2005

EXPONENTIAL STABILITY OF DYNAMIC EQUATIONS ON TIME SCALES

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Received 6 July 2004 and in revised form 16 December 2004

We investigate the exponential stability of the zero solution to a system of dynamic equations on time scales. We do this by defining appropriate Lyapunov-type functions and then formulate certain inequalities on these functions. Several examples are given.

1. Introduction

This paper considers the exponential stability of the zero solution of the first-order vector dynamic equation

$$x^{\Delta} = f(t,x), \quad t \geq 0. \tag{1.1}$$

Throughout the paper, we let $x(t,t_0,x_0)$ denote a solution of the initial value problem (IVP) (1.1),

$$x(t_0) = x_0, \quad t_0 \geq 0,\ x_0 \in \mathbb{R}. \tag{1.2}$$

(For the existence, uniqueness, and extendability of solutions of IVPs for (1.1)-(1.2), see [2, Chapter 8].) Also we assume that $f : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and t is from a so-called "time scale" $\mathbb{T}$ (which is a nonempty closed subset of $\mathbb{R}$). Throughout the paper, we assume that $0 \in \mathbb{T}$ (for convenience) and that $f(t,0) = 0$, for all t in the time scale interval $[0,\infty) := \{t \in \mathbb{T} : 0 \leq t < \infty\}$, and call the zero function the trivial solution of (1.1).

If $\mathbb{T} = \mathbb{R}$, then $x^{\Delta} = x'$ and (1.1)-(1.2) becomes the following IVP for ordinary differential equations

$$x' = f(t,x), \quad t \geq 0, \tag{1.3}$$

$$x(t_0) = x_0, \quad t_0 \geq 0. \tag{1.4}$$

Recently, Peterson and Tisdell [7] used Lyapunov-type functions to formulate some sufficient conditions that ensure all solutions to (1.1)-(1.2) are bounded. Earlier, Raffoul [8] used some similar ideas to obtain boundedness of all solutions of (1.3) and (1.4). Here

Advances in Difference Equations 2005:2 (2005) 133–144
DOI: 10.1155/ADE.2005.133

we use Lyapunov-type functions on time scales and then formulate appropriate inequalities on these functions that guarantee that the trivial solution to (1.1) is exponentially or uniformly exponentially stable on $[0,\infty)$. Some of our results are new even for the special cases $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$.

To understand the notation used above and the idea of time scales, some preliminary definitions are needed.

Definition 1.1. A *time scale* $\mathbb{T}$ is a nonempty closed subset of the real numbers $\mathbb{R}$.

Since we are interested in the asymptotic behavior of solutions near ∞, we assume that $\mathbb{T}$ is unbounded above.

Since a time scale may or may not be connected, the concept of the jump operator is useful.

Definition 1.2. Define the *forward jump operator* $\sigma(t)$ at t by

$$\sigma(t)=\inf\{\tau>t:\tau\in\mathbb{T}\},\quad \forall t\in\mathbb{T}, \tag{1.5}$$

and define the *graininess function* $\mu:\mathbb{T}\to[0,\infty)$ as $\mu(t)=\sigma(t)-t$.

Also let $x^\sigma(t)=x(\sigma(t))$, that is, x^σ is the composite function $x\circ\sigma$. The jump operator σ then allows the classification of points in a time scale in the following way. If $\sigma(t)>t$, then we say that the point t is right scattered; while if $\sigma(t)=t$ then, we say the point t is right dense.

Throughout this work, the assumption is made that $\mathbb{T}$ has the topology that it inherits from the standard topology on the real numbers $\mathbb{R}$.

Definition 1.3. Fix $t\in\mathbb{T}$ and let $x:\mathbb{T}\to\mathbb{R}^n$. Define $x^\Delta(t)$ to be the vector (if it exists) with the property that given $\epsilon>0$, there is a neighborhood U of t with

$$\left|\left[x_i(\sigma(t))-x_i(s)\right]-x_i^\Delta(t)\left[\sigma(t)-s\right]\right|\le\epsilon\left|\sigma(t)-s\right|,\quad \forall s\in U \text{ and each } i=1,\dots,n. \tag{1.6}$$

It is said that $x^\Delta(t)$ is the *(delta) derivative* of $x(t)$ and that x is (delta) differentiable at t.

