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Algebraic Isomorphisms and Spectra of Triangular Limit Algebras

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ABSTRACT. We show that the spectrum of a triangular regular limit algebra (TAF algebra) is an invariant for algebraic isomorphism. Combining this with previous results provides a striking rigidity property: two triangular regular limit algebras are algebraically isomorphic if and only if they are isometrically isomorphic. A consequence of spectral invariance is a structure theorem for isomorphisms between limit algebras.

The proof of the main theorem makes use of a characterization of the completely meet irreducible ideals of a TAF algebra and a dual space formulation of the spectrum.

1. INTRODUCTION

It has long been known that two AF C^*-algebras can be isomorphic even if the embeddings or approximating algebras involved are rather different. This situation was clarified by the celebrated theorem of Elliott [4, 5] which established a complete isomorphism invariant, namely the triple consisting of the K_0 group of the AF C*-algebra, the positive cone, and the scale.

This satisfying situation changes radically, even with quite well-behaved embeddings, when the approximating finite dimensional C^*-algebras are replaced by more general finite dimensional operator algebras. Peters, Poon and Wagner [10] and Power [13, Proposition 8.6] showed that the isomorphism class of such a regular limit algebra is very sensitive to the choice of embeddings. In view of these and other examples, it is clear that finer invariants than those arising from K-theory are needed for classification.

In [11, 12], Power introduced the fundamental relation for limit algebras, now called the *spectrum*, and showed that it is an invariant for *isometric* isomorphism. Moreover, he showed that it is a complete invariant in the triangular case, that is, if $\mathcal{A} \cap \mathcal{A}^*$ is a masa. An AF C^*-algebra C may be coordinatized by viewing it as the C^*-algebra of an AF groupoid $\mathcal{G}_0$, so that $C \simeq C^*(\mathcal{G}_0)$ [14, 16]. When $\mathcal{A} \subseteq C$ is a limit algebra (as developed in Section 2), the spectrum of $\mathcal{A}$ is a suitable subsemigroupoid of $\mathcal{G}_0$. Thus, the spectrum of a limit algebra may be viewed as a coordinization of the algebra and this viewpoint makes connections with the coordinates for operator algebras described in several sources; see, for example, [6, 8, 9].

A question open for almost a decade is whether the spectrum is an invariant for algebraic isomorphism of triangular limit algebras [13, Problem 7.8]. An affirmative answer was obtained for an important special case, triangular limit algebras generated by their order preserving normalisers, using an automatic continuity result of independent interest [2]. In this paper, we provide an affirmative answer to this question for *all* triangular limit algebras. While we use the automatic continuity result, we also employ a description of the spectrum as a subset of the dual space.

It follows from our main result, Theorem 4.1, that if two limit algebras are isomorphic as algebras, then the implementing isomorphism may be replaced with an isometric isomorphism. This is quite different from the situation of weakly closed operator algebras, such as nest subalgebras of $\mathcal{B}(\mathcal{H})$; Larson showed that nest algebras may be similar without being isometrically isomorphic [7]. As a corollary of Theorem 4.1, we obtain a structure theorem for algebraic isomorphisms of triangular limit algebras. Such an isomorphism can be factored as a pair of maps: an isometric isomorphism induced by a homeomorphism between the spectra and an algebraic automorphism which is the identity on the spectrum.

In view of the automatic continuity of algebra isomorphisms obtained by Donsig, Hudson and Katsoulis [2], invariance of the spectrum under Banach algebra isomorphism implies invariance under algebra isomorphism. Our approach to showing the spectrum is a Banach algebra invariant is firstly to extend the work on completely meet irreducible ideals in strongly maximal triangular limit algebras pioneered in [1] and [3]: we characterize the completely meet irreducible ideals in triangular limit algebras and then use this characterization to show that the set of such ideals corresponds to a dense subset of the spectrum. Through this, we deduce that algebra isomorphisms induce maps between such dense subsets.

Secondly, we show that this map extends to a homeomorphism of spectra and for this step we utilize a formulation of the spectrum as a subset of the dual space, much as is done for C^*-algebras by Renault in [15]. In this formulation, the groupoid topology on the spectrum appears naturally as the weak-$*$ topology. Section 2 is devoted to describing the spectrum as functionals. We expect that several of the results from Section 2 can be obtained using the approach taken in [15], however, we have not done so here because of a desire to connect with the approaches to limit algebras in existing literature and to be as self-contained as possible. Section 3 concerns the relationship between the spectrum and the family of completely meet irreducible ideals. The main result, Theorem 4.1, is proved in Section 4.

2. The Spectrum as a subset of the Dual Space

We begin with a precise definition of the limit algebras under consideration.

Given an inclusion $\mathcal{D} \subseteq C$ of C^*-algebras, the (partial isometry) *normalizer* of $\mathcal{D}$ is the set

$$\mathcal{N}(\mathcal{D}) := \{u \in C : u^*u = (u^*u)^2, u\mathcal{D}u^* \subseteq \mathcal{D}, \text{ and } u^*\mathcal{D}u \subseteq \mathcal{D}\}.$$

Suppose for each $n \in \mathbb{N}$, that we have an inclusion $D_n \subseteq A_n \subseteq C_n$ where C_n is a finite dimensional C^*-algebra, D_n is a masa in C_n, and A_n is an algebra, but not necessarily self-adjoint. Thus, each building block algebra A_n is a digraph algebra or finite-dimensional CSL (FDCSL) algebra. In addition, suppose that $\alpha_n : C_n \to C_{n+1}$ is a C*-algebra embedding such that $\alpha_n(A_n) \subseteq A_{n+1}$ and $\alpha_n(\mathcal{N}(D_n)) \subseteq \mathcal{N}(D_{n+1})$. In this case the map $\alpha_n : A_n \to A_{n+1}$ is said to be *regular* with respect to the masa choice. The inductive limit $\mathcal{A} := \varinjlim(A_n, \alpha_n)$ is referred to as a *regular limit algebra,* (or simply as a limit algebra if the context is clear). Such algebras are analogous to AF C*-algebras in that they have natural (partial) matrix unit systems such that embeddings map matrix units to sums of matrix units. The abelian C*-subalgebra $\mathcal{D} := \varinjlim D_n$ is a *regular masa* in C: that is, the span of $\mathcal{N}(\mathcal{D})$ is dense in C. Such a masa is sometimes refered to as a Strătilă-Voiculescu masa (in view of [16]) or as a standard regular masa. In fact the regular limit algebras are precisely the closed subalgebras of AF C*-algebras which contain such a masa. They may also be described intrinsically (without reference to an ambient C*-algebra) as the limits of systems of digraph algebras in which the connecting embeddings are star-extendible and decomposable as direct sums of multiplicity one embeddings. Both approaches are presented in [13].

Let $\mathcal{A} = \varinjlim(A_k, \alpha_k)$ be a regular limit algebra. Without loss of generality (in fact) we assume that the maps α_k are injective for all k and we identify each algebra A_k with its canonical image in $\mathcal{A}$. For each of the digraph algebras A_k we may choose and fix a matrix unit system $C_k = \{x_{ij}^{(k)}\}$ for A_k such that α_k maps matrix units to sums of matrix units. Let D_k be the diagonal masa of A_k given by the matrix unit system and note that we have $\alpha_k(D_k) \subseteq D_{k+1}$. Thus for each k, the elements $\{x_{ij}^{(k)}\}$ form a basis for A_k and $\alpha_k(x_{ij}^{(k)}) = \sum c_{ij}^{(k+1)} x_{ij}^{(k+1)}$ where each coefficient $c_{ij}^{(k+1)}$ belongs to the set $\{0, 1\}$.

Definition. *We define the spectrum of $\mathcal{A}$ to be the set*

$$\operatorname{Spec}(\mathcal{A}) = \left\{\rho \in \mathcal{A}^\# : \|\rho\| = 1 \textit{ and } \rho\Big(\bigcup_{k=1}^{\infty} C_k\Big) \subseteq \{0,1\}\right\}$$

equipped with the relative weak-$*$-topology (i.e., relative to $\sigma(\mathcal{A}^\#, \mathcal{A})$, where $\mathcal{A}^\#$ is the dual Banach space of A).

