2013

Isometric Tuples are Hyperreflexive

Adam H. Fuller
University of Nebraska - Lincoln, afuller7@math.unl.edu

Matthew Kennedy
Carleton University, mkennedy@math.carleton.ca

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ADAM H. FULLER & MATTHEW KENNEDY

ABSTRACT. An n-tuple of operators $(V_1, \ldots, V_n)$ acting on a Hilbert space $\mathcal{H}$ is said to be isometric if the row operator $(V_1, \ldots, V_n) : \mathcal{H}^n \to \mathcal{H}$ is an isometry. We prove that every isometric n-tuple is hyperreflexive, in the sense of Arveson. For $n = 1$, the hyperreflexivity constant is at most 95. For $n \geq 2$, the hyperreflexivity constant is at most 6.

1. INTRODUCTION AND PRELIMINARIES

An n-tuple of operators $(V_1, \ldots, V_n)$ acting on a Hilbert space $\mathcal{H}$ is said to be isometric if the row operator $(V_1, \ldots, V_n) : \mathcal{H}^n \to \mathcal{H}$ is an isometry. This is equivalent to requiring that the operators in the tuple be isometries with pairwise orthogonal ranges, and hence that they satisfy the algebraic relations

$$V_i^* V_j = \delta_{ij} I, \quad 1 \leq i, j \leq n.$$

An isometric tuple is a natural higher-dimensional generalization of an isometry, and these objects appear throughout mathematics and mathematical physics (see, e.g., [Dav01, Ken11, Ken12] and the references therein).

The notion of reflexivity, introduced by Halmos in [H70] and [H71], plays an important role in operator theory. A subspace S of $\mathrm{B}(\mathcal{H})$ is *reflexive* if

$$S = \mathrm{Alg}(\mathrm{Lat}(S)) = \{T \in \mathrm{B}(\mathcal{H}) \mid P^{\perp} T P = 0 \text{ for every } P \in \mathrm{Lat}(S)\}, \tag{1.1}$$

where $\mathrm{Lat}(S)$ denotes the lattice of closed invariant subspaces for S, and where we have identified each subspace in $\mathrm{Lat}(S)$ with the corresponding projection in $\mathrm{B}(\mathcal{H})$. The basic idea is that a reflexive space of operators is completely determined by its invariant subspaces.

The notion of hyperreflexivity was introduced by Arveson in [Arv75] as a quantitative strengthened form of reflexivity. Before giving the formal definition

Indiana University Mathematics Journal ©, Vol. 62, No. 5 (2013)

of hyperreflexivity, it will be convenient to give another, slightly different, characterization of reflexivity. For a subspace $\mathcal{S}$ of $B(\mathcal{H})$, let

$$\beta(T, \mathcal{S}) = \sup\{\|P^{\perp}TP\| : P \in \mathrm{Lat}(\mathcal{S})\}, \quad T \in B(\mathcal{H}).$$

The quantity $\beta(T, \mathcal{S})$ measures the "fit" between an operator T in $B(\mathcal{H})$ and the invariant subspace lattice of $\mathcal{S}$. By (1.1), we see that $\mathcal{S}$ is reflexive precisely when

$$\mathcal{S} = \{T \in B(\mathcal{H}) \mid \beta(T, \mathcal{S}) = 0\}. \tag{1.2}$$

Note that this is equivalent to the existence of, for every T in $B(\mathcal{H})$, a constant $C_T > 0$ (depending on T) such that $\mathrm{dist}(T, \mathcal{S}) \le C_T \beta(T, \mathcal{S})$, where

$$\mathrm{dist}(T, \mathcal{S}) = \inf\{\|T - S\| : S \in \mathcal{S}\}, \quad T \in B(\mathcal{H}).$$

The subspace $\mathcal{S}$ is hyperreflexive if a constant can be chosen independent of T. Specifically, a subspace $\mathcal{S}$ of $B(\mathcal{H})$ is *hyperreflexive* if there is a constant $C > 0$ such that

$$\mathrm{dist}(T, \mathcal{S}) \le C\beta(T, \mathcal{S}), \quad T \in B(\mathcal{H}). \tag{1.3}$$

In this case, we say that $\mathcal{S}$ is hyperreflexive with distance constant at most C. In addition, we follow the standard convention and say that a family of operators is hyperreflexive if the weakly closed algebra generated by the family is hyperreflexive.