Definition 1.4. If $G^\Delta(t)=g(t)$, $t\in\mathbb{T}$, then it is said that G is a *(delta) antiderivative* of g and the *Cauchy (delta) integral* is defined by

$$\int_a^t g(s)\Delta s=G(t)-G(a). \tag{1.7}$$

For a more general definition of the delta integral, see [2, 3].

The following theorem is due to Hilger [5].

Theorem 1.5. *Assume that $g:\mathbb{T}\to\mathbb{R}^n$ and let $t\in\mathbb{T}$.*

(i) *If g is differentiable at t, then g is continuous at t.*

(ii) *If g is continuous at t and t is right scattered, then g is differentiable at t with*

$$g^\Delta(t)=\frac{g(\sigma(t))-g(t)}{\sigma(t)-t}. \tag{1.8}$$

(iii) *If g is differentiable and t is right dense, then*

$$g^{\Delta}(t) = \lim_{s\to t}\frac{g(t)-g(s)}{t-s}. \tag{1.9}$$

(iv) *If g is differentiable at t, then $g(\sigma(t)) = g(t)+\mu(t)g^{\Delta}(t)$.*

We assume throughout that $t_0 \geq 0$ and $t_0 \in \mathbb{T}$. By the time scale interval $[t_0,\infty)$, we mean the set $\{t \in \mathbb{T} : t_0 \leq t < \infty\}$. The theory of time scales dates back to Hilger [5]. The monographs [2, 3, 6] also provide an excellent introduction.

2. Lyapunov functions

In this section, we define what Peterson and Tisdell [7] call a type I Lyapunov function and summarize a few of the results and examples given in [7] relative to what we do here.

Definition 2.1. It is said that $V : \mathbb{R}^n \to \mathbb{R}^+$ is a *"type I" Lyapunov function* on $\mathbb{R}^n$ provided that

$$V(x) = \sum_{i=1}^{n} V_i(x_i) = V_1(x_1)+\cdots+V_n(x_n), \tag{2.1}$$

where each $V_i : \mathbb{R}^+ \to \mathbb{R}^+$ is continuously differentiable and $V_i(0) = 0$.

Peterson and Tisdell [7] proved that if V is a type I Lyapunov function and the function $\dot{V}$ is defined by

$$\dot{V}(t,x) = \int_0^1 \nabla V(x+h\mu(t))\cdot f(t,x)dh, \tag{2.2}$$

where $\nabla = (\partial/\partial x_1,\dots,\partial/\partial x_n)$ is the gradient operator and the "$\cdot$" denotes the usual scalar product, then, if x is a solution to (1.1), it follows that

$$[V(x(t))]^{\Delta} = \dot{V}(t,x(t)). \tag{2.3}$$

Peterson and Tisdell [7] also show that

$$\dot{V}(t,x) = \begin{cases} \displaystyle\sum_{i=1}^{n}\frac{[V_i(x_i+\mu(t)f_i(t,x)) - V_i(x_i)]}{\mu(t)} & \text{when } \mu(t)\neq 0,\\ \nabla V(x)\cdot f(t,x) & \text{when } \mu(t) = 0. \end{cases} \tag{2.4}$$

Sometimes the domain of V will be a subset D of $\mathbb{R}^n$.

Note that $V = V(x)$ and even if the vector field associated with the dynamic equation is autonomous, $\dot{V}$ still depends on t (and x of course) when the graininess function of $\mathbb{T}$ is nonconstant. Several formulas are given in Peterson and Tisdell [7] for $\dot{V}(t,x)$ for various type I Lyapunov functions $V(x)$. In this paper, the only one of these formulas that we will use is that if $V(x) = \|x\|^2$, then

$$\dot{V}(t,x) = 2x\cdot f(t,x)+\mu(t)||f(t,x)||^2. \tag{2.5}$$

It is the second term in (2.5) that usually makes the Lyapunov theory for time scales much more difficult than the continuous case.

3. Exponential stability

In this section, we present some results on the exponential stability of the trivial solution of (1.1). First we give a few more preliminaries.

Definition 3.1. Assume that $g:\mathbb{T}\to\mathbb{R}$. Define and denote $g\in C_{\text{rd}}(\mathbb{T};\mathbb{R})$ as right-dense continuous (rd-continuous) if g is continuous at every right-dense point $t\in\mathbb{T}$ and $\lim_{s\to t^-} g(s)$ exists and is finite at every left-dense point $t\in\mathbb{T}$, where left-dense is defined in the obvious manner.