It is easy to see that $\operatorname{Spec}(\mathcal{A}) \cup \{0\}$ is weak-$*$-closed and hence weak-$*$-compact. Thus $\operatorname{Spec}(\mathcal{A})$ is a locally compact Hausdorff space.

Note that the definition of Spec($\mathcal{A}$) masks an apparent dependence on the choice of matrix unit system for $\mathcal{A}$ or equivalently, on the choice of a regular masa. For triangular algebras, which are our main concern, this is not an issue.

This is not the usual definition of spectrum, and we pause to briefly give the usual definition. Let $M(\mathcal{D})$ be the space of multiplicative linear functionals on $\mathcal{D}$. Associated to any matrix unit $e \in \mathcal{N}(\mathcal{D})$ is a partial homeomorphism $\tilde{e}$ on $M(\mathcal{D})$, with domain $\{x \in M(\mathcal{D}) : x(e^*e) \neq 0\}$ and range $\{x \in M(\mathcal{D}) : x(ee^*) \neq 0\}$, and is defined by $(\tilde{e}(x))(d) = x(ede^*)$, where $x \in \text{Domain}(\tilde{e})$ and $d \in \mathcal{D}$. The graph of e is the set $\mathcal{G}(e) = \{(\tilde{e}(x), x) : x \in \text{Domain}(\tilde{e})\}$. The spectrum of $\mathcal{A}$ is usually defined in the literature as the set,

$$\text{Sp}(\mathcal{A}) := \bigcup\{\mathcal{G}(e) : e \in \mathcal{N}(\mathcal{D}) \cap \mathcal{A}\}.$$

Let τ be the topology on Sp($\mathcal{A}$) generated by the family $\{\mathcal{G}(e)\}_{e\in\mathcal{N}(\mathcal{D})\cap\mathcal{A}}$. A (partially defined) product can be placed on Sp($\mathcal{A}$) by mandating that (x_1, x_2) and (y_1, y_2) are composable if and only if $x_2 = y_1$ and when that occurs, the product is (x_1, y_2). Then Sp($\mathcal{A}$) becomes a topological semigroupoid.

In this section we shall construct a (partially) defined operation on Spec($\mathcal{A}$) and show that the resulting semigroupoid is topologically isomorphic to Sp($\mathcal{A}$).

If e and f are normalizing partial isometries in $\mathcal{N}(\mathcal{D})$, where $\mathcal{D}$ is the canonical masa of $\mathcal{A}$, then we say that e is a *subordinate* of f if $e = fp$ for some projection p in the canonical diagonal. We write $e \ll f$ when e is subordinate to f. A *chain of matrix units* is defined to be a sequence of matrix units which is a sequence of subordinates in consecutive building block algebras. Notice that given a chain (e_k) of matrix units, $\cap_k \mathcal{G}(e_k e_k^*)$ and $\cap_k \mathcal{G}(e_k^* e_k)$ determine unique points y and x in $M(\mathcal{D})$ and hence there is a natural mapping from chains of matrix units to Sp($\mathcal{A}$) given by $(e_k) \mapsto (y, x)$. This map is bijective and hence Sp($\mathcal{A}$) is often viewed as the set of matrix unit chains. We shall see that matrix unit chains provide a link between Spec($\mathcal{A}$) and Sp($\mathcal{A}$) and play an important role in characterizing the spectrum in terms of closed ideals.

We start with an elementary lemma.

Lemma 2.1. *Let M_n be the set of all complex $n \times n$ matrices, let $D_n \subseteq M_n$ be the set of diagonal matrices and suppose $\mathcal{M} \subseteq M_n$ is a D_n-bimodule. Let ρ be a linear functional on $\mathcal{M}$ such that for some matrix unit $E \in \mathcal{M}$, $\|\rho\| = \rho(E)$. Then $\rho(F) = 0$ for all matrix units $F \neq E$ in $\mathcal{M}$.*

Proof. Since ρ has an extension to M_n which does not increase its norm, it suffices to prove the lemma if $\mathcal{M} = M_n$, and this is what we shall do.

If $FF^* \neq EE^*$ and $F^*F \neq E^*E$, then for every $\lambda \in \mathbb{T}$, $1 = \|\lambda F + E\|$ and $\rho(\lambda F + E) = \lambda\rho(F) + \|\rho\|$. If $\rho(F) \neq 0$, then we may choose λ so that $\lambda\rho(F) > 0$, implying $\rho(\lambda F + E) > \|\rho\|$.

Next, if $FF^* = EE^*$ and $F^*F \neq E^*E$, then define $t = \rho(F)/\|\rho\|$ and put

$$T = \frac{\bar{t}F + E}{\sqrt{|t|^2 + 1}}.$$

Again $\|T\| = 1$ and a calculation shows $\rho(T) = \|\rho\| \sqrt{|t|^2+1} > \|\rho\|$ unless $\rho(F) = 0$. Similar considerations show that if $FF^* \neq EE^*$ and $F^*F = E^*E$, then $\rho(F) = 0$. □

Corollary 2.2. *For each functional* $\rho \in \mathrm{Spec}(\mathcal{A})$*, there is an integer N and a unique chain of matrix units* $(e_k)_{k\geq N}$ *such that* e_k *is the unique element of* C_k *satisfying* $\rho(e_k) = 1$.

Conversely, if (e_k) *is a chain of matrix units then there exists a unique functional* $\rho \in \mathrm{Spec}(\mathcal{A})$ *such that* $\rho(e_k) = 1$ *for all k.*

Proof. If ρ is a nonzero functional then there is a smallest integer N so that $\rho|_{A_N} \neq 0$ and, in particular, there is some matrix unit of A_N, e say, so that $\rho(e) = 1$. As ρ restricts to a functional on the algebra A_N and $\rho(e) = \|\rho\|$, Lemma 2.1 shows that e is the unique matrix unit of A_N with $\rho(e) = 1$. Let $e_N = e$. Repeating this argument for each $k \geq N$ gives the sequence. Writing e_k as a sum of matrix units in A_{k+1}, we must have e_{k+1} in this sum and so e_{k+1} is a subordinate of e_k. The converse direction is elementary. □

Since there is a bijective correspondence between the set of matrix unit chains and elements of $\mathrm{Sp}(\mathcal{A})$, Corollary 2.2 implies that there is a bijection $w : \mathrm{Spec}(\mathcal{A}) \to \mathrm{Sp}(\mathcal{A})$. Instead of using this bijection to develop the (partially defined) groupoid multiplication, we construct it directly.

The dual spaces $A_k^\#$ each have a dual basis φ_{ij}^k relative to the basis C_k, thus

$$\varphi_{ij}^{(k)}(x_{mn}^{(k)}) = \begin{cases} 1 & \text{if } (i,j) = (m,n), \\ 0 & \text{otherwise.} \end{cases}$$

The corollary shows that $\mathrm{Spec}(\mathcal{A})$ may in some sense be regarded as the "dual basis" to the matrix unit system $\{x_{ij}^{(k)}\}$. Indeed, if $\rho \in \mathrm{Spec}(\mathcal{A})$ then for some k, the restriction of ρ to A_k is not zero. Let $x \in C_k$ be such that $\rho(x) = 1$ and let φ be the element in the dual basis such that $\varphi(x) = 1$. It follows from Lemma 2.1 that $\rho|_{A_k} = \varphi$.

The following lemma relates the weak-$*$-topology to behavior of the functionals on the family of approximating algebras $\{A_k\}$.

Lemma 2.3. *Suppose* ρ *is in* $\mathrm{Spec}(\mathcal{A})$ *and* (ρ_n) *is a sequence in* $\mathrm{Spec}(\mathcal{A})$. *Then* $\mathrm{w}^*\text{-}\lim \rho_n = \rho$ *if and only if for each* $k \in \mathbb{N}$*, there exists* $N \in \mathbb{N}$ *such that for every* $n \geq N$, $\rho_n|_{A_k} = \rho|_{A_k}$.

