is independent. When $|X| = d - |F|$, $X \cup F$ is a basis for $S_{m,d}(\mathcal{CH})$ if and only if $X \cup F \neq C$, for any $C \in \mathcal{CH}$, which is true if and only if $X \notin \mathcal{CH}(F)$. That is, we get a matroid isomorphic to $S_{m,d-|f|}(\mathcal{CH}(F))$. □

With these in mind, we can now compute the characteristic equation for $S_{m,d}(\mathcal{CH})$. First, note that for a matroid M, the characteristic polynomial is given by

$$\chi_M(t) = \sum_{F \in L(M)} \mu_{L(M)}(\hat{\mathbf{0}}, F) t^{\operatorname{rk} M - \operatorname{rk} F},$$

where $L(M)$ is the lattice of flats for matroid M. The case when $M = U_{m,d}$,

$\chi_M(t)$ is well understood.

$$\chi_{U_{m,d}}(t) = (-1)^d \binom{m+d-1}{d-1} + \sum_{i=0}^{d-1} (-1)^i \binom{m+d}{i} t^{d-i}.$$

Parts of this also arise in $\chi_{S_{m,d}(\mathcal{CH})}$.

Proposition 5. *Let* $c = |\mathcal{CH}|$.

$$\chi_{S_{m,d}(\mathcal{CH})}(t)$$
$$= (-1)^d \binom{m+d-1}{d-1} - (-1)^d c + t(-1)^{d-1} \left(\binom{m+d}{d-1} - c \right) + \sum_{i=0}^{d-2} (-1)^i \binom{m+d}{i} t^{d-i}.$$

It is noteworthy that this characteristic polynomial is the same for all choices of $\mathcal{CH}$ that have the same size. This is due entirely to the symmetric difference condition on $\mathcal{CH}$, which we will utilize in the proof.

Proof of Proposition 5. For convenience, we omit subscripts for χ and μ, since throughout we work in $S_{m,d}(\mathcal{CH})$. The terms of degree at least 2 follows from the uniform matroid case since in $S_{m,d}(\mathcal{CH})$, every set of size at most $d-2$ is still flat, since every set of size $d-1$ is independent. The term of degree one comes from summing $\mu(\hat{\mathbf{0}}, F)$ for flats F of rank $d-1$. Recall that these flats are the elements of $\mathcal{CH}$ and all elements of $\binom{[m+d]}{d-1}$ not contained in any member of $\mathcal{CH}$. When F is one of the latter described flats, it follows from the uniform case that $\mu(\hat{\mathbf{0}}, F) = (-1)^{d-1}$. Note that the number of such flats is $\binom{m+d}{d-1} - c\binom{d}{d-1}$, since the symmetric difference condition on $\mathcal{CH}$ implies that $|C_i \cap C_j| \leq d-2$ for all $C_i, C_j \in \mathcal{CH}$. That is to say that no set of size $d-1$

is contained in two elements of $\mathcal{CH}$. Otherwise, if $C \in \mathcal{CH}$,

$$\begin{aligned}
\mu(\hat{\mathbf{0}}, C) &= -\sum_{\hat{\mathbf{0}} \leq F < C} \mu(\hat{\mathbf{0}}, F) \\
&= -\sum_{i=0}^{d-2} (-1)^i \binom{m+d}{i} \\
&= (-1)^d + d(-1)^{d-1}.
\end{aligned}$$

Thus the coefficient linear term for χ is given by

$$\begin{aligned}
&c(-1)^d + cd(-1)^{d-1} + (-1)^{d-1}\left(\binom{m+d}{d-1} - c\binom{d}{d-1}\right) \\
&= (-1)^{d-1}\binom{m+d}{d-1} - c(-1)^{d-1}.
\end{aligned}$$

The constant term is equal to the negative of the sum over $\mu(\hat{\mathbf{0}}, F)$ for all flats $F \neq [m+d]$. This gives

$$\begin{aligned}
&-\sum_{i=0}^{d-2} (-1)^i \binom{m+d}{i} - (-1)^{d-1}\binom{m+d}{d-1} - c(-1)^d \\
&= -\sum_{i=0}^{d-1} (-1)^i \binom{m+d}{i} - c(-1)^d \\
&= (-1)^d \binom{m+d-1}{d-1} - c(-1)^d.
\end{aligned}$$

□

It will be helpful to restate this proposition in the following way for when we prove Theorem 1.

Corollary 1. *(Proposition 5 restated.)*

$$[t^i]\chi_{S_{m,d}(\mathcal{CH})} = \begin{cases} (-1)^d\binom{m+d-1}{d-1} - c(-1)^d & i = 0 \\ (-1)^{d-1}\binom{m+d}{d-1} - c(-1)^{d-1} & i = 1 \\ (-1)^{d-i}\binom{m+d}{d-i} & 2 \leq i \leq d \end{cases}$$

2.4 Kazhdan-Lusztig Polynomials for Matroids

In 2016, Elias, Proudfoot, and Wakefield [9] defined the *Kazhdan-Lusztig polynomial* for a matroid M, denoted $P_M(t)$, as follows.

1. If $\operatorname{rk} M = 0$, then $P_M(t) = 1$.
2. If $\operatorname{rk} M > 0$, then $\deg P_M(t) < \frac{1}{2}\operatorname{rk} M$.
3. $t^{rkM}P_M(t^{-1}) = \sum_{F \in L(M)} \chi_{M^F}(t)P_{M_F}(t)$.

The fact that this polynomial exists requires proof. One familiar with the definition for the KL polynomial of Coxeter groups will notice similarities between it and the definition for KL polynomials of matroids. The primary difference between them is that while the KL polynomial for Coxeter groups are specifically defined for a *pair* of elements in a Coxeter group, the KL polynomial for matroids is just defined for a matroid. Briefly, this is because any interval in the lattice of flats for a matroid is isomorphic to the lattice of flats of another matroid, but a similar notion is not true for Coxeter groups under the Bruhat order. Hence, one may view KL polynomials for matroid as nicer to work with than those for Coxeter groups.

In the same paper KL polynomials were defined, the following was proven.

Proposition 6. *[9, Proposition 2.11] The constant term of $P_M(t)$ is always 1.*

As mentioned in the introduction, they also conjectured that the coefficients of these polynomials are non-negative, which was recently proven in 2020 via algebro-geometric methods [4].

Chapter 3

Skew Young Tableaux

In this chapter, we introduce the tableaux of interest, along with identities and symmetries involving them. Consider the following shape.

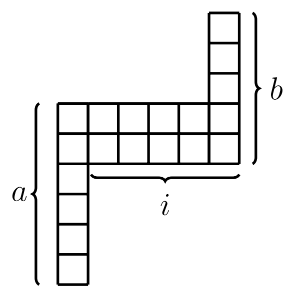


Figure 3.1: The left-most column has height a, followed by $i-1$ columns of height 2, followed by the right-most column of height b.

A *legal filling* of the above shape involves placing each number from $\{1, 2, \ldots, a+2i+b-2\}$ into the squares such that the values in the columns and rows strictly increase going down and right, respectively. Note that this is the same restriction on the entries of a standard Young tableau, but the above shape does not fit the description of the typical Young tableau.

Definition 3. A legal filling of the above shape is a *skew Young tableau.* $\mathrm{Skyt}(a, i, b)$ denotes the set of all legal fillings of this shape, and we let $\mathrm{skyt}(a, i, b) := \#\,\mathrm{Skyt}(a, i, b)$.

For our tableaux to be defined, we need $a, b \geq 2$ and $i \geq 1$, but our formula in Theorem 1 may be used for other non-negative values of a, b, and i. Hence, there are some conventions we have set for the few exceptional values that can occur so that our formula still works.

- If $i = 0$, then $\mathrm{skyt}(a, i, b) := 1$.
- If $i > 0$ and at least one of a or b is less than 2, then $\mathrm{skyt}(a, i, b) := 0$.

We also define a related collection of objects to $\mathrm{Skyt}(a, i, b)$.

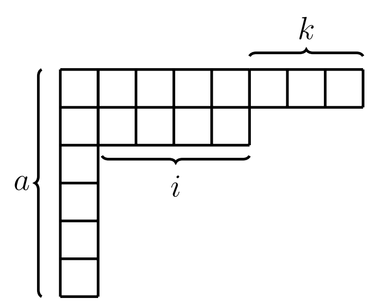
Definition 4. $\overline{\mathrm{Skyt}}(i, b)$ is the subset of $\mathrm{Skyt}(2, i, b)$ so that 1 is always the entry at the top of the left-most column. The size of $\overline{\mathrm{Skyt}}(i, b)$ is denoted $\overline{\mathrm{skyt}}(i, b)$.

By convention, $\overline{\mathrm{skyt}}(i, b) := 0$ if $i = 0$.

3.1 Identities

It will be valuable to have a formula for $\mathrm{skyt}(a, i, b)$ and $\overline{\mathrm{skyt}}(a, i, b)$ explicitly. For this, we will need the following object.

Definition 5. $SYT(a, i, k)$ is the set of standard Young tableaux of the following shape.



That is, $SYT(a,i,k)$ is the set of all fillings using the number in $[a+2i+k]$ so that the values in the columns and rows strictly increase going down and to the right, respectively.

Lemma 1.

$$\text{skyt}(a,i,b) = \sum_{k=0}^{b-2} (-1)^k \binom{a+2i+b-2}{b-k-2} \#SYT(a,i,k),$$

Proof. Observe that one could build $\text{Skyt}(a,i,b)$ by starting with a Young diagram μ with $b-2$ parts of size 1, choosing the elements from $[a+b+2i-2]$ to place in there in increasing order, and then from the remaining numbers, place them in one of $\#SYT(a,i,0)$ ways, giving a tableau λ, and then attaching μ to λ so that the bottom of μ is adjacent to the top right of λ. See the figure below.

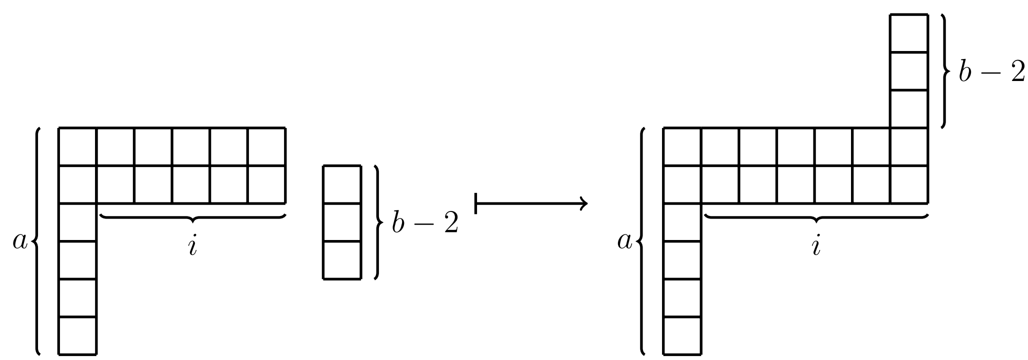


Figure 3.2: Tableaux λ and μ combine to give the shape desired skew-symmetric tableau shape.

Of course, these pieces are only compatible if the entry in the bottom entry of μ is smaller than the top right of λ, so we need to remove the cases not giving legal fillings. By moving the bottom square of μ the right of the top right piece of λ, we have a bijection between this case and having a pair of tableau, one standard Young tableau with $b-3$ parts of size 1 and the other from $SYT(a,i,1)$, that we wish to remove from the possible count. Of course, this will also remove cases where the second to last entry in μ is larger than the last entry, which was not accounted for before since we assumed we placed the entries in μ in increasing order, so we wish to add these cases back in. We can count this in a similar way by counting the number of pairs of standard Young tableaux where one is $b-4$ parts of size 1, and then selecting an element from $SYT(a,i,2)$. Continuing this process gives the right hand side of the desired equality in the statement of Lemma 1, and by an inclusion-exclusion argument, we have also counted the left.

□

A benefit of using $SYT(a,i,k)$ is that there is a known closed formula for its cardinality, thanks to the hook length formula for standard Young tableaux [21]. In particular,

$$\#SYT(a,i,k) = \frac{(a+2i+k)!(k+1)}{(a+i+k)(a+i-1)(a-2)!i!(i+k+1)!}.$$

Hence, one gets the following equivalent statement of the prior lemma.

Lemma 2.

$$\begin{aligned}&\text{skyt}(a,i,b)\\ &= \frac{1}{i!(a-2)!(a+i-1)}\sum_{k=0}^{b-2}(-1)^k\binom{a+b+2i-2}{b-2-k}\frac{(a+2i+k)!(k+1)}{(a+i+k)(i+k+1)!},\end{aligned}$$

This formula for $\text{skyt}(a,i,b)$ is useful in proving the following.