If $g\in C_{\text{rd}}$, then g has a (delta) antiderivative [2, Theorem 1.74]. Now define the so-called set of regressive functions, $\mathcal{R}$ by

$$\mathcal{R}=\{p:\mathbb{T}\longrightarrow\mathbb{R};\, p\in C_{\text{rd}}(\mathbb{T};\mathbb{R}) \text{ and } 1+p(t)\mu(t)\neq 0 \text{ on } \mathbb{T}\}. \tag{3.1}$$

Under the addition on $\mathcal{R}$ defined by

$$(p\oplus q)(t):=p(t)+q(t)+\mu(t)p(t)q(t),\quad t\in\mathbb{T}, \tag{3.2}$$

$\mathcal{R}$ is an Abelian group (see [2, exercise 2.26]), where the additive inverse of p, denoted by $\ominus p$, is defined by

$$(\ominus p)(t):=\frac{-p(t)}{1+\mu(t)p(t)},\quad t\in\mathbb{T}. \tag{3.3}$$

Then define the set of positively regressive functions by

$$\mathcal{R}^+=\{p\in\mathcal{R}:1+p(t)\mu(t)>0 \text{ on } \mathbb{T}\}. \tag{3.4}$$

For $p\in\mathcal{R}$, the generalized exponential function $e_p(\cdot,t_0)$ on a time scale $\mathbb{T}$ can be defined (see [2, Theorem 2.35]) to be the unique solution to the IVP

$$x^\Delta=p(t)x,\qquad x(t_0)=x_0. \tag{3.5}$$

We will frequently use the fact that if $p\in\mathcal{R}^+$, then [2, Theorem 2.48] $e_p(t,t_0)>0$ for $t\in\mathbb{T}$. We will use many of the properties of this (generalized) exponential function, which are summarized in the following theorem (see [2, Theorem 2.36]).

THEOREM 3.2. *If p, $q\in\mathcal{R}$, then for t, s, $r\in\mathbb{T}$,*
(i) $e_0(t,s)\equiv 1$ *and* $e_p(t,t)\equiv 1$;
(ii) $e_p(\sigma(t),s)=(1+\mu(t)p(t))e_p(t,s)$;
(iii) $1/e_p(t,s)=e_{\ominus p}(t,s)$;

(iv) $e_p(t,s) = 1/e_p(s,t) = e_{\ominus p}(s,t)$;
(v) $e_p(t,s)e_p(s,r) = e_p(t,r)$;
(vi) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;
(vii) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s)$, *where* $p \ominus q := p \oplus (\ominus q)$.

It follows from Bernoulli's inequality (see [2, Theorem 6.2]) that for any time scale, if the constant $\lambda \in \mathcal{R}^+$, then

$$0 < e_{\ominus\lambda}(t,t_0) \leq \frac{1}{1+\lambda(t-t_0)}, \quad t \geq t_0. \tag{3.6}$$

It follows that

$$\lim_{t\to\infty} e_{\ominus\lambda}(t,t_0) = 0. \tag{3.7}$$

In particular, if $\mathbb{T} = \mathbb{R}$, then $e_{\ominus\lambda}(t,t_0) = e^{-\lambda(t-t_0)}$ and if $\mathbb{T} = \mathbb{Z}^+$, then $e_{\ominus\lambda}(t,t_0) = (1+\lambda)^{-(t-t_0)}$. For the growth of generalized exponential functions on time scales, see Bodine and Lutz [1]. With all this in mind, we make the following definition.

Definition 3.3. Say that the trivial solution of (1.1) is *exponentially stable on* $[0,\infty)$ if there exist a positive constant d, a constant $C \in \mathbb{R}^+$, and an $M > 0$ such that for any solution $x(t,t_0,x_0)$ of the IVP (1.1)-(1.2), $t_0 \geq 0$, $x_0 \in \mathbb{R}^n$,

$$||x(t,t_0,x_0)|| \leq C(||x_0||,t_0)\left(e_{\ominus M}(t,t_0)\right)^d, \quad \forall t \in [t_0,\infty), \tag{3.8}$$

where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^n$. The trivial solution of (1.1) is said to be *uniformly exponentially stable on* $[0,\infty)$ if C is independent of t_0.

