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Results on Nonorientable Surfaces for Knots and 2-knots

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RESULTS ON NONORIENTABLE SURFACES FOR KNOTS AND 2-KNOTS

by

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RESULTS ON NONORIENTABLE SURFACES FOR KNOTS AND 2-KNOTS

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Advisor: Alex Zupan and Mark Brittenham

A *classical knot* is a smooth embedding of the circle S^1 into the 3-sphere S^3. We can also consider embeddings of arbitrary surfaces (possibly nonorientable) into a 4-manifold, called *knotted surfaces*. In this thesis, we give an introduction to some of the basics of the studies of classical knots and knotted surfaces, then present some results about nonorientable surfaces bounded by classical knots and embeddings of nonorientable knotted surfaces.

First, we generalize a result of Satoh about connected sums of projective planes and twist spun knots. Specifically, we will show that for any odd natural n, the connected sum of the n-twist spun sphere of a knot K and an unknotted projective plane in the 4-sphere becomes equivalent to the same unknotted projective plane after a single trivial stabilization. We additionally provide a fix to a small error in Satoh's proof of the case that K is a 2-bridge knot.

Additionally, we show that the band unknotting number of a classical knot is an upper bound for the unknotting number of any twist spin of the classical knot. This result is also motivated by a result of Satoh which states that for a classical knot, the unknotting number and the bridge number minus one are both upper bounds for the twist spin of the classical knot.

Milnor's conjecture, first proved by Kronheimer and Mrowka in 1993, states that the 4-ball genus of a torus knot $T(p, q)$ is equal to $\frac{(p-1)(q-1)}{2}$. Batson's

conjecture is a nonorientable version of Milnor's conjecture which states that the nonorientable 4-ball genus is equal to the pinch number of a torus knot, i.e. the number of a specific type of (nonorientable) band surgeries needed to obtain the unknot. The conjecture was recently proved to be false by Lobb. We will show that Lobb's counterexample fits into an infinite family of counterexamples.

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I feel incredibly fortunate to be in the position I am in, and in my mind there is no doubt I would not be in this position without the unbelievable amount of support I've received from those around me. This accomplishment belongs to all of you just as much as it does me. Throughout graduate school, I have learned so much from my advisors Alex and Mark. They gave me all of the guidance, direction, and support I ever needed to get to this point. Their incredible amount of knowledge and resourcefulness has been so helpful, and they essentially never let a question go by without an answer. They've also opened up many doors for me, supporting me to go to many conferences and meet many friends and collaborators in my research field.

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CHAPTER 1

Background

1.1 Classical Knots

All work done in this thesis will be in the smooth category. In this chapter we will discuss much of the necessary background for the results in the upcoming chapters. We will assume the reader is comfortable with many of the topics covered in a standard first year graduate level topology course (see [15] and [29]). For an additional resource for some of the following material, see [23] and [11]. We begin by defining the n-dimensional sphere $S^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : x_0^2 + \ldots x_n^2 = 1\}$ and the n-dimensional ball $B^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + \ldots x_n^2 \leq 1\}$. A *classical knot* is a smooth embedding of S^1 into S^3. A *knot diagram* D is a projection of a knot K onto a plane with only double points, along with crossing information at the double points. Two knots are *equivalent* if they are (smoothly) isotopic. Typically, two knots are shown to be equivalent by demonstrating a series of deformations taking one knot to the other (usually working with knot diagrams). Showing that two knots are distinct however, requires the use of knot invariants.

1.1.1 Knot Invariants

A *knot invariant* is a quantity defined for all knots that is the same for equivalent knots. More formally, a knot invariant is simply a well-defined function from the set (of equivalence classes) of knots to another set.

Next we define some common knot invariants. The *unknotting number* of a knot K is the minimum number of times a knot needs to pass through itself (i.e. crossing changes) to obtain the unknot (i.e. a knot which bounds a disk). The *genus* of a knot K is $g(K) = \min\{g(\Sigma) : \Sigma \subset S^3, \partial\Sigma = K, \Sigma \text{ is an orientable surface}\}$ where $g(\Sigma)$ denotes the genus of a surface Σ. Related invariants are the 4-ball genus and nonorientable 4-ball genus. Note that the boundary of B^4 is equal to S^3, so we can consider surfaces embedded in B^4 whose boundaries lie in S^3. The *4-ball genus* of a knot K is $g_4(K) = \min\{g(\Sigma) : \Sigma \subset B^4, \partial\Sigma = K, \Sigma \text{ is a smoothly embedded orientable surface}\}$. Similarly, the *nonorientable 4-ball genus* of a knot K is $\gamma_4(K) = \min\{b_1(\Sigma) : \Sigma \subset B^4, \partial\Sigma = K, \Sigma \text{ is a smoothly embedded nonorientable surface}\}$, where $b_1(\Sigma)$ denotes the first Betti number of a surface Σ, i.e. the rank of the first homology group $H_1(\Sigma)$. We say a knot is *(smoothly) slice* if $g_4(K) = 0$. In other words, a knot is slice if it bounds a smoothly embedded disk in B^4.

1.1.2 Band Surgeries and Cobordisms

A *band surgery* (also sometimes called a *band resolution* or *band move*) is shown in Figures 1.1 and 1.2. Formally, a band surgery is performed by starting with an embedded band $[0,1] \times [0,1]$ in S^3 that meets an oriented link L in exactly the image of $[0,1] \times \{0,1\}$. We then remove the image of $[0,1] \times \{0,1\}$ from L and replace it with (the image of) $\{0,1\} \times [0,1]$ to obtain a new link L'. A

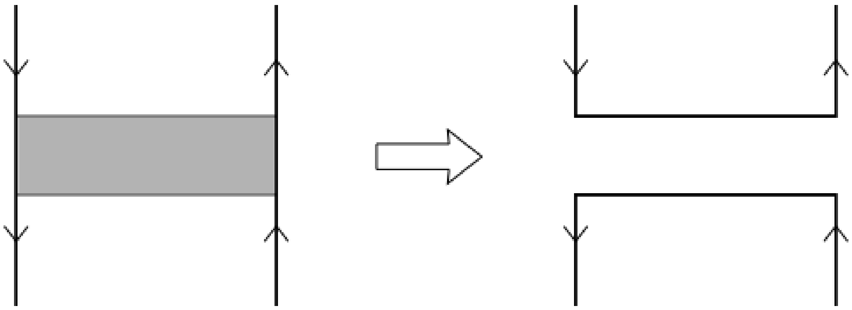

Figure 1.1: An orientation preserving band surgery.

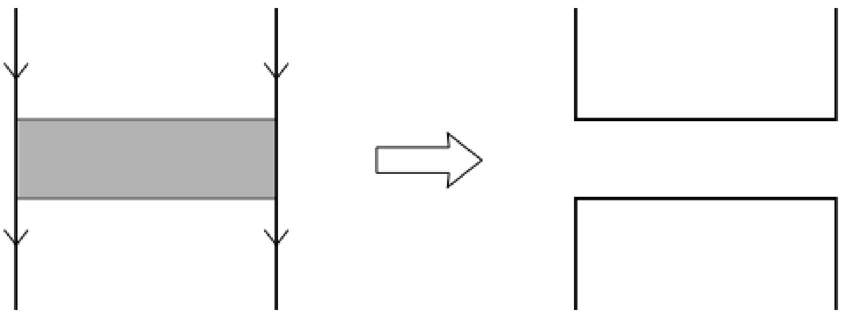

Figure 1.2: An orientation reversing band surgery. The name *reversing* comes from the fact that the band yields an orientation reversing loop in the cobordism the band defines.

band surgery is *orientable* if the band is orientation-preserving (Figure 1.1), and is *nonorientable* if the band is orientation-reversing (Figure 1.2).

Similar to the unknotting number, we can define the *band unknotting number* of a knot K, denoted by $u_b(K)$, to be the minimum number of band surgeries needed to obtain the unknot. Related invariants are $u_2(K)$, the minimum number of *component preserving* band surgeries taking K to the unknot, and $u_0(K)$, the minimum number of *orientation preserving* band surgeries taking K to the unknot. In [2], it was proved that $u_b(K)$ is equal to either $u_2(K)$ or $u_2(K) - 1$, and $u_b(K) = u_2(K)$ if either $u_b(K)$ is odd, $u_2(K)$ is even, or $u_2(K) = 1$. It is also proved in [1] that $u_b(K) \leq u(K) + 1$ and $u_b(K) \leq c(K)/2$ for all K, where $c(K)$ denotes the crossing number of a knot K.

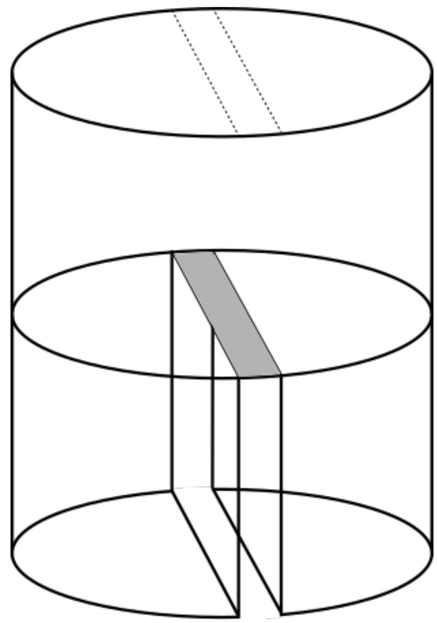

Figure 1.3: A cobordism from an unknot to a two component unlink described by a band. The surface looks like a pair of pants.

A *cobordism* between two closed n-manifolds M_1 and M_2 is an $(n+1)$-manifold W such that $\partial W = M_1 \cup M_2$. Note that we do not require M_i to be connected. A simple example of a cobordism is a cylinder. A cylinder $C = S^1 \times [0,1]$ has two boundary components $\partial C = (S^1 \times 0) \cup (S^1 \times 1)$, hence C is a cobordism between two copies of S^1.

A band surgery on a knot also describes a cobordism. Given a knot K and a band B attached to K, we can perform a band surgery to obtain a knot K'. The cobordism described by this band surgery can be constructed as $(K \times [0, \frac{1}{2}]) \cup ((K \cup B) \times \{\frac{1}{2}\}) \cup (K' \times [\frac{1}{2}, 1])$. See Figure 1.3 for an example of a cobordism described by an orientable band surgery on the unknot.

Returning to reality a bit, we now explore some examples of common knots. Specifically, we examine two of the simplest and most well understood families of knots: 2-bridge knots and torus knots.

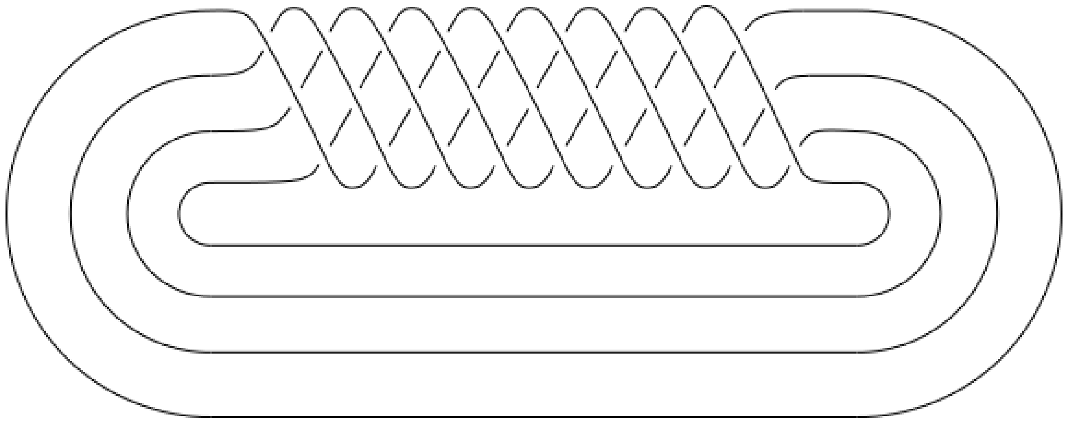

Figure 1.4: The torus knot $T(4,9)$.