Proof. Assume that $\mathrm{w}^*\text{-}\lim \rho_n = \rho$. Then for each matrix unit, $e \in C_k$, we have $\lim \rho_n(e) = \rho(e)$. But functionals in $\mathrm{Spec}\,\mathcal{A}$ are either 0 or 1 on a matrix unit, so the terms of the sequence $\rho_n(e)$ are eventually equal to $\rho(e)$. Fixing k, there are only finitely many elements of C_k hence there exists an N so that for $n \geq N$, $\rho_n(e) = \rho(e)$ for all $e \in C_k$. Thus $\rho_n|_{A_k} = \rho|_{A_k}$. The converse implication is left to the reader. □

Notice that Lemma 2.3 also shows that the bijection w between $\mathrm{Spec}(\mathcal{A})$ and $\mathrm{Sp}(\mathcal{A})$ is a homeomorphism. Indeed, for every matrix unit $e \in \mathcal{N}(\mathcal{D}) \cap \mathcal{A}$, the graph $\mathcal{G}(e)$ is a clopen set. Using the above lemma, it follows readily that a sequence $(x_n, y_n) \in \mathrm{Sp}(\mathcal{A})$ converges if and only if $w^{-1}(x_n, y_n)$ converges in $\mathrm{Spec}(\mathcal{A})$.

Next we define the groupoid product on the spectrum.

If ρ and σ are in $\mathrm{Spec}(\mathcal{A})$ then, for each k, a functional τ_k on A_k is defined by the linear extension of the specification

$$\tau_k(z) = \sum_{xy=z} \rho(x)\sigma(y),$$

for a matrix unit $z \in C_k$, where the sum runs over all x and y in C_k satisfying $xy = z$. Observe that τ_k is non-zero if and only if there are $e, f \in C_k$ with $\rho(e) \neq 0$, $\sigma(f) \neq 0$, and $ef \neq 0$. As e and f are unique, in fact τ_k is 1 on ef and is zero on all other matrix units of A_k.

Lemma 2.4. *With τ_k defined as above, there is an integer K so that for all $k \geq K$, $\tau_k = \tau_{k+1} \circ \alpha_k$. If τ is the functional on $\mathcal{A}$ induced by $(\tau_k)_{k \geq K}$, then τ is in* $\mathrm{Spec}(\mathcal{A}) \cup \{0\}$.

Proof. Let $(e_n)_{n \geq N}$ and $(f_n)_{n \geq M}$ be the chains of matrix units associated to ρ and σ. Notice that $e_{k+1}f_{k+1}$ is a subordinate of $e_k f_k$. In particular, if $e_k f_k = 0$, then $e_n f_n = 0$ for all $n \geq k$. Thus we have two cases, either $e_k f_k$ is eventually zero or $e_k f_k \neq 0$ for all k. In the first case, we choose K to the smallest k so that $e_k f_k = 0$ and $\tau = 0$. In the second, we choose $K = \max\{M, N\}$ and τ is the element of the spectrum associated to $(e_k f_k)_{k \geq K}$. ◻

Definition. *An ordered pair (ρ, σ) with both elements in* $\mathrm{Spec}(\mathcal{A})$ *is called a composable pair if the associated functional τ is in* $\mathrm{Spec}(\mathcal{A})$. *In this case we write $\rho \cdot \sigma$ for τ and we write* $\mathrm{Spec}(\mathcal{A})^{(2)}$ *for the set of composable pairs.*

Note that if $\rho, \sigma \in \mathrm{Spec}(\mathcal{A})$, with associated chains (e_k), (f_k) respectively, then ρ, σ is a composable pair if and only if $e_k f_k \neq 0$ for all (large enough) k. This leads to the continuity and the associativity of the product operation, as in the next lemma.

Proposition 2.5. *The set* $\mathrm{Spec}(\mathcal{A})^{(2)}$ *is closed in the product topology on* $\mathrm{Spec}(\mathcal{A}) \times \mathrm{Spec}(\mathcal{A})$. *Moreover, if (ρ_n, σ_n) is a sequence of composable pairs which converges to (ρ, σ), then*

$$\rho_n \cdot \sigma_n \to \rho \cdot \sigma.$$

Finally, the product is associative: if (ρ_1, ρ_2) and (ρ_2, ρ_3) are composable pairs then $(\rho_1, \rho_2 \cdot \rho_3)$ and $(\rho_1 \cdot \rho_2, \rho_3)$ both belong to $\mathrm{Spec}(\mathcal{A})^{(2)}$ *and $(\rho_1 \cdot \rho_2) \cdot \rho_3 = \rho_1 \cdot (\rho_2 \cdot \rho_3)$.*

Proof. Suppose that $(\rho_n, \sigma_n) \in \mathrm{Spec}(\mathcal{A})^{(2)}$ and $\rho_n \to \rho$, and $\sigma_n \to \sigma$. Let $(e_{k,n})$ and $(f_{k,n})$ be the matrix unit sequences associated with ρ_n and σ_n respectively. Then an application of Lemma 2.3 shows that for any $k \in \mathbb{N}$, the sequences $(e_{k,n})_{n=1}^{\infty}$ and $(f_{k,n})_{n=1}^{\infty}$ are eventually constant. Let $e_k = \lim_{n\to\infty} e_{k,n}$ and $f_k = \lim_{n\to\infty} f_{k,n}$. Note that (e_k) and (f_k) are the matrix unit sequences associated to ρ and σ respectively. Now for each k, the element $e_k f_k \neq 0$ because $e_k f_k = e_{k,n} f_{k,n}$ for large enough n and $(\rho_n, \sigma_n) \in \mathrm{Spec}(\mathcal{A})^{(2)}$. It follows that (ρ, σ) is a composable pair and $\rho_n \cdot \sigma_n \to \rho \cdot \sigma$.

Associativity follows in a similar fashion by using the associativity of the product in the kth approximating algebra A_k. ◻

We now consider the range and source maps in this setting and relate them to the relative w*-topology.

Observe that $\rho \cdot \rho = \rho$ implies that, if $(e_n)_{n\geq N}$ is the chain of matrix units associated to ρ, then $e_n e_n = e_n$ so each e_n is a diagonal matrix unit. Thus, the space of units in $\mathrm{Spec}(\mathcal{A})$ is precisely the set of functionals in the spectrum whose restriction to $\mathcal{D}$ is non-zero. If $\mathcal{A}$ is triangular this is equal to the set of multiplicative functionals on $\mathcal{A}$.

Definition. *The range and source maps* $r, s : \mathrm{Spec}(\mathcal{A}) \to \mathrm{Spec}(\mathcal{D})$ *are given by* $r(\rho) = \tau, s(\rho) = \sigma$ *where* τ, σ *are determined by the sequences* $(e_k e_k^*), (e_k^* e_k)$ *respectively, where* e_k *is the matrix unit chain for* ρ.

Remark. The range and source maps can be defined without reference to matrix unit chains as follows. Given $\rho \in \mathrm{Spec}(\mathcal{A})$, observe that for any $d \in \mathcal{D}$, the functional $d \cdot \rho$ given by $(d \cdot \rho)(x) = \rho(xd)$ is a scalar multiple (depending on d) of ρ. Let $s_d \in \mathbb{C}$ be the scalar such that $d \cdot \rho = s_d \rho$. It is then easy to show that $d \mapsto s_d$ is a multiplicative linear functional which coincides with $s(\rho)$. Such considerations using $(\rho \cdot d)(x) := \rho(dx)$ also give a "coordinate free" definition of the range map.

Using Lemma 2.2, it is routine to verify the following lemma.