Note that if $\mathbb{T} = \mathbb{R}$, then $(e_{\ominus\lambda}(t,t_0))^d = e^{-\lambda d(t-t_0)}$ and if $\mathbb{T} = \mathbb{Z}^+$, then $(e_{\ominus\lambda}(t,t_0))^d = (1+\lambda)^{-d\lambda(t-t_0)}$.

We are now ready to present some results.

Theorem 3.4. *Assume that* $D \subset \mathbb{R}^n$ *contains the origin and there exists a type I Lyapunov function* $V : D \to [0,\infty)$ *such that for all* $(t,x) \in [0,\infty) \times D$,

$$W(\|x\|) \leq V(x) \leq \phi(\|x\|), \tag{3.9}$$

$$\dot{V}(t,x) \leq \frac{\psi(\|x\|) - L(M \ominus \delta)(t)e_{\ominus\delta}(t,0)}{1+\mu(t)M}, \tag{3.10}$$

$$\psi(\phi^{-1}(V(x))) + MV(x) \leq 0, \tag{3.11}$$

where W, ϕ, ψ *are continuous functions such that* $\phi, W : [0,\infty) \to [0,\infty)$, $\psi : [0,\infty) \to (-\infty,0]$, ψ *is nonincreasing,* ϕ *and* W *are strictly increasing;* $L \geq 0$, $\delta > M > 0$ *are constants. Then all solutions of (1.1)-(1.2) that stay in* D *satisfy*

$$||x(t)|| \leq W^{-1}((V(x_0)+L)e_{\ominus M}(t,t_0)), \quad \forall t \geq t_0. \tag{3.12}$$

Proof. Let x be a solution to (1.1)-(1.2) that stays in D for all $t \geq 0$. Consider

$$\begin{aligned}
&[V(x(t))e_M(t,0)]^{\Delta} \\
&\quad = \dot{V}(t,x(t))e_M^{\sigma}(t,0) + V(x(t))e_M^{\Delta}(t,0), \quad \text{using (2.3) and the product rule} \\
&\quad \leq (\psi(\|x(t)\|) - L(M \ominus \delta)(t)e_{\ominus\delta}(t,0))e_M(t,0) + MV(x(t))e_M(t,0), \quad \text{by (3.10)} \\
&\quad = (\psi(\|x(t)\|) - L(M \ominus \delta)(t)e_{\ominus\delta}(t,0) + MV(x(t)))e_M(t,0) \\
&\quad \leq (\psi(\phi^{-1}(V(x(t)))) + MV(x(t)) - L(M \ominus \delta)(t)e_{\ominus\delta}(t,0))e_M(t,0), \quad \text{by (3.9)} \\
&\quad \leq -L(M \ominus \delta)(t)e_{\ominus\delta}(t,0)e_M(t,0), \quad \text{by (3.11)} \\
&\quad = -L(M \ominus \delta)(t)e_{M\ominus\delta}(t,0), \quad \text{by Theorem 3.2.}
\end{aligned} \tag{3.13}$$

Integrating both sides from t_0 to t with $x_0 = x(t_0)$, we obtain, for $t \in [t_0, \infty)$,

$$\begin{aligned}
V(x(t))e_M(t,0) &\leq V(x_0)e_M(t_0,0) - Le_{M\ominus\delta}(t,0) + Le_{M\ominus\delta}(t_0,0) \\
&\leq V(x_0)e_M(t_0,0) + Le_{M\ominus\delta}(t_0,0) \\
&\leq (V(x_0) + L)e_M(t_0,0).
\end{aligned} \tag{3.14}$$

It follows that for $t \in [t_0, \infty)$,

$$V(x(t)) \leq (V(x_0) + L)e_M(t_0,0)e_{\ominus M}(t,0) = (V(x_0) + L)e_{\ominus M}(t,t_0). \tag{3.15}$$

Thus by (3.9),

$$\|x(t)\| \leq W^{-1}((V(x_0) + L)e_{\ominus M}(t,t_0)), \quad t \in [t_0, \infty). \tag{3.16}$$

This concludes the proof. □

We now provide a special case of Theorem 3.4 for certain functions ϕ and ψ.

