Acoustic attenuation coefficients for polycrystalline materials containing crystallites of any symmetry class

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Abstract: This letter provides a theoretical extension to the elastic properties of polycrystals in order to describe elastic wave scattering from grain boundaries. The extension allows the longitudinal and shear attenuation coefficients for scattering to be derived and is valid for polycrystals containing crystallites of any symmetry class. Attenuation curves are given for polycrystalline SiO_2, ZrO_2, and SnF_2, which contain monoclinic crystallites. This work will allow ultrasonic techniques to be applied to new classes of materials containing nontrivial microstructures.

[JM]
Date Received: January 15, 2015 **Date Accepted:** May 12, 2015

Powder sintering processes allow a wide variety of materials, both metallic and non-metallic, to be fused into usable components. The goal of these processes is zero porosity and perfect bonding between all particles. Ultrasonic methods are often used to evaluate material properties nondestructively to ensure part integrity.[1,2] It is well known that ultrasonic attenuation increases if porosity or cracks are present.[3–6] However, baseline attenuation values of perfectly sintered and undamaged materials are typically unknown. A clear understanding of the baseline attenuation values helps quantify the integrity of the sintering process and the final part. In this letter, a generalized approach is used to determine attenuation for a random assembly of particles with arbitrary crystallite symmetry.

The single-crystal elastic constants are defined by $\mathbf{C}$ and can be related to any frame by applying four successive orthogonal transformations, where the rotated tensor is

$$C'_{ijkl} = \sum_{a,b,c,d=1}^{3} a_{ia}a_{jb}a_{kc}a_{ld}C_{abcd}, \tag{1}$$

and $\mathbf{a}(\phi,\theta,\zeta)$ are rotation matrices defined using the Euler angles (ϕ, θ, and ζ).[7] The summation over the repeated indices from 1 to 3 is explicitly shown in Eq. (1). The summation convention over repeated indices in subsequent expressions will be implied. The Voigt average of the single-crystal elastic constants can be obtained by finding the average value of $\mathbf{C}$ for all possible crystallite orientations. Because the average is independent of initial orientation of the single-crystal, the Voigt average may be equivalently performed on $\mathbf{C}'$, which gives

$$\begin{aligned}\langle C_{ijkl}\rangle &= \langle a_{ia}a_{jb}a_{kc}a_{ld}\rangle C_{abcd}\\ &= \frac{C_{abcd}}{8\pi^2}\int_0^{2\pi}\int_0^{2\pi}\int_0^{\pi} a_{ia}a_{jb}a_{kc}a_{ld}\sin\theta \mathrm{d}\theta\mathrm{d}\phi\mathrm{d}\zeta,\end{aligned} \tag{2}$$

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where $\langle\cdots\rangle$ denotes the operation of averaging over all possible crystallite rotations. The variation of the single-crystal elastic moduli from the Voigt average is defined through the variance tensor, which can be written as

$$\begin{aligned}\Xi_{ijkl}^{\alpha\beta\gamma\delta} &= \langle C_{ijkl}C_{\alpha\beta\gamma\delta}\rangle - \langle C_{ijkl}\rangle\langle C_{\alpha\beta\gamma\delta}\rangle \\ &= \left(\langle a_{ia}a_{jb}a_{kc}a_{ld}a_{\alpha m}a_{\beta n}a_{\gamma o}a_{\delta p}\rangle - \langle a_{ia}a_{jb}a_{kc}a_{ld}\rangle\langle a_{\alpha m}a_{\beta n}a_{\gamma o}a_{\delta p}\rangle\right)C_{abcd}C_{mnop}. \qquad (3)\end{aligned}$$

Equations (2) and (3) assume the polycrystal exhibits statistically isotropic elastic symmetry while the crystallites are of any symmetry class. Anisotropy of the polycrystal and crystallites can be considered using Morris' technique.[8] Previous authors[7,9,10] have calculated Ξ by averaging the elastic modulus tensor appropriate to the specific crystal class. Equation (3) is valid for all 32 crystal classes. Equation (3) does not require *a priori* knowledge of the form of the elastic modulus tensor $\mathbf{C}$ in order to define Ξ.

Weaver performs various inner products between $\boldsymbol{\Xi}$ and incoming and scattered wave vectors in order to define longitudinal and shear wave attenuation coefficients of statistically isotropic polycrystalline materials. The incoming wave vectors can be of either longitudinal type ($\hat{\mathbf{p}}$) or two shear wave types ($\hat{\mathbf{p}}_1$ or $\hat{\mathbf{p}}_2$). The incoming wave scatters at grain boundaries and produces scattered longitudinal ($\hat{\mathbf{s}}$) and shear waves ($\hat{\mathbf{s}}_1$ or $\hat{\mathbf{s}}_2$). Figure 1 illustrates the scattering configuration where the angle θ_{ps} is defined as the scattering angle relative to the incoming wave. Weaver[7] showed that the attenuation due to grain scattering for a statistically isotropic polycrystal depended on three inner products between $\boldsymbol{\Xi}$ and the wave vectors $\hat{\mathbf{p}}$, $\hat{\mathbf{p}}_1$, $\hat{\mathbf{p}}_2$, $\hat{\mathbf{s}}$, $\hat{\mathbf{s}}_1$, $\hat{\mathbf{s}}_2$, which are given by