Lemma 2.6. *The map* $\rho \mapsto (s(\rho), r(\rho))$ *is 1-1. Moreover, the maps* $\rho \mapsto s(\rho)$ *and* $\rho \mapsto r(\rho)$ *are (weak-∗-weak-∗) continuous maps of* $\mathrm{Spec}(\mathcal{A})$ *into* $\mathrm{Spec}(\mathcal{A})$.

Proof. To see that $\rho \mapsto (s(\rho), r(\rho))$ is 1-1, suppose $\rho, \sigma \in \mathrm{Spec}(\mathcal{A})$ with $s(\rho) = s(\sigma)$ and $r(\rho) = r(\sigma)$. Using Lemma 2.2, it follows that if (e_k) is the matrix unit chain associated to ρ and (f_k) is the matrix unit chain associated to σ, then $e_k^* e_k = f_k^* f_k$ and $e_k e_k^* = f_k f_k^*$ for all large k. As initial and final projections uniquely determine matrix units in each C_k, $e_k = f_k$ and so ρ and σ define the same functional.

For the second statement, it suffices to show that the maps are sequentially continuous, as the weak-∗ topology on unit ball of $\mathcal{A}$ is metrizable. We consider only the source map; the argument for the range map is similar.

If $\rho_n \in \mathrm{Spec}(\mathcal{A})$ converges weak-∗ to ρ, then there is K so that $\rho|_{A_K} \neq 0$. Fixing $k \geq K$, by Lemma 2.3, there is $N \in \mathbb{N}$ so that $\rho_n|_{A_k} = \rho|_{A_k}$ for every

$n \geq N$. Let e be the unique matrix unit in C_k so that $\rho_n(e) = \rho(e) = 1$ for all $n \geq N$. By Lemma 2.2 and our definition of the source map, $s(\rho_n)(e^*e) = s(\rho)(e^*e) = 1$ for all $n \geq N$. This implies $s(\rho_n)|_{A_k} = s(\rho)|_{A_k}$ and so applying Lemma 2.3 again, $s(\rho_n)$ converges weak-$*$ to $s(\rho)$. ◻

The map that sends $(s(\rho), r(\rho))$ to ρ is almost (weak-$*$-weak-$*$) continuous, and this will be needed for the proof of the main theorem.

Lemma 2.7. *Suppose $(\tau_n)_{n=1}^{\infty}$ is a sequence in* $\text{Spec}(\mathcal{A})$*, let $\sigma, \rho \in \text{Spec}(\mathcal{A})$ and assume that*

$$\text{w}^*\text{-}\lim s(\tau_n) = \sigma \quad \textit{and} \quad \text{w}^*\text{-}\lim r(\tau_n) = \rho.$$

If no subsequence of τ_n converges weakly to 0*, then there exists a unique element $\tau \in \text{Spec}(\mathcal{A})$ such that* $\text{w}^*\text{-}\lim \tau_n = \tau$.

Proof. By compactness of $\text{Spec}(\mathcal{A}) \cup \{0\}$, we see that every subsequence (τ_{n_j}) of (τ_n) has a convergent subsubsequence. Let τ be the limit of such a subsubsequence. By hypothesis, $\tau \neq 0$, and by the second part of Lemma 2.6, we find $s(\tau) = \sigma$ and $r(\tau) = \rho$. By the first part, we see that if τ' is the weak-$*$ limit of another convergent subsubsequence, then $\tau' = \tau$. It follows that the set of cluster points of (τ_n) is the singleton set $\{\tau\}$, whence τ_n converges weak-$*$ to τ. ◻

3. MEET IRREDUCIBLE IDEALS

We maintain the notation of the previous section. In particular $\mathcal{A}$ is a regular limit algebra with building block algebras A_k viewed as subalgebras of $\mathcal{A}$ and $\mathcal{A}$ has an associated matrix unit system. Throughout this section ideals are assumed to be two-sided and closed. Finally, if $X \subseteq \mathcal{B}$ is a non-empty subset of an algebra $\mathcal{B}$, we use the notation $\langle X \rangle_{\mathcal{B}}$, or simply $\langle X \rangle$ when the context is clear, for the ideal in $\mathcal{B}$ generated by X.

Definition. *An ideal $\mathcal{I}$ is said to be meet irreducible if $\mathcal{I} \neq \mathcal{A}$ and whenever $\mathcal{I} = \mathcal{J} \cap \mathcal{K}$, then $\mathcal{I} = \mathcal{J}$ or $\mathcal{I} = \mathcal{K}$. The ideal is completely meet irreducible if, whenever $\mathcal{I} = \bigcap_{\alpha \in \Gamma} \mathcal{J}_\alpha$ is an intersection of a family of ideals, then $\mathcal{I} = \mathcal{J}_\alpha$ for some $\alpha \in \Gamma$. Equivalently, $\mathcal{I}$ is completely meet irreducible if $\mathcal{I}^+$ properly contains $\mathcal{I}$ where $\mathcal{I}^+$ is the intersection of all ideals that properly contain $\mathcal{I}$. Finally $\mathcal{I}$ is said to be strongly completely meet irreducible if the linear dimension of $\mathcal{I}^+/\mathcal{I}$ is one.*

For a finite dimensional building block algebra A, the first two classes of ideals of the definition coincide. One can readily check that in this case, the meet irreducible ideals are of the form $I(e)$, where $I(e)$ is the largest ideal of A not containing a matrix unit $e \in A$. (It is elementary and instructive to consider an algebra of block upper triangular matrices.) In the triangular case, the correspondence $e \to I(e)$ is a bijection from matrix units to meet irreducible ideals. We shall obtain a limit algebra variant of this in which the role of the matrix unit e is played by an element of the spectrum, or, equivalently, by a chain of matrix units. The appropriate ideals for this variant are the completely meet irreducible ideals.

Recall that an ideal $\mathcal{I}$ of a regular limit algebra $\mathcal{A}$ is inductive in the sense that it is the closed union of the ideals $I_k = \mathcal{I} \cap A_k$ in the building block algebras. (See [13].) In particular an ideal is the closed span of the matrix units that it contains.

The next lemma implies that for triangular algebras the completely meet irreducible ideals are in fact strongly meet irreducible.

The following finite dimensional observation will be needed. Let A be a triangular digraph algebra with ideal I and let e, f be partial isometries in $A \setminus I$ which are sums of matrix units of A. If the ideals $\langle e, I\rangle$ and $\langle f, I\rangle$ coincide, then e and f share a subordinate matrix unit.

Lemma 3.1. *Let $\mathcal{A}$ be a triangular regular limit algebra (a TAF algebra) and let $\mathcal{I}$ be a closed ideal of A. Then either $\mathcal{I}^+ = \mathcal{I}$ or $\mathcal{I}^+/\mathcal{I}$ has linear dimension one.*

Proof. Note first that it follows from the inductivity of ideals that $\mathcal{I}^+/\mathcal{I}$ has dimension one if and only if there is a chain of matrix units $(e_k)_{k\geq N}$ such that $\mathcal{I}^+ = \langle e_k, \mathcal{I}\rangle$ for all k.

Suppose, by way of contradiction, that $\mathcal{I}^+/\mathcal{I}$ has dimension greater than 1. Then, for some k, $\mathcal{I}^+ \cap A_k$ contains two distinct matrix units e, f say, of A_k which do not belong to $\mathcal{I}$. We have $\langle e, \mathcal{I}\rangle = \mathcal{I}^+$ and $\langle f, \mathcal{I}\rangle = \mathcal{I}^+$, by the definition of $\mathcal{I}^+$. Also, since ideals are inductive, the matrix units in $\langle e, \mathcal{I}\rangle$ and $\langle f, \mathcal{I}\rangle$ coincide with $\bigcup_{k=1}^{\infty}(\langle e, \mathcal{I}_k\rangle \cap C_k)$ and $\bigcup_{k=1}^{\infty}(\langle f, \mathcal{I}_k\rangle \cap C_k)$ respectively. Thus, for some n, m, we have $f \in \langle e, I_n\rangle$ and $e \in \langle f, I_m\rangle$. Hence for $N = \max\{n, m\}$ we obtain $\langle e, I_N\rangle = \langle f, I_N\rangle$. Since $\mathcal{A}$ is triangular so too is A_N. By the observation in the paragraph preceding the lemma, e and f share at least one subordinate in A_N. This is a contradiction since e and f are distinct matrix units in A_k. ◻

Let $(e_k)_{k\geq N}$ be a matrix unit chain. In analogy with the finite dimensional meet irreducible ideals, define $\mathcal{I}((e_k))$ to be the maximal ideal of $\mathcal{A}$ which contains no matrix unit e_k. Such ideals are easily seen to be meet irreducible, but as we note following the proof of Lemma 3.2, they need not be completely meet irreducible.