Note that the inequality

$$\beta(T, \mathcal{S}) \le \mathrm{dist}(T, \mathcal{S}), \quad T \in B(\mathcal{H}), \tag{1.4}$$

always holds. This is a consequence of the fact that for arbitrary S in $\mathcal{S}$,

$$\|P^{\perp}TP\| = \|P^{\perp}(T - S)P\| \le \mathrm{dist}(T, \mathcal{S}).$$

If $\mathcal{S}$ is hyperreflexive, then, by (1.3) and (1.4), the function $T \to \beta(T, \mathcal{S})$ is equivalent to the distance function $T \to \mathrm{dist}(T, \mathcal{S})$ on $B(\mathcal{H})$. In particular, it follows from (1.2) that $\mathcal{S}$ is reflexive. It was shown in [KL86], however, that the converse is false, and hence that hyperreflexivity is a strictly stronger property than reflexivity.

In addition to Arveson's work, the notion of hyperreflexivity has been studied by many other authors. The work of Christensen in [Chr82] (see also [Chr77]) established that the hyperreflexivity of a von Neumann algebra is equivalent to a positive solution to the Derivation Problem. In [Dav87], Davidson proved that the unilateral shift is hyperreflexive, and in [KP05], Kliś and Ptak generalized this result to a more general class of operators called quasinormal operators.

In [DP99], Davidson and Pitts generalized this result in a different direction, and proved the hyperreflexivity of a class of isometric tuples that can be seen as a higher-dimensional generalization of the unilateral shift. Subsequently, in [Ber98], Bercovici established the hyperreflexivity of a much larger class of algebras, and substantially improved the distance constant from [DP99]. See also the papers [LS75], [Ros82], [KL85], [KL86], [MP05], [DL06], [KP06], [JP06], and [PP12].

Despite these results, hyperreflexivity is still not well understood. In fact, it is still not known whether many interesting and naturally occuring examples are hyperreflexive, or even reflexive. In [DKP01], which was a followup to [DP99], Davidson, Katsoulis and Pitts posed the problem of whether every isometric tuple is hyperreflexive. Recently, in [Ken11] (see also [Ken12]), a partial solution to this problem was given, and it was established that the class of absolutely continuous isometric tuples is hyperreflexive (see Section 3 below for the definition of absolute continuity).

The main result in this paper is the following theorem, which completely resolves the problem from [DKP01], and more generally, provides the first large class of hyperreflexive examples in multivariate operator theory.

Theorem 1.1. *Every isometric n-tuple is hyperreflexive. For $n = 1$, the distance constant is at most 95. For $n \geq 2$, the distance constant is at most 6.*

In addition to this introduction, this paper has two other sections. In Section 2, we consider the classical case of a single isometry. In Section 3, we consider isometric tuples and establish our main result.

2. The Hyperreflexivity of Isometries

In this section, we consider the case of a single isometry. Since an isometry is a special case of a quasinormal operator, the main result from [KP05] implies that every isometry is hyperreflexive. However, we provide a simple proof of this result here, based on the main result in [Dav87], and obtain a better distance constant.

We will require the following classical result, which follows from the Wold decomposition and the Lebesgue decomposition of a measure. The details can be found in, for example, [NFBK10].

Theorem 2.1 (Lebesgue-von Neumann-Wold Decomposition). *Let V be an isometry. Then, V can be decomposed as*

$$V = V_u \oplus V_a \oplus V_s,$$

where V_u is a unilateral shift, V_a is a unitary with an absolutely continuous spectral measure, and V_s is a unitary with a singular spectral measure. Note that absolute continuity and singularity are with respect to Lebesgue measure.

We will also require two results from the literature. The first result, proved by Kraus and Larson in [KL86], and independently by Davidson in [Dav87], shows that, in certain cases, hyperreflexivity is inherited by subspaces. Recall that a weak-* closed subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ has property $\mathbb{A}_1(1)$ if, for every $\varepsilon > 0$, and every weak-* continuous linear functional φ on $\mathcal{S}$, there are vectors x and y in $\mathcal{H}$ such that $\varphi(S) = \langle Sx, y \rangle$ for all S in $\mathcal{S}$, and $\|x\| \|y\| < (1 + \varepsilon)\|\varphi\|$.