$$\begin{aligned}L(\theta_{\text{ps}}) &= \boldsymbol{\Xi}_{\cdots\cdot\hat{\mathbf{p}}\hat{\mathbf{p}}\hat{\mathbf{s}}\hat{\mathbf{s}}}^{\cdots\cdot\hat{\mathbf{p}}\hat{\mathbf{p}}\hat{\mathbf{s}}\hat{\mathbf{s}}} = \Xi_{ijkl}^{\alpha\beta\gamma\delta}\hat{p}_\alpha\hat{p}_i\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\hat{s}_\delta\hat{s}_l \\ &= \frac{1}{1575}\left(L_0 + L_1\cos^2\theta_{\text{ps}} + L_2\cos^4\theta_{\text{ps}}\right), \\ M(\theta_{\text{ps}}) &= \boldsymbol{\Xi}_{\cdots\cdot\hat{\mathbf{p}}\hat{\mathbf{p}}\hat{\mathbf{s}}}^{\cdots\cdot\hat{\mathbf{p}}\hat{\mathbf{p}}\hat{\mathbf{s}}}\mathbf{I} = \boldsymbol{\Xi}_{\cdots\cdot}^{\cdots\cdot}\mathbf{I}_{\hat{\mathbf{p}}\hat{\mathbf{s}}\hat{\mathbf{s}}}^{\hat{\mathbf{p}}\hat{\mathbf{s}}\hat{\mathbf{s}}} = \Xi_{ijkl}^{\alpha\beta\gamma\delta}\hat{p}_\alpha\hat{p}_i\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\delta_{\delta l} \\ &= \Xi_{ijkl}^{\alpha\beta\gamma\delta}\left(\hat{p}_\alpha\hat{p}_i\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\hat{s}_\delta\hat{s}_l + \hat{p}_\alpha\hat{p}_i\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\hat{s}_{1\delta}\hat{s}_{1l} + \hat{p}_\alpha\hat{p}_i\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\hat{s}_{2\delta}\hat{s}_{2l}\right) \\ &= \frac{1}{1575}\left(M_0 + M_1\cos^2\theta_{\text{ps}}\right), \\ N(\theta_{\text{ps}}) &= \boldsymbol{\Xi}_{\cdots\cdot}^{\cdots\cdot}\mathbf{I}_{\hat{\mathbf{p}}\hat{\mathbf{s}}}^{\hat{\mathbf{p}}\hat{\mathbf{s}}}\mathbf{I} = \Xi_{ijkl}^{\alpha\beta\gamma\delta}\delta_{\alpha i}\hat{p}_\beta\hat{p}_j\hat{s}_\gamma\hat{s}_k\delta_{\delta l} \\ &= \Xi_{ijkl}^{\alpha\beta\gamma\delta}\left(\hat{p}_\alpha\hat{p}_i\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\hat{s}_\delta\hat{s}_l + 2\hat{p}_\alpha\hat{p}_i\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\hat{s}_{1\delta}\hat{s}_{1l} + 2\hat{p}_\alpha\hat{p}_i\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\hat{s}_{2\delta}\hat{s}_{2l}\right. \\ &\quad \left. + \hat{p}_{1\alpha}\hat{p}_{1i}\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\hat{s}_{1\delta}\hat{s}_{1l} + \hat{p}_{2\alpha}\hat{p}_{2i}\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\hat{s}_{2\delta}\hat{s}_{2l} + 2\hat{p}_{1\alpha}\hat{p}_{1i}\hat{s}_\beta\hat{s}_j\hat{p}_\gamma\hat{p}_k\hat{s}_{2\delta}\hat{s}_{2l}\right) \\ &= \frac{1}{1575}\left(N_0 + N_1\cos^2\theta_{\text{ps}}\right), \qquad (4)\end{aligned}$$

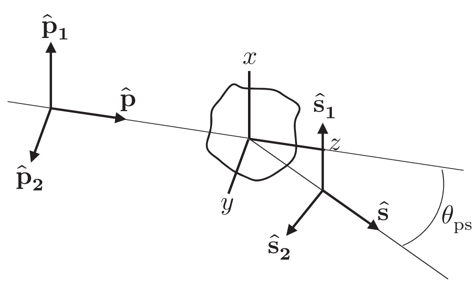

Fig. 1. An incident wave defined by the wave vector $\hat{\mathbf{p}}$, $\hat{\mathbf{p}}_1$, or $\hat{\mathbf{p}}_2$ scatters at an angle θ_{ps} into a scattered wave defined by the wave vectors $\hat{\mathbf{s}}$, $\hat{\mathbf{s}}_1$, or $\hat{\mathbf{s}}_2$.