1.1.3 Torus Knots

First, we discuss torus knots. A *torus knot* is a knot which can be embedded on the standardly embedded torus $T^2 \subset R^3$ or S^3 (or equivalently, a Heegaard torus for S^3, which will be discussed later). Since a knot K which lives on a torus T^2 can be viewed as an element of $\pi_1(T^2)$, and $\pi_1(T^2) = \mathbb{Z}^2$ can be generated by a meridian μ and longitude λ for the torus, then we can express $[K] \in \pi_1(T^2)$ as $[K] = p[\lambda] + q[\mu]$ for some $p, q \in \mathbb{Z}$. We note that p and q must be relatively prime since K is connected (if $\gcd(p,q) > 1$, then K must be a link). Also, for any relatively prime $p, q \in \mathbb{Z}$, we can construct a knot $K \subset T^2$ such that $[K] = p[\lambda] + q[\mu]$. Thus, there is a bijection between torus knots and pairs of relatively prime integers, or $\frac{p}{q} \in \mathbb{Q} \cup \{\frac{1}{0}\}$. If a torus knot corresponds to the reduced fraction $\frac{p}{q}$, we denote this torus knot by $T(p,q)$. See Figure 1.4 for a diagram of the torus knot $T(4,9)$.

1.1.4 2-bridge Knots

A more detailed exposition on the content of this section can be found in [10] and [35]. A knot K is a b-bridge knot if there is a diagram for K which has a height function with b local minima (and maxima), and there are no diagrams for K with *fewer* than b local minima (with respect to some height function). See Figure 1.5 for a diagram of a 2-bridge knot. The family of 2-bridge knots is well understood. All 2-bridge knots can be represented by a diagram, such as in Figure 1.5, which has alternating columns of c_1 right handed twists, c_2 left handed twists, c_3 right handed twists, c_4 left handed twists, and so on (where c_i is any integer, and a negative value of c_i changes the sign of the crossings in that column of twists). Note that a continued fraction expansion $[c_1, \ldots, c_n]$ with each c_i positive corresponds to an alternating knot diagram. Such a diagram corresponds to an $n-$tuple of integers $[c_1, \ldots, c_n]$ (where a negative c_i means that the signs of the twists change. For example, if there were $c_3 = -2$ right handed twists, then this is equivalent to $+2$ left handed twists). This $n-$tuple corresponds to the continued fraction $c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \frac{1}{c_4 + \ldots}}}$ which can be simplified to a fraction $\frac{p}{q}$. Any fraction $\frac{p}{q}$ in simplest terms can be expanded to finitely many continued fraction expansions $c_1 + \frac{1}{c_2 + \frac{1}{c_3 + \frac{1}{c_4 + \ldots}}}$ with each $|c_i| \geq 2$, and the knot diagrams corresponding to these different expansions will represent equivalent knots. Moreover, two 2-bridge knots (or links) with corresponding fractions $\frac{p}{q}$ and $\frac{p'}{q'}$ are equivalent as oriented knots (or links) if and only if $p = p'$ and $q^{\pm 1} = q' \mod 2p$ (for unoriented knots or links, the condition is $p = p'$ and $q^{\pm 1} = q' \mod p$). See [10] for the complete statement and [35] for a proof.

Another way to classify 2-bridge knots is by classifying them as unions of rational tangles. A *rational tangle* (τ, B) is a pair of disjoint arcs $\tau = \alpha \cup \beta$ embedded in a 3-ball B with $\partial\tau = \partial\tau \cap \partial B = \{x_1, x_2, x_3, x_4\}$ for some distinct points $x_i \in \partial B$ such that there exists an isotopy of τ rel boundary taking τ into ∂B. We can view the torus T^2 as a 2-fold branched double cover of

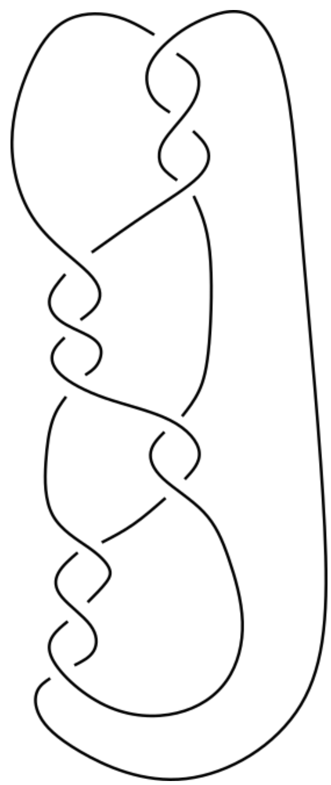

Figure 1.5: The 2-bridge knot with continued fraction expansion $[3, 3, -2, 3]$.

∂B branched over the four points $\{x_1, x_2, x_3, x_4\}$. Each rational tangle (τ, B) can be associated to a fraction $p/q \in \mathbb{Q} \cup \{1/0\}$ by isotoping τ into ∂B and taking a lift of τ to a simple closed curve in T^2, which (after fixing a basis for $H_1(T^2, \mathbb{Z})$) is canonically associated to a unique fraction $p/q \in \mathbb{Q} \cup \{1/0\}$. We sometimes use the notation $\tau\,(p/q)$ to denote the rational tangle with fraction p/q.

We can associate a rational tangle to a given continued fraction expansion in a natural way as well. Take the trivial tangle in a 3-ball consisting of two horizontal arcs. We label the boundary points x_1, x_2, x_3, x_4 counterclockwise such that the top horizontal arc has boundary points x_1 and x_2 and the bottom arc has boundary points x_3 and x_4. For the continued fraction expansion $[c_0, \ldots, c_n]$, we add c_0 (horizontal) half twists (imagine grabbing the points x_1

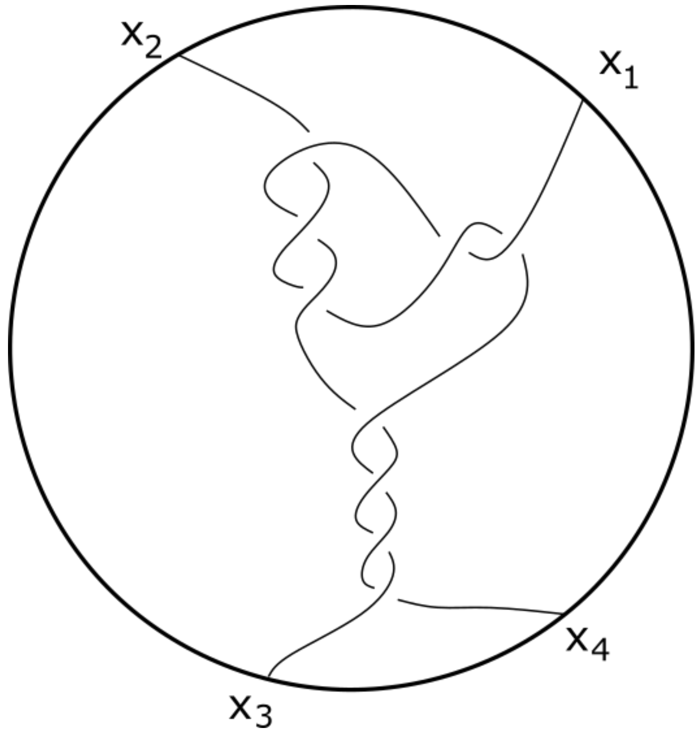

Figure 1.6: The rational tangle corresponding to the continued fraction expansion [3,2,4].

and x_4 on the right side of the diagram and giving them c_0 half twists), then c_1 (vertical) half twists to the bottom two points x_3 and x_4 of the diagram, then c_2 (horizontal) half twists to the right side of the diagrm, c_3 half twists to the bottom of the diagram, and so on. See Figure 1.6 for a diagram of the rational tangle corresponding to the continued fraction expansion [3,2,4].

Assume $\tau \subset \partial B$ and identify ∂B with two unit squares $[0,1] \times [0,1]$ glued along their boundary so that $\partial\tau$ is identified with the four corners, $\alpha = [0,1] \times \{0\}$ lifts to a longitude, and $\beta = \{0\} \times [0,1]$ lifts to a meridian. If τ cannot be isotoped into one of the unit squares (i.e. p/q is not equal to $0/1$, $1/0$, or $\pm 1/1$), then we can compute the associated fraction p/q by counting intersections between τ with α and β (after removing any bigons between τ and α or β). If $|\tau \cap \alpha| = n$ and $|\tau \cap \beta| = m$, then $p = n + 1$ and $q = m + 1$.

The following is described in greater detail in [12]. Any matrix $A \in SL(2,\mathbb{Z})$ defines a homeomorphism of T^2 which takes a p/q curve to an r/s curve, where $A \cdot \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} r \\ s \end{bmatrix}$. For any such matrix A, there also exists a homeomorphism of $\partial B \cong S^2$ fixing the four points $\partial\tau$ setwise (we can think of this as a homeomorphism of the four punctured sphere) which extends to a homeomorphism of B and takes a p/q tangle to an r/s tangle. Let $B_1 \cup_S B_2$ be a genus 0 Heegaard splitting of S^3. Then K is a 2-bridge knot if and only if K can be isotoped to a knot K' such that $(K' \cap B_i, B_i)$ is a rational tangle for each i. So K' can be written as $K' = \tau\,(p_1/q_1) \cup \tau\,(p_2/q_2)$ for some $p_i/q_i \in \mathbb{Q} \cup \{1/0\}$. Then a matrix $A \in SL(2,\mathbb{Z})$ defines a homeomorphism of $B_1 \cup B_2$ taking $\tau\,(p_1/q_1) \cup \tau\,(p_2/q_2)$ to $\tau\,(r_1/s_1) \cup \tau\,(r_2/s_2)$ where $A \cdot \begin{bmatrix} p_i \\ q_i \end{bmatrix} = \begin{bmatrix} r_i \\ s_i \end{bmatrix}$ for each i. So K' is equivalent to the knot $\tau\,(r_1/s_1) \cup \tau\,(r_2/s_2)$.

While we have discussed the fraction of a rational tangle, we have not discussed the fraction of a *union* of rational tangles. However, we can simply reduce to the previous cases as follows. If $K = \tau(\frac{p}{q}) \cup \tau(\frac{r}{s})$, then K is a 2-bridge knot and hence has an associated fraction. One way to determine this fraction is to find a matrix $A \in SL(2,\mathbb{Z})$ such that $A \cdot \begin{bmatrix} r \\ s \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Then $A \cdot \begin{bmatrix} p \\ q \end{bmatrix} = \begin{bmatrix} p' \\ q' \end{bmatrix}$ for some integers p' and q', and $\frac{p'}{q'}$ is the fraction associated to K.

1.2 Handlebodies and Heegaard Splittings

An *n-dimensional k handle* attached to an n-manifold X is a copy of $D^k \times D^{n-k}$ glued along $(\partial D^k) \times D^{n-k} = S^{k-1} \times D^{n-k}$ in ∂X. We call $(\partial D^k) \times 0 = S^{k-1} \times 0$ the *attaching sphere* for the k-handle. Similarly, the *belt sphere* is $0 \times \partial(D^{n-k}) = 0 \times S^{n-k-1}$. The *core* is $D^k \times 0$ and the *co-core* is $0 \times D^{n-k}$. For example, a 3-dimensional 1-handle is copy of $D^1 \times D^2$ glued along $S^0 \times D^2$, i.e. glued along two disjoint copies of D^2. Intuitively, a 1-handle is a cylinder which is glued along the top and bottom disks (or a 'soup can' glued along its top and bottom). A 3-dimensional 2-handle is also a cylinder $D^2 \times D^1$, but is glued along $S^1 \times D^1$, an annulus. Intuitively we can think of this as a soup can glued along the its 'label' (the outer annulus of the can). A 3-dimensional 3-handle is a copy of $D^3 \times D^0 = D^3 \times \{*\}$ glued along $\partial D^3 \times \{*\} = S^2 \times \{*\}$. So we can think of attaching a 3-dimensional 3-handle as just filling in a ball along any unfilled sphere S^2 in our manifold. A 3-dimensional 0-handle is a copy of $D^0 \times D^3 = \{*\} \times D^3$ glued along $\partial D^0 \times D^3 = \emptyset$. In other words, a 0-handle is just a ball with no attaching region.