Theorem 3.5. *Assume that $D \subset \mathbb{R}^n$ contains the origin and there exists a type I Lyapunov function $V : D \to [0,\infty)$ such that for all $(t,x) \in [0,\infty) \times D$,*

$$\lambda_1(t)\|x\|^p \leq V(x) \leq \lambda_2(t)\|x\|^q, \tag{3.17}$$

$$\dot{V}(t,x) \leq \frac{-\lambda_3(t)\|x\|^r - L(M \ominus \delta)(t)e_{\ominus\delta}(t,0)}{1 + M\mu(t)}, \tag{3.18}$$

$$V(x) - V^{r/q}(x) \leq 0, \tag{3.19}$$

where $\lambda_1(t)$, $\lambda_2(t)$, and $\lambda_3(t)$ are positive functions, where $\lambda_1(t)$ is nondecreasing; p, q, r are positive constants; L is a nonnegative constant, and $\delta > M := \inf_{t\geq 0}\lambda_3(t)/[\lambda_2(t)]^{r/q} > 0$. Then the trivial solution of (1.1) is exponentially stable on $[0,\infty)$.

Proof. As in the proof of Theorem 3.4, let x be a solution to (1.1)-(1.2) that stays in D for all $t \geq 0$. Since $M = \inf_{t\geq 0}\lambda_3(t)/[\lambda_2(t)]^{r/q} > 0$, $e_M(t,0)$ is well defined and positive. Since $\lambda_3(t)/[\lambda_2(t)]^{r/q} \geq M$, we have

$$\begin{aligned}
&\left[V(x(t))e_M(t,0)\right]^\Delta \\
&\quad = \dot{V}(t,x(t))e_M^\sigma(t,0) + V(x(t))e_M^\Delta(t,0), \quad \text{using (2.3) and the product rule} \\
&\quad \leq \left(-\lambda_3(t)\|x(t)\|^r - L(M\ominus\delta)(t)e_{\ominus\delta}(t,0)\right)e_M(t,0) + MV(x(t))e_M(t,0), \quad \text{by (3.18)} \\
&\quad \leq \left(\frac{-\lambda_3(t)}{[\lambda_2(t)]^{r/q}}V^{r/q}(x(t)) + MV(x(t)) - L(M\ominus\delta)(t)e_{\ominus\delta}(t,0)\right)e_M(t,0), \quad \text{by (3.17)} \\
&\quad \leq \left(M\left(V(x(t)) - V^{r/q}(x(t))\right) - L(M\ominus\delta)(t)e_{\ominus\delta}(t,0)\right)e_M(t,0) \\
&\quad \leq -L(M\ominus\delta)(t)e_{M\ominus\delta}(t,0), \quad \text{by (3.19).}
\end{aligned} \tag{3.20}$$

Integrating both sides from t_0 to t with $x_0 = x(t_0)$, and by invoking condition (3.17) and the fact that $\lambda_1(t) \geq \lambda_1(t_0)$, we obtain

$$\|x(t)\| \leq \lambda_1^{-1/p}(t)\left(\left(V(x_0)+L\right)e_{\ominus M}(t,t_0)\right)^{1/p} \tag{3.21}$$

$$\leq \lambda_1^{-1/p}(t_0)\left(\left(V(x_0)+L\right)e_{\ominus M}(t,t_0)\right)^{1/p}, \quad \forall t \geq t_0. \tag{3.22}$$

This concludes the proof. □

Remark 3.6. In Theorem 3.5, if $\lambda_i(t) = \lambda_i$, $i = 1,2,3$, are positive constants, then the trivial solution of (1.1) is uniformly exponentially stable on $[0,\infty)$. The proof of this remark follows from Theorem 3.5 by taking $\delta > \lambda_3/[\lambda_2]^{r/q}$ and $M = \lambda_3/[\lambda_2]^{r/q}$.

The next theorem is an extension of [4, Theorem 2.1].

Theorem 3.7. *Assume that $D \subset \mathbb{R}^n$ contains the origin and there exists a type I Lyapunov function $V : D \to [0,\infty)$ such that for all $(t,x) \in [0,\infty)\times D$,*

$$\lambda_1\|x\|^p \leq V(x), \tag{3.23}$$

$$\dot{V}(t,x) \leq \frac{-\lambda_3 V(x) - L(\varepsilon\ominus\delta)(t)e_{\ominus\delta}(t,0)}{1+\varepsilon\mu(t)}, \tag{3.24}$$

where λ_1, λ_3, p, $\delta > 0$, $L \geq 0$ are constants and $0 < \varepsilon < \min\{\lambda_3,\delta\}$. Then the trivial solution of (1.1) is uniformly exponentially stable on $[0,\infty)$.