where the coefficients L_0, L_1, L_2, M_0, M_1, N_0, and N_1 are combinations of single-crystal elastic constants and are easily found using the definition in Eq. (3) and performing the respective inner products. During this procedure, we make use of Voigt's reduced notation for the single-crystal elastic constants, $C_{ijkl} = c_{IJ}$ where pairs of indices ($ij \rightarrow I$) and ($kl \rightarrow J$) are defined according to $11 \rightarrow 1$, $12 \rightarrow 6$, $13 \rightarrow 5$, $22 \rightarrow 2$, $23 \rightarrow 4$, and $33 \rightarrow 3$. We assume the existence of the minor ($C_{ijkl} = C_{jikl} = C_{ijlk}$) and major ($C_{ijkl} = C_{klij}$) elastic symmetry relations, which reduces to the maximum number of 21 independent elastic constants of the triclinic crystal class. The coefficients of the inner products are found to be

$$
\begin{aligned}
L_0 = {} & 8c_{11}^2 + 83c_{12}^2 + 83c_{13}^2 + 240c_{14}^2 + 60c_{15}^2 + 60c_{16}^2 + 8c_{22}^2 + 83c_{23}^2 + 60c_{24}^2 + 240c_{25}^2 + 60c_{26}^2 \\
& + 8c_{33}^2 + 60c_{34}^2 + 60c_{35}^2 + 240c_{36}^2 + 92c_{44}^2 + 240c_{45}^2 + 240c_{46}^2 + 92c_{55}^2 + 240c_{56}^2 + 92c_{66}^2 \\
& + 4c_{11}c_{12} + 4c_{11}c_{13} - 74c_{12}c_{13} + c_{11}c_{22} - 26c_{11}c_{23} + 4c_{12}c_{22} - 74c_{12}c_{23} - 26c_{13}c_{22} \\
& - 74c_{13}c_{23} + 60c_{14}c_{24} + 60c_{15}c_{25} - 60c_{16}c_{26} + c_{11}c_{33} - 26c_{12}c_{33} + 4c_{22}c_{23} + 4c_{13}c_{33} \\
& + 60c_{14}c_{34} - 60c_{15}c_{35} + 60c_{16}c_{36} + 28c_{11}c_{44} + c_{22}c_{33} + 52c_{12}c_{44} + 4c_{23}c_{23} + 52c_{13}c_{44} \\
& - 60c_{24}c_{34} + 60c_{25}c_{35} - 120c_{15}c_{46} - 120c_{16}c_{45} + 60c_{26}c_{36} - 32c_{11}c_{55} - 32c_{22}c_{44} \\
& + 52c_{12}c_{55} - 68c_{23}c_{44} - 68c_{13}c_{55} - 240c_{14}c_{56} - 240c_{25}c_{46} - 120c_{26}c_{45} - 32c_{11}c_{66} \\
& + 28c_{22}c_{55} - 32c_{33}c_{44} - 68c_{12}c_{66} + 52c_{23}c_{55} + 52c_{13}c_{66} - 120c_{24}c_{56} - 120c_{35}c_{46} \\
& - 240c_{36}c_{45} - 32c_{22}c_{66} - 32c_{33}c_{55} + 52c_{23}c_{66} - 120c_{34}c_{56} + 28c_{33}c_{66} - 56c_{44}c_{55} \\
& - 56c_{44}c_{66} - 56c_{55}c_{66},
\end{aligned}
$$

$$
\begin{aligned}
L_1 = {} & 92c_{11}^2 - 98c_{12}^2 - 98c_{13}^2 - 300c_{14}^2 + 300c_{15}^2 + 300c_{16}^2 + 92c_{22}^2 - 98c_{23}^2 + 300c_{24}^2 \\
& - 300c_{25}^2 + 300c_{26}^2 + 92c_{33}^2 + 300c_{34}^2 + 300c_{35}^2 - 300c_{36}^2 - 72c_{44}^2 - 240c_{45}^2 - 240c_{46}^2 \\
& - 72c_{55}^2 - 240c_{56}^2 - 72c_{66}^2 + 36c_{11}c_{12} + 36c_{11}c_{13} + 104c_{12}c_{13} - 86c_{11}c_{22} - 84c_{11}c_{23} \\
& + 36c_{12}c_{22} + 104c_{12}c_{23} - 84c_{13}c_{22} + 104c_{13}c_{23} + 240c_{14}c_{24} + 240c_{15}c_{25} + 480c_{16}c_{26} \\
& - 86c_{11}c_{33} - 84c_{12}c_{33} + 36c_{22}c_{23} + 36c_{13}c_{33} + 240c_{14}c_{34} + 480c_{15}c_{35} + 240c_{16}c_{36} \\
& - 88c_{11}c_{44} - 86c_{22}c_{33} - 152c_{12}c_{44} + 36c_{23}c_{33} - 152c_{13}c_{44} + 480c_{24}c_{34} + 240c_{25}c_{35} \\
& + 240c_{15}c_{46} + 240c_{16}c_{45} + 240c_{26}c_{36} + 32c_{11}c_{55} + 32c_{22}c_{44} - 152c_{12}c_{55} + 328c_{23}c_{44} \\
& + 328c_{13}c_{55} + 960c_{14}c_{56} + 960c_{25}c_{46} + 240c_{26}c_{45} + 32c_{11}c_{66} - 88c_{22}c_{55} + 32c_{33}c_{44} \\
& + 328c_{12}c_{66} - 152c_{23}c_{55} - 152c_{13}c_{66} + 240c_{24}c_{56} + 240c_{35}c_{46} + 960c_{36}c_{45} + 32c_{22}c_{66} \\
& + 32c_{33}c_{55} - 152c_{23}c_{55} + 240c_{34}c_{56} - 88c_{33}c_{66} + 96c_{44}c_{55} + 96c_{44}c_{66} + 96c_{55}c_{66},
\end{aligned}
$$

