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The Classification of Limits of 2n-cycle Algebras

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The Classification of Limits of 2n-cycle Algebras

Allan P. Donsig & S.C. Power

Abstract. We obtain a complete classification of the locally finite algebras $\mathcal{A}_0 = \underrightarrow{\operatorname{alg\,lim}}\, A_k$ and the operator algebras $\mathcal{A} = \varinjlim A_k$ associated with towers $A_1 \subseteq A_2 \subseteq A_3 \cdots$ consisting of $2n$-cycle algebras, where $n \geq 3$, with the inclusions of rigid type. The complete isomorphism invariant is essentially the triple $(K_0(\mathcal{A}), H_1(\mathcal{A}), \Sigma(\mathcal{A}))$ where $K_0(\mathcal{A})$ is viewed as a scaled ordered group, $H_1(\mathcal{A})$ is a partial isometry homology group and $\Sigma(\mathcal{A}) \subseteq K_0(\mathcal{A}) \oplus H_1(\mathcal{A})$ is the $2n$-cycle joint scale.

1. Introduction

In this paper, we obtain a complete classification of the locally finite algebras $\mathcal{A}_0 = \underrightarrow{\operatorname{alg\,lim}}\, A_k$ and the operator algebras $\mathcal{A} = \varinjlim A_k$ associated with towers

$$A_1 \subseteq A_2 \subseteq A_3 \cdots$$

consisting of $2n$-cycle algebras, where $n \geq 3$, with the inclusions of rigid type. The complete isomorphism invariant is essentially the triple

$$(K_0(\mathcal{A}), H_1(\mathcal{A}), \Sigma(\mathcal{A}))$$

where $K_0(\mathcal{A})$ is viewed as a scaled ordered group, $H_1(\mathcal{A})$ is a partial isometry homology group and $\Sigma(\mathcal{A}) \subseteq K_0(\mathcal{A}) \oplus H_1(\mathcal{A})$ is the $2n$-cycle joint scale.

Recall that a cycle algebra is a finite dimensional digraph algebra, or incidence algebra, whose reduced digraph is a cycle. For example, the basic 6-cycle algebra in M_6 has the form

$$\begin{bmatrix} * & * & & & & * \\ & * & & & & \\ & * & * & * & & \\ & & & * & & \\ & & & * & * & * \\ & & & & & * \end{bmatrix}$$

and its digraph is a hexagon with alternating edge directions and a loop edge at each vertex. These algebras are of interest in that they are the simplest family of finite dimensional complex algebras with non-zero homology. We focus on the natural embeddings between cycle algebras known as rigid embeddings and which have the property that they are determined by K_0 and H_1. More precisely, a rigid embedding φ is determined up to inner conjugacy by $K_0\varphi \oplus H_1\varphi$, where $H_1\varphi$ is the induced homomorphism between the first integral simplicial homology groups of the simplicial complexes affiliated to the digraphs of the algebras. For cycle algebras, the map $H_1\varphi$ is simply a group homomorphism $\mathbb{Z} \to \mathbb{Z}$. This K_0H_1 uniqueness is in analogy with the fact that embeddings between finite dimensional semisimple complex algebras are determined up to inner conjugacy by K_0.

Limit homology groups for direct limits of digraph algebras were introduced in [2] and an intrinsic formulation as stable partial-isometry homology groups was given in [8]. The classification of cycle algebras was first considered in [7, Chapter 11], in the restricted context of direct limits of 4-cycle algebras with homologically limited embeddings. These results were then extended to limits of 4-cycle algebras with general rigid embeddings in [4], and the classification of the operator algebras up to *regular* isomorphism was obtained in terms of the triple $(K_0(\mathcal{A}), H_1(\mathcal{A}), \Sigma(\mathcal{A}))$. In [9], the second author showed that this classification was, in fact, up to star extendible isomorphism for all locally finite algebras and for the operator algebras arising from direct systems of 4-cycle algebras in the somewhat more tractable odd case. The current paper is concerned with higher cycles and, building on the ideas of [4] and [9], we classify, up to star-extendible isomorphism, both the locally finite algebras and the operator algebras arising from direct limits of $2n$-cycle algebras, $n \geq 3$, with rigid embeddings.

Cohomology has been considered in the context of non-selfadjoint algebras for many years; recent references include [5, 6]. We remark that in addition to homological augmentations of K_0 invariants one can also consider scaled Grothendieck group invariants for regular systems of digraph algebras and their operator algebras. This topic is developed in [10].