The ideals $I(e_k)$ in each A_k are not simply related to $\mathcal{I}((e_k))$, and in particular, it may happen that no ideal $I(e_k)$ is contained in $\mathcal{I}((e_k))$. An example of this phenomenon is the following. Let T_n be the $n \times n$ upper triangular matricies and consider the inclusion $T_3 \to T_6$ given by the (non order preserving) embedding

$$\begin{bmatrix} a & x & z \\ & b & y \\ & & c \end{bmatrix} \to \begin{bmatrix} a & x & 0 & 0 & z & 0 \\ & b & 0 & 0 & y & 0 \\ & & a & x & 0 & z \\ & & & b & 0 & y \\ & & & & c & 0 \\ & & & & & c \end{bmatrix}.$$

The matrix unit $e_{25}^{(2)}$ is a subordinate of $e_{23}^{(1)}$, yet the ideal $I(e_{23}^{(1)})$ in A_1 is not contained in $I(e_{25}^{(2)})$. By repeating such embeddings, it is possible to obtain limit algebras $\mathcal{A} = \varinjlim(A_k, \alpha_k)$ and a sequence of matrix units (e_k) so that $\mathcal{I}((e_k)) \cap A_k$ is properly contained in $I(e_k)$ for each k. In particular, notice that $I(e_j)$ is not properly contained in $\mathcal{I}((e_k))$ for any j.

We now give a second important lemma. The proof is closely related to the proof of the characterization of completely meet irreducible ideals in a strongly maximal TAF algebra given in [3, Theorem 5.1].

Lemma 3.2. *Let $\mathcal{A}$ be triangular and let $\mathcal{I} \subseteq \mathcal{A}$ be an ideal. Then the following conditions are equivalent.*

(i) *$\mathcal{I}^+/\mathcal{I}$ is one dimensional.*
(ii) *There is a matrix unit chain $(e_k)_{k \geq N}$ such that $e_m - e_{m+1} \in \mathcal{I}((e_k))$ for $m \geq N$ and $\mathcal{I} = \mathcal{I}((e_k))$.*

Proof. Suppose $\mathcal{I} = \mathcal{I}((e_k))$ and that $e_m - e_{m+1} \in \mathcal{I}((e_k))$ for $m \geq N$. Let $\mathcal{J}$ be an ideal that contains $\mathcal{I}$ properly. By the definition of $\mathcal{I}$, the ideal $\mathcal{J}$ contains e_{j+1} for some j. By hypothesis, it follows that $e_j - e_{j+1}$ belongs to $\mathcal{I}$ and so e_j belongs to $\mathcal{J}$. Repeating this argument we obtain that $e_{j+1}, e_j, \dots, e_N$ belong to $\mathcal{J}$. Since this holds for every ideal properly containing $\mathcal{I}$, we conclude that $e_N \in \mathcal{I}^+$ and so $\mathcal{I}^+ \neq \mathcal{I}$. Now (i) follows from the previous lemma.

On the other hand, suppose that $\mathcal{I}$ is a completely meet irreducible ideal. Then $\mathcal{I}^+/\mathcal{I}$ is one dimensional and it follows that $\mathcal{I}^+ \cap A_k$ contains at most one matrix unit which is not in $\mathcal{I} \cap A_k$. To see this, observe that the map from $A_k/(A_k \cap \mathcal{I}) \to \mathcal{A}/\mathcal{I}$ given by $a + (A_k \cap \mathcal{I}) \mapsto a + \mathcal{I}$ is well defined and injective. Hence $\dim(A_k/(A_k \cap \mathcal{I})) \leq \dim(\mathcal{A}/\mathcal{I}) = 1$. Thus there is a matrix unit chain $(e_k)_{k \geq N}$ such that e_k belongs to $\mathcal{I}^+$ and such that $\mathcal{I}^+ = \mathbb{C}e_k + \mathcal{I}$ for all $k \geq N$. In particular (ii) holds and the proof is complete. ◻

The conditions of Lemma 3.2 need not always hold: it is straightforward to construct a regular limit algebra for which there is a chain (e_k) such that both $\mathcal{I}((e_k))$ and $\mathcal{I}((e_k))^+$ are zero. This also shows that in general $\mathcal{I}((e_k))$ need not be completely meet irreducible.

We remark that meet irreducible ideals need not be of the form $\mathcal{I}((e_k))$ for a matrix unit chain. As an example, we construct such a meet irreducible ideal in the 2^∞ refinement embedding algebra. Define A_k to be T_{2^k}, upper triangular 2^k by 2^k matrices, and define $\alpha_k : A_k \to A_{k+1}$ by $(a_{ij}) \mapsto (a_{ij}I_2)$. Let $\{e_{ij}^{(k)}\}$ be the set of matrix units of A_k, and consider the sequence of matrix units $f_k := e^{(k)}_{2^{k-2}+1, 2^k - 2^{k-2}}$, $k \geq 2$. Although f_{k+1} is not a subordinate of f_k, there is a largest ideal $\mathcal{J}$ not containing any f_k: observe that

$$\mathcal{J} = \overline{\operatorname{Span}}\{e_{i,j}^{(k)} : i < 2^{k-2} + 1 \text{ or } j > 2^k - 2^{k-2}\}.$$

It is not hard to verify directly that $\mathcal{J}$ is meet irreducible but not completely meet irreducible; one can also apply the general characterizations of such ideals given in Theorems 1.2 and 5.1 of [3].

Next, we extend this characterization of completely meet irreducible ideals to TAF algebras. The following terminology is used in [3].

Definition. *A sequence $(e_k)_{k\geq N}$ of matrix units from $\mathcal{A}$ is called an MI-chain if the following two conditions hold for all k.*

(i) $e_k \in \mathcal{A}_k$.
(ii) *e_{k+1} is in the ideal of A_k generated by e_k.*
Furthermore, the sequence is called a CMI-chain if in addition
(iii) *The ideal in $\mathcal{A}$ generated by $e_k - e_{k+1}$ does not contain e_j, for any $j \geq N$.*

These three conditions imply that the sequence (e_k) is necessarily a matrix unit chain. It follows that the matrix unit chain appearing in condition (ii) of Lemma 3.2 is equivalent to the third condition in the definition of CMI-chains. Therefore, the chains occurring in Lemma 3.2 are precisely the CMI-chains.

Thus we have the following result as a corollary. The special case of strongly maximal TAF algebras was obtained in [3].

Theorem 3.3. *An ideal $\mathcal{I}$ in a TAF algebra is completely meet irreducible if and only if it is of the form $\mathcal{I}((e_k))$ for some CMI-chain (e_k).*

In view of the correspondence between matrix unit chains and points of the spectrum given in Corollary 2.2, the ideals $\mathcal{I}((e_k))$ can also be specified in terms of the spectrum. Indeed, $\mathcal{I}((e_k))$ is the maximal ideal, $\mathcal{I}(\rho)$ say, which is annihilated by the functional ρ in the spectrum corresponding to the chain (e_k). (See the propositions below.) An algebraic description of these "spectral ideals" would provide an alternative approach to the invariance of the spectrum. However, Lemma 3.2 gives an algebraic determination of those spectral points ρ corresponding to completely meet irreducible ideals and, in Theorem 3.7 below, we see that for triangular algebras these spectral points are dense. This, as we see in the next section, will be sufficient to pin down the spectrum in algebraic terms.