$$
\begin{aligned}
L_2 = {} & 12c_{11}^2 + 47c_{12}^2 + 47c_{13}^2 + 140c_{14}^2 + 40c_{15}^2 + 40c_{16}^2 + 12c_{22}^2 + 47c_{23}^2 + 40c_{24}^2 + 140c_{25}^2 \\
& + 40c_{26}^2 + 12c_{33}^2 + 40c_{34}^2 + 40c_{35}^2 + 140c_{36}^2 + 108c_{44}^2 + 320c_{45}^2 + 320c_{46}^2 + 108c_{55}^2 \\
& + 320c_{56}^2 + 108c_{66}^2 - 24c_{11}c_{12} - 24c_{11}c_{13} - 46c_{12}c_{13} - 11c_{11}c_{22} + 46c_{11}c_{23} - 24c_{12}c_{22} \\
& - 46c_{12}c_{23} + 46c_{13}c_{22} - 46c_{13}c_{23} - 140c_{14}c_{24} - 140c_{15}c_{25} + 60c_{16}c_{26} - 11c_{11}c_{33} \\
& + 46c_{12}c_{33} - 24c_{22}c_{23} - 24c_{13}c_{33} - 140c_{14}c_{34} + 60c_{15}c_{35} - 140c_{16}c_{36} - 68c_{11}c_{44} \\
& - 11c_{22}c_{33} + 68c_{12}c_{44} - 24c_{23}c_{33} + 68c_{13}c_{44} + 60c_{24}c_{34} - 140c_{25}c_{35} + 200c_{15}c_{46} \\
& + 200c_{16}c_{45} - 140c_{26}c_{36} + 32c_{11}c_{55} + 32c_{22}c_{44} + 68c_{12}c_{55} - 132c_{23}c_{44} - 132c_{13}c_{55} \\
& - 400c_{14}c_{56} - 400c_{25}c_{46} + 200c_{26}c_{45} + 32c_{11}c_{66} - 68c_{22}c_{55} + 32c_{33}c_{44} - 132c_{12}c_{66} \\
& + 68c_{23}c_{55} + 68c_{13}c_{66} + 200c_{24}c_{56} + 200c_{35}c_{46} - 400c_{36}c_{45} + 32c_{22}c_{66} + 32c_{33}c_{55} \\
& + 68c_{23}c_{66} + 200c_{34}c_{56} - 68c_{33}c_{66} - 104c_{44}c_{55} - 104c_{44}c_{66} - 104c_{55}c_{66},
\end{aligned}
$$

$$
\begin{aligned}
M_0 =\,& 38c_{11}^2+158c_{12}^2+158c_{13}^2+450c_{14}^2+210c_{15}^2+210c_{16}^2+38c_{22}^2+158c_{23}^2+210c_{24}^2\\
&+450c_{25}^2+210c_{26}^2+38c_{33}^2+210c_{34}^2+210c_{35}^2+450c_{36}^2+212c_{44}^2+540c_{45}^2+540c_{46}^2\\
&+212c_{55}^2+540c_{56}^2+212c_{66}^2+4c_{11}c_{12}+4c_{11}c_{13}-134c_{12}c_{13}-14c_{11}c_{22}-56c_{11}c_{23}\\
&+4c_{12}c_{22}-134c_{12}c_{23}-56c_{13}c_{22}-134c_{11}c_{23}+120c_{14}c_{24}+120c_{15}c_{25}-60c_{16}c_{26}\\
&-14c_{11}c_{33}-56c_{12}c_{33}+4c_{22}c_{23}+4c_{13}c_{33}+120c_{14}c_{34}-60c_{15}c_{35}+120c_{16}c_{36}+28c_{11}c_{44}\\
&-14c_{22}c_{33}+82c_{12}c_{44}+4c_{23}c_{33}+82c_{13}c_{44}-60c_{24}c_{34}+120c_{25}c_{35}-180c_{15}c_{46}\\
&-180c_{16}c_{45}+120c_{26}c_{36}-62c_{11}c_{55}-62c_{22}c_{44}+82c_{12}c_{55}-68c_{23}c_{44}-68c_{13}c_{55}\\
&-300c_{14}c_{56}-300c_{25}c_{46}-180c_{26}c_{45}-62c_{11}c_{66}+28c_{22}c_{55}-62c_{33}c_{44}-68c_{12}c_{66}\\
&+82c_{23}c_{55}+82c_{13}c_{66}-180c_{24}c_{56}-180c_{35}c_{46}-300c_{36}c_{45}-62c_{22}c_{66}-62c_{33}c_{55}\\
&+82c_{23}c_{66}-180c_{34}c_{56}+28c_{33}c_{66}-116c_{44}c_{55}-116c_{44}c_{66}-116c_{55}c_{66},
\end{aligned}
$$