Lemma 3.

$$\overline{\text{skyt}}(i,b) = \frac{1}{(i+1)!}\sum_{k=0}^{b-2}(-1)^k\binom{b+2i-1}{b-2-k}\frac{(2i+k+2)!(k+1)}{(i+k+2)!},$$

Proof. Observe that $\overline{\text{Skyt}}(i,b) \subseteq \text{Skyt}(2,i,b)$. In particular, note that the tableaux in $T := \text{Skyt}(2,i,b) \setminus \overline{\text{Skyt}}(i,b)$ are those where the value 1 appears at the top of the rightmost column. One can achieve a bijection between $\text{Skyt}(2,i,b-1)$ and T: For any tableaux α in $\text{Skyt}(2,i,b-1)$, increase each numerical value in α by 1, and then extend the rightmost column by adding one cell at the top of the column, placing the number 1 in this position. Reversing this process recovers α. Hence,

$$\begin{aligned}
\overline{\text{skyt}}(i,b) &= \text{skyt}(2,i,b) - \text{skyt}(2,i,b-1)\\
&= \frac{1}{(i+1)!}\sum_{k=0}^{b-2}(-1)^k\left(\binom{b+2i}{b-2-k} - \binom{b-1+2i}{b-3-k}\right)\frac{(2i+k+2)!(k+1)}{(i+k+2)!}\\
&= \frac{1}{(i+1)!}\sum_{k=0}^{b-2}(-1)^k\binom{b+2i-1}{b-2-k}\frac{(2i+k+2)!(k+1)}{(i+k+2)!},
\end{aligned}$$

by Pascal's identity. □

One can obtain two formulas for $\text{skyt}(a,i,b)$ and $\overline{\text{skyt}}(i,b)$ that avoid alternating sums. We will need a few integral identities to produce these formulas. These identities can be found in Appendix A, but are referenced as they are needed in the proofs that follow. Throughout, $(x)^{(n)}$ is the rising factorial $(x)(x+1)\cdots(x+n-1)$ for integers x and n.

We start with the formula for $\text{skyt}(a,i,b)$.

Lemma 4.

$$\text{skyt}(a,i,b) = \binom{a+i-2}{i}\binom{a+b+2i-2}{b+i-1}\sum_{k=0}^{b-2}\frac{\binom{b+i-k-3}{i-1}}{\binom{a+i+k}{k+1}}$$

Proof. One can rewrite Lemma 2 as

$$\begin{aligned}&\text{skyt}(a,i,b)\\ &= \frac{(a+b+2i-2)!}{i!(a-2)!(a+i-1)(b-2)!}\sum_{k=0}^{b-2}(-1)^k\binom{b-2}{k}\frac{1}{(a+i+k)(k+2)^{(i)}}.\end{aligned} \tag{3.1}$$

We can recover this sum for $\text{skyt}(a,i,b)$ by applications of integrals to a polynomial. Let

$$f(x,y) = \frac{(a+b+2i-2)!xy^{a+i-1}(1-xy)^{b-2}}{i!(a-2)!(a+i-1)(b-2)!}.$$

Our integrals are broken up into three parts.

(a) First find $g(x)$, where $g(x) := \int_0^1 f(x,y)\ dy$; then

(b) find $h_{i-1}(x_{i-1}) := \int_0^{x_{i-1}} h_{i-2}(x_{i-2})\ dx_{i-2}$, where $h_1(x_1) := \int_0^{x_1} g(x_0)\ dx_0$ and $x_0, x_1, \ldots, x_{i-1}$ are i variables; then

(c) evaluate $\int_0^1 h_{i-1}(x_{i-1})\ dx_{i-1}$.

It is not difficult to show that, if $(1-xy)^{b-2}$ is written using the binomial expansion, part (c) will give the equation for $\text{skyt}(a,i,b)$ found in equation (3.1) above. To get the statement of Lemma 4, we apply these three steps to $f(x,y)$ directly as written.

First, we use Corollary 3 to do part (a).

$$\begin{aligned} g(x) &:= \int_0^1 f(x,y)\, dy \\ &= \frac{(a+b+2i-2)!(a+i-1)!}{i!(a-2)!(a+i-1)} \sum_{k=0}^{b-2} \frac{(1-x)^{b-k-2}x^{k+1}}{(a+i+k)!(b-k-2)!}. \end{aligned}$$

To complete parts (b) and (c) we apply Proposition 10 to get

$$\begin{aligned} &\int_0^1 h_{i-1}(x_{i-1})\, dx_{i-1} \\ &= \frac{(a+b+2i-2)!(a+i-1)!}{i!(a-2)!(a+i-1)(i-1)!(b+i-1)!} \sum_{k=0}^{b-2} \frac{(b+i-k-3)!(k+1)!}{(a+i+k)!(b-k-2)!} \end{aligned}$$

This gets us a manifestly positive sum, and all that is left to get our desired result is to perform some algebraic manipulations. The terms $(b+i-k-3)!$, $(b-k-2)!$, and $(i-1)!$ combine to give $\binom{b+i-k-3}{i-1}$. Then combine $(a+i-1)!$, $(k+1)!$, and $(a+i+k)!$ to get $\binom{a+i+k}{k+1}$. Then scaling by $\frac{(a+i-2)!}{(a+i-2)!}$ allows us to group the remaining factors into binomial coefficients giving

$$\binom{a+i-2}{i}\binom{a+b+2i-2}{b+i-1} \sum_{k=0}^{b-2} \frac{\binom{b+i-k-3}{i-1}}{\binom{a+i+k}{k+1}}. \qquad \square$$

Remark 3.

1. *While having a manifestly positive formula for* $\mathrm{skyt}(a,i,b)$ *is nice, it is unfortunate that, in general, the terms of the sum in Lemma 4 are not necessarily integers, even if you scale them by* $\binom{a+i-2}{i}$ *and* $\binom{a+b+2i-2}{b+i-1}$.

2. *It will be useful to rewrite Lemma 4 using a common denominator. We can do this by rewriting the binomials in the sum using the falling factorial* $(x)_{(n)} := x(x-1)\cdots(x-n+1)$. *Rewriting the sum gives*

$$\sum_{k=0}^{b-2} \frac{\binom{b+i-k-3}{i-1}}{\binom{a+i+k}{k+1}}$$
$$= \sum_{k=0}^{b-2} \frac{(b+i-k-3)_{b-k-2}(k+1)!}{(b-k-2)!(a+i+k)_{(k+1)}}$$
$$= \frac{\sum_{k=0}^{b-2}(b+i-k-3)_{b-k-2}(k+1)!(b-2)_{(k)}(a+i+b-2)_{(b-k-2)}}{(b-2)!(a+i+b-2)_{(b-1)}}$$

We will find this version useful later, though it is not as concise as the original formula.

Using similar methods, we can find a formula for $\overline{\text{skyt}}(i,b)$ which not only avoids an alternating sum, but is in fact a single term.

Lemma 5.

$$\overline{\text{skyt}}(i,b) = \frac{2(b+2i-1)!}{(i+1)!(i-1)!(b-2)!(b+i)(b+i-2)}$$

Proof. One can rewrite Lemma 5 as

$$\overline{\text{skyt}}(i,b) = \frac{(b+2i-1)!}{(i+1)!(b-2)!} \sum_{k=0}^{b-2}(-1)^k \binom{b-2}{k} \frac{(2i+k+2)}{(k+2)^{(i+1)}}. \tag{3.2}$$

We can recover this sum for $\overline{\text{skyt}}(a,i,b)$ by applications of a derivative and integrals to a polynomial. Let

$$f(x,y) = \frac{(b+2i-1)!xy^{2i+2}(1-xy)^{b-2}}{(i+1)!(b-2)!}.$$

We break up our plan for applications of a derivative and integrals into three steps.

(a) First solve $g(x) := \frac{d}{dy} f(x,y) \bigg|_{y=1}$; then

(b) find $h_i(x_i) := \int_0^{x_i} h_{i-1}(x_{i-1})\ dx_{i-1}$, where $h_1(x_1) := \int_0^1 g(x_0)\ dx_0$ and $x_0, x_1, \ldots, x_i$ are $i+1$ variables; then finally

(c) find $\int_0^1 h_i(x_i)\ dx_i$.

Using the binomial theorem to expand $(1-xy)^{b-2}$, one may rewrite $f(x,y)$ as

$$f(x,y) = \frac{(b+2i-1)!}{(i+1)!(b-2)!} \sum_{k=0}^{b-2} \binom{b-2}{k} (-1)^k x^{k+1} y^{2i+k+2}.$$

With $f(x,y)$ in this form, observe that applying steps (a), (b), and (c) in order will directly give the equation for $\overline{\text{skyt}}$ in equation (3.2) above. We will show that if one leaves $f(x,y)$ as originally defined and applies the three aforementioned steps, one gets the right-hand side of the equation in Lemma 5, hence proving the desired result.

First, for step (a) observe that

$$\begin{aligned} g(x) &= \frac{d}{dy} f(x,y)\bigg|_{y=1} \\ &= \frac{2(i+1)(b+2i-1)!}{(i+1)!(b-2)!} x(1-x)^{b-2} - \frac{(b-2)(b+2i-1)!}{(i+1)!(b-2)!} x^2(1-x)^{b-3}. \end{aligned}$$

We do steps (b) and (c) simultaneously due to Proposition 10. This gives

$$\begin{aligned}
&\int_0^1 h_i(x_i)\ dx_i \\
&= \frac{2(i+1)(b+2i-1)!}{(i+1)!(b-2)!}\frac{(b-2+i)!}{i!(b+i)!} - \frac{(b-2)(b+2i-1)!}{(i+1)!(b-2)!}\frac{2(b-3+i)!}{i!(b+i)!} \\
&= \frac{2(b+2i-1)!(b+i-3)![(i+1)(b-2+i)-(b-2)]}{i!(i+1)!(b+i)!(b-2)!} \\
&= \frac{2i(b+2i-1)!(b+i-3)!(b+i-1)}{i!(i+1)!(b+i)!(b-2)!} \\
&= \frac{2(b+2i-1)!}{(i+1)!(i-1)!(b-2)!(b+i)(b+i-2)} \square
\end{aligned}$$

We now prove a couple identities that will be essential for proving the main result (Theorem 1) with some useful tableaux identities—one involving skyt, the other involving $\overline{\text{skyt}}$—whose proofs will be similar. We first discuss the identity pertaining to skyt.

Lemma 6. *If $i \geq 1$, then*

$$\begin{aligned}
0 &= (-1)^{d-i}\binom{m+d}{d-i} \\
&+ \sum_{j=0}^{d-1}\sum_{k=0}^{i}(-1)^{j-i+k}\binom{j}{j-i+k}\binom{m+d}{j}\text{skyt}(m+1,k,d-j-2k+1).
\end{aligned}$$

In proving this Lemma, it will be first useful to have the following result, which is in a sense the dual to Lemma 1.

Lemma 7.

$$\begin{aligned}
&\#SYT(m+1,k,d-2k-p-1) \\
&= \sum_{j=0}^{d-2k-1}(-1)^{d-1+j}\binom{m+d-p}{j-p}\text{skyt}(m+1,k,d-j-2k+1).
\end{aligned}$$

Proof. The proof is a similar inclusion-exclusion proof to what was provided in Lemma 1. Starting with the term for $j = d - 2k - 1$, consider choosing a pair of tableau. The first tableaux is a row with $d-2k-p-1$ squares, with entries selected from $[m+d]$. We call this tableau μ. To get the second tableau, which we call λ choose an element of $\text{Skyt}(m+1, k, 2)$, using the numbers in $[m+d]$ not in the entries of μ.

Our goal now is to attach the left block of μ (whose entry is denoted i) to the right of the top right block of λ (whose entry is denoted j) in order to build an element of $SYT(m+1, k, d-2k-p-1)$. This only works, of course, if $j < i$. The cases where $j > i$ are in bijection with picking a pair of tableau similar to the ones selected before, but now with a row with $d - 2k - p - 2$ entries and an element of $\text{Skyt}(m+1, k, 3)$. But here, there will be a scenario where the top right of the element of $\text{skyt}(m+1, k, 4)$ will be smaller than the left of the row with $d - 2k - p - 2$ entries, but these are counted with a pair of an element from $\text{Skyt}(m+1, k, 3)$ and a row with $d - 2k - p - 3$ entries. Continuing this alternating sum gives the desired result. □

Proof of Lemma 6. First, we note that

$$\binom{j}{j-i+k}\binom{m+d}{j} = \binom{m+d-i+k}{j-i+k}\binom{m+d}{i-k}.$$

Turning our attention to the double sum in the statement of Lemma 6, we may use the above identity to push the summand indexed by j past one of the binomial coefficients. The right side of Lemma 6 becomes

$$(-1)^{d-i}\binom{m+d}{d-i}+$$
$$\sum_{k=0}^{i}(-1)^{k-i}\binom{m+d}{i-k}\sum_{j=0}^{d-1}(-1)^{j}\binom{m+d-i+k}{j-i+k}\,\mathrm{skyt}(m+1,k,d-j-2k+1).$$

Note that if $k>0$, terms for $j>d-2k-1$ are zero as the last input to skyt will be less than 2, so taking $p=i-k$ in Lemma 7 gives that the sum over j is equal to $(-1)^{d-1}\#SYT(m+1,k,d-k-i-1)$. So we have now reduced the right side of Lemma 6 to

$$(-1)^{d-i}\binom{m+d}{d-i}+(-1)^{d-i-1}\sum_{k=0}^{i}(-1)^{k}\binom{m+d}{i-k}\#SYT(m+1,k,d-k-i-1).$$

Utilizing the hook-length formula, one can verify that

$$\begin{aligned}&\binom{m+d}{i-k}\#SYT(m+1,k,d-k-i-1)\\&=\frac{(m+d)!}{(m+d-i)(m-1)!(d-i)!i!}\,\frac{d-i-k}{m+k}\binom{i}{k}.\end{aligned}$$

Hence, the desired result follows when we show

$$\begin{aligned}&\binom{m+d}{d-i}\frac{(m+d-i)(m-1)!(d-i)!i!}{(m+d)!}\\&=\sum_{k=0}^{i}(-1)^{k}\binom{i}{k}\frac{d-i-k}{m+k}.\end{aligned}\tag{3.3}$$

Consider the function

$$f(x,y)=\sum_{k=0}^{i}\binom{i}{k}x^{m+k-1}y^{d-i-k}.$$

On the one hand, taking the derivative of $f(x, y)$ with respect to y, evaluating at $y = 1$, then integrating with respect to x with lower limit 0 and upper limit -1, we recover the ride side of equation (3.3). On the other hand, we can find a closed form for $f(x, y)$ first:

$$\begin{aligned} f(x,y) &= \sum_{k=0}^{i} (-1)^k \binom{i}{k} x^{m+k-1} y^{d-i-k} \\ &= x^{m-1} y^{d-i} \sum_{k=0}^{i} \binom{i}{k} \left(\frac{x}{y}\right)^k \\ &= x^{m-1} y^{d-i} \left(1 + \frac{x}{y}\right)^i \end{aligned}$$

Using this explicit version of $f(x, y)$, define $g(x)$ by

$$g(x) := \frac{d}{dy} f(x,y)|_{y=1} = (d-i)x^{m-1}(1+x)^i - x^m i (1+x)^{i-1}.$$

Hence, $\int_0^{-1} g(x)\ dx$ should yield the desired result a closed form for the right side of equation (3.3). So long as $i \geq 1$, we can apply Proposition 5 to get

$$\begin{aligned} &\frac{(d-i)(-1)^m i!}{m(m+1)\cdots(m+i)} - \frac{(-1)^{m+1} i!}{(m+1)(m+2)\cdots(m+i)} \\ &= \frac{(-1)^m i!(m-1)!(d-i+m)}{(m+i)!}. \end{aligned}$$

One can verify this is the left side of equation (3.3). □

We have an essentially equivalent identity for $\overline{SkYT}$.