It is fortuitous that, in one important respect, the analysis of $2n$-cycle algebras for $n \geq 3$ is much simpler than that of 4-cycle algebras: star-extendible isomorphisms between algebraic direct limits are necessarily induced by a commuting diagram of regular linking maps between the given towers. A *regular* morphism is a direct sum of multiplicity one embeddings. The existence of irregular morphisms between regular limit algebras was first observed in [3]. On the other hand, the K_0-groups of towers of $2n$-cycle algebras are much less readily identifiable, except in some cases of independent interest, most notably, when $\mathcal{A} \cap \mathcal{A}^*$ is a direct sum of matroid algebras.

Organisation. In the next section, we study $2n$-cycle algebras and the rigid embeddings between them, culminating in the theorem that locally finite algebras are star-extendibly isomorphic if and only if there is an intertwining rigid diagram. In the third section, we extend this theorem from locally finite algebras to norm-closed direct limits. The last section introduces the classifying invariants and obtains the main classification theorem. Finally we illustrate this with an application to a family of limit algebras for which the only variations are ones of homology.

2. Towers of cycle algebras

We first set out notation and recall some terminology.

A *digraph* is a finite directed graph with no multiple directed edges of the same orientation. To a transitive, reflexive digraph G on the vertices $v_1, \dots, v_n$, we associate the complex algebra $A(G)$ spanned by those standard matrix units $e_{i,j} \in M_n(\mathbb{C})$ associated with edges in G from v_j to v_i. These subalgebras of $M_n(\mathbb{C})$ are precisely those which contain the diagonal algebra.

For convenience we restrict attention henceforth to those morphisms between digraph algebras $A(G_1) \to A(G_2)$ which are star-extendible, in the sense of being restrictions of star-algebra homomorphisms $C^*(A(G_1)) \to C^*(A(G_2))$.

A *$2n$-cycle digraph algebra* is a digraph algebra $A(G)$ for which the reduced digraph of G, call it G_r, is isomorphic to the connected graph on $2n$ vertices with the $2n$ edges (i,i) for $i = 1, \dots, 2n$ and $2n$ edges between successive vertices, with alternating orientations. We may assume that the vertex labeled 1 is a range vertex rather than a source vertex. Denote this directed graph by D_{2n}.

The elements of a $2n$-cycle algebra thus have a block matrix staircase form:

$$a = \begin{bmatrix} a_{11} & a_{12} & & & & & & a_{1,2n} \\ & a_{22} & & & & & & \\ & a_{32} & a_{33} & a_{34} & & & & \\ & & & a_{44} & & & & \\ & & & a_{54} & a_{55} & a_{56} & & \\ & & & & & \ddots & & \\ & & & & & & \ddots & a_{2n-1,2n} \\ & & & & & & & a_{2n,2n} \end{bmatrix}.$$

A *partial isometry* is an element u for which u^*u (and hence uu^*) is a projection (a selfadjoint idempotent). A convenient consequence of the star-extendibility of an embedding $\varphi : A_1 \to A_2$ between digraph algebras is that, for each partial isometry $v \in A_1$ with vv^* and v^*v in A_1, the image $\varphi(v)$ is a partial isometry with its initial and final projections in A_2. In $2n$-cycle algebras with $n \geq 3$, such partial isometries have a particularly clear form. We remark that the next lemma does not hold for 4-cycle algebras, which complicates considerably the analysis of their limit algebras.

Lemma 2.1. *Suppose $a = \big(a_{ij}\big)$ is a block matrix in a $2n$-cycle algebra A, $n \geq 3$. If a is a partial isometry with initial and final projections in A, then each entry a_{ij} is a partial isometry.*

Proof. The $(1,1)$ block entry of a^*a is $a_{11}^*a_{11}$. Thus, by hypothesis $a_{11}^*a_{11}$ is a projection and so a_{11} is a partial isometry. By symmetry, a_{kk} is a partial isometry for all k. It follows that the block matrix

$$b = \begin{bmatrix} a_{12} & & & & a_{1,2n} \\ a_{32} & a_{34} & & & \\ & a_{54} & a_{56} & & \\ & & \ddots & \ddots & \\ & & & a_{2n-1,2n-2} & a_{2n-1,2n} \end{bmatrix}$$

is a partial isometry, and that its initial and final projections are block diagonal. Since the $(1,1)$ entry of bb^* is zero, it follows that $a_{12}a_{32}^* = 0$ and hence that a_{12} and a_{32} have orthogonal initial projections. Similarly, the $(1,2n)$ entry of b^*b being zero implies that a_{12} and $a_{1,2n}$ have orthogonal final projections. Since b is a partial isometry, this double orthogonality forces a_{12} to be a partial isometry. By symmetry, each $a_{i\,j}$ is a partial isometry, for all i and j. □

Notice that, if one tries to apply this proof to a 4-cycle algebra, then the matrix b has no zeros and so there are no orthogonality relations.