$$
\begin{aligned}
M_1 =\,& 124c_{11}^2-96c_{12}^2-96c_{13}^2-300c_{14}^2+420c_{15}^2+420c_{16}^2+124c_{22}^2-96c_{23}^2+420c_{24}^2-300c_{25}^2\\
&+420c_{26}^2+124c_{33}^2+420c_{34}^2+420c_{35}^2-300c_{36}^2+36c_{44}^2+60c_{45}^2+60c_{46}^2+36c_{55}^2+60c_{56}^2\\
&+36c_{26}^2+2c_{11}c_{12}+2c_{11}c_{13}+108c_{12}c_{13}-112c_{11}c_{22}-28c_{11}c_{23}+2c_{12}c_{22}+108c_{12}c_{23}\\
&-28c_{13}c_{22}+108c_{13}c_{23}+60c_{14}c_{24}+60c_{15}c_{25}+600c_{16}c_{26}-112c_{11}c_{33}-28c_{12}c_{33}\\
&+2c_{22}c_{23}+2c_{13}c_{33}+60c_{14}c_{34}+600c_{15}c_{35}+60c_{16}c_{36}-196c_{11}c_{44}-112c_{22}c_{33}\\
&-134c_{12}c_{44}+2c_{23}c_{33}-134c_{13}c_{44}+600c_{24}c_{34}+60c_{25}c_{35}+540c_{15}c_{46}+540c_{16}c_{45}\\
&+60c_{26}c_{36}+74c_{11}c_{55}+74c_{22}c_{44}-134c_{12}c_{55}+316c_{23}c_{44}+316c_{13}c_{55}+900c_{14}c_{56}\\
&+900c_{25}c_{46}+540c_{26}c_{45}+74c_{11}c_{66}-196c_{22}c_{55}+74c_{33}c_{44}+316c_{12}c_{66}-134c_{23}c_{55}\\
&-134c_{13}c_{66}+540c_{24}c_{56}+540c_{35}c_{46}+900c_{36}c_{45}+74c_{22}c_{66}+74c_{33}c_{55}-134c_{23}c_{66}\\
&+540c_{34}c_{56}-196c_{33}c_{66}+12c_{44}c_{55}+12c_{44}c_{66}+12c_{55}c_{66},
\end{aligned}
$$

$$
\begin{aligned}
N_0 =\,& 91c_{11}^2+301c_{12}^2+301c_{13}^2+840c_{14}^2+525c_{15}^2+525c_{16}^2+91c_{22}^2+301c_{23}^2+525c_{24}^2+840c_{25}^2\\
&+525c_{26}^2+91c_{33}^2+525c_{34}^2+525c_{35}^2+840c_{36}^2+539c_{44}^2+1365c_{45}^2+1365c_{46}^2+539c_{55}^2\\
&+1365c_{56}^2+539c_{66}^2-42c_{11}c_{12}-42c_{11}c_{13}-238c_{12}c_{13}-28c_{11}c_{22}-42c_{11}c_{23}-42c_{12}c_{22}\\
&-238c_{12}c_{23}-42c_{13}c_{22}-238c_{13}c_{23}-210c_{16}c_{26}-28c_{11}c_{33}-42c_{12}c_{33}-42c_{22}c_{23}\\
&-42c_{13}c_{33}-210c_{15}c_{35}-14c_{11}c_{44}-28c_{22}c_{33}+154c_{12}c_{44}-42c_{23}c_{33}+154c_{13}c_{44}\\
&-210c_{24}c_{34}-210c_{15}c_{46}-210c_{16}c_{45}-119c_{11}c_{55}-119c_{22}c_{44}+154c_{12}c_{55}-56c_{23}c_{44}\\
&-56c_{13}c_{55}-420c_{14}c_{56}-420c_{25}c_{46}-210c_{26}c_{45}-119c_{11}c_{66}-14c_{22}c_{55}-119c_{33}c_{44}\\
&-56c_{12}c_{66}+154c_{23}c_{55}+154c_{13}c_{66}-210c_{24}c_{56}-210c_{35}c_{46}-420c_{36}c_{45}-119c_{22}c_{66}\\
&-119c_{33}c_{55}+154c_{23}c_{66}-210c_{34}c_{56}-14c_{33}c_{66}-287c_{44}c_{55}-287c_{44}c_{66}-287c_{55}c_{66}, \text{and}
\end{aligned}
$$