Proof. Let x be a solution to (1.1)-(1.2) that stays in D for all $t \in [0,\infty)$. Since $\varepsilon \in \mathcal{R}^+$, $e_\varepsilon(t,0)$ is well defined and positive. Now consider

$$\begin{aligned}
&[V(x(t))e_\varepsilon(t,0)]^\Delta \\
&\quad = \dot{V}(t,x(t))e_\varepsilon^\sigma(t,0) + \varepsilon V(x(t))e_\varepsilon(t,0) \\
&\quad \le \big(-\lambda_3 V(x(t)) - L(\varepsilon \ominus \delta)(t)e_{\ominus\delta}(t,0)\big)e_\varepsilon(t,0) + \varepsilon V(x(t))e_\varepsilon(t,0), \quad \text{by (3.24)} \\
&\quad = e_\varepsilon(t,0)\big[\varepsilon V(x(t)) - \lambda_3 V(x(t)) - L(\varepsilon \ominus \delta)(t)e_{\ominus\delta}(t,0)\big] \\
&\quad \le -e_\varepsilon(t,0)L(\varepsilon \ominus \delta)(t)e_{\ominus\delta}(t,0) \\
&\quad = -L(\varepsilon \ominus \delta)(t)e_{\varepsilon\ominus\delta}(t,0).
\end{aligned} \tag{3.25}$$

Integrating both sides from t_0 to t, we obtain

$$\begin{aligned}
V(x(t))e_\varepsilon(t,0) &\le V(x_0)e_\varepsilon(t_0,0) - Le_{\varepsilon\ominus\delta}(t,0) + Le_{\varepsilon\ominus\delta}(t_0,0) \\
&\le V(x_0)e_\varepsilon(t_0,0) + Le_{\varepsilon\ominus\delta}(t_0,0) \\
&\le (V(x_0)+L)e_\varepsilon(t_0,0).
\end{aligned} \tag{3.26}$$

Dividing both sides of the above inequality by $e_\varepsilon(t,0)$, we obtain

$$\begin{aligned}
V(x(t)) &\le (V(x_0)+L)e_\varepsilon(t_0,0)e_{\ominus\varepsilon}(t,0) \\
&= (V(x_0)+L)e_{\ominus\varepsilon}(t,t_0).
\end{aligned} \tag{3.27}$$

The proof is completed by invoking condition (3.23). □

4. Examples

We now present some examples to illustrate the theory developed in Section 3.

Example 4.1. Consider the IVP

$$x^\Delta = ax + bx^{1/3}e_{\ominus\delta}(t,0), \qquad x(t_0) = x_0, \tag{4.1}$$

where $\delta > 0$, a, b are constants, $x_0 \in \mathbb{R}$, and $t_0 \in [0,\infty)$. If there is a constant $0 < M < \delta$ such that

$$(2a + a^2\mu(t) + 1)(1 + M\mu(t)) \le -M, \tag{4.2}$$

$$\left(\frac{2}{3}(\mu(t)b^2)^{3/2} + \frac{|2b + 2ab\mu(t)|^3}{3}\right)(1 + M\mu(t)) \le -L(M \ominus \delta)(t), \tag{4.3}$$

for some constant $L \ge 0$ and all $t \in [0,\infty)$, then the trivial solution of (4.1) is uniformly exponentially stable.

Proof. We will show that under the above assumptions, the conditions of Remark 3.6 are satisfied. Choose $D = \mathbb{R}$ and $V(x) = x^2$, then (3.17) holds with $p = q = 2$, $\lambda_1 = \lambda_2 = 1$. Now from (2.5),

$$\begin{aligned}\dot V(t,x) &= 2x \cdot f(t,x) + \mu(t)\|f(t,x)\|^2\\ &= 2x\left(ax + bx^{1/3}e_{\ominus\delta}(t,0)\right) + \mu(t)\left(ax + bx^{1/3}e_{\ominus\delta}(t,0)\right)^2\\ &\le \left(2a + a^2\mu(t)\right)x^2 + |2b + 2ab\mu(t)|\,x^{4/3}e_{\ominus\delta}(t,0) + b^2\mu(t)x^{2/3}\left(e_{\ominus\delta}(t,0)\right)^2.\end{aligned} \tag{4.4}$$