Lemma 8. *Suppose $i \geq 1$. Then*

$$(-1)^{d-i-1} i \binom{d}{i+1} + \sum_{j=0}^{d-1} \sum_{k=1}^{i} (-1)^{j-i+k} \binom{j}{j-i+k} \binom{d}{j} \overline{\text{skyt}}(k, d-j-2k+1) = \begin{cases} 0 & i > 1 \\ (-1)^{d-2} & i = 1 \end{cases}.$$

The proof for this is very much similar to Lemma 6, but especially due to the dependence on the value of i, it is worth at least outlining aspects of the proof.

Here is the corresponding version of Lemma 7, which has the same proof of Lemma 7.

Lemma 9.

$$\begin{aligned} &\#SYT(2, k, d-2k-p-1) \\ &= \sum_{j=0}^{d-2k-1} (-1)^{d-1+j} \binom{m+d-p}{j-p} \overline{\text{skyt}}(k, d-j-2k+1). \end{aligned}$$

Proof of Lemma 8. The start of this proof works similarly to the proof of Lemma 6. First, note that

$$\binom{j}{j-i+k} \binom{d}{j} = \binom{d-i+k}{j-i+k} \binom{d}{i-k},$$

and so we rewrite the right side of Lemma 8 as

$$(-1)^{d-i-1} i \binom{d}{i+1}$$
$$+ \sum_{k=1}^{i} (-1)^{k-i} \binom{d}{i-k} \sum_{j=0}^{d-1} (-1)^j \binom{d-i+k}{j-i+k} \overline{\text{skyt}}(k, d-j-2k+1).$$

The inner sum equals $(-1)^{d-1}\#SYT(2,k,d-k-i-1)$ by Lemma 9. Hence the right hand side of Lemma 8 becomes

$$(-1)^{d-i-1} i \binom{d}{i+1} + (-1)^{d-i-1} \sum_{k=1}^{i} (-1)^k \binom{d}{i-k} \#SYT(2,k,d-k-i-1).$$

It is worth noting this can not be recovered from the proof of Lemma 6, and so at this point, the proof of this Lemma diverges slightly though we employ similar strategies.

We again apply the hook-length formula to get

$$\binom{d}{i-k} \#SYT(2,k,d-k-i-1) = \frac{d!}{i!(d-i+1)!} \frac{(d-i+k+1)(d-k-i)}{k+1} \binom{i}{k}$$

and hence proving the Lemma for the case where $i > 1$ is equivalent to showing

$$i \binom{d}{i+1} \frac{i!(d-i+1)!}{d!} = \sum_{k=1}^{i} (-1)^{k+1} \frac{(d-i+k+1)(d-k-i)}{k+1} \binom{i}{k}. \quad (3.4)$$

This time, we define a function $f(x,y,z)$ so that

$$f(x,y,z) = \sum_{k=0}^{i} \binom{i}{k} x^{d-i+k+1} y^{d-k-i} z^k.$$

When we differentiate f in x and y and evaluate both at 1, then integrate with respect to z from 0 to -1, and remove the term corresponding to $k = 0$, we recover the right side of equation (3.4). On the other hand, we can find f explicitly:

$$\begin{aligned} f(x,y,z) &= \sum_{k=0}^{i} \binom{i}{k} x^{d-i+k+1} y^{d-k-i} z^k \\ &= x^{d-i+1} y^{d-i} \sum_{k=0}^{i} \binom{i}{k} \left(\frac{xz}{y}\right)^k \\ &= x^{d-i+1} y^{d-i} \left(1 + \frac{xz}{y}\right)^i . \end{aligned}$$

Define $g(z)$ so that

$$\begin{aligned} g(z) :&= \frac{d}{dx}\left(\frac{d}{dy} f(x,y,z)|_{y=1}\right)|_{x=1} \\ &= (d-i+1)(d-i)(1+z)^i - 2i(1+z)^{i-1}z - i(i-1)(1+z)^{i-2}z^2. \end{aligned}$$

(Note here that it is important we are in the case where $i > 1$ due to the exponent on the last term of $g(z)$.) We can directly apply Proposition 5 to get that

$$\begin{aligned} \int_0^{-1} g(z)\ dz &= -(d-i+1)(d-i)\frac{i!}{(i+1)!} - 2i\frac{(i-1)!}{(i+1)!} + i(i-1)\frac{2(i-2)!}{(i+1)!} \\ &= -\frac{i!(d-i+1)(d-i)}{(i+1)!}. \end{aligned}$$

This will not be the right side of equation (3.4), because we have to remove the $k = 0$ term appearing in the sum in equation (3.4) first. This term is

$-(d-i+1)(d-i)$, so we get

$$\frac{i!(d-i+1)(d-i)i}{(i+1)!},$$

which one can verify is the left of equation (3.4).

The case for $i=1$ can be simplified from the $i>1$ case since

$$\begin{aligned}&(-1)^{d-i-1}\sum_{k=1}^{i}(-1)^k\binom{d}{i-k}\#SYT(2,k,d-k-i-1)\\&=(-1)^{d-1}\#SYT(2,1,d-3).\end{aligned}$$

Hence, when $i=1$, the left side of the equality in the statement of Lemma 8 becomes

$$\begin{aligned}&(-1)^{d-2}\binom{d}{2}+(-1)^{d-1}\#SYT(2,1,d-3)\\&=(-1)^{d-1}\left(\frac{(d+1)(d-2)-d(d-1)}{2}\right)\\&=(-1)^{d-2}.\end{aligned}$$

□

3.2 Symmetry

This combinatorial realization does more than provide a manifestly positive and integral interpretation for these coefficients. In [15], Gedeon, Proudfoot, and Young define a new polynomial called the equivariant KL polynomial for the uniform matroid, and use it to observe a surprising symmetry in the coefficients of the equivariant KL polynomial for uniform matroid. Let $C_{m,d}^{i}$ be the ith coefficient of the equivariant KL polynomial for the uniform matroid of rank d on $m+d$ elements. The authors of [15] showed that $C^i_{m,d} = C^i_{d-2i,m+2i}$, remarking that they see “no philosophical reason why this symmetry should exist” [15, Remark 3.5]. They are able to use $C^i_{m,d}$ to recover $c^i_{m,d}$, and so the same symmetry is true for the latter. We recover this symmetry by observing symmetry in our skew symmetric tableaux.

Lemma 10.

$$\mathrm{skyt}(a,i,b) = \mathrm{skyt}(b,i,a)$$

Proof. Given $\alpha \in \mathrm{Skyt}(a,i,b)$, define $\bar{\alpha} \in \mathrm{Skyt}(b,i,a)$ in the following way. Let n be the maximum value for the entries of the elements of $\mathrm{Skyt}(a,i,b)$, and hence also $\mathrm{Skyt}(b,i,a)$. Replace each number i in α with $n+1-i$, and rotate the shape 180 degrees, so that the shape corresponds to the elements of $\mathrm{Skyt}(b,i,a)$. This map is necessarily an involution.

This process is also well defined. Let x and y be two positions in α containing entries $i, j \in [n]$ respectively. Suppose x and y are positioned so that the entry in x is required to be smaller than the entry in y. This is to say that x is to the right or above y (or both). This also gives us that $i < j$. Our above map replaces the entries of x and y with $n+1-i$ and $n+1-j$, and then rotates α giving us $\bar{\alpha}$. When we do this, if x was above y, it is now below, and likewise with being to the right versus left. Regardless, there relative locations now require the value of y to be less than x, which is indeed true since $i < j$, giving this map is indeed well-defined. The figure below gives an example of this map. □

In light of this, we have the following corollary to Theorem 1.

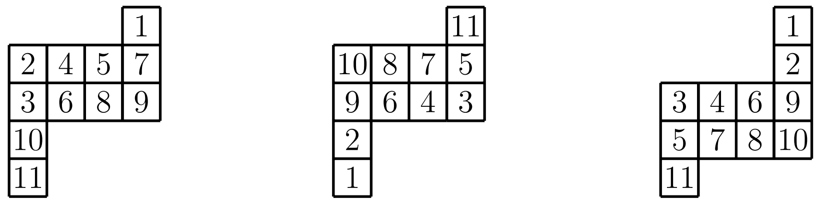


Figure 3.3: The left most tableau is an element of Skyt$(4,3,3)$, the middle tableau replaces each entry i of the left with $11+1-i$, and then rotating gives us the tableau on the right, an element of Skyt$(3,3,4)$.

Corollary 2. *Let $c^i_{m,d}$ be the ith coefficient of the Kazhdan-Luzstig polynomial of the uniform matroid of rank d on $m+d$ elements. Then*

$$c^i_{m,d} = c^i_{d-2i,m+2i}.$$

Chapter 4

The Kazhdan-Lusztig Polynomials for Sparse Paving Matroids

This chapter is dedicated to justifying the combinatorial formula given in Theorem 1. We restate this part here for convenience, as its own Theorem.

Theorem 2. *Let $c^i_{m,d}(\mathcal{CH})$ be the i-th coefficient for the Kazhdan-Lusztig polynomial for the sparse paving matroid $S_{m,d}(\mathcal{CH})$. Then*

$$c^i_{m,d}(\mathcal{CH}) = \operatorname{skyt}(m+1, i, d-2i+1) - |\mathcal{CH}| \cdot \overline{\operatorname{skyt}}(i, d-2i+1).$$

Remark 4. *For some values of m, d, and i, we need to use our conventions set in place for $\operatorname{skyt}(a, i, b)$ and $\overline{\operatorname{skyt}}(a, i, b)$ in chapter 3 for our formula to truly work.*

- *[9, Proposition 2.11] shows that the degree 0 term always has coefficient 1. That is, when $i = 0$, our formula must always return 1.*
- *When $d = 0$ we are forced to have $P_{S_{m,d}(\mathcal{CH})}(t) = 1$.*
- *When $0 < d < 3$, the degree requirement on Kazhdan-Lusztig polynomials forces $P_{S_{m,d}(\mathcal{CH})}(t)$ to have degree 0. Namely, in this case, we have $P_{S_{m,d}(\mathcal{CH})}(t) = 1$, again by [9, Proposition 2.11].*

- *When $m = 0$, note that $\mathcal{CH}$ is forced to be empty and $S_{m,d}(\mathcal{CH})$ becomes $U_{0,d}$. It is shown in [9, Proposition 2.7] that $P_{M_1 \oplus M_2}(t) = P_{M_1}(t)P_{M_2}(t)$ for matroids M_1 and M_2. With this, one can verify that $P_{U_{0,d}}(t) = 1$ by seeing that $P_{U_{0,1}}(t) = 1$ based on the $d < 3$ discussion above.*

In all cases, our conventions guarantee we get the right values. Besides these cases, our conventions are not needed for our formula, and we are guaranteed that $S_{m,d}(\mathcal{CH})$ has more interesting structure than that of the boolean lattice.

The following technical result will be crucial in demonstrating why the formula given in Theorem 2 only depends on $|\mathcal{CH}|$, and not the relationship between the elements of $\mathcal{CH}$.