A k-handle is important to the notion of a *handle decomposition* of a manifold, i.e. a decomposition of an n-manifold X as a union of n-dimensional handles built on top of each other. We start by decomposing ∂X into a lower and upper boundary, $\partial X = \partial_- X \sqcup \partial_+ X$ (either or both may be empty) and then attach handles along $\partial_- X \times \{1\} \subset \partial_- X \times [0, 1]$. We can introduce some number of 0-handles, then glue on 1-handles, 2-handles, and so on, to obtain our manifold X (we do not need to glue in increasing order of the index k, but it can always be arranged so that this is the case). It is important to note that all smooth n-manifolds have a handle decomposition.

The following is explained in further detail in [14]. Recall that an n-dimensional k-handle is a copy of $D^k \times D^{n-k}$ attached along $S^{k-1} \times D^{n-k}$. In a handle decomposition of X with $\partial X = \partial_- X \sqcup \partial_+ X$, we start with $\partial_- X \times [0, 1]$, introduce some 0-handles, then glue some number of 1-handles, then 2-handles, and so on until we obtain X. However, we could instead start with $\partial_+ X \times [0, 1]$ and attach each handle in decreasing order (starting with n handles, then $(n-1)$-handles, and so on). By attaching each handle along the complement of the attaching region in the boundary of the handle, we can view each n-dimensional k-handle as an $(n-k)$-handle. Note that doing this reverses the roles of core and co-core. We often refer to this as turning the handle decomposition of X upside down, or attaching upside down handles.

A widely studied example relating to the existence of handle decompositions is Heegaard splittings of 3-manifolds. An n-dimensional *handlebody* is a connected n-manifold consisting of only n-dimensional 0-handles and 1-handles. Equivalently, a handlebody is an n-dimensional thickening (or regular neighborhood) of a graph. A *Heegaard splitting* of a 3-manifold M is a decomposition of M into two handlebodies $M = V \cup W$ such that $V \cap W = \partial V \cap \partial W$ is a genus n orientable surface. Any closed, orientable 3-manifold has a Heegaard splitting. One way to see this is to use the fact that any such 3-manifold admits a triangulation. The 1-skeleton of this triangulation can be viewed as a graph G_1, and we can create a dual graph G_2 by assigning a vertex to each 3-cell of the triangulation and an edge between vertices corresponding to 3-cells which meet along a 2-cell. A regular neighborhood of G_1 is one handlebody V, and the complement W of V is a regular neighborhood of G_2, and hence $V \cup W$ is a Heegaard splitting of the triangulated 3-manifold.

1.3 Knotted Surfaces

We now move on to the theory of knotted surfaces. Many of the definitions will be similar to those given for classical knots in S^3, so we invite the reader to compare the definitions for further understanding. A *knotted surface* is a smooth embedding of a surface Σ into S^4 (or some other manifold). A *2-knot* is a knotted sphere, i.e. a smooth embedding of S^2 into S^4. Two knotted surfaces are *equivalent* if there is a smooth isotopy of S^4 taking one surface to the other.

Note that knotted surfaces may or may not be orientable. An orientable knotted surface Σ is *unknotted* if Σ bounds a solid handlebody in S^4 (similar to how a classical knot is unknotted if it bounds a disk). Defining a notion of what is means to be unknotted for nonorientable surfaces is somewhat trickier, however. We will discuss this in section 1.3.1.

1.3.1 Examples and Descriptions of Knotted Surfaces

One widely studied example of a nontrivial knotted surface is the spin of a classical knot. Informally, imagine taking a knot K in S^3 and cutting it at a point, then gluing the two loose ends to an axis of rotation to obtain a knotted arc K^+ in $\mathbb{R}^3_+$ (thought of as a sheet in $\mathbb{R}^4 \cong \mathbb{R}^3_+ \times S^1/((x,\theta) \sim (x,\psi)$ for all $x \in \partial\mathbb{R}^3_+, \theta, \psi \in S^1)$. Spinning this knotted arc around the axis of rotation will trace out a surface, called the *spin* of the classical knot K. If while rotating, we additionally twist the knotted arc a full n rotations, we trace out a different surface, called the *n-twist spin* of the classical knot K. Note that spinning an arc around an axis traces out a sphere, so we sometimes instead refer to the n-twist spin of a knot K as the *n-twist spun K sphere*, denoted by $\tau^n K$.

While sometimes we describe knotted surfaces with a description of how they are built, such as our definition of $\tau^n K$, we often prefer to have a diagram of the surface. One of the most common types of diagrams is a banded unlink diagram. A *banded unlink diagram* is a diagram of an unlink L and some number of bands such that surgery on the bands yields another unlink L'. A banded unlink diagram describes a cobordism from L to L' as described in Section 1.1. This cobordism has two unlinks as its boundary, and hence can be capped off uniquely with disks to obtain a closed surface Σ in S^4. Thus, a banded unlink diagram describes a unique closed surface. In Figure 1.7, two banded unlink diagrams are shown for embeddings of the projective plane. These embeddings are distinct, however, and can be distinguished by their Euler number, which is related to their self intersection number (see [28] for a definition). A nonorientable surface is *unknotted* if it is isotopic to a connected sum of some number of these two standard embeddings. We note that Euler number is additive under connected sum, so any unknotted nonorientable surface is specified by its (nonorientable) genus and Euler number. We later in this paper denote the unknotted nonorientable surface with genus g and Euler number e by $P_g(e)$. For a quick example, $P_3(2)$ is isotopic to the connected sum of three unknotted projective planes, two of which have Euler number 2, and the third with Euler number -2.

Another common diagram used to describe a knotted surface is a *broken surface diagram*. Choose a projection of $\mathbb{R}^4$ onto $\mathbb{R}^3$ and consider a knotted surface Σ embedded in $\mathbb{R}^4$. We can perturb Σ (or the projection) slightly so that the image of Σ in $\mathbb{R}^3$ under the projection is consists of only triple points, double points, branch points, and nonsingular points. We indicate

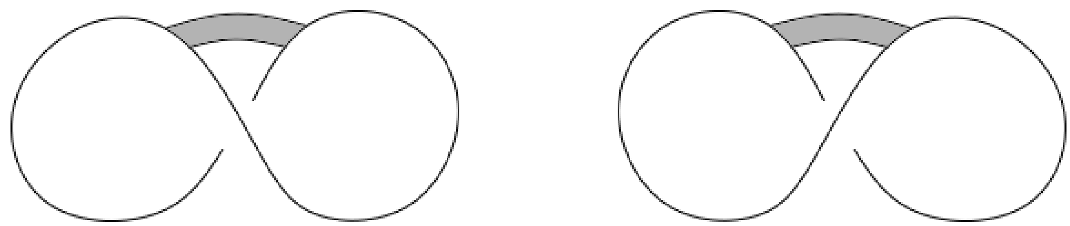

Figure 1.7: Banded unlink diagrams for the two standard embeddings of the projective plane, distinguished by their Euler numbers 2 and -2, respectively.

crossing information in a similar way to how we indicate crossing information at double points in classical knot diagrams - by drawing a break in the under strand. At a curve of double points where two sheets of Σ intersect, we indicate which sheet lies below by "breaking" the sheet at the double point curve.

1.3.2 Surgery and Handle Addition

In Section 1.1, band surgery was defined. Band surgery is a special case of surgery on an n-manifold. Surgery is closely related to the attachment of n-dimensional k-handles. If $D^k \times D^{n-k}$ is an n-dimensional k-handle which meets an $(n-1)$-manifold X in $(\partial D^k) \times D^{n-k} = S^{k-1} \times D^{n-k}$, we can perform *surgery* on the k-handle by replacing $X \cap (D^k \times D^{n-k}) = S^{k-1} \times D^{n-k}$ with $D^k \times (\partial D^{n-k}) = D^k \times S^{n-k-1}$, the other portion of the boundary of the k-handle $D^k \times D^{n-k}$. This viewpoint of surgery is especially useful when considering n-manifolds X embedded in higher dimensional m-manifolds Y, such as classical knots and knotted surfaces. If the k-handle is required to be embedded in Y (which it usually is), then performing surgery on X along this k-handle results in a new n-manifold X' which is also embedded in Y.

Band surgery as defined in Section 1.1 is exactly surgery on classical knot $K \subset S^3$ along a 2-dimensional 1-handle $D^1 \times D^1$. In the same vein, we will consider surgery along 1-handles for knotted surfaces as well. Specifically, we

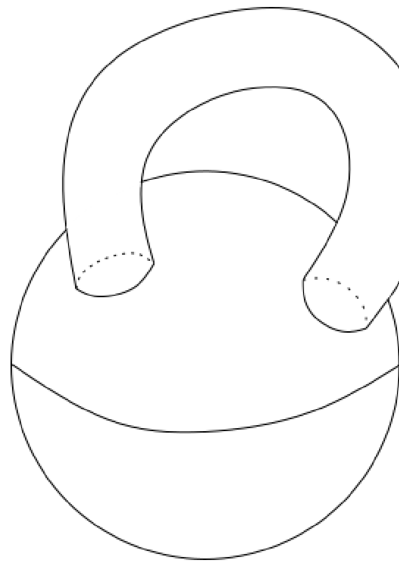

Figure 1.8: Adding a 1-handle to an unknotted sphere.

will often consider surgery on a knotted surface (a 2-manifold Σ embedded in a 4-manifold X, usually $X = S^4$) along a 3-dimensional 1-handle $D^1 \times D^2$. That is, if $D^1 \times D^2$ meets Σ in $S^0 \times D^2$ (two disjoint disks), we alter Σ by replacing $\Sigma \cap D^1 \times D^2 = S^0 \times D^2$ with $D^1 \times (\partial D^2) = D^1 \times S^1$. We often denote 1-handle addition to a knotted surface Σ along the 1-handle h by $\Sigma + h$. See Figure 1.8 for an example of 1-handle addition on an unknotted sphere.

Interestingly, 1-handle addition is an unknotting operation for knotted surfaces in S^4 [16]. One way to see this is to leverage the fact that any knotted surface Σ bounds an orientable 3-manifold X (see [11] or [37]). Then X has a handle decomposition consisting of a single 0-handle, some 1-handles, and some 2-handles. We can then view the 2-handles as upside down 1-handles attached to Σ, so the union of Σ and these upside down 2-handles bounds the 0- and 1-handles of X. Hence, surgery on Σ along the upside down 2-handles yields a new surface Σ' which bounds a handlebody.

1.4 Knot Groups

We begin with a discussion on group presentations. Let A be a finite set (our *alphabet*). The *Kleene star on* A is the set $A^* = \{a_1a_2\dots a_n | a_i \in A, n \in \mathbb{N}_0\}$. Additionally, we define the sets $A^{-1} = \{a^{-1} | a \in A\}$ and $A^{\pm 1} = A \cup A^{-1}$. The *free group on* A is the group $F(A) = (A^{\pm 1})^* / \sim$, where $\sim$ is the smallest equivalence relation on A^* such that $wxx^{-1}u \sim wu \sim wx^{-1}xu$ for all $x \in A$ and all $w, u \in A^*$. The group operation $\cdot : F(A) \times F(A) \to F(A)$ is given by $[w] \cdot [v] = [wv]$, where $w, v \in A^{\pm 1}$ and wv is the concatenation of the words w and v. Let R be a subset of A^* and $\langle R \rangle^N$ the normal subgroup generated by R. Then we denote by $\langle A|R \rangle$ the quotient group $F(A)/\langle R \rangle^N$. The group $\langle A|R \rangle$ is generated by the elements of A, and we say the elements of R are *relators* for the group $\langle A|R \rangle$. If a group G is isomorphic to $\langle A|R \rangle$, we say $\langle A|R \rangle$ is a *group presentation* for the group G. If G can be presented by a group $\langle A|R \rangle$ where A is a finite set, we say G is finitely generated. If G can be presented by a group $\langle A|R \rangle$ where both A and R are finite sets, we say G is finitely presented.