Theorem 2.2 (Kraus-Larson, Davidson). *Let S be a hyperreflexive subspace of* $\mathrm{B}(\mathcal{H})$ *with distance constant at most C. Suppose, furthermore, that S has property* $\mathbb{A}_1(1)$. *Then, any weak-* closed subspace of S is hyperreflexive with distance constant at most $2C + 1$.*

The second result, proved by Kliś and Ptak in [KP06], shows that hyperreflexivity is preserved under taking direct sums.

Theorem 2.3 (Kliś-Ptak). *Let $\{\mathcal{A}_n \mid n \in \mathbb{N}\}$ be a family of hyperreflexive subspaces with hyperreflexivity constants bounded by C. Then, the direct sum $\mathcal{A} = \bigoplus_{n\in\mathbb{N}} \mathcal{A}_n$ is hyperreflexive with distance constant at most $2 + 3C$.*

We are now able to prove the main result in this section.

Proposition 2.4. *Every isometry is hyperreflexive with distance constant at most* 95.

Proof. Let V be an isometry with Lebesgue-von Neumann-Wold decomposition $V = V_u \oplus V_a \oplus V_s$ as in Theorem 2.1. Then, it is clear that

$$\mathrm{W}(V) \subseteq \mathrm{W}(V_u) \oplus \mathrm{W}^*(V_a) \oplus \mathrm{W}^*(V_s).$$

We will first show that $W(V_u)$ is hyperreflexive with distance constant at most 15. By [Dav87], unilateral shifts of multiplicity 1 are hyperreflexive, and by [KP05], this distance constant is at most 13. Now, suppose V_u has multiplicity α and acts on the Hilbert space $\mathcal{H} = \bigoplus \mathcal{H}_i$. Let

$$\mathcal{M} = \bigoplus \mathcal{B}(\mathcal{H}_i) \subseteq \mathcal{B}(\mathcal{H}).$$

Then, $\mathcal{M}$ is an injective von Neumann algebra with abelian commutant $\mathcal{M}'$. Denote by Φ the expectation from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{M}$. Take any $T \in \mathcal{B}(\mathcal{H})$. Following the methods of [Ros82], and noting that $\beta_{\mathcal{M}}(T) \le \beta_{W(V_u)}(T)$, we see that

$$\|T - \Phi(T)\| \le 2\beta_{W(V_u)}(T).$$

We have now that

$$\begin{aligned}
\operatorname{dist}(T, W(V_u)) &\le \operatorname{dist}(T - \Phi(T), W(V_u)) + \operatorname{dist}(\Phi(T), W(V_u)) \\
&\le \|T - \Phi(T)\| + \operatorname{dist}(\Phi(T), W(V_u)) \\
&\le 2\beta_{W(V_u)}(T) + 13\beta_{W(V_u)}(T) = 15\beta_{W(V_u)}(T).
\end{aligned}$$

Since $\mathrm{W}^*(V_a)$ and $\mathrm{W}^*(V_s)$ are abelian von Neuman algebras, they are both hyperreflexive with distance constant at most 2 by [Ros82]. Again, since $\mathrm{W}^*(V_a)$ and $\mathrm{W}^*(V_s)$ are abelian von Neumann algebras, they also have property $\mathbb{A}_1(1)$ (see, e.g., Theorem 60.1 of [Con00]). Therefore, by Theorem 2.2, $\mathrm{W}(V_a)$ and $\mathrm{W}(V_s)$ are hyperreflexive with distance constant at most 5.

By Theorem 2.3, the algebra $\mathrm{W}(V_u) \oplus \mathrm{W}^*(V_a) \oplus \mathrm{W}^*(V_s)$ is hyperreflexive with distance constant at most 47. Finally, by Theorem 2.2 we get that $W(V)$ is hyperreflexive with distance constant at most 95. ∎

3. The Hyperreflexivity of Isometric Tuples

In this section, we will prove that every isometric tuple is hyperreflexive, which is the main result of this paper. No straightforward generalization of the approach taken in Section 2 will suffice here, because the structure of an isometric tuple can be substantially more complicated than the structure of an isometry. We will utilize the structure theorem for isometric tuples from [Ken12]. Before stating that result, we need to introduce some terminology.