First we clarify the relationship between completely meet irreducible ideals $\mathcal{I}$ and their associated spectrum functionals. We write $\mathfrak{M}$ for the set of completely meet irreducible ideals.

Proposition 3.4. *Let $\mathcal{I} \in \mathfrak{M}$ and suppose that $\rho \in (\mathcal{I}^+)^\#$ satisfies $\rho|_{\mathcal{I}} = 0$. Then there is a unique extension $\tilde{\rho}$ of ρ to $\mathcal{A}$ such that $\|\tilde{\rho}\| = \|\rho\|$.*

Proof. The result is obvious when $\rho = 0$. We establish the proposition when $\|\rho\| = 1$ and the general case follows immediately. So assume that $\|\rho\| = 1$ and that $\tilde{\rho}$ is a Hahn-Banach extension of ρ to $\mathcal{A}$ such that $\|\tilde{\rho}\| = 1$.

Let K be the smallest integer so that $A_K \cap \mathcal{I} \neq A_K \cap \mathcal{I}^+$. Since $\dim(\mathcal{I}^+/\mathcal{I}) = 1$, we see that for $k \geq K$, there is a unique matrix unit $e_k \in (A_k \cap \mathcal{I}^+)$ which not in $A_k \cap \mathcal{I}$. Since e_k is a sum of matrix units in $\mathcal{A}_{k+1} \cap \mathcal{I}^+$ and $e_k \notin \mathcal{A}_{k+1} \cap \mathcal{I}$, we observe that $e_k - e_{k+1} \in \mathcal{I}$. Hence there is a non-zero complex number λ such that $\lambda = \rho(e_k) = \rho(e_j)$ for $i, j \geq K$. It is clear that $|\lambda| \leq 1$ and we claim that actually, $|\lambda| = 1$. To see this, note that if $j \geq K$ and $y \in \mathcal{I}^+ \cap \mathcal{A}_j$ has norm 1, then we may find $v \in \mathcal{I} \cap \mathcal{A}_j$ and $\alpha \in \mathbb{C}$ with $|\alpha| \leq 1$ and $y = \alpha e_j + v$. Then $|\rho(y)| = |\alpha\lambda| \leq |\lambda|$. Since

$$1 = \|\rho\| = \sup_{j \geq K} \left\| \rho|_{\mathcal{I}^+ \cap \mathcal{A}_j} \right\|,$$

we find $|\lambda| = 1$.

Since $\tilde{\rho}$ extends ρ, we have for each $k \geq K$, $\|\tilde{\rho}\| = |\tilde{\rho}(e_k)| = 1$. By Lemma 2.1, $\tilde{\rho}$ vanishes on all matrix units other than the elements of the sequence (e_k). In particular, $\tilde{\rho}$ is unique. ◻

As a corollary we obtain the following result:

Proposition 3.5. *For each $J \in \mathfrak{M}$, there exists a unique element $\varphi_J \in$ Spec$(\mathcal{A})$ such that φ_J annihilates J but does not annihilate J^+.*

We now identify the functionals $s(\varphi_{\mathcal{I}})$ and $r(\varphi_{\mathcal{I}})$ when $\mathcal{I} \in \mathfrak{M}$. Notice that since $\mathcal{I}^+$ and $\mathcal{I}$ are ideals, the operator R_a of right multiplication by a induces a linear operator $\tilde{R}_a$ on $\mathcal{I}^+/\mathcal{I}$. As $\dim(\mathcal{I}^+/\mathcal{I}) = 1$, there exists a scalar $\lambda(a)$ such that for each $x \in \mathcal{I}^+$, $\tilde{R}_a(x + \mathcal{I}) = xa + \mathcal{I} = \lambda(a)x + \mathcal{I}$. It is clear that the map $a \mapsto \lambda(a)$ is a multiplicative linear functional on $\mathcal{A}$ and thus has norm one. Let (e_k) be the CMI-chain associated with $\mathcal{I}$. Then taking $x = e_k \in \mathcal{I}^+$ and $a = e_k^* e_k$ we find $x - xa = 0$, whence $\lambda(e_k^* e_k) = 1$. By Proposition 2.1 we see that the restrictions of λ and $s(\varphi_{\mathcal{I}})$ agree on $\mathcal{A}_k$ for every k. Hence $\lambda = s(\varphi_{\mathcal{I}})$. Similar considerations apply to the operator of left multiplication by a and $r(\varphi_{\mathcal{I}})$. Thus we have established the first part of the following result.

Proposition 3.6. *Suppose that $\mathcal{I} \in \mathfrak{M}$ and $a \in \mathcal{A}$. Then $s(\varphi_{\mathcal{I}})(a)$ is the unique scalar which satisfies*

$$s(\varphi_{\mathcal{I}})(a)\, x - xa \in \mathcal{I} \quad \textit{for every } x \in \mathcal{I}^+.$$

Similarly, $r(\varphi_{\mathcal{I}})(a)$ is the unique scalar satisfying

$$r(\varphi_{\mathcal{I}})(a)\, x - ax \in \mathcal{I} \quad \textit{for every } x \in \mathcal{I}^+.$$

Moreover, if $\mathcal{I}$ and $\mathcal{J}$ belong to $\mathfrak{M}$ and $\varphi_{\mathcal{I}} = \varphi_{\mathcal{J}}$ then $\mathcal{I} = \mathcal{J}$.

Proof. It remains to show that the map $\mathcal{I} \mapsto \varphi_{\mathcal{I}}$ is one-to-one. Let (e_k) and (f_k) be the CMI-chains associated to $\mathcal{I}$ and $\mathcal{J}$ respectively. If $\varphi_{\mathcal{I}} = \varphi_{\mathcal{J}}$, then $s(\varphi_{\mathcal{I}}) = s(\varphi_{\mathcal{J}})$. In particular, $s(\varphi_{\mathcal{I}})(f_k^* f_k) = 1$, so we have $x - x f_k^* f_k \in \mathfrak{J}$ for all $x \in \mathcal{I}^+$. Taking $x = e_k$ we obtain $e_k - e_k(f_k^* f_k) \in \mathfrak{J}$. As $e_k(f_k^* f_k) = 0$ unless $f_k^* f_k = e_k^* e_k$, we conclude that $e_k^* e_k = f_k^* f_k$. Similar considerations using $r(\varphi_{\mathcal{I}})$ and $f_k f_k^*$ show that $e_k e_k^* = f_k f_k^*$. Thus $e_k = f_k$, and we conclude that $\mathcal{I}$ and $\mathcal{J}$ have the same CMI-chains. Thus $\mathcal{I} = \mathcal{J}$. ◻

To summarize, we now have an injective map $i : \mathfrak{M} \to \mathrm{Spec}(\mathcal{A})$ given by $\mathcal{J} \to \varphi_{\mathcal{J}}$. In general, this map is not onto because there are spectral ideals which are not completely meet irreducible. However, the range is dense. This fact is given, in slightly different form in [1, Theorem 3]. Because this fact is integral to our study, we give another proof here for the convenience of the reader.

Theorem 3.7 ([1])**.** *The range of the map* $i : \mathfrak{M} \to \mathrm{Spec}(\mathcal{A})$ *is weak-∗-dense in* $\mathrm{Spec}(\mathcal{A})$*.*

Proof. Let $\rho \in \mathrm{Spec}(\mathcal{A})$. Then there exists $N \in \mathbb{N}$ so that $\rho|_{A_N}$ is non-zero. Fix $n \geq N$, and let $e \in C_n$ be the (unique) matrix unit so that $\rho|_{A_n}(e) = 1$. By Lemma 2.3, it suffices to construct $J \in \mathfrak{M}$ so that $\varphi_J(e) = 1$.