To further simplify the above inequality, we make use of Young's inequality, which says that for any two nonnegative real numbers w and z, we have

$$wz \le \frac{w^e}{e} + \frac{z^f}{f}, \quad \text{with } \frac{1}{e} + \frac{1}{f} = 1,\ e, f > 1. \tag{4.5}$$

Thus, for $e = 3/2$ and $f = 3$, we get

$$\begin{aligned}x^{4/3}|2b + 2ab\mu(t)|\,e_{\ominus\delta}(t,0) &\le \left[\frac{\left(x^{4/3}\right)^{3/2}}{3/2} + \frac{|2b + 2ab\mu(t)|^3\left(e_{\ominus\delta}(t,0)\right)^3}{3}\right]\\ &= \frac{2}{3}x^2 + \frac{|2b + 2ab\mu(t)|^3\left(e_{\ominus\delta}(t,0)\right)^3}{3},\\ x^{2/3}b^2\mu(t)\left(e_{\ominus\delta}(t,0)\right)^2 &\le \frac{\left(x^{2/3}\right)^3}{3} + \frac{\left(b^2\mu(t)\left(e_{\ominus\delta}(t,0)\right)^2\right)^{3/2}}{3/2}\\ &= \frac{x^2}{3} + \frac{2}{3}\left(\mu(t)b^2\right)^{3/2}\left(e_{\ominus\delta}(t,0)\right)^3.\end{aligned} \tag{4.6}$$

Thus, putting everything together, we arrive at

$$\begin{aligned}\dot V(t,x) &\le \left(2a + \mu(t)a^2 + 1\right)x^2 + \left[\frac{2}{3}\left(\mu(t)b^2\right)^{3/2} + \frac{|2b + 2ab\mu(t)|^3}{3}\right]\left(e_{\ominus\delta}(t,0)\right)^3\\ &\le \left(2a + \mu(t)a^2 + 1\right)x^2 + \left[\frac{2}{3}\left(\mu(t)b^2\right)^{3/2} + \frac{|2b + 2ab\mu(t)|^3}{3}\right]e_{\ominus\delta}(t,0).\end{aligned} \tag{4.7}$$

Dividing and multiplying the right-hand side by $(1 + M\mu(t))$, we see that (3.18) holds under the above assumptions with $r = 2$ (note that $\lambda_3 = M$). Also, since $r = q = 2$, (3.19) is satisfied. Therefore all the hypotheses of Remark 3.6 are satisfied and we conclude that the trivial solution of (4.1) is uniformly exponentially stable. We next look at the three special cases of (4.1) when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}_0$, and $\mathbb{T} = h\mathbb{N}_0 = \{0, h, 2h, \dots\}$.

Case 4.2. If $\mathbb{T} = \mathbb{R}$, then $\mu(t) = 0$ and it is easy to see that if we assume that $a < -1/2$, then (4.2) is true if we take $M = -(2a + 1) > 0$. For $L = 8|b|^3/3(\delta - M)$, condition (4.3) is satisfied. Hence in this case we conclude that if $a < -1/2$ and $\delta > -(2a + 1)$, then the trivial solution to (4.1) is uniformly exponentially stable.

Case 4.3. If $\mathbb{T} = \mathbb{N}_0$, then $\mu(t) = 1$ and condition (4.2) cannot be satisfied for positive M.

To get around this, we will adjust the steps leading to inequality (4.7) as follows:

$$\begin{aligned}
x^{4/3}\,|2b+2ab\mu(t)|\,e_{\ominus\delta}(t,0) &\le |2b+2ab\mu(t)|\left[\frac{(x^{4/3})^{3/2}}{3/2}+\frac{(e_{\ominus\delta}(t,0))^3}{3}\right]\\
&= \frac{2}{3}|2b+2ab\mu(t)|x^2+\frac{|2b+2ab\mu(t)|}{3}(e_{\ominus\delta}(t,0))^3,\\
x^{2/3}b^2\mu(t)(e_{\ominus\delta}(t,0))^2 &\le b^2\mu(t)\left[\frac{(x^{2/3})^3}{3}+\frac{((e_{\ominus\delta}(t,0))^2)^{3/2}}{3/2}\right]\\
&= b^2\mu(t)\frac{x^2}{3}+\frac{2}{3}\mu(t)b^2(e_{\ominus\delta}(t,0))^3.
\end{aligned} \tag{4.8}$$