Let $\mathcal{F}_n$ denote the full Fock space over $\mathbb{C}^n$. That is,

$$\mathcal{F}_n = \mathbb{C} \oplus \mathbb{C}^n \oplus (\mathbb{C}^n)^{\otimes 2} \oplus (\mathbb{C}^n)^{\otimes 3} \oplus \cdots .$$

For a fixed orthonormal basis $\{\xi_1, \dots, \xi_n\}$ of $\mathbb{C}^n$, let $L_1, \dots, L_n$ denote the left creation operators on $\mathcal{F}_d$. That is,

$$L_i(\xi_{j_1} \otimes \cdots \otimes \xi_{j_k}) = \xi_i \otimes \xi_{j_1} \otimes \cdots \otimes \xi_{j_k}, \quad 1 \leq i, j_1, \dots, j_k \leq n.$$

The *unilateral n-shift* is the isometric tuple $L = (L_1, \dots, L_n)$, and the *noncommutative analytic Toeplitz algebra $\mathcal{L}_n$* is the weakly closed algebra $\mathrm{W}(L)$ generated by $L_1, \dots, L_n$.

The motivation for these names is the fact that, for $n = 1$, L can be identified with the classical unilateral shift, and $\mathcal{L}_n$ can be identified with the classical algebra H^∞ of bounded analytic functions on the complex unit disk. The study of these objects was initiated by Popescu in [Pop91] and [Pop96], where they were shown to possess a great deal of analytic structure. They were also studied in detail by Davidson and Pitts in [DP98] and [DP99], and more recently in [Ken11] and [Ken12]. It has become clear in recent years that these objects play a central role in multivariate operator theory (see, e.g., [Dav01]).

Definition 3.1. Let $V = (V_1, \dots, V_n)$ be an isometric n-tuple, for $n \geq 2$. Then,

(1) V is a *unilateral shift* if it is unitarily equivalent to an ampliation of the unilateral n-shift;
(2) V is *absolutely continuous* if the weakly closed algebra $\mathrm{W}(V_1, \dots, V_n)$ is isomorphic to the noncommutative analytic Toeplitz algebra $\mathcal{L}_n$;
(3) V is *singular* if the weakly closed algebra $\mathrm{W}(V_1, \dots, V_n)$ is a von Neumann algebra;
(4) V is of *dilation type* if it has no summand that is absolutely continuous or singular.

This definition merits a few remarks. First, the existence of singular isometric tuples was an open problem for some time before it was established by Read in [Rea05] (see also [Dav06] for an exposition). Second, isometric tuples of dilation type do not appear in the classical setting of a single isometry. In the higher-dimensional setting, they arise as the minimal isometric dilation of contractive tuples (see [Ken12] for more details).

The following theorem is the Lebesgue-von Neumann-Wold decomposition of an isometric tuple from [Ken12].

Theorem 3.2 (Lebesgue-von Neumann-Wold Decomposition). *Let $V = (V_1, \dots, V_n)$ be an isometric n-tuple. For $n \geq 2$, V can be decomposed as*

$$V = V_u \oplus V_a \oplus V_s \oplus V_d, \tag{3.1}$$

where V_u is a unilateral n-shift, V_a is an absolutely continuous n-tuple, V_s is a singular n-tuple, and V_d is an isometric n-tuple of dilation type.

It follows from the results in [Ken11] and [Ken12] that an isometric tuple V is absolutely continuous if and only if $V_s = 0$ and $V_d = 0$ in the decomposition (3.1). The following result was also obtained in [Ken12].

Proposition 3.3. *For $n \geq 2$, every absolutely continuous n-tuple is hyperreflexive with distance constant at most 3.*

To handle the case of singular isometric tuples and isometric tuples of dilation type, we will need a better understanding of the von Neumann algebra generated by certain isometric tuples. An isometric tuple $V = (V_1, \dots, V_n)$ is said to be *unitary* if the row operator $(V_1, \dots, V_n) : \mathcal{H}^n \to \mathcal{H}$ is surjective. Since, in the classical case, a unitary is precisely a surjective isometry, this is a natural higher-dimensional generalization of the notion of a unitary.