Lemma 11. *Let $c, i \in \mathbb{N} \cup \{0\}$. For $I \subseteq [c]$, let x_I be a variable. Let $g(k)$ and $h(k)$ are functions varying in k. Then*

$$-\sum_{\substack{J \subseteq [c] \\ |J| \geq 2}} (-1)^{|J|} x_J \sum_{k=0}^{i} g(k) = \sum_{\emptyset \subsetneq I \subseteq [c]} \sum_{\substack{I \subseteq J \subseteq [c] \\ |J| \geq 2}} (-1)^{|J|-|I|} x_J \sum_{k=0}^{i} (\ g(k) - |I| h(k)\ ),$$

Proof. We show that the term with x_J on both sides of the statement of the lemma is the same for every $J \subseteq [c]$, where $|J| \geq 2$. We start with the coefficient of x_J on the right side. We note that the terms with x_J appear for each I that is contained in J, where $|I| \geq 1$. Hence, the term with x_J on the right hand side of the statement of the Lemma is

$$\begin{aligned}
&\sum_{\ell=1}^{|J|}(-1)^{|J|-\ell}x_J\binom{|J|}{\ell}\sum_{k=0}^{i}(\ g(k)-\ell h(k)\ )\\
&=x_J(-1)^{|J|}\sum_{\ell=1}^{|J|}\binom{|J|}{\ell}(-1)^{\ell}\sum_{k=0}^{i}(\ g(k)-\ell h(k)\ )\\
&=x_J(-1)^{|J|}\left(\sum_{k=0}^{i}g(k)\sum_{\ell=1}^{|J|}(-1)^{\ell}\binom{|J|}{\ell}-\sum_{k=0}^{i}h(k)\sum_{\ell=1}^{|J|}(-1)^{\ell}\ell\binom{|J|}{\ell}\right)\\
&=x_J(-1)^{|J|}\left(-\sum_{k=0}^{i}g(k)\right),
\end{aligned}$$

since we know in general we have the identities $\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{\ell}=0$ for $n\geq 1$ and $\sum_{\ell=0}^{n}(-1)^{\ell}\binom{n}{\ell}\ell=0$ for $n\geq 2$. Note that the there is exactly one time where x_J appears exactly once, and the corresponding term is $-x_J(-1)^{|J|}\sum_{k=0}^{i}g(k)$. □

We now prove the desired formula for $c^i_{m,d}(\mathcal{CH})$.

Proof of Theorem 2. Let $M := S_{m,d}(\mathcal{CH})$, and set $c := |\mathcal{CH}|$. Recall that the definition for the Kazhdan-Lusztig polynomial is that it satisfies the following recurrence,

$$t^{rkM}P_M(t^{-1})=\sum_{F\text{ a flat}}\chi_{M^F}(t)P_{M_F}(t),$$

which may be rewritten as

$$t^{rkM}P_M(t^{-1})-P_M(t)=\sum_{F\text{ a non-empty flat}}\chi_{M^F}(t)P_{M_F}(t).$$

Recall that $\deg P(t)<\frac{1}{2}d$, and so the power of each monomial in $t^dP_M(t^{-1})$

is strictly larger than $\frac{1}{2}d$. Hence, our goal is to show that for $0 \leq i < \frac{1}{2}d$ we have

$$\begin{aligned}
&-\operatorname{skyt}(m+1, i, d-2i+1) + c \cdot \overline{\operatorname{skyt}}(i, d-2i+1) \\
&\qquad = [t^i] \sum_{F \text{ a non-empty flat}} \chi_{M^F}(t) P_{M_F}(t).
\end{aligned} \tag{4.1}$$

Using our work from Proposition 4, and consolidating common factors involving the various flats in $\mathcal{CH}$, we can rewrite the right of equation (4.1) to be

$$\begin{aligned}
&[t^i]\chi_{S_{m,d}(\mathcal{CH})} + c[t^i]\chi_{U_{1,d-1}} P_{U_{m-1,1}} \\
&+ \sum_{\substack{\emptyset \subsetneq F \subsetneq C \\ \text{For some } C \in \mathcal{CH}}} [t^i]\chi_{U_{0,|F|}} P_{S_{m,d-|F|}(\mathcal{CH}(F))} + \sum_{\substack{\emptyset \subsetneq F \subsetneq [m+d] \\ F \not\subseteq C \ \forall C \in \mathcal{CH}}} [t^i]\chi_{U_{0,|F|}} P_{U_{m,d-|F|}},
\end{aligned} \tag{4.2}$$

where the first term corresponds to the case where $F = [m+d]$, and the second where $F \in \mathcal{CH}$.

By Corollary 1, we are required to break this up into three case: $i = 0$, $i = 1$, and $2 \leq i < d/2$ if we are to write this out explicitly. Note we can write everything explicitly except $P_{S_{m,d-|F|}(\mathcal{CH}(F))}$. Hence, we proceed by induction on the matroid rank d, noting that $d > d - |F|$ since for the corresponding summand F is never empty.

We now define some notation in order to rewrite the summations appearing in (4.2). Let $I \subseteq [c]$ and $C_i \in \mathcal{CH}$. We define $C_I := \bigcap_{i \in I} C_i$ and denote $c_I := |C_I|$. By convention, $C_\emptyset = [m+d]$. Recall that $\mathcal{CH}(F) := \{C \setminus F : C \in \mathcal{CH} \text{ such that } F \subseteq C\}$. Let j be an integer and define the following sum indexed by J:

$$\Phi_j(I) := \sum_{I \subseteq J \subseteq [c]} (-1)^{|J|-|I|} \binom{c_J}{j}.$$

If j is selected appropriately, $\Phi_j(I)$ counts the number of flats of rank j contained in C_I, but not in any C_J so that $C_J \subseteq C_I$. Hence, F is a flat counted by $\Phi_j(I)$ if and only if $\mathcal{CH}(F) = \{C_i \setminus F : i \in I\}$. What we will leverage from this is that $|\mathcal{CH}(F)| = |I|$.

We can now rewrite equation (4.2). We use the Kronecker delta function $\delta(i,j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$ to combine the cases for $i = 1$ and $2 \leq i < d/2$.

$i = 0:$

$$(-1)^d \binom{m+d-1}{d-1} - c(-1)^d + c(-1)^{d-1}\binom{d-1}{d-2}$$
$$+ \sum_{j=1}^{d-2} \sum_{\emptyset \subsetneq I \subseteq [c]} \Phi_j(I)(-1)^j(\operatorname{skyt}(m+1, 0, d-j+1) - |I| \cdot \overline{\operatorname{skyt}}(0, d-j+1))$$
$$+ \sum_{j=1}^{d-1} \Phi_j(\emptyset)(-1)^j \operatorname{skyt}(m+1, 0, d-j+1)$$

$i > 0:$

$$(-1)^{d-i}\binom{m+d-1}{d-i} - c(-1)^{d-1}\delta(i,1) + c(-1)^{d-1-i}\binom{d}{d-1-i}$$
$$+ \sum_{j=1}^{d-2} \sum_{\emptyset \subsetneq I \subseteq [c]} \Phi_j(I) \sum_{k=0}^{i} (-1)^{j-i+k}\binom{j}{j-i+k} \operatorname{skyt}(m+1, k, d-j-2k+1)$$
$$- \sum_{j=1}^{d-2} \sum_{\emptyset \subsetneq I \subseteq [c]} \Phi_j(I) \sum_{k=0}^{i} (-1)^{j-i+k}\binom{j}{j-i+k} |I|\, \overline{\operatorname{skyt}}(k, d-j-2k+1)$$
$$+ \sum_{j=1}^{d-1} \Phi_j(\emptyset) \sum_{k=0}^{i} (-1)^{j-i+k}\binom{j}{j-i+k} \operatorname{skyt}(m+1, k, d-j-2k+1)$$

In both cases, the sum running from $j = 1$ to $j = d - 2$ is the summand in equation (4.2) over $\emptyset \subsetneq F \subsetneq C$ for $C \in \mathcal{CH}$, since the flats contained in C have size at most $d - 2$. The other sum running from $j = 1$ to $j = d - 1$ corresponds to the summand in equation (4.2) over $\emptyset \subsetneq F \subsetneq [m+d]$ such that $F \nsubseteq C$ for all $C \in \mathcal{CH}$.

To simplify things further, first, note that

$$\Phi_{d-1}(\emptyset) = \binom{m+d}{d-1} - c\binom{d}{d-1}.$$

By construction, $\Phi_{d-1}(\emptyset)$ counts the rank $d - 1$ flats contained in no element of $\mathcal{CH}$. Recall that the only rank $d - 1$ flats are those not contained in any circuit-hyperplane.

Next, note that many terms from the two sums running over j in both the $i = 0$ and $i > 0$ case will cancel as a result of Lemma 11. Fix $j \leq d - 2$ and suppose $J \subseteq [c]$. Set

- $x_J := \binom{c_J}{j}$,
- $g(k) := (-1)^{j-i+k}\binom{j}{j-i+k}\,\text{skyt}(m+1, k, d-j-2k+1)$, and
- $h(k) := (-1)^{j-i+k}\binom{j}{j-i+k}\,\overline{\text{skyt}}(k, d-j-2k+1)$.

This allows us to rewrite our two cases in the following way.

$i = 0$:

$$(-1)^d\binom{m+d-1}{d-1} - c(-1)^d + c(-1)^{d-1}\binom{d-1}{d-2}$$
$$+\sum_{j=1}^{d-2}\sum_{\emptyset\subsetneq I\subseteq[c]}\sum_{I\subseteq J\subseteq[c]}(-1)^{|J|-|I|}x_J(\ g(0) - |I|h(0)\ )$$
$$+\sum_{j=1}^{d-1}\sum_{\emptyset\subseteq J\subseteq[c]}(-1)^{|J|}x_J g(0)$$

$i > 0:$

$$
\begin{aligned}
&(-1)^{d-i}\binom{m+d}{d-i} - c(-1)^{d-1}\delta(i,1) + c(-1)^{d-1-i}\binom{d}{d-1-i} \\
&+\sum_{j=1}^{d-2}\sum_{\emptyset\subsetneq I\subseteq[c]}\sum_{I\subseteq J\subseteq[c]}(-1)^{|J|-|I|}x_J\sum_{k=0}^{i} g(k)-|I|h(k) \\
&+\sum_{j=1}^{d-1}\sum_{\emptyset\subseteq J\subseteq[c]}(-1)^{|J|}x_J\sum_{k=0}^{i} g(k)
\end{aligned}
$$

The following argument works for both the $i = 0$ and $i > 0$ case, so we speak of both simultaneously as if they were one. Let A correspond to the sum indexed by j where j is at most $d-2$. Likewise define B to be the sum indexed by j where j is at most $d-1$. By Lemma 11, the terms where $|J| \geq 2$ in A will cancel all terms where $|J| \geq 2$ in B. What remains in A are the terms where $|J| = 1$, that is, the terms where $J = I$ and $|I| = 1$. There are c such terms, each contributing $\binom{d}{j}$, as the members of $\mathcal{CH}$ have cardinality d. For B, when $j \leq d-2$, the only terms that remain are those where $|J|$ equals 0 or 1. This gives $c+1$ terms: one contributing $\binom{m+d}{j}$, and c terms contributing $-\binom{d}{j}$. Combining this with our identity for $\Phi_{d-1}(\emptyset)$ given above, we get the following simplification.

$i = 0:$

$$
\begin{aligned}
&(-1)^{d}\binom{m+d-1}{d-1} - c(-1)^{d} + c(-1)^{d-1}\binom{d-1}{d-2} \\
&+c\sum_{j=1}^{d-2}\binom{d}{j}(-1)^{j}(\mathrm{skyt}(m+1,0,d-j+1)-\overline{\mathrm{skyt}}(0,d-j+1)) \\
&+\sum_{j=1}^{d-1}\left(\binom{m+d}{j}-c\binom{d}{j}\right)(-1)^{j}\,\mathrm{skyt}(m+1,0,d-j+1)
\end{aligned}
$$

$i > 0:$

$$
\begin{aligned}
&(-1)^{d-i}\binom{m+d}{d-i} - c(-1)^{d-1}\delta(i,1) + c(-1)^{d-1-i}\binom{d}{d-1-i} \\
&+ c\sum_{j=1}^{d-2}\binom{d}{j}\sum_{k=0}^{i}(-1)^{j-i+k}\binom{j}{j-i+k}\operatorname{skyt}(m+1,k,d-j-2k+1) \\
&- c\sum_{j=1}^{d-2}\binom{d}{j}\sum_{k=0}^{i}(-1)^{j-i+k}\binom{j}{j-i+k}\overline{\operatorname{skyt}}(k,d-j-2k+1) \\
&+ \sum_{j=1}^{d-1}\binom{m+d}{j}\sum_{k=0}^{i}(-1)^{j-i+k}\binom{j}{j-i+k}\operatorname{skyt}(m+1,k,d-j-2k+1) \\
&- \sum_{j=1}^{d-1}c\binom{d}{j}\sum_{k=0}^{i}(-1)^{j-i+k}\binom{j}{j-i+k}\operatorname{skyt}(m+1,k,d-j-2k+1).
\end{aligned}
$$

For the case for $i = 0$, note that our conventions simplify the formula to be

$$
\begin{aligned}
&(-1)^{d}\binom{m+d-1}{d-1} - c(-1)^{d} + c(-1)^{d-1}\binom{d-1}{d-2} \\
&+ c\sum_{j=1}^{d-2}\binom{d}{j}(-1)^{j} + \sum_{j=1}^{d-1}\left(\binom{m+d}{j} - c\binom{d}{j}\right)(-1)^{j} \\
&= (-1)^{d}\binom{m+d-1}{d-1} - c(-1)^{d} + c(-1)^{d-1}\binom{d-1}{d-2} \\
&- c(-1)^{d-1}\binom{d}{d-1} + \sum_{j=1}^{d-1}\binom{m+d}{j}(-1)^{j} \\
&= (-1)^{d}\binom{m+d-1}{d-1} - 1 + \sum_{j=0}^{d-1}\binom{m+d}{j}(-1)^{j}.
\end{aligned}
$$

We have seen these terms before—as a porism to Proposition 5, this prior sum simplifies to be -1 (see the last lines of the proof to the Proposition).