$$
\begin{aligned}
N_1 =\,& 147c_{11}^2-133c_{12}^2-133c_{13}^2-420c_{14}^2+525c_{15}^2+525c_{16}^2+147c_{22}^2-133c_{23}^2+525c_{24}^2\\
&-420c_{25}^2+525c_{26}^2+147c_{33}^2+525c_{34}^2+525c_{35}^2-420c_{36}^2+63c_{44}^2+105c_{45}^2+105c_{46}^2+63c_{55}^2\\
&+105c_{56}^2+63c_{66}^2-14c_{11}c_{12}-14c_{11}c_{13}+154c_{12}c_{13}-126c_{11}c_{22}-14c_{11}c_{23}-14c_{12}c_{22}\\
&+154c_{12}c_{23}-14c_{13}c_{22}+154c_{13}c_{23}+630c_{16}c_{26}-126c_{11}c_{33}-14c_{12}c_{33}-14c_{22}c_{23}\\
&-14c_{13}c_{33}+630c_{15}c_{35}-238c_{11}c_{44}-126c_{22}c_{33}-182c_{12}c_{44}-14c_{23}c_{33}-182c_{13}c_{44}\\
&+630c_{24}c_{34}+630c_{15}c_{46}+630c_{16}c_{45}+77c_{11}c_{55}+77c_{22}c_{44}-182c_{12}c_{55}+448c_{23}c_{44}\\
&+448c_{13}c_{55}+1260c_{14}c_{56}+1260c_{25}c_{46}+630c_{26}c_{45}+77c_{11}c_{66}-238c_{22}c_{55}+77c_{33}c_{44}\\
&+448c_{12}c_{66}-182c_{23}c_{55}-182c_{13}c_{66}+630c_{24}c_{56}+630c_{35}c_{46}+1260c_{36}c_{45}+77c_{22}c_{66}\\
&+77c_{33}c_{55}-182c_{23}c_{66}+630c_{34}c_{56}-238c_{33}c_{66}+21c_{44}c_{55}+21c_{44}c_{66}+21c_{55}c_{66}.
\end{aligned}
\tag{5}
$$

The inner products are valid for all 32 crystal symmetry classes. They reduce to the published expressions for crystallites of cubic,[7] hexagonal,[9] and orthorhombic[10] symmetries. Inner products for all 32 crystal symmetries can be obtained using the respective symmetry relations.[11] For example, the coefficients L_0, L_1, L_2, M_0, M_1, N_0, and N_1 for hexagonal crystallite symmetry are reduced from the triclinic class,

$$\begin{aligned}
L_0 &= 8c_{11}^2 - 40c_{11}c_{12} + 8c_{11}c_{13} + 16c_{11}c_{33} - 64c_{11}c_{44} + 140c_{12}^2 - 200c_{12}c_{13} - 40c_{12}c_{33} \\
&\quad + 160c_{12}c_{44} + 92c_{13}^2 + 8c_{13}c_{33} - 32c_{13}c_{44} + 8c_{33}^2 - 64c_{33}c_{44} + 128c_{44}^2, \\
L_1 &= 112c_{11}^2 + 240c_{11}c_{12} - 248c_{11}c_{13} - 216c_{11}c_{33} - 16c_{11}c_{44} - 280c_{12}^2 + 360c_{12}c_{13} - 40c_{12}c_{33} \\
&\quad - 400c_{12}c_{44} - 92c_{13}^2 + 72c_{13}c_{33} + 352c_{13}c_{44} + 92c_{33}^2 + 64c_{33}c_{44} - 48c_{44}^2, \\
L_2 &= 72c_{11}^2 - 200c_{11}c_{12} + 112c_{11}c_{13} - 56c_{11}c_{33} - 176c_{11}c_{44} + 140c_{12}^2 - 160c_{12}c_{13} + 80c_{12}c_{33} \\
&\quad + 240c_{12}c_{44} + 48c_{13}^2 - 48c_{13}c_{33} - 128c_{13}c_{44} + 12c_{33}^2 + 64c_{33}c_{44} + 112c_{44}^2, \\
M_0 &= 53c_{11}^2 - 70c_{11}c_{12} - 22c_{11}c_{13} - 14c_{11}c_{33} - 184c_{11}c_{44} + 245c_{12}^2 - 350c_{12}c_{13} - 70c_{12}c_{33} \\
&\quad + 280c_{12}c_{44} + 182c_{13}^2 + 8c_{13}c_{33} + 28c_{13}c_{44} + 38c_{33}^2 - 124c_{33}c_{44} + 308c_{44}^2, \\
M_1 &= 219c_{11}^2 + 70c_{11}c_{12} - 186c_{11}c_{13} - 322c_{11}c_{33} - 232c_{11}c_{44} - 245c_{12}^2 + 350c_{12}c_{13} + 70c_{12}c_{33} \\
&\quad - 280c_{12}c_{44} - 84c_{13}^2 + 4c_{13}c_{33} + 364c_{13}c_{44} + 124c_{33}^2 + 148c_{33}c_{44} + 84c_{44}^2, \\
N_0 &= 679c_{11}^2/4 - 525c_{11}c_{12}/2 - 14c_{11}c_{13} - 63c_{11}c_{33} - 553c_{11}c_{44} + 1855c_{12}^2/4 - 630c_{12}c_{13} \\
&\quad - 35c_{12}c_{33} + 595c_{12}c_{44} + 364c_{13}^2 - 84c_{13}c_{33} + 196c_{13}c_{44} + 91c_{33}^2 \\
&\quad - 238c_{33}c_{44} + 791c_{44}^2, \text{ and} \\
N_1 &= 1043c_{11}^2/4 - 238c_{11}c_{13} - 371c_{11}c_{33} - 301c_{11}c_{44} + 175c_{12}c_{11}/2 - 112c_{13}^2 - 28c_{13}c_{33} \\
&\quad + 532c_{13}c_{44} + 490c_{12}c_{13} + 147c_{33}^2 + 154c_{33}c_{44} + 105c_{12}c_{33} + 147c_{44}^2 - 385c_{12}c_{44} \\
&\quad - 1365c_{12}^2/4. \qquad (6)
\end{aligned}$$