Proposition 3.4. *For $n \geq 2$, the commutant of the von Neumann algebra generated by a unitary n-tuple is injective.*

Proof. Let $U = (U_1, \dots, U_n)$ be a unitary n-tuple acting on a Hilbert space $\mathcal{H}$, and let $\mathcal{M} = \mathrm{W}^*(U_1, \dots, U_n)$ denote the von Neumann algebra generated by U. Define a completely positive map $\Gamma : \mathrm{B}(\mathcal{H}) \to \mathrm{B}(\mathcal{H})$ by

$$\Gamma(T) = \sum_{i=1}^{n} U_i T U_i^*, \quad T \in \mathrm{B}(\mathcal{H}),$$

and let

$$\mathcal{F}(\Gamma) = \{T \in \mathrm{B}(\mathcal{H}) \mid \Gamma(T) = T\}$$

denote the set of fixed points of Γ. Let $\mathcal{M}'$ denote the commutant of $\mathcal{M}$. We will first show that $\mathcal{M}' = \mathcal{F}(\Gamma)$.

Suppose first that T belongs to $\mathcal{M}'$. Then,

$$\Gamma(T) = \sum_{i=1}^{n} U_i T U_i^* = T \sum_{i=1}^{n} U_i U_i^* = T,$$

and hence $T \in \mathcal{F}(\Gamma)$. Now suppose that T belongs to $\mathcal{F}(\Gamma)$. Then,

$$T = \Gamma(T) = \sum_{i=1}^{n} U_i T U_i^*.$$

Therefore, for each $1 \le j \le n$, multiplying T on the left by U_j^* gives $U_j^* T = TU_j^*$, and multiplying T on the right by U_j gives $TU_j = U_j T$. Hence, $T \in \mathcal{M}'$. Thus, we see that $\mathcal{M}' = \mathcal{F}(\Gamma)$.

By Lemma 2 of [Arv72], there is a completely positive and contractive idempotent map $\Phi : \mathrm{B}(\mathcal{H}) \to \mathrm{B}(\mathcal{H})$ with the property that $\operatorname{ran}\Phi = \mathcal{F}(\Gamma)$. We note that the existence of the map Φ has become a standard tool in the theory of completely positive maps. It can be realized as the strong operator limit

$$\Phi(X) = \lim_{k\to\infty} \Gamma^k(X), \quad X \in \mathrm{B}(\mathcal{H}).$$

Note that, since U is unitary, $\Gamma(I) = I$. Therefore, $\Phi(I) = I$, which gives $\|\Phi\| = 1$. It follows that Φ is a projection from $\mathrm{B}(\mathcal{H})$ to $\mathcal{M}'$, and hence that $\mathcal{M}'$ is injective. ◻

We note that, in addition to the direct proof given here, the previous result can also be obtained using some deep results from the theory of C*-algebras and von Neumann algebras. Indeed, the C*-algebra generated by a unitary n-tuple is isomorphic to the Cuntz algebra $\mathcal{O}_n$, and hence is nuclear [Cu77]. Therefore, the results in [CE78] imply that the von Neumann algebra generated by a unitary n-tuple is injective, and it follows from the theory of injective von Neumann algebras that the commutant is injective.

Corollary 3.5. *For $n \ge 2$, every singular isometric n-tuple is hyperreflexive with distance constant at most* 4.

Proof. By definition, the weakly closed algebra generated by a singular isometric tuple is a von Neumann algebra, and by Proposition 3.4, the commutant of this von Neumann algebra is injective. By [Chr77], this algebra is hyperreflexive with distance constant at most 4. ◻

We now have everything we need to prove the main result.

Theorem 3.6. *Every isometric n-tuple is hyperreflexive. For $n = 1$, the distance constant is at most* 95. *For $n \ge 2$, the distance constant is at most* 6.

Proof. For $n = 1$, the result follows from Proposition 2.4. Therefore, we let $V = (V_1, \dots, V_n)$ be an isometric n-tuple of dilation type acting on a Hilbert space $\mathcal{H}$, for $n \ge 2$. Let $\mathcal{M}$ denote the von Neumann algebra $\mathrm{W}^*(V)$ generated by V, and let S be the weakly closed algebra $\mathrm{W}(V)$, that is, the free semigroup algebra, generated by V. By the structure theorem from [DKP01], there is a projection P in S, with range coinvariant for S, such that $SP = \mathcal{M}P$.