We note that while two words $u, v \in A^*$ may be distinct, they may be equal in the group $G = \langle A|R \rangle$. We use the notation $u =_G v$ to mean that u and v are equal in the group G. For a quick example illustrating this, consider the group $\mathbb{Z}^2 = \langle x, y | xy = yx \rangle$. The words $u = xyxy$ and $v = x^2y^2$ are distinct when considered only as words in A^*, but are equal in $\mathbb{Z}^2$ since $u = xyxy =_{\mathbb{Z}^2} x(xy)y = x^2y^2 = v$. So $u =_{\mathbb{Z}^2} v$.

The *fundamental group* of a knot is simply the fundamental group of the exterior of the knot. More precisely, if an n-manifold M^n is embedded in an $n+2$-manifold X^{n+2}, then the group of M^n is $\pi_1(X \setminus \nu M)$, where νM denotes a regular neighborhood of M in X. For classical knots, M is an embedded copy of S^1 and typically $X = S^3$. For knotted surfaces, M is any 2-dimensional surface and X is any 4-manifold, typically S^4.

The *peripheral subgroup* of $M^n \subset X^{n+2}$ is the group $P = \pi_1(\partial\nu M)$, where $\partial\nu M$ denotes the boundary of a regular neighborhood of M. Note that if we choose the basepoint of $\pi_1(X \setminus M)$ to be a point $x \in \partial\nu M$, then there is a natural way to view $\pi_1(\partial\nu M)$ as a subgroup of $\pi_1(X \setminus M)$. The *positive peripheral subgroup* of $M^n \subset X^{n+2}$ is the subgroup P^+ of $\pi_1(\partial\nu(X \setminus M))$ corresponding to orientation preserving loops. Note that if M and X are both orientable, then $P^+ = P$.

It is well known that the abelianization of the fundamental group of a topological space X is isomorphic to the first homology group of X. It is also well known that the fundamental group of any knot exterior has a Wirtinger presentation (see [15]); in particular, there exists a generator x such that all other generators of the group are conjugate to x. This implies that the abelianization of the fundamental group of a knot exterior is isomorphic to $\mathbb{Z}$. In fact, the fundamental group of any knot exterior $S^3 \setminus K$ or $S^4 \setminus \Sigma$, where Σ is an orientable surface, is isomorphic to $\mathbb{Z}$. A proof that the first homology group of $S^4 \setminus \Sigma$ is isomorphic to $\mathbb{Z}$ comes from Alexander duality. In particular, Theorem 3.44 of [15] (page 254) implies that $H_1(S^4 \setminus \Sigma; \mathbb{Z}) \cong H^2(\Sigma; \mathbb{Z}) \cong \mathbb{Z}$ for an orientable surface Σ. A loop γ in the boundary of a knot exterior $S^{n+2} \setminus \nu M$ is called a *meridian* if the image of $[\gamma]$ in $H_1(S^{n+2} \setminus \nu M^n, \mathbb{Z}) \cong \mathbb{Z}$ under the abelianization map is a generator (where M is orientable). Alternatively, γ is a meridian of the knot K (or knotted surface Σ) if γ bounds a disk D^2 which meets K (or Σ) transversely in a single point.

CHAPTER 2

On 2-knots and Connected Sums with Projective Planes

Much of the material in this chapter appears in [27].

2.1 Introduction

One of the earliest examples of knotted surfaces in S^4 is Artin's spun knot [3], later generalized to twist spun knots by Zeeman in [38] (see Subsection 1.3.1). These knots have easily computable fundamental groups and canonical broken surface diagrams associated to them (see [32]); as such, they both provide a good starting point for many interesting questions about knotted surfaces. Problem 4.58 on Kirby's list asks whether the connected sum of an unknotted projective plane with an odd twist spun knot is always equivalent (via a diffeomorphism of S^4) to the unknotted projective plane; these (connected sums of) knots all have associated fundamental group $\mathbb{Z}_2$ (although for even twist spun knots, their groups are typically not cyclic). By Freedman's topological s-cobordism theorem (see [13]), it follows that there is a pairwise homeomorphism between $(S^4, \mathbb{RP}^2)$ and $(S^4, K\#\mathbb{RP}^2)$, where K is any odd twist spun knot. By Theorem 1 of [8], it follows that these knots become smoothly isotopic after enough internal stabilizations (i.e. connected sums with an unknotted torus $T^2 \subset S^4$). While it is still unknown whether the knots are diffeomorphic without stabilizing, in this chapter we generalize a result of Satoh to show that they become smoothly isotopic after a single (trivial) internal stabilization for any knot K. This falls in line with each of the examples given in [8], where one internal stabilization is all that is needed to make a pair of exotically embedded surfaces smoothly isotopic. We invite the reader to compare these results with those in [5], [4], and [7] where one *external* stabilization is all that is needed to make certain exotic smooth structures on 4-manifolds diffeomorphic.

2.2 Preliminaries

Let K^+ be a knotted arc in $\mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 | z \geq 0\}$ with endpoints in the boundary. Recall from Chapter 1 that we can define the n-twist spin of K, $\tau^n K$, by spinning the knotted arc K^+ once around an axis of twisting while simultaneously twisting K^+ a full n rotations (imagining the knotted portion of K^+ as living within a ball which rotates while spinning around the axis). If instead we replace K^+ with a knot K that does not intersect the boundary of $\mathbb{R}^3_+$, we obtain the *n-twist spun K torus*, which we denote by $\sigma^n K$ to be consistent with the notation used in [34] (cf. [9]). If h is a 1-handle whose core is contained in the axis of spinning of $\tau^n K$, then $\tau^n K + h \cong \sigma^n K$, where $\tau^n K + h$ denotes performing surgery along the 1-handle h (see Figure 2.1).

Let $P_g(e)$ be an unknotted and non-orientable surface knot in S^4 specified by its genus g and Euler number e as in 1.3.1 (cf. [19]). The main theorem of this chapter is the following:

Theorem 2.1. *Let K be a classical knot and n be a natural number. If either*

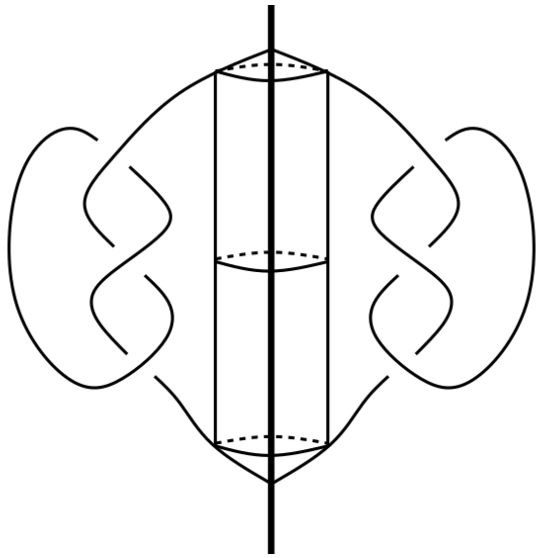

Figure 2.1: A cross-section of a projection of the spun trefoil, along with the 1-handle h whose core is contained in the axis of spinning.

n is odd or K is a 2-bridge knot, then

$$\tau^n K \# P_3(\pm 2) \cong \tau^{n+2} K \# P_3(\pm 2).$$

Note that Theorem 2.1 implies that for all odd n, $\tau^n K \# P_3(\pm 2) \cong P_3(\pm 2)$, since $\tau^1 K$ is an unknotted sphere for any knot K. A proof of Theorem 2.1 in the case that K is a 2-bridge knot and n is any natural number is presented in [34], of which the proof of Theorem 2.1 heavily draws from; however, this proof has a minor error, which we will point out and show how to fix.

2.3 Proof of the Main Theorem

The following lemma is proved by Satoh in [34].

Lemma 2.2. *For any classical knot K and any natural number n,*

$$\sigma^n K \# P_1(\pm 2) \cong \sigma^{n+2} K \# P_1(\pm 2).$$

We present the following lemma.

Lemma 2.3. *Let h be the 1-handle attached to $\tau^n K \# P_1(\pm 2)$ whose core is contained in the axis of twisting of $\tau^n K$. Suppose that either n is odd or K is a 2-bridge knot. Then h is isotopic to a trivial 1-handle.*

The proof in [34] states that if h is merely attached to $\tau^n K$ (instead of $\tau^n K \# P_1(\pm 2)$), then according to [9], h is trivial (for K a 2-bridge knot and n arbitrary); however, this is false in general. While Theorem 14 of [9] did show that h is trivial for $n = 1, 2$, Theorem 15 of [9] showed that if K is the trefoil or figure-8 knot, then h is nontrivial for all $n \neq 1, 2$.

Proof. Let K be a classical knot in S^3 and let $\langle A | R \rangle$ be a Wirtinger presentation for the knot group $\pi_1(S^3 \setminus K)$. Write $A = \{a_1, \ldots, a_k\}$. In [32], it is shown that

$$\langle A | R \cup \{a_1^n a_i a_1^{-n} = a_i | i = 2, \ldots k\} \rangle$$

is a presentation for the surface knot group $\pi_1(S^4 \setminus \tau^n K)$. Note that taking the connected sum of a surface knot S with $P_1(\pm 2)$ results in a knot whose group is obtained by adding the relation $a^2 = 1$ for some meridional generator a of $\pi_1(S^4 \setminus S)$. Thus,

$$\langle A | R \cup \{a_1^2 = 1\} \cup \{a_1^n a_i a_1^{-n} = a_i | i = 2, \ldots k\} \rangle$$

is a presentation for $\pi_1(S^4 \setminus (\tau^n K \# P_1(\pm 2)))$.

Depending on the parity of n, this presentation can be simplified; since $a_1^2 = 1$, then the relation $a_1^n a_i a_1^{-n} = a_i$ is equivalent to $a_1 a_i a_1^{-1} = a_i$ for n odd, and equivalent to the trivial relation $a_i = a_i$ for n even. For n odd, we see that this presentation is equivalent to the presentation

$$\langle A | R \cup \{a_1^2 = 1\} \cup \{a_1 a_i = a_i a_1 | i = 2, \dots k\}\rangle.$$

Since the relators of R imply that all of the generators are conjugate to a_1, this presentation is equivalent to

$$\langle a_1 | a_1^2 = 1\rangle \cong \mathbb{Z}_2.$$

It was proved in [8] that this implies every handle attached to $\tau^n K \# P_1(\pm 2)$ is trivial. For n even, the presentation is equivalent to the presentation

$$\langle A | R \cup \{a_1^2 = 1\}\rangle \cong \pi_1(S^3 \setminus K) / < \mu^2 >^N$$

where μ is a meridional generator for $\pi_1(S^3 \setminus K)$ and $< \mu^2 >^N$ is the normal closure of the subgroup generated by μ^2.

For each n, let λ_n denote the image of a preferred longitude $\lambda' \in \pi_1(S^3 \setminus K)$ of the knot K under the inclusion $(\iota_n)_* : (B^3 \setminus K^+) \times 0 \to S^4 \setminus \tau^n K$, where (B^3, K^+) is the 3-ball, knotted arc pair (here we are using the equivalent definition of $\tau^n K$ as given in [9]). Additionally, for each $n \in \mathbb{N}$, write $G_{1,n} = \pi_1(S^4 \setminus \tau^n K)$ and $G_{2,n} = \pi_1(S^4 \setminus \tau^n K \# P_1(\pm 2))$.