For cubic crystallites, the values in Eq. (5) reduce to the expected values of[7]

$$\begin{gathered}
L_0 = 27(c_{11} - c_{12} - 2c_{44})^2 = 27\nu^2, \quad L_1 = 18(c_{11} - c_{12} - 2c_{44})^2 = 18\nu^2, \\
L_2 = 3(c_{11} - c_{12} - 2c_{44})^2 = 3\nu^2, \\
M_0 = 72(c_{11} - c_{12} - 2c_{44})^2 = 72\nu^2, \quad M_1 = 36(c_{11} - c_{12} - 2c_{44})^2 = 36\nu^2, \\
N_0 = 189(c_{11} - c_{12} - 2c_{44})^2 = 189\nu^2, \quad N_1 = 63(c_{11} - c_{12} - 2c_{44})^2 = 63\nu^2. \qquad (7)
\end{gathered}$$

The longitudinal and shear attenuation coefficients are written in terms of attenuation components that describe the relative scattering losses of the incoming wave to either scattered wave type,[7]

$$\alpha_L = \alpha_{LL} + \alpha_{LT} \quad \text{and} \quad \alpha_L = \alpha_{TT} + \alpha_{TL}. \qquad (8)$$

The components are given by[7]

$$\alpha_{LL} = \frac{\rho^2\omega^4\bar{d}^3}{\left(c_{33}^0\right)^4}\int_0^\pi \chi^{LL}(\theta_{\rm ps})L(\theta_{\rm ps})\sin\theta_{\rm ps}{\rm d}\theta_{\rm ps},$$
$$\alpha_{LT} = \frac{\rho^2\omega^4\bar{d}^3}{\left(c_{33}^0\right)^{3/2}\left(c_{55}^0\right)^{5/2}}\int_0^\pi \chi^{LT}(\theta_{\rm ps})\left[M(\theta_{\rm ps}) - L(\theta_{\rm ps})\right]\sin\theta_{\rm ps}{\rm d}\theta_{\rm ps},$$
$$\alpha_{TL} = \frac{\rho^2\omega^4\bar{d}^3}{2\left(c_{33}^0\right)^{5/2}\left(c_{55}^0\right)^{3/2}}\int_0^\pi \chi^{LT}(\theta_{\rm ps})\left[M(\theta_{\rm ps}) - L(\theta_{\rm ps})\right]\sin\theta_{\rm ps}{\rm d}\theta_{\rm ps},$$
$$\alpha_{TT} = \frac{\rho^2\omega^4\bar{d}^3}{2\left(c_{33}^0\right)^4}\int_0^\pi \chi^{TT}(\theta_{\rm ps})\left[N(\theta_{\rm ps}) - 2M(\theta_{\rm ps}) + L(\theta_{\rm ps})\right]\sin\theta_{\rm ps}{\rm d}\theta_{\rm ps}, \quad (9)$$