If $P = 0$, then V is analytic, and the result follows from Proposition 3.3. On the other hand, if $P = I$, then V is singular, and the result follows from Corollary 3.5. Hence, we can suppose that $P \neq 0$ and $P \neq I$. Note that SP and $SP^\perp$ are both weakly closed algebras, and that we can write $S = SP + SP^\perp$. Note also that if

S and R belong to $\mathcal{S}$, then SP and $RP^\perp$ also belong to $\mathcal{S}$. Hence, for T in $\mathrm{B}(\mathcal{H})$, we have

$$\begin{aligned}
\operatorname{dist}(T,\mathcal{S})^2 &= \inf\{\|T-S\|^2 : S\in\mathcal{S}\}\\
&= \inf\{\|TP-SP+TP^\perp-SP^\perp\|^2 : S\in\mathcal{S}\}\\
&= \inf\{\|TP-SP+TP^\perp-RP^\perp\|^2 : S,R\in\mathcal{S}\}\\
&= \inf\{\|PT^*-PS^*+P^\perp T^*-P^\perp R^*\|^2 : S,R\in\mathcal{S}\}\\
&= \inf\{\|TP-SP\|^2+\|TP^\perp-RP^\perp\|^2 : S,R\in\mathcal{S}\}\\
&= \inf\{\|TP-SP\|^2 : S\in\mathcal{S}\}+\inf\{\|TP^\perp-RP^\perp\|^2 : R\in\mathcal{S}\}\\
&= \operatorname{dist}(TP,\mathcal{S}P)^2+\operatorname{dist}(TP^\perp,\mathcal{S}P^\perp)^2.
\end{aligned}$$

To show that $\mathcal{S}$ is hyperreflexive, we will bound $\operatorname{dist}(TP,\mathcal{S}P)$ and $\operatorname{dist}(TP^\perp,\mathcal{S}P^\perp)$ separately.

First, we consider the value of $\operatorname{dist}(TP,\mathcal{S}P)$. Suppose that V is a unitary n-tuple. By the argument from the proof of Corollary 3.5, the von Neumann algebra $\mathcal{M}$ is hyperreflexive with distance constant at most 4. Therefore,

$$\operatorname{dist}(TP,\mathcal{S}P)=\operatorname{dist}(TP,\mathcal{M}P)=\operatorname{dist}(TP,\mathcal{M})\le 4\beta(TP,\mathcal{M}). \tag{3.2}$$

For x and y in $\mathcal{H}$, let xy^* denote the linear functional on $\mathrm{B}(\mathcal{H})$ defined by

$$(xy^*)(T)=(Tx,y),\quad T\in\mathrm{B}(\mathcal{H}).$$

Then, we have the inequality

$$\begin{aligned}
\beta(TP,\mathcal{M}) &= \sup\{|\langle TPx,y\rangle| : xy^*\in\mathcal{M}_\perp,\ \|x\|\,\|y\|\le 1\}\\
&= \sup\{|\langle TP(Px),y\rangle| : (Px)y^*\in\mathcal{M}_\perp,\ \|x\|\,\|y\|\le 1\}\\
&\le \sup\{|\langle TPx,y\rangle| : xy^*\in(\mathcal{M}P)_\perp,\ \|x\|\,\|y\|\le 1\}\\
&= \beta(TP,\mathcal{M}P),
\end{aligned} \tag{3.3}$$

and similarly,

$$\begin{aligned}
\beta(TP,\mathcal{S}P) &= \sup\{|\langle TPx,y\rangle| : xy^*\in(\mathcal{S}P)_\perp,\ \|x\|\,\|y\|\le 1\}\\
&= \sup\{|\langle T(Px),y\rangle| : (Px)y^*\in\mathcal{S}_\perp,\ \|x\|\,\|y\|\le 1\}\\
&\le \sup\{|\langle Tx,y\rangle| : xy^*\in\mathcal{S}_\perp,\ \|x\|\,\|y\|\le 1\}\\
&= \beta(T,\mathcal{S}).
\end{aligned} \tag{3.4}$$

Putting (3.2), (3.3), and (3.4) together gives

$$\operatorname{dist}(TP,\mathcal{S}P)\le 4\beta(TP,\mathcal{M})\le 4\beta(TP,\mathcal{M}P)\le 4\beta(T,\mathcal{S}). \tag{3.5}$$

The case when V is not necessarily unitary follows from the fact that P is contained in the von Neumann algebra generated by the unitary part.