In a general digraph algebra A, we refer to a partial isometry v as a *regular partial isometry* if pvq is a partial isometry for each pair of central projections p, q in $A \cap A^*$. In particular, such a partial isometry can be written as a sum of rank one partial isometries in A. Note that a regular embedding sends regular partial isometries to regular partial isometries.

We call an embedding *locally regular* if it maps regular partial isometries to regular partial isometries. This is a strictly weaker property, as we now show.

Example 2.2. Consider the upper triangular realisation of $A(D_4)$ spanned by the diagonal matrix units and the 4-cycle $\{e_{13}, e_{14}, e_{24}, e_{23}\}$. Define φ from $A(D_4)$ to $A(D_4) \otimes M_4$ by mapping this rank-one 4-cycle in $A(D_4)$ to the 4-cycle $\{v_1, v_2, v_3, v_4\}$ where, as 2×2 block matrices, the v_i have the following upper right blocks and are otherwise zero:

$$\left[\begin{array}{cc|cc|cc|cc}
1 & 0 & & & 0 & 0 & & \\
0 & 0 & & & 1 & 0 & & \\ \hline
 & & & & & & & \\
 & & & & & & & \\ \hline
0 & 1 & & & 0 & 0 & & \\
0 & 0 & & & 0 & 1 & & \\ \hline
 & & & & & & & \\
 & & & & & & &
\end{array}\right], \quad \frac{1}{\sqrt{2}}\left[\begin{array}{cc|cc|cc|cc}
 & & 1 & 0 & & & 1 & 0 \\
 & & 1 & 0 & & & -1 & 0 \\ \hline
 & & & & & & & \\
 & & & & & & & \\ \hline
 & & 0 & 1 & & & 0 & 1 \\
 & & 0 & 1 & & & 0 & -1 \\ \hline
 & & & & & & & \\
 & & & & & & &
\end{array}\right],$$

and

$$\frac{1}{\sqrt{2}}\left[\begin{array}{cc|cc|cc|cc}
 & & & & & & & \\
 & & & & & & & \\ \hline
 & & 1 & 0 & & & 1 & 0 \\
 & & 1 & 0 & & & -1 & 0 \\ \hline
 & & & & & & & \\
 & & & & & & & \\ \hline
 & & 0 & 1 & & & 0 & 1 \\
 & & 0 & 1 & & & 0 & -1
\end{array}\right], \quad \left[\begin{array}{cc|cc|cc|cc}
 & & & & & & & \\
 & & & & & & & \\ \hline
1 & 0 & & & 0 & 0 & & \\
0 & 0 & & & 1 & 0 & & \\ \hline
 & & & & & & & \\
 & & & & & & & \\ \hline
0 & 1 & & & 0 & 0 & & \\
0 & 0 & & & 0 & 1 & &
\end{array}\right].$$

Since the v_i are regular partial isometries with orthogonal initial and final projections, φ is locally regular and star-extendible. However, the product $v_1^* v_2$ is easily seen to be not locally regular. It follows that φ is not regular.

2.1. ***Rigid embeddings.*** The digraph D_{2n} has $2n$ automorphisms which induce $2n$ automorphisms of $A(D_{2n})$, denoted $\theta_1, \theta_2, \cdots, \theta_{2n}$. For definiteness, we let θ_1 be the identity, θ_2 be the reflection which fixes the vertex 1, θ_3 be the shift which maps each vertex k to $(k-2) \mod 2n$, $\theta_{2k-1} = \theta_3^{k-1}$, for $1 \leq k \leq n$, and $\theta_{2k} = \theta_2\theta_{2k-1}$, for $2 \leq k \leq n$.

Suppose that A is a $2n$-cycle algebra and $i : A(D_{2n}) \to A$ is a multiplicity one star-extendible embedding, which is *proper*, in the sense that $i(e_{jj})$, $1 \leq j \leq 2n$, are inequivalent projections. Then the embeddings $i \circ \theta_1, i \circ \theta_2, \ldots, i \circ \theta_{2n}$ are representatives for the $2n$ inner unitary equivalence classes of the proper multiplicity one injections. We say that a star-extendible embedding $\varphi : A(D_{2n}) \to A(D_{2n}) \otimes M_n(\mathbb{C})$ is *rigid* if it decomposes as direct sum of multiplicity one proper embeddings.