where

$$\chi^{LL} = \left[4 + \frac{2\rho\omega^4\bar{d}^2}{c_{33}^0}(1-\cos\theta_{\rm ps})\right]^{-2},$$
$$\chi^{TT} = \left[4 + \frac{2\rho\omega^2\bar{d}^2}{c_{55}^0}(1-\cos\theta_{\rm ps})\right]^{-2}, \quad \text{and}$$
$$\chi^{TL} = \chi^{LT} = \left(4 + \frac{\rho\omega^2\bar{d}^2}{c_{33}^0} + \frac{\rho\omega^2\bar{d}^2}{c_{55}^0} - \frac{2\rho\omega^2\bar{d}^2}{\sqrt{c_{33}^0 c_{55}^0}}\cos\theta_{\rm ps}\right)^{-2}, \quad (10)$$

contain the spatial Fourier transforms of the two-point spatial correlation functions.[7] The constants c_{33}^0 and c_{55}^0 are Voigt averages obtained from Eq. (2) and given by

$$c_{33}^0 = \frac{1}{15}(3c_{11} + 2c_{12} + 2c_{13} + 2c_{22} + 2c_{23} + 3c_{33} + 4c_{44} + 4c_{55} + 4c_{66}),$$
$$c_{55}^0 = \frac{1}{15}(c_{11} - c_{12} - c_{13} + c_{22} - c_{23} + c_{33} + 3c_{44} + 3c_{55} + 3c_{66}). \quad (11)$$

The Voigt averages c_{33}^0 and c_{55}^0 define the elastic moduli of the statistically isotropic bulk material. They are reduced for higher symmetry cases by applying the crystallite symmetry relations.[11] Thus, $\mathbf{C}^0$ has isotropic symmetry[11] and $c_{33}^0 = c_{13}^0 + 2c_{55}^0$.

Example calculations are used to show the longitudinal and shear attenuation coefficients of polycrystalline SiO_2, ZrO_2, and SnF_2 in Fig. 2. Such materials are assumed to contain crystallites of monoclinic symmetry. The inner products defined in Eq. (5) and the Voigt averages in Eq. (11) were reduced from triclinic to monoclinic symmetry using the symmetry relations found in Brugger.[11] For this case, the 13 independent single-crystal elastic constants are c_{11}, c_{12}, c_{13}, c_{15}, c_{22}, c_{23}, c_{25}, c_{33}, c_{35}, c_{44}, c_{46}, c_{55}, and c_{66}. The values of the single-crystal elastic constants for the specific materials were obtained from the Landolt-Börnstein tables.[12] Figure 2 illustrates the dimensionless longitudinal and shear attenuation coefficients for polycrystalline SiO_2, ZrO_2, and SnF_2. The attenuation coefficients are plotted against the frequency ($\omega = 2\pi f$), density (ρ), and mean grain diameter ($\bar{d}$). SiO_2 has a larger longitudinal attenuation coefficient than ZrO_2, and SnF_2, but a smaller shear coefficient.

Figure 2, for the first time, gives theoretical attenuation estimates for materials containing crystallites with elastic symmetry lower than orthorhombic. Equation (9) can easily be evaluated for any material where the particles have known single-crystal elastic constants. The formulation of Eqs. (2) and (3) has large implications on other

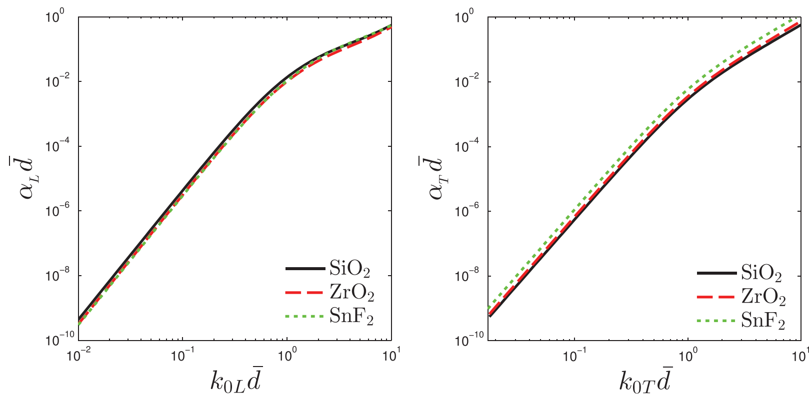

Fig. 2. Dimensionless attenuation coefficients ($\alpha_L \bar{d}$ and $\alpha_T \bar{d}$) as a function of dimensionless wave number ($k_{0L}\bar{d} = \sqrt{\rho}\omega\bar{d}/\sqrt{c_{33}^0}$ and $k_{0T}\bar{d} = \sqrt{\rho}\omega\bar{d}/\sqrt{c_{55}^0}$) for polycrystalline SiO_2, ZrO_2, and SnF_2 having grains of monoclinic symmetry. The dimensionless wave number gives the attenuation dependence on frequency ($\omega = 2\pi f$), density (ρ), Voigt average elastic constants $\mathbf{C}^0$, and mean grain diameter ($\bar{d}$).

models that allow for bulk anisotropy, multiple phases, and materials with elongated grains.[8,13,14] Equations (2) and (3) can also enter and extend the models of ultrasonic backscatter and radiative transfer.[15–17]

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