Now restrict to the case where K is a 2-bridge knot and n is even. Note that $\langle A | R \cup \{a_1^2 = 1\}\rangle$ is a group presentation for each $G_{2,n}$ (and hence they are all isomorphic); as such, we will instead write G_2 for each $G_{2,n}$ (where n is even).

By Theorem 14 of [9], λ_2 is trivial. Now, each λ_n can be represented by the same word $w \in A^*$ (specifically, $\lambda' \in \pi_1(S^3 \setminus K)$ can be represented by a word in $w \in A^*$. Now take quotients to get that $\bar{w} =_{G_{1,n}} \lambda_n$, where $\bar{w}$ is the group element that the word w represents in $G_{1,n}$). Furthermore, w also represents the image λ_n' of each λ_n under the quotient map $q_n : G_{1,n} \to G_{2,n} = G_2$ that sends a_1^2 to the identity. The relations for these groups are all equivalent and hence $\lambda_n' =_{G_2} \lambda_m'$ for all even m and n. Since λ_2 is trivial, then $\lambda_n' =_{G_2} 1$ for all even n.

Let P^+ be the positive peripheral subgroup of $\pi_1(S^4 \setminus \tau^n K \# P_1(\pm 2))$. (As usual the peripheral subgroup is the fundamental group of the boundary of the knotted surface exterior. There is a natural homomorphism from the peripheral subgroup to the fundamental group of the surface. Recall from Section 1.4 that the *positive peripheral subgroup* is the subgroup that corresponds to orientation preserving loops.) If B is a 1-handle attached to a surface Σ with oriented core C, we note that the pair (B, C) specifies an element of $\pi_1(S^4 \setminus \Sigma)$. It was proved in [19] that two one handles with oriented cores (B, C) and (B', C') attached to a non-orientable surface knot are equivalent if and only if $P^+(B, C)P^+ = P^+(B', C')P^+$ (we use the notation HgH to denote the double coset $HgH = \{hgk : h, k \in H\}$). If we drag the 1-handle h with its core c (arbitrarily oriented) along the knot K in $(B^3, K) \times \{0\} \subset \tau^n K$ so that both of the basepoints of its core are near the south pole, we can see that the core of h is equivalent to the longitude λ in $\pi_1(S^3 \setminus K)$ or its inverse. Since the image of λ (under the above maps) in $\pi_1(S^4 \setminus \tau^n K \# P_1(\pm 2))$ is trivial, we see that $P^+(h, c)P^+ = P^+$ and hence h is trivial.

□

With the hard part out of the way, we now prove Theorem 2.1. This proof is essentially the same as the one given in [34].

Proof. Assume either n is odd or K is a 2-bridge knot. By Lemma 2.3,

$$\tau^n K \# P_3(\pm 2) \cong (\tau^n K \# P_1(\pm 2)) + h \cong \sigma^n K \# P_1(\pm 2).$$

Then by Lemma 2.2,

$$\sigma^n K \# P_1(\pm 2) \cong \sigma^{n+2} K \# P_1(\pm 2) \cong \tau^{n+2} K \# P_3(\pm 2).$$

□

CHAPTER 3

Unknotting Numbers of Twist Spun Knots

3.1 Introduction

In [33], Satoh proved that both $u(K)$ and $b(K)-1$ are upper bounds for the (4-dimensional) unknotting number of the twist spin of a classical knot K, where $b(K)$ is the bridge number of K and $u(K)$ is the (3-dimensional) unknotting number of K. Here we will show that the band unknotting number of K is also an upper bound for the unknotting number of a twist spin of K. Recall that the *band unknotting number* of a knot K, denoted $u_b(K)$, is the minimum number of band surgeries applied to K needed to obtain the unknot. Related are the invariants $u_2(K)$, the minimum number of *component preserving* band surgeries needed to obtain the unknot, and $u_0(K)$, the minimum number of *orientation preserving* band surgeries needed to obtain the unknot. The invariant u_0 will be the main focus of this chapter, but we will give some brief background on the other invariants as well.

The band unknotting number $u_b(K)$ was first formally introduced in [1]. It was proved in [1] that $u_b(K) \leq u(K) + 1$ and $u_b(K) \leq c(K)/2$ for all K. It was also shown in [2] that $u_b(K)$ is equal to either $u_2(K)$ or $u_2(K) - 1$ for all K. The inequalities $u_b(K) \leq u_2(K)$ and $u_b(K) \leq u_0(K)$ are immediate from their definitions (since u_0 and u_2 simply place restrictions on the bands taking K to the unknot). We note that if K is a knot, then a band surgery on K is orientation reversing if and only if the result of the band surgery is a knot, i.e. any component preserving band surgery is necessarily orientation reversing. Thus, since orientation respecting band surgeries change the number of components of a knot or link by one, then $u_0(K)$ is necessarily even.

3.2 Main Theorem

Let S be a knotted surface in S^4. In [16], it was proved that there exists a finite number of mutually disjoint 1-handles $h_i = D^1 \times D^2$ with $S \cap h_i = (\partial D^1) \times D^2$ such that surgery on S along the collection of h_i yields an unknotted surface S' (i.e. S' bounds a solid handlebody in S^4). Recall from Chapter 1 that the *unknotting number* $u(S)$ is defined to be the minimum number of 1-handles for S that yields an unknotted surface.

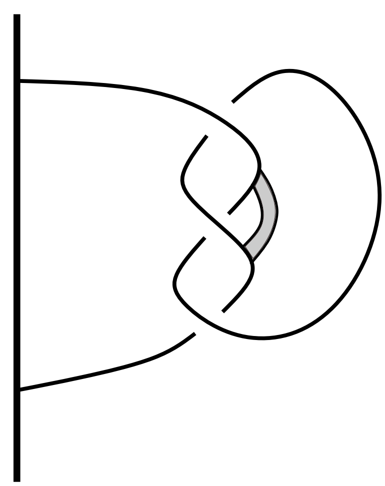

Figure 3.1: A banded spin diagram for a twist spun trefoil.

Let $b = D^1 \times D^1$ be a band attached to K^+ along $\partial D^1 \times D^1$, where the knotted arc K^+ is obtained from the knot K by removing a small neighborhood

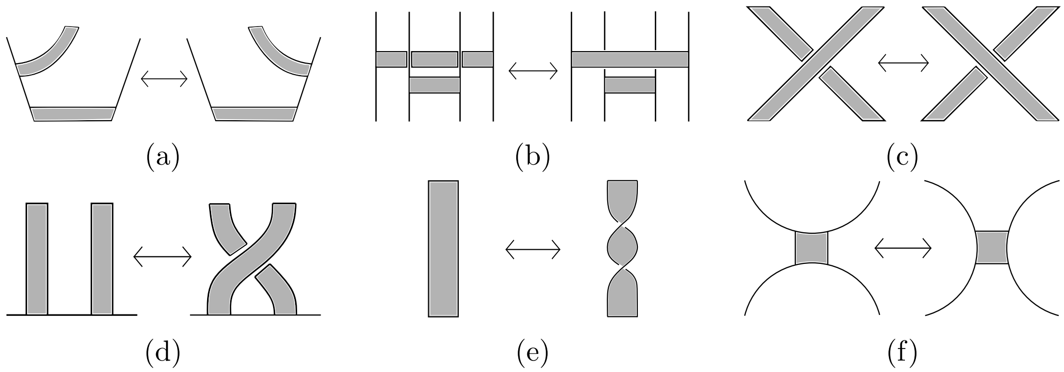

Figure 3.2: Band equivalence moves between banded spin diagrams

of a point from K. Spinning the band b along with the knotted arc K^+ and taking its continuous trace throughout the interval $[0, 1/2]$ describes a 1-handle h associated to b (for simplicity, the reader may assume the twisting of K^+ takes place only on the interval $[1/2, 1]$). Thus, a diagram consisting of a knotted arc K^+ together with a collection of bands $\{b_i\}$ (along with a number n to denote the number of twists, which will usually be omitted) can be used to describe the surface obtained by performing surgery on the n-twist spin of a knot K (denoted $\tau^n K$) along the 1-handles $\{h_i\}$ associated to the bands $\{b_i\}$ (see [33] for a more detailed description). In [33] it was proved that any two such diagrams that are related by a sequence of the band equivalence moves (a)-(f) as in Figure 3.2 (as well as isotopies) define equivalent knotted surfaces. Here we assume the diagrams to be identical outside of the portions pictured. We prove the following.

Theorem 3.1. *For any classical knot K and any $n \in \mathbb{Z}$, we have the inequality $u(\tau^n K) \leq u_0(K)$.*

Proof. Suppose B is a set of $m = u_0(K)$ orientation preserving bands such that $K + B$ is the unknot U, and let H be the m 1-handles described by B. We note that since the bands are orientation preserving, then the 1-handles described by the bands are orientation preserving 1-handles as well, and the surface obtained via these 1-handle additions is also orientable (if the bands were orientation reversing, then we would find an orientation reversing loop in the resulting surface). Then if K^+ is obtained by removing a neighborhood of a point in K, we also get that $K^+ + B$ is an unknotted tangle. Then $K^+ \cup B$ is a diagram for $\tau^n K + H$, and is equivalent via band surgery moves (move (f) in Figure 3.2) to $U^+ \cup B'$ for a set of bands B'. Since $U^+ \cup B'$ is a diagram for an unknotted S^2 with m 1-handles attached, then by Lemma 2.7 from [16], all of the 1-handles are trivial. Thus, $\tau^n K + H$ is an unknotted surface.

□

CHAPTER 4

Counterexamples to Batson's Conjecture

Much of the material in this chapter appears in [26].

4.1 Introduction

Batson's conjecture is a nonorientable analogue of Milnor's conjecture about the (orientable) 4-ball genus of torus knots. Batson's conjecture was recently proved to be false by Lobb in [25]. In this chapter, we provide a new infinite family of counterexamples of the form $T(4n, (2n \pm 1)^2)$. See Figure 4.1 for one such counterexample.

Milnor's conjecture states that the unknotting number of the (p, q) torus knot $T(p, q)$ is equal to $\frac{(p-1)(q-1)}{2}$. However, since the 4-ball genus $g_4(K)$ of a knot K is a lower bound for the unknotting number of K, it suffices to show that $g_4(T(p, q)) = \frac{(p-1)(q-1)}{2}$. This was eventually verified in [21] and [22] using powerful tools from gauge theory. It is natural to ask if there is a similar formulation for the nonorientable 4-ball genus γ_4 for torus knots. Indeed, in [6], it is shown that $\gamma_4(T(2k, 2k-1))$ is equal to the *pinch number* of $T(2k, 2k-1)$, and it was conjectured that $\gamma_4(T(p, q))$ is equal to the pinch number of $T(p, q)$ for all (relatively prime) p, q. This is Batson's conjecture.

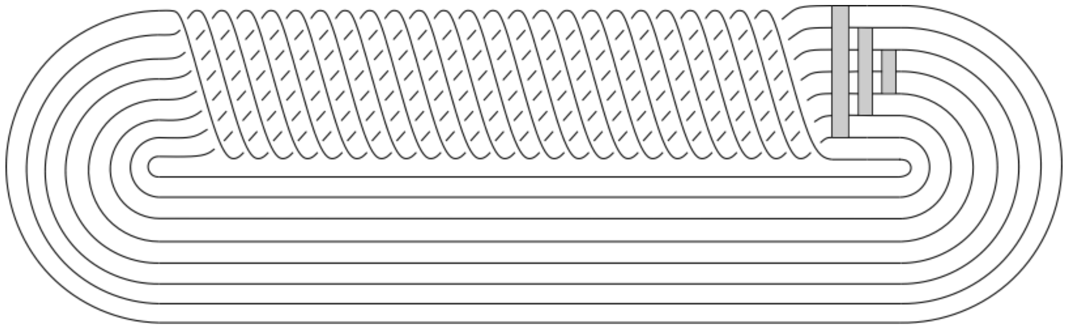

Figure 4.1: The knot $K_2 = T(8, 25)$ along with three orientation reversing bands. Surgery on these bands yields the slice knot 10_3. Labeling the strands 1-8 from top to bottom, the bands attach strands 1 to 7, 2 to 6, and 3 to 5.