For $m \geq n$, define

$$F_m := \{u \in C_m : u \ll e \text{ and } u \notin \langle e - u \rangle_{\mathcal{A}_m}\}.$$

We first show that $F_m \neq \emptyset$. Let U be the set of elements of C_m subordinate to e and notice that $\langle e \rangle_{A_m} = \langle U \rangle_{A_m}$. As A_m is a triangular digraph algebra, there is a partial ordering on the minimal projections of $A_m \cap A_m^*$ given by $p \preceq q$ if and only if $pA_m q \neq (0)$. Let $u \in U$ be chosen so that for all $u_1 \in U$, either: a) $uu^* \preceq u_1 u_1^*$, or b) uu^* is not comparable to $u_1 u_1^*$ (with respect to $\preceq$). Then $u \in F_m$, whence $F_m \neq \emptyset$.

Let $G_m = \{\varphi \in \mathrm{Spec}(\mathcal{A}) : \varphi(f) = 1 \text{ for some } f \in F_m\}$. We shall show that $\{G_m\}_{m=n}^{\infty}$ is a decreasing sequence of non-empty compact sets. Since F_m is non-empty, so is G_m. Notice also that G_m is a closed subset of the compact set $\mathcal{G}(e)$, so G_m is compact.

Next, suppose that $m \geq k \geq n$ and suppose that $f \in C_m$ and $g \in C_k$ satisfy $f \ll g \ll e$. Notice that if $g \notin F_k$, then $f \notin F_m$. Indeed, if $g \notin F_k$ then $g \in \langle e - g \rangle_{\mathcal{A}_k} \subseteq \langle e - g \rangle_{\mathcal{A}_m} \subseteq \langle e - f \rangle_{\mathcal{A}_m}$; hence $f \in \langle e - f \rangle_{\mathcal{A}_m}$, so $f \notin F_m$. Thus if $f \in F_m$, then f is subordinate to some $g \in F_k$. Therefore, if $\varphi \in G_m$, then there exists $f \in F_m$ with $\varphi(f) = 1$. If $g \in F_k$ satisfies $f \ll g$, then clearly $\varphi(g) = 1$, whence $\varphi \in G_k$. Thus $G_m \subseteq G_k$.

Let $\rho \in \mathrm{Spec}(\mathcal{A})$ be such that $\rho \in \bigcap_m G_m$ and let $(e_k)_{k \geq n}$ be the matrix unit chain for ρ. By construction $e_n = e$ and $e_m \in F_m$ for all $m \geq n$. Thus, $(e_k)_{k \geq n}$ is a CMI-chain. Putting $\mathcal{I} = \mathcal{I}((e_k))$, we see $\mathcal{I} \in \mathfrak{M}$ and $\rho = \varphi_{\mathcal{I}}$, as desired. □

We remark that for certain limit algebras determined by order preserving systems, the map i of the theorem above is surjective. In such limit algebras, there is a simpler approach to the invariance of spectrum; see [1].

4. Spectral Invariance

We prove the main result, the invariance of the spectrum under isomorphism:

Theorem 4.1. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are two triangular limit algebras. If $\vartheta : \mathcal{A} \to \mathcal{B}$ is an algebra isomorphism, then there is a unique bicontinuous semigroupoid isomorphism $\gamma : \mathrm{Spec}(\mathcal{A}) \to \mathrm{Spec}(\mathcal{B})$ such that $\gamma(\rho) = \rho \circ \vartheta^{-1}$ for every multiplicative linear functional ρ on $\mathcal{A}$.*

We call γ the spectral isomorphism induced by ϑ and denote it $\gamma(\vartheta)$.

By [2, Theorem 1.4], we know that algebraic isomorphisms are automatically continuous and so map closed ideals to closed ideals. Hence ϑ induces a lattice-preserving bijection between the lattice of closed ideals of $\mathcal{A}$ and that of $\mathcal{B}$. Moreover ϑ maps the set $\mathfrak{M}$ of completely meet irreducible ideals of $\mathcal{A}$ to the corresponding set $\mathfrak{N}$ for $\mathcal{B}$. The following technical lemma will be needed to determine the spectral continuity properties of this map.

Lemma 4.2. *Suppose $\mathcal{A}$ and $\mathcal{B}$ are triangular limit algebras, $\vartheta : \mathcal{A} \to \mathcal{B}$ is an algebra isomorphism, and $\mathcal{I} \in \mathfrak{M}$. If $\varphi_{\mathcal{I}}$ is the element of $\mathrm{Spec}(\mathcal{A})$ associated to $\mathcal{I}$ and $\psi_{\vartheta(\mathcal{I})}$ is the element of $\mathrm{Spec}(\mathcal{B})$ associated to $\vartheta(\mathcal{I})$, then there exists $\lambda \in \mathbb{C}$ so that*

$$\psi_{\vartheta(\mathcal{I})}|_{\vartheta(\mathcal{I})^+} = \lambda \cdot \varphi_{\mathcal{I}} \circ \vartheta^{-1}|_{\vartheta(\mathcal{I})^+} \quad \textit{and} \quad \frac{1}{\|\vartheta^{-1}\|} \leq |\lambda| \leq \|\vartheta\|\,.$$

Moreover, if $\mathcal{I}$ is a maximal ideal, then $\lambda = 1$.

Proof. Note that $\vartheta(\mathcal{I})^+ = \vartheta(\mathcal{I}^+)$. Both $\psi_{\vartheta(\mathcal{I})}$ and $\varphi_{\mathcal{I}} \circ \vartheta^{-1}$ are nonzero functionals on $\vartheta(\mathcal{I}^+)$ which are zero on $\vartheta(\mathcal{I})$. Since $\vartheta(\mathcal{I}^+)/\vartheta(\mathcal{I})$ is one-dimensional, there exists a non-zero $\lambda \in \mathbb{C}$ such that

$$\psi_{\vartheta(\mathcal{I})}|_{\vartheta(\mathcal{I})^+} = \lambda \cdot \varphi_{\mathcal{I}} \circ \vartheta^{-1}|_{\vartheta(\mathcal{I})^+}. \tag{4.1}$$

Let (e_k) be the CMI-chain associated to $\mathcal{I}$. Then $\vartheta(e_k) \in \vartheta(\mathcal{I}^+)$, and we have

$$(\varphi_{\mathcal{I}} \circ \vartheta^{-1})\left(\frac{\vartheta(e_k)}{\|\vartheta(e_k)\|}\right) = \frac{1}{\|\vartheta(e_k)\|} \geq \frac{1}{\|\vartheta\|},$$

so we find

$$\frac{1}{\|\vartheta\|} \leq \left\|\varphi_{\mathcal{I}} \circ \vartheta^{-1}|_{\vartheta(\mathcal{I})^+}\right\| \leq \left\|\vartheta^{-1}\right\|. \tag{4.2}$$

Since the norm of $\psi_{\vartheta(\mathcal{I})}|_{\vartheta(\mathcal{I})^+}$ is one, equality (4.1) yields, $\|\varphi_{\mathcal{I}} \circ \vartheta^{-1}|_{\vartheta(\mathcal{I})^+}\| = |\lambda|^{-1}$. When $\mathcal{I}$ is a maximal ideal, $\varphi_{\mathcal{I}} \circ \vartheta^{-1}|_{\vartheta(\mathcal{I})^+} = \varphi_{\mathcal{I}} \circ \vartheta^{-1}$ is a multiplicative linear functional on $\mathcal{B}$, whence $\lambda = 1$. For general $\mathcal{I} \in \mathfrak{M}$, the inequalities in (4.2) yield $1/\|\vartheta^{-1}\| \leq |\lambda| \leq \|\vartheta\|$. ∎

Proof of Theorem 4.1: Suppose $\mathcal{A} = \varinjlim(A_k, \alpha_k)$ and $\mathcal{B} = \varinjlim(B_c, \beta_c)$.