We expected this—the constant term of the KL polynomial is always 1 [9, Proposition 2.11]. Moreover, by the conventions we have taken, note that

$$-\operatorname{skyt}(m+1,0,d+1)+c\cdot\overline{\operatorname{skyt}}(m+1,0,d+1)=-1,$$

as desired.

For the case where $i>0$, we have done all the hard work previously—now what is left is to simply rewrite the equation to be able to utilize the identities. We first combine the parts with the $\binom{d}{j}$ and skyt to rewrite it as

$$\begin{aligned}
&(-1)^{d-i}\binom{m+d}{d-i}-c(-1)^{d-1}\delta(i,1)+c(-1)^{d-1-i}\binom{d}{d-1-i}\\
&-c\sum_{j=1}^{d-2}\binom{d}{j}\sum_{k=0}^{i}(-1)^{j-i+k}\binom{j}{j-i+k}\overline{\operatorname{skyt}}(k,d-j-2k+1)\\
&+\sum_{j=1}^{d-1}\binom{m+d}{j}\sum_{k=0}^{i}(-1)^{j-i+k}\binom{j}{j-i+k}\operatorname{skyt}(m+1,k,d-j-2k+1)\\
&-c\binom{d}{d-1}\sum_{k=0}^{i}(-1)^{d-1-i+k}\binom{d-1}{d-1-i+k}\operatorname{skyt}(m+1,k,2-2k)
\end{aligned}$$

For this last term, observe that $\operatorname{skyt}(m+1,k,2-2k)$ takes on two values: 1 if $k=0$, and 0 otherwise (since $k>1$ implies $2-2k<2$). The same casework allows us to extend the index for the summand involving $\overline{\operatorname{skyt}}$ to $d-1$, since in either of the cases $\overline{\operatorname{skyt}}(k,2-2k)=0$. Note further that if we extend our sums over j so that they start at $j=0$, since $\binom{0}{0-i+k}=\delta(i,k)$, the extra terms we get are $\operatorname{skyt}(m+1,i,d-2i+1)$ and $-c\cdot\overline{\operatorname{skyt}}(m+1,i,d-2i+1)$. But recall that our goal is to show the above sum equals $-\operatorname{skyt}(m+1,i,d-2i+1)+c\cdot\overline{\operatorname{skyt}}(i,d-2i+1)$, so effectively all we have done by allowing j to start at 0 was change the problem to show the following is true:

$$\begin{aligned}0 =&(-1)^{d-i}\binom{m+d}{d-i} - c(-1)^{d-1}\delta(i,1)\\&+ c(-1)^{d-1-i}\binom{d}{d-1-i} - c(-1)^{d-1-i}\binom{d}{d-1}\binom{d-1}{d-1-i}\\&- c\sum_{j=0}^{d-1}\binom{d}{j}\sum_{k=0}^{i}(-1)^{j-i+k}\binom{j}{j-i+k}\overline{\text{skyt}}(k, d-j-2k+1)\\&+ \sum_{j=0}^{d-1}\binom{m+d}{j}\sum_{k=0}^{i}(-1)^{j-i+k}\binom{j}{j-i+k}\text{skyt}(m+1, k, d-j-2k+1).\end{aligned}$$

Note that $\binom{d}{d-1-i} - \binom{d}{d-1}\binom{d-1}{d-1-i} = -i\binom{d}{i+1}$ and $\overline{\text{skyt}}(k, d-j-2k+1) = 0$ when $k = 0$, and so this equality, and hence the theorem, is true by the identities from Lemmas 6 and 8 found in Appendix A. □

The formula suggests that having fewer bases may lead to weakly smaller coefficients, but this is not generally true. Let $\mathcal{B}(M)$ denote the set of bases of a matroid M. Define matroids M and N so that

$$\mathcal{B}(M) = \binom{[9]}{5} \setminus \binom{[6]}{5}$$

and

$$\mathcal{B}(N) = \binom{[9]}{5} \setminus \{[4] \cup \{x\} : x \in \{5, 6, 7, 8, 9\}\}.$$

Observe that $|\mathcal{B}(M)| = 120 < 121 = |\mathcal{B}(N)|$ and yet $P_M(t) = 99t^2 + 103t + 1$ and $P_N(t) = 106t^2 + 101t + 1$.

Rather than comparing the number of bases two matroids have, we believe the right thing to do is compare the sets of bases themselves. The following conjecture captures this idea. It is supported directly by Theorem 1 and by various computer computations.

Conjecture 2. *Let M and N be matroids of equal rank such that the ground set of M is contained in the ground set of N. Let $c_i(M)$ and $c_i(N)$ be the ith coefficient of their Kazhdan-Lusztig polynomial respectively. If $\mathcal{B}(M) \subseteq \mathcal{B}(N)$, then $c_i(M) \leq c_i(N)$.*

Note that the prior example is not a counter example as neither $\mathcal{B}(M) \subseteq \mathcal{B}(N)$ nor $\mathcal{B}(N) \subseteq \mathcal{B}(M)$ is true. This conjecture suggests that one can alternatively prove the positivity for all Kazhdan-Lusztig polynomials, by proving it for matroids whose collection of bases is minimal, that is, no subcollection of its set of bases defines a matroid.

Chapter 5

Non-Negativity for Sparse Paving Matroids

With the formula for Theorem 1 proven, we now move to showing that this formula is always non-negative. First, we discuss the special case where $\mathcal{CH}$ is a disjoint family, in which we can reinterpret the coefficients in a manifestly positive way. First, define a distinguished subset of $\mathrm{Skyt}(m+1, i, d-2i+1)$, which we denote $\mathrm{Skyt}_\rho(m+1, i, d-2i+1)$. Every $\alpha \in \mathrm{Skyt}_\rho(m+1, i, d-2i+1)$ must satisfy at least one of the following conditions.

- the top entry of the right-most column of α is 1; or
- the bottom entry of the right-most column is greater than $d+\rho$; or
- the third entry (from the top) of the left-most column is less than $d+1$.

We then have the following proposition.

Proposition 7.

$$\mathrm{skyt}_\rho(m+1, i, d-2i+1) = \mathrm{skyt}(m+1, i, d-2i+1) - \rho \cdot \overline{\mathrm{skyt}}(m+1, i, d-2i+1).$$

To prove this, we will need the following.

Lemma 12. *Let $m \geq 1$, and $A := \{0, 1, \ldots, m-1\}$. Then there exists an inclusion*

$$\iota : A \times \mathrm{Skyt}(2, i, b) \hookrightarrow \mathrm{Skyt}(m+1, i, b).$$

Proof. Let $\alpha \in \mathrm{Skyt}(2, i, b)$. There is a natural way of viewing α as an element of $\mathrm{Skyt}(m+1, i, b)$—attach to α a column of $m-1$ squares, placing in them the largest possible numbers (in increasing order) of the entries appearing in an element of $\mathrm{Skyt}(m+1, i, b)$. For future reference, we refer to the $1 \times (m-1)$ column as μ, and refer to this described image of α as $\bar{\alpha}$. Note the following facts:

- The entry in the bottom right corner of α is the largest number in the tableau. This number is $n := 2 + 2(i-1) + b = 2i + b$.
- n is smaller than every entry in μ, and every entry in μ is larger than every element in α. The elements of μ are $\{n+1, n+2, \ldots, n+m-1\}$.

We define an action on the locations of the numbers in $\bar{\alpha}$ by the elements of A, which we denote $i \cdot \bar{\alpha}$ for $i \in A$. The element $i \cdot \bar{\alpha} \in \mathrm{Skyt}(m+1, i, b)$ is defined by starting with $\bar{\alpha}$, removing $n+i$ from μ and placing it where n is, shifting all entries of μ down, and then placing n at the top of μ. The action is well-defined by the itemized facts above.

Hence, we may define the map $\iota : (i, \alpha) \mapsto i \cdot \bar{\alpha}$. To see why this map is an inclusion, simply note that any two distinct $\alpha, \beta \in \mathrm{Skyt}(2, i, b)$ *must* disagree in a location other than the bottom right corner, as both are required to have n there. This position will never change value by ι. Then it is immediate that ι sends (i, α) and (j, β) to different elements since the outputs of both will still disagree in the position that α and β did. □

Proof of Proposition 7. Let $\alpha \in \overline{\text{skyt}}(m+1, i, d-2i+1)$. Hence, in particular, 1 is at the top of the left column and the largest possible elements are in the left tail (by tail, we mean the the entries starting at the third entry from the top). Let $i \in \{0, 1, \ldots, \rho - 1\}$. Let $i \cdot \alpha$ denote the tableaux that is α has a 1 in the top left position, has $d + 1 + i$ at the bottom of the right column, and the elements of $\{d + 1, d + 2, \ldots, m + d\} \setminus \{d + 1 + i\}$ in the left tail.

Let $\mathcal{S} = \{i \cdot \alpha : \alpha \in \overline{\text{skyt}}(m + 1, i, d - 2i + 1), i \in \{0, 1, \ldots, \rho - 1\}\}$. Our work from Lemma 12 gives

$$\#\mathcal{S} = \rho \cdot \overline{\text{skyt}}(m + 1, i, d - 2i + 1).$$

Hence, $\text{skyt}(m+1, i, d-2i+1) - \rho \cdot \overline{\text{skyt}}(m+1, i, d-2i+1)$ counts the number of elements in $\text{Skyt}(m+1, i, d-2i+1) \setminus \mathcal{S}$. Such elements are exactly described by the elements in $\text{Skyt}_\rho(m + 1, i, d - 2i + 1)$. □

It is now equivalent to state Theorem 1 as our primary result.

Theorem 3 (Theorem 1)**.** *Let $c^i_{m,d}(\mathcal{CH})$ be the ith coefficient for the Kazhdan-Lusztig polynomial for the matroid $S_{m,d}(\mathcal{CH})$ so that $\mathcal{CH}$ is a disjoint family. Then*

$$c^i_{m,d}(\mathcal{CH}) = \text{skyt}_\rho(m + 1, i, d - 2i + 1).$$

We now move on to the more general case of sparse paving matroids where $\mathcal{CH}$ is not necessarily disjoint. Unfortunately, there still is no manifestly non-negative expression in terms of tableaux. Instead, we show directly that our formula from Theorem 1 is non-negative by relying on the bounds given in Appendix B for $|\mathcal{CH}|$, our formulas for $\text{skyt}(a, i, b)$ and $\overline{\text{skyt}}(i, b)$ given in Chapter 3, and some standard algebra and calculus tools. The details for this proof will be rather technical, and our proof will need a few cases, so the proof serves more as an outline, leaving most of the work to separate Lemmas and Propositions. Throughout the proofs of this chapter, we use the falling factorial $(x)_{(n)} := x(x-1)\cdots(x-n+1)$. We will also regularly use the fact $\deg P_M(t) < \frac{1}{2}\operatorname{rk} M$. That is, if d is the rank of a matroid M, and i is the power of some term in the Kazhdan-Lusztig polynomial $P_M(t)$, then we must have $i < d/2$.

Theorem 4. *Let $S_{m,d}(\mathcal{CH})$ be a sparse paving matroid. Then*

$$\operatorname{skyt}(m+1, i, d-2i+1) - |\mathcal{CH}| \cdot \overline{\operatorname{skyt}}(i, d-2i+1) \geq 0.$$

Proof. We are able to take care of most of the cases simultaneously. For convenience, using our notation from Theorem 2, let

$$c^i_{m,d}(\mathcal{CH}) = \operatorname{skyt}(m+1, i, d-2i+1) - |\mathcal{CH}| \cdot \overline{\operatorname{skyt}}(i, d-2i+1).$$

Recall that $|\mathcal{CH}| \leq \frac{2}{m+d+2}\binom{m+d}{d}$ by Theorem 6. Hence,

$$c^i_{m,d}(\mathcal{CH}) \geq \operatorname{skyt}(m+1, i, d-2i+1) - \frac{2}{m+d+2}\binom{m+d}{d} \cdot \overline{\operatorname{skyt}}(i, d-2i+1).$$

Then by Lemma 13, this expression is non-negative for $i \geq 3$, $m \geq 3$, and for all possible d. That is, for $d > 2i$

This leaves a small number of more specific cases left, which need to be addressed independently. We first note that the cases for $m = 0$ and $i = 0$ are taken care of by Remark 4.

When $m = 1$, notice that any pair of basis elements have symmetric difference 2, and so $|\mathcal{CH}| \leq 1$. In this case our desired result is immediate since by definition, we may view $\overline{\text{Skyt}}(i, d-2i+1)$ as a subset of $\text{Skyt}(2, i, d-2i+1)$.