Batson's conjecture can be thought of in the following way: a *pinch move* on a torus knot is done by performing surgery on a (blackboard framed) band between two adjacent strands in a standard diagram of a torus knot (see Figure 4.2), called a *pinch band*. Note that a pinch band is necessarily nonorientable. Since the band embeds on the same torus as the torus knot that the band is attached to, then after performing the band surgery, the resulting knot still lives on a torus (and in fact is a less complicated torus knot). Thus, we can always find a sequence of pinch moves resulting in the unknot (called a *pinch sequence*). See Figure 4.2 for the pinch sequence for $T(4, 9)$. Capping off the trace of this sequence of pinch moves with a disk yields a nonorientable surface bounded by the original torus knot. Batson's conjecture essentially asserts that there is no shorter sequence of non-orientable band surgeries taking a torus knot to the unknot (or more generally, a slice knot).

Lobb disproved Batson's conjecture in [25] by showing that $T(4, 9)$ has pinch number two and nonorientable 4-ball genus one. More specifically, Lobb found a nonorientable band surgery which takes $T(4, 9)$ to the Stevedore's knot, which is slice (whereas two pinch moves are needed to obtain the unknot). It was noted in [17] that from this (or any other) counterexample, one can easily

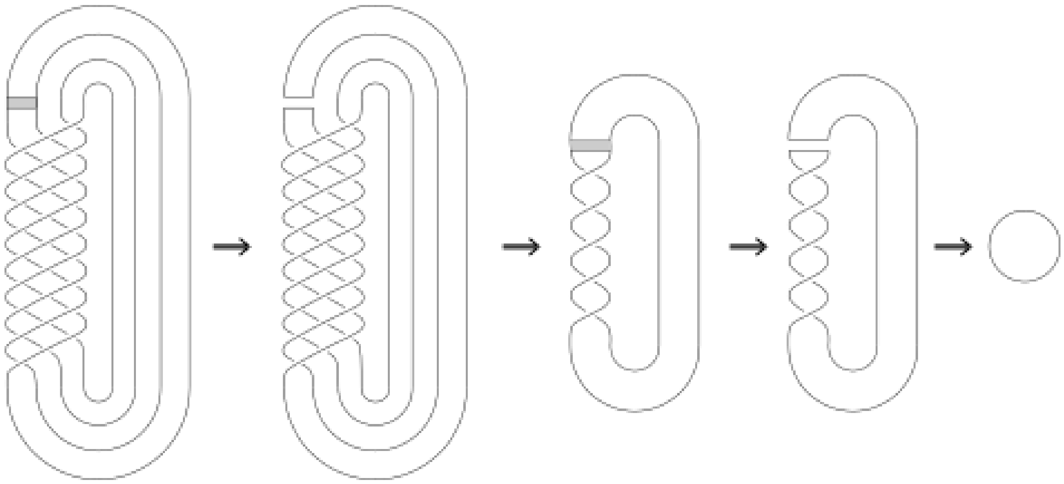

Figure 4.2: A sequence of two pinch moves taking $T(4,9)$ to the unknot. Lobb showed that there is a single band surgery (not pictured) on $T(4,9)$ which yields Stevedore's knot, which is slice.

work backwards to find an infinite sequence of torus knots $\ldots \to T(p_n, q_n) \to T(p_{n-1}, q_{n-1}) \to \ldots \to T(p_0, q_0) = T(4,9)$ where $T(p_i, q_i) \to T(p_{i-1}, q_{i-1})$ denotes a pinch move taking $T(p_i, q_i)$ to $T(p_{i-1}, q_{i-1})$. Then each $T(p_i, q_i)$ has pinch number $i+2$ and nonorientable 4-ball genus at most $i+1$. It is reasonable to wonder if other counterexamples exist which cannot be obtained in this way. In this chapter, we provide a partial answer to this question by providing two new infinite families $\{K_n\}$ and $\{J_n\}$ of counterexamples to Batson's conjecture.

Theorem 4.1. *The torus knots* $K_n = T(4n, (2n+1)^2)$ *(for* $n \geq 1$*) and* $J_n = T(4n, (2n-1)^2)$ *(for* $n \geq 2$*) have pinch number* $2n$ *and nonorientable 4-ball genus at most* $2n-1$*. Specifically, there are* $2n-1$ *band surgeries for* K_n *and* J_n *(which are not all orientation preserving) that yield the slice 2-bridge knot with continued fraction expansion* $[-(2n+2), -2n]$ *and* $[-(2n-2), -2n]$ *respectively.*

We note that Lobb's example $T(4,9)$ appears in these families as K_1. We also note that the knot $J_3 = T(12, 25)$ has a pinch sequence which passes through $T(4, 9)$. In fact, we will show that for each $n > 2$, four pinch moves applied to J_n yields the knot K_{n-2} (Corollary 4.4). We note that $J_2 = T(8, 9)$ does not have a pinch sequence through any nontrivial K_n. We also show that no pinch sequence starting at any K_n passes through any other K_m (Corollary 4.5). The knot $K_2 = T(8, 25)$, which motivated these families of counterexamples, is shown in Figure 4.1, along with three nonorientable band surgeries which yield the slice knot 10_3 in the knot table.

4.2 Preliminaries

Recall that for a knot $K \subset S^3 = \partial B^4$, we define the *4-ball genus* $g_4(K)$ to be the minimum genus of a smooth orientable surface properly embedded in B^4 bounded by K. Similarly, we define the *nonorientable 4-ball genus* $\gamma_4(K)$ to be the minimum value of $b_1(\Sigma)$ for a smooth nonorientable surface $\Sigma \subset B^4$ bounded by K, where $b_1(\Sigma)$ denotes the first Betti number of the surface Σ (i.e. the rank of the first homology group of Σ).

One way to find upper bounds for $g_4(K)$ and $\gamma_4(K)$ for a knot K is to find a set of band surgeries for K that take K to the unknot (or more generally, a slice knot). A sequence of band surgeries taking K to K' describes a cobordism W between K and K', and if K' bounds a disk in B^4, then we can cap off W with that disk to obtain a surface Σ in B^4 with $\partial\Sigma = K$. The resulting surface Σ is orientable if and only if each of the surgeries are performed on orientation preserving bands.

Recall that torus knot is a knot which embeds on a standardly embedded torus $T^2 \subset S^3$. Also recall that a *pinch band* B for a torus knot $K \subset T^2$ is a band attached to K that also embeds on the same torus T^2, and a *pinch move* is surgery on a pinch band (see Figure 4.2). Since a pinch band embeds on the same torus as K, then surgery along a pinch band results in another torus knot. A simple number-theoretic way to compute the resulting torus knot is described in [17] (included in this paper as Lemma 4.3 for convenience), which can be used to compute the number of pinch moves needed to take a torus knot K to the unknot. We call this the *pinch number* of the torus knot K. We note that Lemma 4.3 also implies that the torus knot K' obtained from a pinch move on a torus knot K is unique, and hence there is a unique sequence of pinch moves from any given torus knot to the unknot. (Note that we could also prove uniqueness by observing that any two pinch bands can be isotoped to each other by sliding the band along the torus.)

There is a natural way to find an orientable surface smoothly embedded in B^4 that is bounded by a given torus knot $T(p,q)$. In a sense, Milnor's conjecture states that the surface realizing $g_4(T(p,q))$ is the most natural one. Batson's conjecture essentially asks if the same is true for $\gamma_4(T(p,q))$: is $\gamma_4(T(p,q))$ equal to the pinch number of $T(p,q)$? One can also ask, for which knots does Batson's conjecture hold? Batson proved his conjecture holds for torus knots of the form $T(2k,2k-1)$, and later Jabuka and Van Cott in [17] gave a combinatorial way to tell which knots satisfy Batson's conjecture.

Recall from Section 1.1.4, a rational tangle (τ, B) is a pair of disjoint arcs $\tau = \alpha \cup \beta$ embedded in a 3-ball B with $\partial\tau = \partial\tau \cap \partial B = \{x_1, x_2, x_3, x_4\}$ for some distinct points $x_i \in \partial B$ such that there exists an isotopy of τ rel boundary taking τ onto ∂B. Each rational tangle can be associated to a fraction $p/q \in \mathbb{Q} \cup \{1/0\}$, and this fraction can be computed by counting intersections between τ and curves α and β in ∂B which lift to a meridian and longitude of a torus (viewing the torus as the branched double cover of ∂B branched over $\{x_1, x_2, x_3, x_4\}$). We also recall that if $A \in SL(2,\mathbb{Z})$ and $K = \tau(p_1/q_1) \cup \tau(p_2, q_2)$ is a 2-bridge knot expressed as a union of two rational tangles, then K is equivalent to the 2-bridge knot $K' = \tau(r_1/s_1) \cup \tau(r_2, s_2)$ where $A \cdot \begin{bmatrix} p_i \\ q_i \end{bmatrix} = \begin{bmatrix} r_i \\ s_i \end{bmatrix}$ for each i.

4.3 The Counterexamples

Our counterexamples are given by the knots $K_n = T(4n, (2n+1)^2)$ and $J_n = T(4n, (2n-1)^2)$. The knot $K_1 = T(4, 9)$ is Lobb's example, and the knot $K_2 = T(8, 25)$ is shown in Figure 4.1.

Theorem 4.1. *The torus knots $K_n = T(4n, (2n+1)^2)$ (for $n \geq 1$) and $J_n = T(4n, (2n-1)^2)$ (for $n \geq 2$) have pinch number $2n$ and nonorientable 4-ball genus at most $2n-1$. Specifically, there are $2n-1$ band surgeries for K_n and J_n (which are not all orientation preserving) that yield the slice knot with continued fraction expansion $[-(2n+2), -2n]$ and $[-(2n-2), -2n]$ respectively.*

Theorem 4.1 is a direct consequence of Proposition 4.2 and Proposition 4.6, which we will now state and prove.

Proposition 4.2. *A torus knot of the form $T(4n, (2n \pm 1)^2)$ has pinch number $2n$ (with the exception of $T(4, 1)$, which is unknotted).*

To prove the proposition, we use the following lemma from [17]:

Lemma 4.3. *A pinch move applied to a torus knot $T(p, q)$ yields the torus knot $T(|p-2t|, |q-2h|)$, where t and h are the smallest nonnegative integers satisfying $t \equiv -q^{-1} \mod p$ and $h \equiv p^{-1} \mod q$.*

Note that if a pinch move on $T(p,q)$ yields the knot $T(r,s) = T(|p-2t|,|q-2h|)$, then a pinch move on $T(q,p)$ yields the knot $T(s,r)$: Indeed, if t and h are the smallest nonnegative integers satisfying $t \equiv -q^{-1} \mod p$ and $h \equiv p^{-1} \mod q$ and t' and h' are the smallest nonnegative integers satisfying $t' \equiv q^{-1} \mod p$ and $h' \equiv -p^{-1} \mod q$, then $t' = p - t$ and $h' = q - h$. Then $(p-2t)+(p-2t') = (p-2t)+(p-2(p-t)) = 0$, so $p-2t = -(p-2t')$. Similarly, $q-2h = -(q-2h')$. We get that a pinch move on $T(q,p)$ yields the knot $T(|q-2h'|,|p-2t'|) = T(|q-2h|,|p-2t|) = T(s,r)$. We are now ready to prove Proposition 4.2.