In general, a *rigid* embedding between $2n$-cycle algebras, $\varphi : A_1 \to A_2$, is one for which $\varphi \circ \eta$ is rigid whenever $\eta : A(D_{2n}) \to A_1$ is proper.

We now come to the crucial lemma in the analysis of rigid embeddings. The case $n = 2$ was proved in Lemma 3.2 of [9] using a similar, albeit simpler, argument.

Lemma 2.3. *Let $\varphi : A_1 \to A_2$ and $\psi : A_2 \to A_3$ be locally regular (star-extendible) embeddings between $2n$-cycle algebras, where $n \geq 3$. If $\psi \circ \varphi$ is rigid, then φ and ψ are rigid.*

Proof. Suppose for simplicity of notation, that $n = 3$. The following argument does not depend on the length of the cycle, and so it suffices to prove the lemma.

Let A_1, A_2, A_3 be 6-cycle algebras and let $\varphi : A_1 \to A_2$ and $\psi : A_2 \to A_3$ be locally regular embeddings whose composition, $\psi \circ \varphi$, is rigid.

We may assume that φ and ψ are unital and that $A_1 = A(D_6)$ as the general case follows readily from this. It is convenient to change our block matrix form for 6-cycle algebras from a staircase pattern to the following one:

$$\begin{bmatrix} * & & & * & & * \\ & * & & * & * & \\ & & * & & * & * \\ & & & * & & \\ & & & & * & \\ & & & & & * \end{bmatrix}$$

Denote the partial isometries of the usual 6-cycle in $A(D_6)$ by $E_1, E_2, \dots, E_6$ where these have the positions

$$\begin{bmatrix} E_1 & & E_6 \\ E_2 & E_3 & \\ & E_4 & E_5 \end{bmatrix}$$

in the off-diagonal block of $A(D_6)$. Since the composition $\psi \circ \varphi$ maps each E_i to an element supported in the off diagonal blocks of A_3, it follows that φ acts similarly. Thus, with respect to the block structure of A_2, we can write

$$\varphi(E_i) = \begin{bmatrix} a_i & & f_i \\ b_i & c_i & \\ & d_i & e_i \end{bmatrix}$$

for $i = 1, 2, \dots, 6$. By the local regularity hypothesis, each of the entries $a_i, \dots, f_i$ is a partial isometry. Moreover, by star-extendibility, the initial projection of a_i is orthogonal to that of b_i and, trivially, to those of $c_i, \dots, f_i$.

Similarly, if $F_1, \dots, F_6$ is a rank-one six-cycle in A_2 then, with respect to the block structure of A_3, we can write

$$\psi(F_i) = \begin{bmatrix} \alpha_i & & \lambda_i \\ \beta_i & \gamma_i & \\ & \delta_i & \epsilon_i \end{bmatrix}$$

for $i = 1, \dots, 6$. As with φ, local regularity implies that $\alpha_i, \dots, \lambda_i$ are partial isometries and star-extendibility implies that, for example, α_i and β_i have orthogonal initial projections.

Fix matrix unit systems $\{e_{i,j}\}, \{f_{i,j}\}$ for A_1 and A_2, and note that we can assume that φ (resp. ψ) maps matrix units in the self-adjoint algebra $A_1 \cap A_1^*$ (resp. $A_2 \cap A_2^*$) to sums of matrix units in A_2 (resp. A_3) and also that the restrictions $\varphi | A_1 \cap A_1^*$ and $\psi | A_2 \cap A_2^*$ are standard embeddings. (This may be arranged by replacing φ, ψ by inner conjugate maps.) We may assume for definiteness that $A_2 = A(D_6) \otimes M_k$ and that $F_1, \dots F_6$ are the matrix units that appear in the top left position of their block. Thus a_1 can be written as

$$a_1 = \sum a_{i,j}^1 f_{i,1} F_1 f_{4k+1,j}$$

where (i,j) range over the set of indices for the block containing F_1 and so

$$\psi(a_1) = \sum a_{i,j}^1 \psi(f_{i,1}) \psi(F_1) \psi(f_{4k+1,j})$$

Since the matrix units $f_{i,1}, f_{4k+1,j}$ lie in the selfadjoint subalgebra $A_2 \cap A_2^*$, their images under ψ are sums of matrix units and the matrix above is identifiable, after conjugation by a permutation unitary, with