When $m = 2$, it is necessary to find a better bound on the size of $\mathcal{CH}$. It is not too much work to show that $|\mathcal{CH}| \leq \frac{d+2}{2}$ by using the symmetric difference condition on $\mathcal{CH}$. It is easier to work with the complements of the elements in $\mathcal{CH}$, which are elements of $\binom{[d+2]}{2}$. Then it is equivalent in this case to count the size of the largest disjoint family in $\binom{[d+2]}{2}$. So in the case of $m = 2$ we have

$$c^i_{m,d}(\mathcal{CH}) \geq \text{skyt}(m+1, i, d-2i+1) - \frac{d+2}{2} \cdot \overline{\text{skyt}}(i, d-2i+1),$$

and so to prove our desired result in this case we need only prove

$$\text{skyt}(m+1, i, d-2i+1) - \frac{d+2}{2} \cdot \overline{\text{skyt}}(i, d-2i+1) \geq 0.$$

We do this for $i \geq 1$, leaving the details to Lemma 14.

Now we move on to the remaining values of i, noting we need only show them for $m \geq 3$. When $i = 1$, one can get the following closed formula for $\text{skyt}(m+1, i, d-2i+1)$. We get

$$\text{skyt}(m+1, 1, d-1) = \binom{m+d}{d-1} - m - d$$

by Proposition 8. Also, note that $\overline{\text{skyt}}(1, d-1) = d-1$, which can be seen by using Lemma 5, or by simply observing that only numbers in $\{2, 3, 4, \ldots, d\}$ may appear below the position containing 1 in $\overline{\text{skyt}}(1, d-1)$. It is also important to note that when $i = 1$, $d \geq 3$. Then to get our desired result in this case, we can combine Theorem 2 and Theorem 5 and instead show that

$$\binom{m+d}{d-1} - m - d - \frac{1}{m+1}\binom{m+d}{d}(d-1) \geq 0.$$

Lemma 15 is able to show this for $d \geq 3$ when $m \geq 4$, but only for $d \geq 4$ when $m = 3$. This leaves the case when $m = 3$ and $d = 3$ to be done explicitly. Note that

$$\text{skyt}(4,1,2) = 9$$

and

$$\overline{\text{skyt}}(1,2) = 2,$$

which can be easily verified by any of our formulas from Chapter 3, or by hand. Then non-negativity follows from the fact that in the special case of $m = d = 3$, we can guarantee $|\mathcal{CH}| \leq 4$, which one verify via a constructive argument.

When $i = 2$, we can use a similar strategy that we used for the $i \geq 3$ and $m \geq 3$ case described in Lemma 13. However, there will be a bit more involved here, and so we leave the details of this final case to Lemma 16.

□

Remark 5. *In the case of $m = d = 3$, it is worth noting that finding the bound $|\mathcal{CH}| \leq 4$ was necessary. Both bounds for $|\mathcal{CH}|$ given by Theorem 5 or Theorem 6 give $|\mathcal{CH}| \leq 5$, and $9 - 5 \cdot 2 = -1$. So in this special case, we needed to get a better bound on $|\mathcal{CH}|$ than what either of our two bounds could provide.*

Lemma 13. *Let i and m both be at least 3. Then*

$$\operatorname{skyt}(m+1,i,d-2i+1)-\frac{2}{m+d+2}\binom{m+d}{d}\overline{\operatorname{skyt}}(i,d-2i+1)\geq 0.$$

Proof. One can rewrite the sum in Lemma 4 using Remark 3. After doing this, letting $a=m+1$ and $b=d-2i+1$, the $k=0$ term in the formula for $\operatorname{skyt}(m+1,i,d-2i+1)$ is

$$\begin{aligned}
A &:= \binom{m+i-1}{i}\binom{m+d}{d-i}\frac{(d-i-2)_{(d-2i-1)}(m+d-i)_{(d-2i-1)}}{(d-2i-1)!(m+d-i)_{(d-2i)}}\\
&= \frac{(m+i-1)!(m+d)!(d-i-2)_{(d-2i-1)}(m+d-i)_{(d-2i-1)}}{i!(m-1)!(d-i)!(m+i)!(d-2i-1)!(m+d-i)_{(d-2i)}}\\
&= \frac{(m+d)!(d-i-2)_{(d-2i-1)}}{i!(m-1)!(d-i)!(m+i)(d-2i-1)!(m+i+1)}.
\end{aligned}$$

Utilizing Lemma 5, we have

$$\begin{aligned}
B &:= \frac{2}{m+d+2}\binom{m+d}{d}\overline{\operatorname{skyt}}(i,d-2i+1)\\
&= \frac{4(m+d)!}{m!(m+d+2)(i+1)!(i-1)!(d-2i-1)!(d-i+1)(d-i-1)}.
\end{aligned}$$

Note that

$$\operatorname{skyt}(m+1,i,d-2i+1)-\frac{2}{m+d+2}\binom{m+d}{d}\overline{\operatorname{skyt}}(i,d-2i+1)\geq A-B,$$

so it suffices to show $A-B\geq 0$. Recall that $i<d/2$. Put another way, this says that $d-i>i>i-1$. Hence, we may combine $A-B$ in the following way.

$$
\begin{aligned}
&A - B \\
&= A\frac{m(i+1)(m+d+2)(d-i+1)(d-i-1)}{m(i+1)(m+d+2)(d-i+1)(d-i-1)} \\
&- B\frac{(d-i)_{(d-2i+1)}(m+i)(m+i+1)}{(d-i)_{(d-2i+1)}(m+i)(m+i+1)} \\
&= \frac{(m+d)!(m+i-1)!p(m,i,d)}{m!(m+d+2)(i+1)!(d-i)!(m+i+1)!(d-i+1)(d-i-1)(d-2i-1)!}
\end{aligned}
$$

where

$$
\begin{aligned}
p(m,i,d) =& (d-i-2)_{(d-2i-1)} m(i+1)(m+d+2)(d-i+1)(d-i-1) \\
&- 4(d-i)_{(d-2i+1)}(m+i)(m+i+1).
\end{aligned}
$$

Hence, it suffices to show that $p(m,i,d) \geq 0$. We can, in fact, reduce the problem further by simplifying $p(m,i,d)$. Observe that

$$
p(m,i,d) = (d-i-1)_{d-2i}[m(i+1)(m+d+2)(d-i+1) - 4(m+i)(m+i+1)(d-i)],
$$

so it now suffices to show

$$
q(m,i,d) := m(i+1)(m+d+2)(d-i+1) - 4(m+i)(m+i+1)(d-i) \geq 0.
$$

We show this for $m, i \geq 3$ by viewing q as a function of m. The desired result follows from the following three claims for q as a function of m.

1. q is quadratic and concave up;

2. the critical point of q is negative; and

3. $q(m, i, d) \geq 0$ for $m = 3$.

Showing these are elementary exercises in algebra and calculus, so we just highlight the important parts.

For claim (1), note that the coefficient of m^2 in $q(m, i, d)$ is $(i+1)(d-i+1) - 4(d-i)$, and that we assume $d > 2i$ and $i \geq 3$. Hence this coefficient is non-negative.

For claim (2), it suffices to show the coefficient of m in $q(m, i, d)$ is positive. This coefficient is

$$(i+1)(d+2)(d-i+1) - 4(i+1)(d-i) - 4i(d-i).$$

Using the fact that $d > 2i$, one can show this is an increasing function in d and is non-negative when $d = 2i$.

For claim (3), it suffices to show $q(3, i, d)$ is an increasing function in d and that $q(3, i, 2i)$ is non-negative. This works out similarly to claim (2).

□

Lemma 14. *Let $i \geq 1$ and $m = 2$. Then*

$$\operatorname{skyt}(m+1, i, d-2i+1) - \frac{d+2}{2}\overline{\operatorname{skyt}}(i, d-2i+1) \geq 0.$$

Proof. As in Lemma 13, keeping in mind that $m = 2$, set

$$A := \frac{(d+2)!(d-i-2)_{(d-2i-1)}}{i!(d-i)!(i+2)(d-2i-1)!(i+3)}$$

and

$$B := \frac{d+2}{2}\overline{\text{skyt}}(i, d-2i+1)$$
$$= \frac{d!(d+2)}{(i+1)!(i-1)!(d-2i-1)!(d-i+1)(d-i-1)}.$$

It follows from the proof of Lemma 13 that $\text{skyt}(m+1, i, d-2i+1) \geq A$ for $m = 2$, and so the desired result follows if we show $A - B \geq 0$. Observe that

$$A - B = A\frac{(i+1)(d-i+1)(d-i-1)}{(i+1)(d-i+1)(d-i-1)} - B\frac{(i+2)(i+3)(d-i)_{(d-2i+1)}}{(i+2)(i+3)(d-i)_{(d-2i+1)}}$$
$$= \frac{d!(d+2)p(i,d)}{(i+3)!(d-i)!(d-2i-1)!(d-i+1)(d-i-1)},$$

where

$$p(i,d) := (d-i-2)_{(d-2i-1)}(d+1)(i+1)(d-i+1)(d-i-1) - (i+2)(i+3)(d-i)_{(d-2i+1)}.$$

Hence, it suffices to show that $p(i, d)$ is non-negative. One can factor $p(i, d)$ to reduce the problem further:

$$p(i,d) = (d-i-1)_{(d-2i)}[(d+1)(i+1)(d-i+1) - (i+2)(i+3)(d-i)],$$

and so it suffices to show that

$$q(i,d) := (d+1)(i+1)(d-i+1) - (i+2)(i+3)(d-i)$$

is non-negative. Since in the context of Kazhdan-Lusztig polynomials we have $d > 2i$, we may set $d = 2i + j$ for $j \geq 1$. Then $q(i, 2i + j)$ is quadratic in j and we have the following values of $[j^\ell]q(i, 2i + j)$:

$$[j^2]q(i, 2i + j) = i + 1$$

$$[j^1]q(i, 2i + j) = 2i^2 - 4$$

$$\text{Remaining terms: } i^3 - 2i + 1$$

When $i \geq 2$, all three values are individually positive positive. If $i = 1$, then

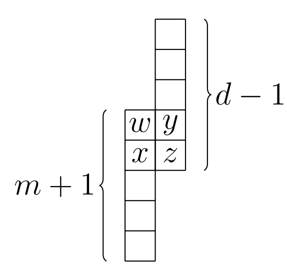
$$q(1, j + 2) = 2j^2 - 2j$$

which is non-negative for all $j \geq 1$, giving our desired result. □

Proposition 8.

$$\text{skyt}(m + 1, 1, d - 1) = \binom{m + d}{d - 1} - m - d.$$

Proof. Note that if $\alpha \in \text{Skyt}(m + 1, 1, d - 1)$, it is made up of two "tails", one of length $m + 1$ extending down, and the other of length $d - 1$ extending up, so that the two tails overlap in exactly two positions. See the below figure for a schematic of α, with some entries labeled.



Note that there are $m + d$ positions in these tableaux, and we require that

$w < y$ and $x < z$. Now, pick an element $S \in \binom{[m+d]}{d-1}$. The number of elements of $\mathrm{Skyt}(m+1,1,d-1)$ is equivalent to the number of S that appear as the right tail in an element in $\mathrm{Skyt}(m+1,1,d-1)$, as the entries of one tail determine the entries of the other. It is easiest to count the complement, that is, the S that will *not* appear as the the right tail in an element of $\mathrm{Skyt}(m+1,1,d-1)$. These are the S that force $w > y$, $x > z$, or both. We leave it to the reader to verify that the complement has size $m+d$.

□

Lemma 15. *We have*

$$\binom{m+d}{d-1} - m - d - \frac{1}{m+1}\binom{m+d}{d}(d-1) \geq 0$$

for $d \geq 3$ when $m \geq 4$, and $d \geq 4$ when $m = 3$.

Proof. We start by rewriting of our expression of interest.

$$\begin{aligned}
&\binom{m+d}{d-1} - m - d - \frac{1}{m+1}\binom{m+d}{d}(d-1) \\
&= \frac{d}{m+1}\binom{m+d}{d} - m - d - \frac{1}{m+1}\binom{m+d}{d}(d-1) \\
&= \frac{1}{m+1}\binom{m+d}{d}(d-(d-1)) - m - d \\
&= \frac{1}{m+1}\binom{m+d}{d} - m - d \\
&= \frac{(m+d)_{(d-1)}}{d!} - m - d \\
&= (m+d)\left(\frac{(m+d-1)_{(d-2)}}{d!} - 1\right).
\end{aligned}$$

Hence, if

$$f(m,d) := \frac{(m+d-1)_{(d-2)}}{d!}$$

it suffices to show $f(m,d) \geq 1$. As a function in m, $f(m,d)$ is increasing. Also,

$$f(4,d) = \frac{(d+3)_{(d-2)}}{d!} = \frac{(d+3)!}{5!d!} = \frac{1}{20}\binom{d+3}{3}.$$

See that $f(4,d)$ is increasing in d and also $f(4,3) = 1$. So when $m \geq 4$, we have our desired result for $d \geq 3$. When $m = 3$, observe we have

$$f(3,d) = \frac{(d+2)_{(d-2)}}{d!} = \frac{(d+2)!}{4!d!} = \frac{1}{12}\binom{d+2}{2}.$$

See that $f(3,d)$ is increasing in d, and $f(3,4) = \frac{15}{12}$. □

Lemma 16. *If $m \geq 3$, we have*

$$c^2_{m,d}(\mathcal{CH}) \geq 0.$$

Proof. It will be important to remember that since $i = 2$, we have $d \geq 5$ by the degree requirement on Kazhdan-Lusztig polynomials.