Now we consider the value of $\operatorname{dist}(TP^\perp, \mathcal{S}P^\perp)$. To obtain a bound, we could appeal to Theorem 2.3, but by calculating directly, we can obtain a better distance constant. By Proposition 6.2 of [Ken12], the compression $P^\perp \mathcal{S}|_{\operatorname{ran}(P^\perp)}$ is absolutely continuous. Hence by Proposition 3.3, the algebra $\mathcal{S}P^\perp$ is also hyperreflexive with distance constant at most 3. Since the range of $P^\perp$ is invariant for $\mathcal{S}$, we have $SP^\perp = P^\perp SP^\perp$ for every S in $\mathcal{S}$. Hence,

$$\begin{aligned}
\operatorname{dist}(TP^\perp, \mathcal{S}P^\perp) &= \inf\{\|TP^\perp - SP^\perp\| : S \in \mathcal{S}\} \\
&\le \inf\{\|PTP^\perp\| + \|P^\perp TP^\perp - SP^\perp\| : S \in \mathcal{S}\} \\
&= \|PTP^\perp\| + \inf\{\|P^\perp TP^\perp - SP^\perp\| : S \in \mathcal{S}\} \\
&= \|PTP^\perp\| + \operatorname{dist}(P^\perp TP^\perp, \mathcal{S}P^\perp).
\end{aligned} \tag{3.6}$$

But since $P^\perp \in \operatorname{Lat}(\mathcal{S})$, $\|PTP^\perp\| \le \beta(T, \mathcal{S})$, and since $\mathcal{S}P^\perp$is hyperreflexive with distance constant at most 3, we have

$$\operatorname{dist}(P^\perp TP^\perp, \mathcal{S}P^\perp) \le 3\beta(P^\perp TP^\perp, \mathcal{S}P^\perp). \tag{3.7}$$

By an argument similar to (3.3) and (3.4), $\beta(P^\perp TP^\perp, \mathcal{S}P^\perp) \le \beta(T, \mathcal{S})$. Hence, putting (3.6) and (3.7) together gives

$$\operatorname{dist}(TP^\perp, \mathcal{S}P^\perp) \le \beta(T, \mathcal{S}) + 3\beta(P^\perp TP^\perp, \mathcal{S}P^\perp) \le 4\beta(T, \mathcal{S}). \tag{3.8}$$

Combining (3.5) and (3.8), we see that $\mathcal{S}$ is hyperreflexive with distance constant at most $4\sqrt{2} < 6$. ∎

Remark. We note that by replacing (3.6) in the proof of Theorem 3.6 with the inequality

$$\operatorname{dist}(TP^\perp, \mathcal{S}P^\perp) \le (\operatorname{dist}(PTP^\perp, P\mathcal{S}P^\perp)^2 + \operatorname{dist}(P^\perp TP^\perp, P^\perp \mathcal{S}P^\perp)^2)^{1/2},$$

we could obtain the slightly better distance constant $\sqrt{26} \approx 5.1$.

Acknowledgement. The authors are grateful to Ken Davidson for many helpful comments and suggestions which, in particular, helped us improve the distance constants obtained in Theorem 3.6. The authors are also grateful to Dilian Yang for correcting an early draft of this paper.

The second author has been partially supported by an NSERC grant.

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Adam H. Fuller:
Department of Mathematics
University of Nebraska-Lincoln
Lincoln, NE, 68588-0130, USA
E-mail: afuller7@math.unl.edu

Matthew Kennedy:
School of Mathematics and Statistics
Carleton University
1125 Colonel By Drive
Ottawa, Ontario K1S 5B6, Canada
E-mail: mkennedy@math.carleton.ca

Key words and phrases: Invariant subspace, reflexivity, hyperreflexivity, distance formula, isometric tuple, free semigroup algebra.

2000 Mathematics Subject Classification: 47A15, 47L05, 47A45, 47B20, 47B48.

Received: August 19, 2012.