Proof. We first prove a slightly stronger statement: a pinch move applied to a torus knot $T(p,q) = T((2n\pm 1)^2 - 2k(n\pm 1), 4n-2k)$ yields the torus knot $T((2n\pm 1)^2 - 2(k+1)(n\pm 1), 4n-2(k+1))$ (note that we are interchanging the roles of p and q before applying the pinch move, which does not affect the resulting knot). Note that $(2n\pm 1)^2 - 2k(n\pm 1)$ can be rewritten as $(4n-2k)(n\pm 1)+1$.

We first compute $t = -(4n-2k)^{-1} \mod (4n-2k)(n\pm 1)+1$. Since $(4n-2k)\cdot(n\pm 1) \equiv -1 \mod (4n-2k)(n\pm 1)+1$, then $t = n\pm 1$. Then

$$|p-2t| = |(4n-2k)(n\pm 1)+1-2(n\pm 1)| = (4n-2(k+1))(n\pm 1)+1.$$

Next we compute $h = ((4n-2k)(n\pm 1)+1)^{-1} \mod 4n-2k$. Since $(4n-2k)(n\pm 1)+1 \equiv 1 \mod 4n-2k$, then $h = 1$. Then $|q-2h| = |(4n-2k)-2| = 4n-2(k+1)$. Thus, a pinch move applied to a torus knot $T(p,q) = T((2n\pm 1)^2 - 2k(n\pm 1), 4n-2k)$ yields the torus knot $T((2n\pm 1)^2 - 2(k+1)(n\pm 1), 4n-2(k+1))$.

Now, we repeatedly apply $2n$ pinch moves to the knot $T((2n\pm 1)^2, 4n)$ to get the knot $T((4n-2(2n))(n\pm 1)+1, 4n-2(2n)) = T(1,0)$, the unknot. Note that if only $2n-1$ pinch moves are applied, we still have a nontrivial knot

$$T((4n-2(2n-1))(n\pm 1)+1, 4n-2(2n-1)) = T(2(n\pm 1)+1, 2)$$

(with the exception of $T(2(1-1)+1,2) = T(1,2)$). Thus, the pinch number of $T((2n\pm 1)^2, 4n)$ is $2n$.

□

The previous proof is perhaps a bit tedious, so we would like to present a proof which is much quicker and uses an interesting computation tool using continued fraction expansions. This proof method was suggested to the author by Cornelia Van Cott. It was proven in [18] (Proposition 2.3) that a pinch move applied to a torus knot $T(p,q)$, where the fraction $\frac{p}{q}$ has continued fraction expansion $[c_0, c_1, \ldots, c_k]$ (i.e. $\frac{p}{q} = c_0 + \frac{1}{c_1 + \frac{1}{c_2 + \ldots}}$), yields the torus knot $T(r,s)$ where $\frac{r}{s}$ has continued fraction expansion $[c_0, c_1, \ldots, c_k - 2]$. We note that the identities $[c_0, c_1, \ldots, c_n, 0] = [c_0, c_1, \ldots, c_{n-1}]$ and $[c_0, c_1, \ldots, c_n, 1] = [c_0, c_1, \ldots, c_n + 1]$ hold (these identities can be seen by looking at the knot diagrams these continued fraction expansions represent).

Proof. The knot $T(p,q) = T(4n, (2n\pm 1)^2$ satisfies

$$\frac{p}{q} = \frac{4n}{4n^2 \pm 4n + 1} = 0 + \frac{1}{(n\pm 1) + \frac{1}{4n}}.$$

Hence $T(p,q)$ corresponds to the continued fraction expansion $[0, n\pm 1, 4n]$. Thus, according to [18, Proposition 2.3], applying k pinch moves to $T(p,q)$ (for $k < n$) yields the torus knot with corresponding continued fraction expansion

$[0, n \pm 1, 4n - 2k]$. Then applying $k = 2n$ pinch moves yields the torus knot with corresponding continued fraction expansion $[0, n \pm 1, 0] = [0]$, i.e. the unknot. Note that applying $2n - 1$ pinch moves yields the nontrivial knot with corresponding continued fraction expansion $[0, n \pm 1, 2]$. Thus, the pinch number of $T(4n, (2n \pm 1)^2)$ is $2n$.

□

With the previous lemma in mind, the following tables show a sequence of pinch sequences for K_n and J_n respectively. For clarity, we abbreviate all pinch moves $T(p, q) \to T(r, s)$ by $(p, q) \to (r, s)$.

$$
\begin{array}{rcccccccccccc}
K_1 = (4,9) & \to & (2,5) & \to & (0,1) & & & & & & & & \\
K_2 = (8,25) & \to & (6,19) & \to & (4,13) & \to & (2,7) & \to & (0,1) & & & & \\
K_3 = (12,49) & \to & (10,41) & \to & (8,33) & \to & (6,25) & \to & (4,17) & \to & (2,9) & \to & (0,1) \\
K_4 = (16,81) & \to & (14,71) & \to & (12,61) & \to & (10,51) & \to & (8,41) & \to & (6,31) & \to & \\
 & & & & & & & \to & (4,21) & \to & (2,11) & \to & (0,1) \\
K_5 = (20,121) & \to & (18,109) & \to & (16,97) & \to & (14,85) & \to & (12,73) & \to & (10,61) & \to & \\
 & & & \to & (8,49) & \to & (6,37) & \to & (4,25) & \to & (2,13) & \to & (0,1)
\end{array}
$$

Table 4.1: Pinch sequences for $K_n = T(4n, (2n + 1)^2)$.

$$
\begin{array}{rcccccccccccc}
J_2 = (8,9) & \to & (6,7) & \to & (4,5) & \to & (2,3) & \to & (0,1) & & & & \\
J_3 = (12,25) & \to & (10,21) & \to & (8,17) & \to & (6,13) & \to & (4,9) & \to & (2,5) & \to & (0,1) \\
J_4 = (16,49) & \to & (14,43) & \to & (12,37) & \to & (10,31) & \to & (8,25) & \to & (6,19) & \to & \\
 & & & & & & & \to & (4,13) & \to & (2,7) & \to & (0,1) \\
J_5 = (20,81) & \to & (18,73) & \to & (16,65) & \to & (14,57) & \to & (12,49) & \to & (10,41) & \to & \\
 & & & \to & (8,33) & \to & (6,25) & \to & (4,17) & \to & (2,9) & \to & (0,1)
\end{array}
$$

Table 4.2: Pinch sequences for $J_n = T(4n, (2n - 1)^2)$.

The reader may notice from the tables that after four pinch moves to each J_n, we obtain the knot K_{n-2} (where $K_0 = T(0, 1)$). We also notice in the table that the pinch sequence starting at each K_n does not pass through any other K_m. Indeed, both of these statements follow in general as a corollary to (the proof of) Proposition 4.2.

Corollary 4.4. *Applying a sequence of four pinch moves to J_n yields the knot K_{n-2}.*

Proof. It can be seen from either of the proofs of Proposition 4.2 that applying 4 pinch moves to the knot $J_n = T(4n, (2n-1)^2)$ yields the knot $T(4n - 2 \cdot 4, (4n - 2 \cdot 4)(n-1) + 1)$. Since

$$(4n - 2 \cdot 4)(n-1) + 1 = 4n^2 - 12n + 9 = (2(n-2)+1)^2$$

then a sequence of four pinch moves to J_n yields the knot $T(4(n-2), (2(n-2)+1)^2) = K_{n-2}$.

□

As noted earlier, Jabuka and Van Cott mentioned that infinitely many counterexamples to Batson's conjecture can be found by working backwards and finding a pinch sequence that passes through $T(4, 9)$. In a sense, Corollary 4.4 says that all of the counterexamples J_n can be obtained from K_{n-2} by working backwards in the same way, with the exception of $J_2 = T(8, 9)$ (since K_0 is unknotted). The next corollary to Proposition 4.2 says that the family $\{K_n\}$ consists of knots which cannot be obtained from each other in the same way.

Corollary 4.5. *Each nontrivial K_n cannot be obtained from any other K_m from a sequence of pinch moves.*

Proof. If K_n is obtained from some other K_m from a sequence of ℓ pinch moves, the proof of Proposition 4.2 also shows that $K_n = T(4n, 4n(n+1)+1)$ can be written in the form

$$T(4m-2\ell,(4m-2\ell)(m+1)+1).$$

Then $4m-2\ell=4n$ for some $\ell\in\mathbb{Z}$. This implies ℓ is even, so we can instead write $\ell=2k$ and say that $4m-4k=4n$ or $m-k=n$ for some $k\in\mathbb{Z}$.

We also have the equation

$$(4m-4k)(m+1)+1=4n(n+1)+1.$$

This implies

$$\begin{aligned} n(n+k+1) &= n(n+1) \\ n^2+nk+n &= n^2+n \\ nk &= 0. \end{aligned}$$

Since $n\neq 0$, then $k=0$, so $\ell=0$, and $n=m$. So each K_n can not be obtained from K_m from a sequence of pinch moves.

□

We now know that for each n, K_n and J_n have pinch number $2n$ (except for the unknotted J_1). Next we will show that the nonorientable 4-ball genus of each K_n and J_n is bounded above by $2n-1$ by finding a set of $2n-1$ band surgeries for K_n and J_n that yield the unknot. Any given band is an orientation reversing band for K_n or J_n, hence the set of band surgeries describes a nonorientable surface Σ bounded by K_n or J_n with $b_1(\Sigma)=2n-1$.

Proposition 4.6. *There is a set of* $2n-1$ *bands for the knot* $T(4n,(2n\pm1)^2)$ *such that surgery on the set of bands yields the slice 2-bridge knot which has continued fraction expansion* $[-(2n\pm2),-2n]$.

Proof. We consider $T(4n,(2n\pm1)^2)$ as the closure of a braid on $4n$ strands. Label the strands from 1 to $4n$ reading left to right. The braid has $m=n\pm1$ full twists, as well as strand $4n$ crossing over strands $4n-1$ through 1 (reading top to bottom). We attach blackboard framed bands from strand k to $4n-k$ for each $k=1,\dots 2n-1$. The situation is depicted in Figure 4.3. After performing surgery on the bands, we can isotope the braid to have two sets of $2n-1$ semicircles opposite each other. We can pull the bottom half of the semicircles through the m full twists to obtain the diagram in Figure 4.4.

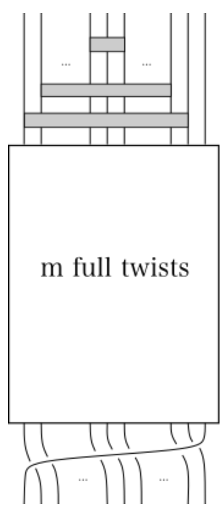

Figure 4.3: A braid whose closure is $T(4n,(2n\pm1)^2)$, along with $2n-1$ bands.

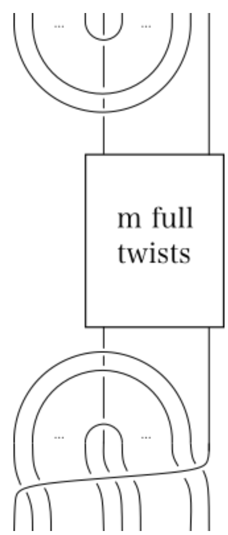

Figure 4.4: We obtain this figure after performing band surgery and an isotopy.