To show our desired result, we will need two separate cases. First suppose $m \geq d$. Note then we already have $m \geq 3$ since $d \geq 5$. As in Lemma 13, accounting for the fact that in this case $i = 2$, let

$$\begin{aligned} A :&= \frac{(m+d)!(d-4)_{(d-5)}}{2(m-1)!(d-2)!(m+2)(d-5)!(m+3)} \\ &= \frac{(m+d)!(d-4)!}{2(m-1)!(d-2)!(m+2)(d-5)!(m+3)} \\ &= \frac{(m+d)!(d-4)m(m+1)}{2(m+3)!(d-2)!}. \end{aligned}$$

Also similarly to Lemma 13, but using the bound from Theorem 5 for $|\mathcal{CH}|$, let

$$B := \frac{1}{m+1}\binom{m+d}{d}\frac{2\cdot d!}{6(d-5)!(d-1)(d-3)} = \frac{(m+d)!(d-2)(d-4)}{3(m+1)!(d-1)!}.$$

A combination of Theorem 2, Theorem 5, and the proof of Lemma 13 implies that

$$c^2_{m,d}(\mathcal{CH}) \geq A - B,$$

and so we show $A - B \geq 0$ when $m \geq d$. Notice that

$$A - B = \frac{(m+d)!(d-4)f(m,d)}{6(m+3)!(d-1)!},$$

where

$$f(m,d) := 3m(m+1)(d-1) - 2(d-2)(m+2)(m+3).$$

Hence, it suffices to show that $f(m,d) \geq 0$ to show that $A - B \geq 0$. Since we are assuming $m \geq d$, we set $m = d + j$, for $j \geq 0$. Then $f(d+j,d)$ is quadratic in j and we have

$$[j^2]f(d+j,d) = d+1$$
$$[j]f(d+j,d) = 2d^2 - 5d + 17$$
$$\text{Remaining terms of } f(d+j,d) : d^3 - 6d^2 + 5d + 24$$

Each of these are positive when $d = 5$. In fact, the $[j^2]$ term is clearly positive when $d \geq 5$. The $[j]$ term is increasing for $d \geq \frac{5}{2}$. For the remaining terms, note that the derivative is $3d^2 - 12d + 5$, which increases so long as $d \geq 2$, and is already positive at $d = 5$. This means that the derivative remains positive for $d \geq 5$, and so the original function remains increasing. Hence, this shows that $c^2_{m,d}(\mathcal{CH}) \geq 0$ so long as $m \geq d$.

Now we show the same result holds when $d \geq m$. To do this, we reuse A as above, and redefine B using our bound from Theorem 6.

$$B := \frac{2}{m+d+2}\binom{m+d}{d}\frac{2 \cdot d!}{6(d-5)!(d-1)(d-3)} = \frac{2(m+d)!(d-2)(d-4)}{3(m+d+2)m!(d-1)!}$$

For similar reasons as before, $c^2_{m,d}(\mathcal{CH}) \geq 0$ if $A - B \geq 0$. Note that

$$A - B = \frac{(m+d)!(d-4)(m+1)g(m,d)}{6(m+3)!(d-1)!},$$

where

$$g(m,d) := 3m(m+d+2)(d-1) - 4(d-2)(m+2)(m+3).$$

Observe that g is a concave up quadratic function in d. If one expands the function, its vertex can be seen to occur at

$$d = \frac{m^2+17m+24}{6m}.$$

However, note that this value is less than m so long as $m \geq 5$ since

$$\frac{m^2+17m+24}{6m} \leq m \text{ if and only if } -5m^2+17m+24 \leq 0.$$

Hence, this says that $g(m,d)$ is increasing in d when $d \geq m \geq 5$. Also, when $m = 3$ the vertex for g is at approximately $d = 4.67$ and when $m = 4$ the vertex for g is at $d = 4.5$. We know that $d \geq 5$ regardless of its relation to m, so we have in fact shown that g is increasing in d for any $m \geq 3$ when $d \geq m$.

Moreover, one can verify

$$g(m,m) = 2(m^3 - 6m^2 + 5m + 24) \geq 0$$

so long as $m \geq 5$. Also, note that $g(3,5) = 0$ and $g(4,5) = 24$. Hence $g(m,d)$ is always non-negative for $d \geq m$ when $m \geq 3$. □

Appendix A

Integral Identities

Proposition 9. *[18, Identity 2.110.8] Let a, b be positive integers. Then*

$$\int y^a(1-xy)^b \; dy = a!b!\sum_{k=0}^{b} \frac{(1-xy)^{b-k}y^{a+k+1}x^k}{(a+k+1)!(b-k)!}.$$

Corollary 3. *Let a, b be positive integers. Then*

$$\int_0^1 y^a(1-xy)^b \; dy = a!b!\sum_{k=0}^{b} \frac{(1-x)^{b-k}x^k}{(a+k+1)!(b-k)!}.$$

Corollary 4. *Let a, b be positive integers. Then*

$$\int_0^y x^a(1-x)^b \; dx = a!b!\sum_{k=0}^{b} \frac{(1-y)^{b-k}y^{a+k+1}}{(a+k+1)!(b-k)!}$$

Corollary 5. *For positive integers a and b,*

$$\int_0^{-1} x^a(1+x)^b \; dx = \frac{(-1)^{a+1}b!}{(a+1)(a+2)\cdots(a+b+1)}.$$

Proposition 10. *Let $x_0, x_1, \ldots, x_i$ be a list of $i+1$ variables. Set $h_1(x_1) = \int_0^{x_1} x_0^a(1-x_0)^b \; dx_0$, and for $i > 1$ define $h_i(x_i) = \int_0^{x_i} h_{i-1}(x_{i-1}) \; dx_{i-1}$.*

Then

$$\int_0^1 h_i(x_i)\ dx_i = \frac{a!(b+i)!}{i!(a+b+i+1)!}.$$

Proof. Using Corollary 4 i times, we get the following expression for $h_i(x_i)$:

$$\sum_{k_1=0}^{b}\sum_{k_2=0}^{b-k_1}\sum_{k_3=0}^{b-k_1-k_2}\cdots\sum_{k_i=0}^{b-\sigma}\frac{a!b!x_i^{a+\sigma+k_i+i}(1-x_i)^{b-\sigma-k_i}}{(a+\sigma+k_i+i)!(b-\sigma-k_i)!} \tag{A.1}$$

where $\sigma = k_1 + k_2 + \cdots + k_{i-1}$. Noting that

$$\int_0^1 x_i^{a+\sigma+k_i+i}(1-x_i)^{b-\sigma-k_i}\ dx_i = \frac{(a+\sigma+k_i+i)!(b-\sigma-k_i)!}{(a+b+i+1)!}.$$

we may use (A.1) to write

$$\begin{aligned}
&\int_0^1 h_i(x_i)\ dx_i \\
&= \sum_{k_1=0}^{b}\sum_{k_2=0}^{b-k_1}\sum_{k_3=0}^{b-k_1-k_2}\cdots\sum_{k_i=0}^{b-\sigma}\frac{a!b!(a+\sigma+k_i+i)!(b-\sigma-k_i)!}{(a+\sigma+k_i+i)!(b-\sigma-k_i)!(a+b+i+1)!} \\
&= \frac{a!b!}{(a+b+i+1)!}\sum_{k_1=0}^{b}\sum_{k_2=0}^{b-k_1}\sum_{k_3=0}^{b-k_1-k_2}\cdots\sum_{k_i=0}^{b-\sigma}1,
\end{aligned}$$

which simplifies using Proposition 11 to

$$\frac{a!b!}{(a+b+i+1)!}\binom{b+i}{i} = \frac{a!(b+i)!}{i!(a+b+i+1)!}.$$

□

Proposition 11.

$$\sum_{k_1=0}^{b}\sum_{k_2=0}^{b-k_1}\sum_{k_3=0}^{b-k_1-k_2}\cdots\sum_{k_i=0}^{b-\sigma}1 = \binom{b+i}{i},$$

where $\sigma = k_1 + k_2 + \cdots + k_{i-1}$.

Proof. It is helpful to first reindex the summations so that they start at 1 instead of 0. Then the identity holds from counting the below set in two ways.

$$\bigcup_{x_1 \in [b+1]} \bigcup_{x_2 \in [b+2] \setminus [x_1]} \bigcup_{x_3 \in [b+3] \setminus [x_2]} \cdots \bigcup_{x_i \in [b+i] \setminus [x_{i-1}]} \{x_1, x_2, \ldots, x_i\}.$$

$\square$

Appendix B

Bounds on $|\mathcal{CH}|$

Our proof for the non-negativity of Theorem 1 will be purely computational. Hence, since $|\mathcal{CH}|$ is a part of our formula, having bounds on this value will be useful. We will give two particularly important bounds.

The first bound is given as follows.

Theorem 5.

$$|\mathcal{CH}| \leq \frac{1}{m+1}\binom{m+d}{d}.$$

This can be recovered in multiple settings. One can find an outline of a matroid theory argument in [3, Lemma 2.7]. However, this bound also happens to be a standard coding theory result. Recall that for $S_{m,d}(\mathcal{CH})$, the circuit-hyperplanes $\mathcal{CH}$ is a subset of elements in $\binom{[m+d]}{d}$ so that any pair has symmetric difference at least 4. One could equivalently describe such a set as a binary constant-weight code with hamming distance 4. In this context, the bound in Theorem 5 gives a bound on the size of a code with these conditions, as shown in [1, Theorem 12]. In fact, [1] proves a more arbitrary bound accounting for any lower bound on symmetric difference, not just 4. It is also worth noting that the proofs for this bound given in both [1] and [3] are in fact different, even when both are in the language of matroid theory.

While this bound will serve useful, there will be times where it will not be sufficient for our purposes. Unlike the prior bound, we found no literature to support the bound that follows.

Theorem 6.

$$|\mathcal{CH}| \leq \frac{2}{m+d+2}\binom{m+d}{d}.$$

Remark 6. *These two bounds have an interesting relationship. First, observe that*

$$\frac{1}{m+1}\binom{m+d}{d} > \frac{2}{m+d+2}\binom{m+d}{d} \text{ if and only if } d > m.$$

A take-away here is that both bounds are necessary to get a good bound for $|\mathcal{CH}|$*. Excitingly, when* $m = d$*, not only do these bounds agree, but they equal the* m*th Catalan number* C_m*, where*

$$C_m = \frac{1}{m+1}\binom{2m}{m}.$$

To prove Theorem 6, we will utilize a graph theory technique known as *discharging*. First, though, it is necessary to make clear the connection between sparse paving matroids and graphs. Let $J(n,d)$ be a graph with vertex set $\binom{[n]}{d}$, where vertices are adjacent if and only if their symmetric difference is size 2. This graph is best known as the *Johnson Graph*. The symmetric difference condition on $\mathcal{CH}$ implies that $\mathcal{CH}$ is an independent set in $J(m+d,d)$, that is, a set of vertices with no edges between them. So finding an upper bound on $|\mathcal{CH}|$ is equivalent to a bound on the size of an independent set in $J(m+d,d)$.

There are some final graph theory notation conventions we give before providing the proof of Theorem 6. Let A and B be vertices in $J(n,d)$. To indicate A and B are adjacent we write $A \sim B$. When an edge has vertex A as an endpoint, we say that edge is *incident* to A. By $N(A)$ we mean the induced graph on the vertices adjacent to A in $J(n,d)$. That is, $N(A)$ is the subgraph of $J(n,d)$ where for all vertices $B, C \in N(A)$, we have $B \sim C$ in $N(A)$ if and only if $B \sim C$ in $J(n,d)$.

Proof of Theorem 6. Let $I \subseteq \binom{[n]}{d}$ be an independent set of vertices in $J(n,d)$. We will describe an assignment of weights to edges of $J(n,d)$ based on I. Start with a weight of 0 on all edges of $J(n,d)$. If $A \in I$ we add a weight of 1 to each edge incident with A. Furthermore, A adds a weight of 1/2 to all edges in $N(A)$. Note that there are $d(n-d)$ vertices of $N(A)$ since every neighbor B of A is specified uniquely by $B = (A \setminus \{a_B\}) \cup \{x_B\}$ where $a_B \in A$ and $x_B \in A^c$. Two vertices $B, C \in N(A)$ are adjacent iff $a_B = a_C$ or $x_B = x_C$. This implies that the graph induced on $N(A)$ is regular of degree $d - 1 + (n - d - 1) = n - 2$. Thus A assigns a total weight of

$$w = d(n-d) + \frac{1}{2} \cdot \frac{d(n-d)(n-2)}{2} = d(n-d)\Big(1 + \frac{n-2}{4}\Big)$$

to edges of the graph.

We will now show that no edge of $J(n,d)$ receives a total weight of more than 1 from this assignment. First, note that no edge is incident with two elements of I, for they would be adjacent. Similarly, if an edge is incident with $A \in I$ it cannot also be an edge in $N(A')$ for any $A' \in I$ for then we would have $A \sim A'$, a contradiction. Thus it only remains to prove that if AB is an edge then there exist at most two elements A' of I that have $A, B \in N(A')$.

Let us consider what common neighbors of A and B look like. We know that $C = A \cap B$ has size $d-1$ and for some $x, y \in [n]$ we have $A = C \cup \{x\}$ and $B = C \cup \{y\}$. Consider now $A' \in N(A) \cap N(B)$. If $C \subseteq A'$ then $A' = C \cup z$ for some $z \neq x, y$ in C^c. We call such common neighbors *type* 1. Now if a neighbor A' of A is not of type 1 then it has the form $(C \setminus \{c\}) \cup \{x, z\}$ for some $c \in C$ and $z \notin A$. But the only way such a set can also be a neighbor of B is to have $z = y$. Thus all other common neighbors of A and B are *type* 2 common neighbors: those of the form $(C \setminus \{c\}) \cup \{x, y\}$.