Hence, inequality (4.7) becomes

$$\begin{aligned}
\dot V(t,x) \le{}& \left(2a+\mu(t)a^2+\frac{2}{3}|2b+2ab\mu(t)|+\frac{\mu(t)b^2}{3}\right)x^2\\
&+\left[\frac{|2b+2ab\mu(t)|+(2/3)\mu(t)b^2}{3}\right]e_{\ominus\delta}(t,0).
\end{aligned} \tag{4.9}$$

Now, if $\mathbb{T} = \mathbb{N}_0$, then $\mu(t) = 1$ and so from this last inequality, given $\delta > 0$, we want to find $0 < M < \delta$ and $L \ge 0$ such that

$$\left(2a+a^2+\frac{2}{3}|2b+2ab|+\frac{b^2}{3}\right)(1+M) \le -M, \tag{4.10}$$

$$\frac{|2b+2ab|+(2/3)b^2}{3}(1+M) \le -L(M\ominus\delta)(t) = \frac{\delta-M}{1+\delta}L. \tag{4.11}$$

Note that condition (4.10) is satisfied for all $M > 0$ sufficiently small if

$$2a+a^2+\frac{2}{3}|2b+2ab|+\frac{b^2}{3} < 0. \tag{4.12}$$

For such a $0 < M < \delta$, if we take

$$L = \frac{2(3|b||1+a|+b^2)(1+M)(1+\delta)}{9(\delta-M)}, \tag{4.13}$$

then (4.3) is satisfied (note that for each $\delta > 0$, we can find such an M so our result holds for all δ). In conclusion, we have for the case $\mathbb{T} = \mathbb{N}_0$ that if (4.12) holds, then the trivial solution of (4.1) is uniformly exponentialy stable. In particular if $a = -4/5$ and $b = 1/5$, then (4.12) is satisfied.

Case 4.4. If $\mathbb{T} = h\mathbb{N}_0 = \{0,h,2h,\dots\}$, then $\mu(t) = h$ and in this case by (4.2) and (4.3), we want to find $0 < M < \delta$ and $L \ge 0$ such that

$$(2a+a^2h+1) \le \frac{-M}{(1+Mh)}, \tag{4.14}$$

$$\frac{|2b+2abh|^3+(2/3)(hb^2)^{3/2}}{3}(1+hM) \le -L(M\ominus\delta)(t) = \frac{\delta-M}{1+h\delta}L. \tag{4.15}$$

Note that (4.14) is satisfied for all $M > 0$ sufficiently small provided that $h > 0$ satisfies

$$ha^2 + 2a + 1 < 0. \tag{4.16}$$

Now the polynomial

$$p(a) := ha^2 + 2a + 1 \tag{4.17}$$

will have distinct real roots

$$\begin{aligned} a_1(h) &= \frac{(-1 - \sqrt{1-h})}{h}, \\ a_2(h) &= \frac{(-1 + \sqrt{1-h})}{h} \end{aligned} \tag{4.18}$$

if $0 < h < 1$. Therefore if $0 < h < 1$ and $a_1(h) < a < a_2(h)$, then

$$ha^2 + 2a + 1 < 0 \tag{4.19}$$

as desired. Now, for such an h, if we let

$$L = \frac{\left[\left((2/9)(hb^2)^{3/2} + |2b + 2abh|^3/3\right)(1 + Mh)\right]}{\delta - M}(1 + \delta h), \tag{4.20}$$

then (4.15) is satisfied. Putting this all together, we get that if $0 < h < 1$ and

$$a_1(h) < a < a_2(h), \tag{4.21}$$

then the trivial solution of (4.1) is uniformly exponentially stable. □

Remark 4.5. It is interesting to note that

$$\begin{aligned} \lim_{h \to 0+} a_2(h) &= \lim_{h \to 0+} \frac{(-1 + \sqrt{1-h})}{h} = \frac{-1}{2}, \\ \lim_{h \to 0+} a_1(h) &= \lim_{h \to 0+} (-1 - \sqrt{1-h})h = -\infty, \end{aligned} \tag{4.22}$$

recalling that if $\mathbb{T} = \mathbb{R}$, then for $-\infty < a < -1/2$, the zero solution to (4.1) is uniformly exponentially stable.

Acknowledgment

The second author gratefully acknowledges the support of the Summer Research Fellowships at the University of Dayton.

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