After pulling the bottom half of the semicircles underneath the last strand (and looking at the braid closure of our previous diagram), we obtain the knot diagram in Figure 4.5. Note that this is a 2-bridge knot, as there is a

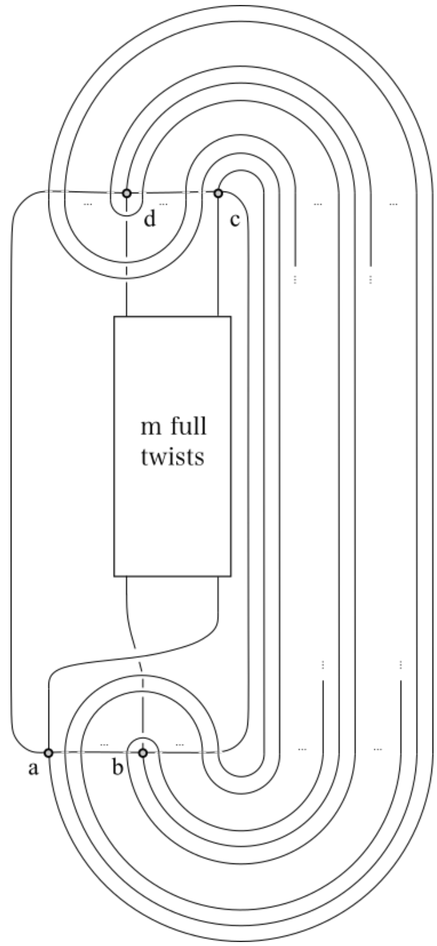

Figure 4.5: After band surgeries and isotopies, we find a 2-sphere, represented by the box with rounded corners, which splits the knot into two rational tangles.

2-sphere S (also shown in Figure 4.5) which intersects the knot in the four points labeled a, b, c, and d and splits the knot into two trivial tangles. Recall that there is a correspondence between 2-strand trivial tangles and rational numbers (as described above). The first trivial tangle consists of m full twists and an additional half twist, i.e. $2m + 1$ half twists. We can compute the slope of this first tangle to be $\frac{1}{2m+1}$, since we can isotope it onto a unit square $[0, 1] \times [0, 1]$ such that the tangle intersects the square in its four corners along with $2m$ intersection points on the left and right sides of the square.

For the second tangle, we first isotope the tangle by pulling the points labeled c and d clockwise around the knot to obtain the picture in Figure 4.6. Now, if we draw arcs horizontally from a to b, b to c, c to d, and d to a (moving right, passing the point at infinity, then appearing from the left side of the page), we can count intersection points of these arcs with the tangle. In fact, the number of intersection points in Figure 4.6 between the tangle and any of the four arcs is equal to $2n-1$, exactly the number of bands we performed surgery on. We can isotope the tangle to remove one intersection point between the tangle and the arcs from b to c and from d to a, as shown in Figure 4.7. Since the tangle intersects the arcs from a to b and c to d in $2n-1$ points and the arcs from b to c and d to a in $2n-2$ points, the slope of this tangle is $\frac{2n}{2n-1}$ (see [39] for more details and exposition on this construction of a rational tangle).

We now have that our 2-bridge knot is the union of tangles whose rational slopes are $\frac{1}{2m+1}$ (where $m = n \pm 1$) and $\frac{2n}{2n-1}$. That is, we can write K as $\tau(\frac{1}{2m+1}) \cup \tau(\frac{2n}{2n-1})$. Consider the matrix $A = \begin{bmatrix} 1 & 0 \\ -2m-1 & 1 \end{bmatrix} \in SL(2,\mathbb{Z})$.

Observe that $\begin{bmatrix} 1 & 0 \\ -2m-1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2m+1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and

$$\begin{aligned} \begin{bmatrix} 1 & 0 \\ -2m-1 & 1 \end{bmatrix} \begin{bmatrix} 2n \\ 2n-1 \end{bmatrix} &= \begin{bmatrix} 1 & 0 \\ -2(n\pm 1)-1 & 1 \end{bmatrix} \begin{bmatrix} 2n \\ 2n-1 \end{bmatrix} \\ &= \begin{bmatrix} 2n \\ -4n(n\pm 1)-1 \end{bmatrix}. \end{aligned}$$

Then, from the discussion in section 4.2 (see also [12]), K is equivalent to

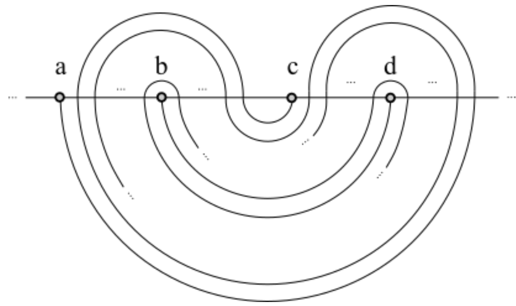

Figure 4.6: The knot in Figure 4.5 is a union of two tangles. We isotope the second tangle by pulling the points labeled c and d counterclockwise around the knot to obtain this diagram.

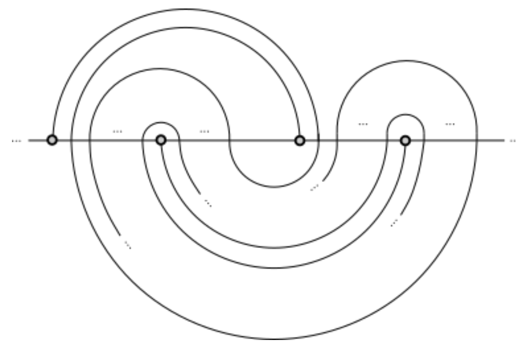

Figure 4.7: After an isotopy, the tangle intersects the horizontal arcs from a to b and c to d in $2n-1$ points and the arcs from b to c and d to a in $2n-2$ points.

the knot $\tau\left(\frac{1}{0}\right) \cup \tau\left(\frac{2n}{-4n(n\pm1)-1}\right)$, also sometimes referred to as the denominator closure of the rational tangle $\tau\left(\frac{2n}{-4n(n\pm1)-1}\right)$ (see [20]). We can compute the continued fraction expansion as

$$\frac{2n}{-4n(n\pm 1)-1} = \frac{1}{\frac{-4n(n\pm1)-1}{2n}} = \frac{1}{-(2n\pm 2)+\frac{1}{-2n}}.$$

Siebenman states in [36] that Casson, Gordon, and Conway showed all knots of this form are slice (see also [24] for a classification of slice 2-bridge knots). For convenience, see Figure 4.8 to see a slice band for the knot with continued fraction expansion $[n+2, n]$.

□

4.4 Further Questions

As noted earlier, Milnor's conjecture asserts that the most efficient way to obtain g_4 for a torus knot is also the nicest or most natural way. The pinch bands which motivated Batson's conjecture were in a sense the nicest band

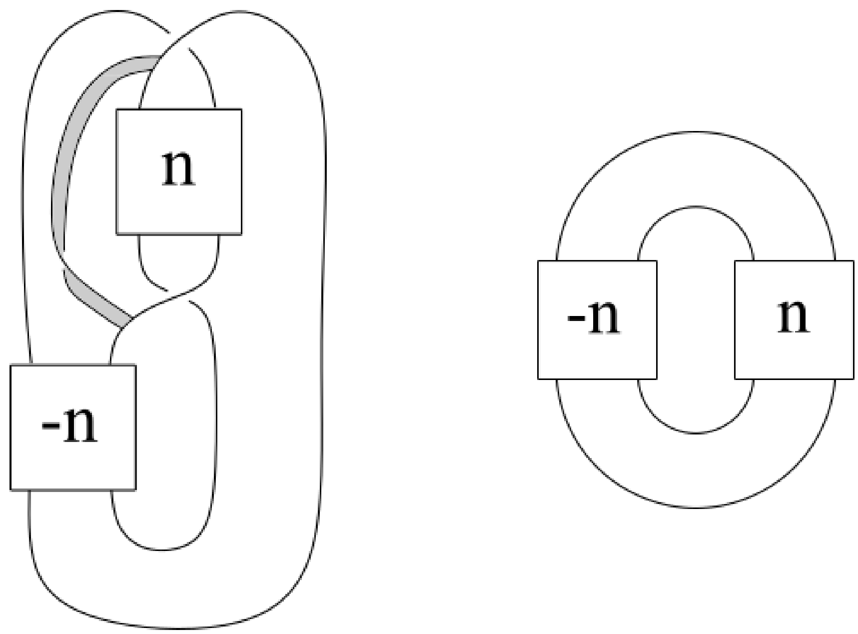

Figure 4.8: A slice band for the knot with continued fraction expansion $[n+2, n]$. After the band surgery, we obtain the two component unlink.

surgeries one can find to obtain the unknot. However, the bands found for our families of counterexamples also have a nice symmetry property to them. Are there counterexamples that do not have a certain amount of symmetry to them? Or is this the best that we can do? We ask the following.

Question 4.7. *Are there other examples of torus knots with nonorientable 4-ball genus less than its pinch number?*

While we expect the answer to this question to be yes, it would be significant if the answer were no. Presently, all known counterexamples to Batson's conjecture have nonorientable 4-ball genus only one less than the pinch number, hence showing the answer to Question 4.7 is no would prove that Batson's conjecture was only off by one - that is, all torus knots have nonorientable 4-ball genus equal to either its pinch number or its pinch number minus one. Should the answer to Question 4.7 be yes, it is natural to wonder how large the difference between the pinch number and nonorientable 4-ball genus can be.

Question 4.8. *Can the difference between the pinch number and the nonorientable 4-ball genus of a torus knot be arbitrarily large?*

Batson verified his conjecture for $T(2k, 2k-1)$ using a lower bound for γ_4. Osváth, Stipsicz and Szabó provided another lower bound $\nu - \frac{1}{2}\sigma$ for γ_4 in [30], where ν is the nu-invariant (see [31] for a definition) and σ is the signature, and Jabuka and Van Cott gave a combinatorial way to compute this lower bound in certain cases in [17]. In particular, Jabuka and Van Cott in [17] gave a categorization of when $\nu - \frac{1}{2}\sigma$ is equal to the pinch number minus one for a torus knot. The *sign* of a pinch move $T(p, q) \to T(r, s)$ is the sign of the value $p - 2t$ or $q - 2h$, where t and h are the smallest nonnegative integers such that $t \equiv -q^{-1} \mod p$ and $h \equiv p^{-1} \mod q$. Let $p, q > 1$ be relatively prime with q odd, and let $T(p, q) = T(p_n, q_n) \to T(p_{n-1}, q_{n-1}) \to \ldots \to T(p_0, q_0) = U$ (with $q_0 = 1$) be a sequence of pinch moves from $T(p, q)$ to the unknot U. Jabuka and Van Cott proved that if p is even, then $\nu(T(p, q)) - \frac{1}{2}\sigma(T(p, q)) = n - 1$ if and only if there is exactly one index k such that the sign of the pinch move $T(p_k, q_k) \to T(p_{k-1}, q_{k-1})$ is negative (Proposition 5.1(b) of [17]).

From the proof of Lemma 4.3, we see that the sign of every pinch move $T((2n\pm1)^2 - 2k(n\pm1), 4n-2k) \to T((2n\pm1)^2 - 2(k+1)(n\pm1), 4n-2(k+1))$ is equal to the sign of $p - 2t = 4n - 2(k+1))(n\pm1) + 1$, which is positive (except for when $p - 2t = 0$, in which case $q - 2h$ can still be seen to be positive). Following the discussion prior to Lemma 4.3, reversing the roles of p and q (since to apply Jabuka and Van Cott's result, we require q to be odd), we see that the sign of every pinch move $T(4n - 2k, (2n \pm 1)^2 - 2k(n \pm 1)) \to T(4n -$

$2(k+1), (2n \pm 1)^2 - 2(k+1)(n \pm 1))$ is negative. Thus, $(\nu - \frac{1}{2}\sigma)(K_n) < 2n-1$ for each $n > 0$, and $(\nu - \frac{1}{2}\sigma)(J_n) < 2n-1$ for each $n > 1$ (the exception $n = 1$ occurs since J_1 is already unknotted). It would be interesting to see if other lower bounds can be used to compute the exact value of γ_4 for K_n and J_n. We hence ask the following.

Question 4.9. *Is the nonorientable 4-ball genus of K_n and J_n equal to $2n-1$?*

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