For each ideal $J \in \mathfrak{M}$ and $K \in \mathfrak{N}$, let $\varphi_J \in \mathrm{Spec}(\mathcal{A})$ and $\psi_K \in \mathrm{Spec}(\mathcal{B})$ be the elements in the spectra obtained from Theorem 3.5. By the remarks at the beginning of this section, ϑ induces a map $\gamma : \mathfrak{M} \to \mathfrak{N}$ with $\gamma(\varphi_J) = \psi_{\vartheta(J)}$. If J is a maximal ideal, then $J^+ = \mathcal{A}$ and Lemma 4.2 shows that $\gamma(\varphi_J) = \varphi_J \circ \vartheta^{-1}$. By the continuity of ϑ, γ is weak-$*$-continuous on the subset of functionals in $\mathfrak{M}$ corresponding to maximal ideals in $\mathcal{A}$.

Our goal is to show that γ is continuous on all of $\mathfrak{M}$. We remark that it is not generally true that $\gamma(\varphi_J) = \varphi_J \circ \vartheta^{-1}$ for $J \in \mathfrak{M}$.

Notice that if $J \in \mathfrak{M}$, then Proposition 3.6 shows that

$$s(\psi_{\vartheta(J)}) = s(\varphi_J) \circ \vartheta^{-1} \quad \text{and} \quad r(\psi_{\vartheta(J)}) = r(\varphi_J) \circ \vartheta^{-1}.$$

Thus for each $J \in \mathfrak{M}$, we have

$$s(\gamma(\varphi_J)) = \gamma(s(\varphi_J)) \quad \text{and} \quad r(\gamma(\varphi_J)) = \gamma(r(\varphi_J)). \tag{4.3}$$

Suppose that $J_n \in \mathfrak{M}$ and $\rho \in \mathrm{Spec}(\mathcal{A})$ satisfy

$$\rho = \mathrm{w}^*\text{-}\lim \varphi_{J_n}.$$

By the first part of Lemma 2.6, $s(\rho) = \mathrm{w}^*\text{-}\lim s(\varphi_{J_n})$ and similarly for the range. As all of these functionals correspond to maximal ideals in $\mathfrak{M}$ and γ is continuous on such linear functionals, we have by (4.3),

$$\mathrm{w}^*\text{-}\lim s(\gamma(\varphi_{J_n})) = \mathrm{w}^*\text{-}\lim \gamma(s(\varphi_{J_n})) = \gamma(s(\rho)). \tag{4.4}$$

Similarly,

$$\mathrm{w}^*\text{-}\lim r(\gamma(\varphi_{J_n})) = \mathrm{w}^*\text{-}\lim \gamma(r(\varphi_{J_n})) = \gamma(r(\rho)). \tag{4.5}$$

We claim that no subsequence of $\gamma(\varphi_{J_n}) = \psi_{\vartheta(J_n)}$ converges weak-$*$ to 0. To see this, choose $M \in \mathbb{N}$ so large that $\varphi_J|_{A_M} \neq 0$ and fix $k \geq M$. Let $e_k \in C_k$ be the matrix unit so that $\rho(e_k) = 1$. By weak-$*$-convergence, there is $N \in \mathbb{N}$ so that for $n \geq N$ we have $\varphi_{J_n}(e_k) = 1$. Since φ_{J_n} takes the value 1 only on one element of C_k, $e_k \in J_n^+$ for all $n \geq N$. Thus $\vartheta(e_k) \in \vartheta(J_n^+)$. By Lemma 4.2, there exist non-zero scalars $\lambda_n \in \mathbb{C}$ so that

$$\psi_{\vartheta(J_n)}|_{\vartheta(J_n)^+} = \lambda_n \cdot \varphi_{J_n} \circ \vartheta^{-1}|_{\vartheta(J_n)^+},$$

and so we conclude that

$$\psi_{\vartheta(J_n)}(\vartheta(e_k)) = \lambda_n \quad \text{for all } n \geq N.$$

But the estimates in Lemma 4.2 show that $|\lambda_n| \geq \|\vartheta^{-1}\|^{-1}$. As k is fixed, no subsequence of $\psi_{\vartheta(J_n)}$ converges weak-$*$ to 0.

By Lemma 2.7, there is a unique element $\tau \in \operatorname{Spec}(\mathcal{B})$ such that $\text{w}^*\text{-}\lim \gamma(\varphi_{J_n}) = \tau$ and we define $\gamma(\rho) := \tau$. Moreover, we have

$$s(\gamma(\rho)) = \text{w}^*\text{-}\lim s(\gamma(\varphi_{J_n})) \quad \text{and} \quad r(\gamma(\rho)) = \text{w}^*\text{-}\lim r(\gamma(\varphi_{J_n})). \tag{4.6}$$

Combining (4.6) with (4.4) and (4.5) yields,

$$s(\gamma(\rho)) = \gamma(s(\rho)) \quad \text{and} \quad r(\gamma(\rho)) = \gamma(r(\rho)). \tag{4.7}$$

The unit ball of $\mathcal{A}$ is metrizable in the weak-$*$ topology, and thus our construction shows γ is continuous on $\operatorname{Spec}(\mathcal{A})$. Symmetric arguments using ϑ^{-1} instead of ϑ show that γ is invertible and has continuous inverse. Therefore, γ is a homeomorphism of $\operatorname{Spec}(\mathcal{A})$ onto $\operatorname{Spec}(\mathcal{B})$.

Suppose that ρ and σ are composable elements of $\operatorname{Spec}(\mathcal{A})$. Then $s(\rho) = r(\sigma)$, $s(\rho\sigma) = s(\sigma)$ and $r(\rho\sigma) = r(\rho)$. By (4.7), we find that $\gamma(\rho)$ and $\gamma(\sigma)$ are composable elements of $\operatorname{Spec}(\mathcal{B})$. Therefore, $s(\gamma(\rho\sigma)) = s(\gamma(\sigma)) = s(\gamma(\rho)\gamma(\sigma))$ and $r(\gamma(\rho\sigma)) = r(\gamma(\rho)) = r(\gamma(\rho)\gamma(\sigma))$. By Lemma 2.6,

$$\gamma(\rho\sigma) = \gamma(\rho)\gamma(\sigma),$$

so γ is a semigroupoid homomorphism.

Finally, we show γ is unique. To do this, first notice that for $\rho \in \operatorname{Spec}(\mathcal{A})$, $r(\rho)$ is the unique multiplicative linear functional on $\mathcal{A}$ such that $(r(\rho), \rho)$ is a composable pair. Now suppose $\mu : \operatorname{Spec}(\mathcal{A}) \to \operatorname{Spec}(\mathcal{B})$ is another bicontinuous semigroupoid isomorphism with $\mu(\rho) = \rho \circ \vartheta^{-1}$ whenever ρ is a multiplicative linear functional on $\mathcal{A}$. Since both μ and γ are semigroupoid isomorphisms, we find

$$r(\mu(\rho)) = \mu(r(\rho)) \quad \text{and} \quad \gamma(r(\rho)) = r(\gamma(\rho)).$$

But μ and γ agree on multiplicative linear functionals, so $\mu(r(\rho)) = \gamma(r(\rho))$, and hence for every $\rho \in \operatorname{Spec}(\mathcal{A})$,

$$r(\mu(\rho)) = r(\gamma(\rho)).$$

Similarly, $s(\mu(\rho)) = r(\gamma(\rho))$. By Lemma 2.6, $\mu(\rho) = \gamma(\rho)$. ◻

A bicontinuous semigroupoid isomorphism between spectra induces an isometric isomorphism between the corresponding limit algebras, by Corollary 7.7 of [13]. Thus we have the following structural result for isomorphisms between limit algebras as a corollary.

Theorem 4.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be triangular limit algebras. If $\mathcal{A}$ and $\mathcal{B}$ are (algebraically) isomorphic, then they are isometrically isomorphic. Moreover, every algebra isomorphism $\vartheta : \mathcal{A} \to \mathcal{B}$ between triangular limit algebras can be factored as $\tau \circ \xi$ where $\tau : \mathcal{A} \to \mathcal{B}$ is a star-extendible isomorphism and $\xi : \mathcal{A} \to \mathcal{A}$ is an algebra automorphism such that γ_ξ is the identity map on* $\operatorname{Spec}(\mathcal{A})$.

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