Now we simply note that the type 1 common neighbors of A and B are all pairwise adjacent to one-another in $J(n, d)$, as are the type 2 common neighbors. That means at most one type 1 neighbor and at most one type 2 neighbor may be in I. Thus the edge AB receives a weight of 1/2 from at most one type 1 common neighbor, and weight 1/2 from at most one type 2 common neighbor, for a total weight of at most 1.

Now we simply compute as follows. Each member of the independent set I assigns total weight w to the edges of $J(n, d)$, and each edge of $J(n, d)$ receives total weight at most 1 from the elements of I, so

$$|I|\, w = |I|\, d(n-d)\Big(1 + \frac{n-2}{4}\Big) \leq \binom{n}{d}\frac{d(n-d)}{2} = e(J(n,d)),$$

thus

$$\begin{aligned} |I|\left(1 + \frac{n-2}{4}\right) &\leq \binom{n}{d}\frac{1}{2} \\ |I|\,(n+2) &\leq 2\binom{n}{d} \\ |I| &\leq \frac{2}{n+2}\binom{n}{d}. \end{aligned}$$

□

Appendix C

A Generating Function Proof for the Kazhdan-Lusztig Polynomial for Uniform Matroids

Let $c^i_{m,d}$ be the ith coefficient of the Uniform matroid of rank d on $m+d$ elements. Let $a = m+1$ and $b = d-2i+1$. We instead prove $c^i_{a-1,b+2i-1} = \text{skyt}(a,i,b)$. In Chapter 3, we prove Lemma 10 which gives the identity $\text{skyt}(a,i,b) = \text{skyt}(b,i,a)$. Using this, it suffices to show that $c^i_{a-1,b+2i-1} = \text{skyt}(b,i,a)$. To that end, first observe

$$\begin{aligned}
&\#SYT(b,i,k)\binom{a+2i+b-2}{a-k-2}\\
&= \frac{1}{(b+i-1)}\frac{(b+2i+k)!(k+1)}{(b-2)!i!(b+i+k)(i+k+1)!}\frac{(a+2i+b-2)!}{(a-k-2)!(2i+k+b)!}\\
&= \frac{1}{(b+i-1)}\frac{(k+1)(a+2i+b-2)!}{(b-2)!i!(b+i+k)(i+k+1)!(a-k-2)!}\\
&= \frac{1}{(b+i-1)}\binom{a+2i+b-2}{i}\frac{(k+1)(a+i+b-2)!(b+i+k-1)!}{(b-2)!(b+i+k)!(i+k+1)!(a-k-2)!}\\
&= \frac{1}{(b+i-1)}\binom{a+2i+b-2}{i}(k+1)\binom{a+i+b-2}{b+i+k}\binom{b+i+k-1}{b-2},
\end{aligned}$$

Utilizing [13, Theorem 1.4], we may write

$$c^i_{a-1,b+2i-1} = \frac{1}{b+i-1}\binom{b+2i+a-2}{i}\sum_{h=0}^{a-2}\binom{b+i+h-1}{h+i+1}\binom{i-1+h}{h}.$$

Hence, it suffices to show that

$$\sum_{k=0}^{a-2}(-1)^k(k+1)\binom{a+i+b-2}{b+i+k}\binom{b+i+k-1}{b-2} = \sum_{h=0}^{a-2}\binom{b+i+h-1}{h+i+1}\binom{i-1+h}{h}.$$

We instead prove that

$$\sum_{a,b\geq 0} x^a y^b \sum_{k=0}^{a-2}(-1)^k(k+1)\binom{a+i+b-2}{b+i+k}\binom{b+i+k-1}{b-2}$$
$$= \sum_{a,b\geq 0} x^a y^b \sum_{h=0}^{a-2}\binom{b+i+h-1}{h+i+1}\binom{i-1+h}{h}.$$

We take advantage of the following two things:

1. $\displaystyle\sum_{k=0}^{\infty}\binom{k+i}{k}x^k = \frac{1}{(1-x)^{i+1}}$

2. $\displaystyle\sum_{k\geq 0}(-1)^k(k+1)z^k = \frac{d}{dz}z\sum_{k\geq 0}(-1)^k z^k = \frac{d}{dz}\frac{z}{1+z} = \frac{1}{(1+z)^2}$

Now on the one hand, we have

$$\begin{aligned}
&\sum_{a,b,k\geq 0} x^a y^b (-1)^k (k+1)\binom{a+i+b-2}{b+i+k}\binom{b+i+k-1}{b-2} \\
&= \sum_{b,k\geq 0} y^b (-1)^k (k+1)\binom{b+i+k-1}{b-2}\sum_{a\geq 0} x^a \binom{a+i+b-2}{b+i+k} \\
&= \sum_{b,k\geq 0} y^b (-1)^k (k+1)\binom{b+i+k-1}{b-2} x^{k+2} \sum_{a\geq k+2} x^{a-k-2}\binom{a+i+b-2}{a-k-2} \\
&= \frac{x^2}{(1-x)^{i+1}}\sum_{k\geq 0}(-1)^k (k+1)\left(\frac{x}{1-x}\right)^k \sum_{b\geq 0}\binom{b+i+k-1}{i+k+1}\left(\frac{y}{1-x}\right)^b \\
&= \frac{x^2}{(1-x)^{i+1}}\sum_{k\geq 0}(-1)^k (k+1)\left(\frac{x}{1-x}\right)^k \left(\frac{y}{1-x}\right)^2 \sum_{b\geq 0}\binom{b+i+k+1}{i+k+1}\left(\frac{y}{1-x}\right)^b \\
&= \frac{x^2 y^2}{(1-x)^{i+3}(1-\frac{y}{1-x})^{i+2}}\sum_{k\geq 0}(-1)^k (k+1)\left(\frac{x}{1-x-y}\right)^k \\
&= \frac{x^2 y^2}{(1-x)^{i+3}(1-\frac{y}{1-x})^{i+2}}\frac{1}{(1+\frac{x}{1-x-y})^2} \\
&= \frac{x^2 y^2}{(1-x)(1-x-y)^i (1-y)^2}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\sum_{a,b} x^a y^b \sum_{h=0}^{a-2} \binom{b+i+h-1}{h+i+1} \binom{i-1+h}{h} \\
&= \sum_{a\geq 0} x^a \sum_{h=0}^{a-2} \binom{i-1+h}{h} \sum_{b\geq 0} y^b \binom{b+i+h-1}{h+i+1} \\
&= \sum_{a\geq 0} x^a \sum_{h=0}^{a-2} \binom{i-1+h}{h} y^2 \sum_{b\geq 0} y^b \binom{b+i+h+1}{h+i+1} \\
&= \frac{y^2}{(1-y)^{i+2}} \sum_{h\geq 0} \binom{i-1+h}{h} \frac{1}{(1-y)^h} \sum_{a\geq h+2} x^a \\
&= \frac{x^2y^2}{(1-y)^{i+2}(1-x)} \sum_{h\geq 0} \binom{i-1+h}{h} \left(\frac{x}{1-y}\right)^h \\
&= \frac{x^2y^2}{(1-y)^{i+2}(1-x)} \frac{1}{(1-\frac{x}{1-y})^i} \\
&= \frac{x^2y^2}{(1-y)^2(1-x)(1-y-x)^i}.
\end{aligned}$$

□

Bibliography

[1] E. Agrell, A. Vardy, and K. Zeger. Upper bounds for constant-weight codes. *IEEE Trans. Inform. Theory*, 46(7):2373–2395, 2000.

[2] A. Björner and F. Brenti. *Combinatorics of Coxeter groups*, volume 231 of *Graduate Texts in Mathematics*. Springer, New York, 2005.

[3] J. E. Bonin. Basis-exchange properties of sparse paving matroids. *Adv. in Appl. Math.*, 50(1):6–15, 2013.

[4] T. Braden, J. Huh, J. P. Matherne, N. Proudfoot, and B. Wang. Singular hodge theory for combinatorial geometries. *arXiv:2010.06088*, 2020.

[5] T. Braden and A. Vysogorets. Kazhdan-Lusztig polynomials of matroids under deletion. *Electron. J. Combin.*, 27(1):Paper No. 1.17, 17, 2020.

[6] F. Brenti. Twisted incidence algebras and Kazhdan-Lusztig-Stanley functions. *Adv. Math.*, 148(1):44–74, 1999.

[7] F. Brenti. P-kernels, IC bases and Kazhdan-Lusztig polynomials. *J. Algebra*, 259(2):613–627, 2003.

[8] H. H. Crapo and G.-C. Rota. On the foundations of combinatorial theory. II. Combinatorial geometries. *Studies in Appl. Math.*, 49:109–133, 1970.

[9] B. Elias, N. Proudfoot, and M. Wakefield. The Kazhdan-Lusztig polynomial of a matroid. *Adv. Math.*, 299:36–70, 2016.

[10] B. Elias and G. Williamson. The Hodge theory of Soergel bimodules. *Ann. of Math. (2)*, 180(3):1089–1136, 2014.

[11] B. Elias and G. Williamson. The Hodge theory of Soergel bimodules. *Ann. of Math. (2)*, 180(3):1089–1136, 2014.

[12] L. Ferroni and L. Vecchi. Matroid relaxations and Kazhdan-Lusztig non-degeneracy.

[13] A. L. L. Gao, L. Lu, M. H. Y. Xie, A. L. B. Yang, and P. B. Zhang. The Kazhdan-Lusztig polynomials of uniform matroids. *Adv. in Appl. Math.*, 122:102117, 24, 2021.

[14] A. L. L. Gao, M. H. Y. Xie, and A. L. B. Yang. The equivariant inverse Kazhdan-Lusztig polynomials of uniform matroids. *arXiv:1605.01685*, 2016.

[15] K. Gedeon, N. Proudfoot, and B. Young. The equivariant Kazhdan-Lusztig polynomial of a matroid. *J. Combin. Theory Ser. A*, 150:267–294, 2017.

[16] K. Gedeon, N. Proudfoot, and B. Young. Kazhdan-Lusztig polynomials of matroids: a survey of results and conjectures. *Sém. Lothar. Combin.*, 78B:Art. 80, 12, 2017.

[17] K. R. Gedeon. Kazhdan-Lusztig polynomials of thagomizer matroids. *Electron. J. Combin.*, 24(3):Paper No. 3.12, 10, 2017.

[18] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products.* Elsevier/Academic Press, Amsterdam, seventh edition, 2007. Translated from the Russian, Translation edited and with a preface by Alan Jeffrey and Daniel Zwillinger, With one CD-ROM (Windows, Macintosh and UNIX).

[19] T. K. Karn and M. D. Wakefield. Stirling numbers in braid matroid Kazhdan-Lusztig polynomials. *Adv. in Appl. Math.*, 103:1–12, 2019.

[20] L. Lu, M. Xie, and A. Yang. Kazhdan-Lusztig polynomials of fan matroids, wheel matroids and whirl matroid. *arXiv:1802.03711*, 2018.

[21] P. A. MacMahon. *Combinatory analysis. Vol. I, II (bound in one volume).* Dover Phoenix Editions. Dover Publications, Inc., Mineola, NY, 2004. Reprint of ıt An introduction to combinatory analysis (1920) and ıt Combinatory analysis. Vol. I, II (1915, 1916).

[22] D. Mayhew, M. Newman, D. Welsh, and G. Whittle. On the asymptotic proportion of connected matroids. *European J. Combin.*, 32(6):882–890, 2011.

[23] P. Nelson. Almost all matroids are nonrepresentable. *Bull. Lond. Math. Soc.*, 50(2):245–248, 2018.

[24] R. Pendavingh and J. van der Pol. On the number of matroids compared to the number of sparse paving matroids. *Electron. J. Combin.*, 22(2):Paper 2.51, 17, 2015.

[25] P. Polo. Construction of arbitrary Kazhdan-Lusztig polynomials in symmetric groups. *Represent. Theory*, 3:90–104, 1999.

[26] N. Proudfoot. The algebraic geometry of Kazhdan-Lusztig-Stanley polynomials. *EMS Surv. Math. Sci.*, 5(1-2):99–127, 2018.

[27] N. Proudfoot. Equivariant Kazhdan-Lustzig polynomials of q-niform matroids. *Algebr. Comb.*, 2(4):613–619, 2019.

[28] N. Proudfoot, M. Wakefield, and B. Young. Intersection cohomology of the symmetric reciprocal plane. *J. Algebraic Combin.*, 43(1):129–138, 2016.

[29] R. P. Stanley. Subdivisions and local h-vectors. *J. Amer. Math. Soc.*, 5(4):805–851, 1992.

[30] R. P. Stanley. Polygon dissections and standard Young tableaux. *J. Combin. Theory Ser. A*, 76(1):175–177, 1996.

[31] M. Wakefield. Partial flag incidence algebras. *arXiv:1605.01685*, 2016.

[32] M. H. Y. Xie and P. B. Zhang. Equivariant Kazhdan-Lusztig polynomials of thagomizer matroids. *Proc. Amer. Math. Soc.*, 147(11):4687–4695, 2019.