$$\left[\begin{array}{ccc|ccc|ccc}
\alpha_1 \otimes a_1 & & 0 & & & & \lambda_1 \otimes a_1 & & 0 \\
0 & 0 & & & & & 0 & 0 & \\
 & 0 & 0 & & & & & 0 & 0 \\
\hline
\beta_1 \otimes a_1 & & 0 & \gamma_1 \otimes a_1 & & 0 & & & \\
0 & 0 & & 0 & 0 & & & & \\
 & 0 & 0 & & 0 & 0 & & & \\
\hline
 & & & \delta_1 \otimes a_1 & & 0 & \epsilon_1 \otimes a_1 & & 0 \\
 & & & 0 & 0 & & 0 & 0 & \\
 & & & & 0 & 0 & & 0 & 0
\end{array}\right]$$

Thus, after conjugation by a permutation unitary, we can arrange $\psi \circ \varphi$ so that, for each i, $\psi \circ \varphi(E_i)$ has off-diagonal part

$$\left[\begin{array}{ccc|ccc|ccc}
\alpha_1 \otimes a_i & & \alpha_6 \otimes f_i & & & & \lambda_1 \otimes a_i & & \lambda_6 \otimes f_i \\
\alpha_2 \otimes b_i & \alpha_3 \otimes c_i & & & & & \lambda_2 \otimes b_i & \lambda_3 \otimes c_i & \\
& \alpha_4 \otimes d_i & \alpha_5 \otimes e_i & & & & & \lambda_4 \otimes d_i & \lambda_5 \otimes e_i \\
\hline
\beta_1 \otimes a_i & & \beta_6 \otimes f_i & \gamma_1 \otimes a_i & & \gamma_6 \otimes f_i & & & \\
\beta_2 \otimes b_i & \beta_3 \otimes c_i & & \gamma_2 \otimes b_i & \gamma_3 \otimes c_i & & & & \\
& \beta_4 \otimes d_i & \beta_5 \otimes e_i & & \gamma_4 \otimes d_i & \gamma_5 \otimes e_i & & & \\
\hline
& & & \delta_1 \otimes a_i & & \delta_6 \otimes f_i & \epsilon_1 \otimes a_i & & \epsilon_6 \otimes f_i \\
& & & \delta_2 \otimes b_i & \delta_3 \otimes c_i & & \epsilon_2 \otimes b_i & \epsilon_3 \otimes c_i & \\
& & & & \delta_4 \otimes d_i & \delta_5 \otimes e_i & & \epsilon_4 \otimes d_i & \epsilon_5 \otimes e_i
\end{array}\right].$$

As $(S \otimes T)^* = S^* \otimes T^*$, it follows that if S_i and T_i are partial isometries, $i = 1, 2$, and if S_1 and S_2 have orthogonal initial projections then so do $S_1 \otimes T_1$ and $S_2 \otimes T_2$. Similar statements apply for final projections and for T_1 and T_2.

In particular, consider the initial projection of $\alpha_1 \otimes a_1$. As $\psi \circ \varphi$ is rigid, the (proper) 6-cycle $E_1, \dots, E_6$ is sent to a direct sum of (proper) rank-one 6-cycles in A_3. As $\alpha_1 \otimes a_1$ is a restriction of $\psi \circ \varphi(E_1)$, its initial projection is the initial projection of some restriction of $\psi \circ \varphi(E_2)$. By the rigidity of $\psi \circ \varphi$, this latter restriction must be some combination of $\beta_1 \otimes a_2, \dots, \beta_6 \otimes f_2$. However, the initial projection of a_1 is orthogonal to those of c_2, d_2, e_2, and f_2, and α_1 has initial projection orthogonal to that of β_1, so the only possibility is that this latter restriction is a subprojection of $\beta_2 \otimes b_2$. In fact, we must have equality, since we can argue reciprocally.

Since $\alpha_1 \otimes a_1$ and $\beta_2 \otimes b_2$ have coincidental initial projections it follows that a_1 and b_2 have coincidental initial projections. Continuing in this way for $\beta_2 \otimes b_2$ and $\gamma_3 \otimes c_3$, we have that $a_1, b_2, c_3, d_4, e_5, f_6$ is a 6-cycle of partial isometries in A_2. This argument can be applied to each a_i and so the $\varphi(E_i)$ is a direct sum of 6-cycles, showing φ is rigid.

Returning to $\alpha_1 \otimes a_1$ and $\beta_2 \otimes b_2$, we also have that α_1 and β_1 have the same initial projection. Repeating the argument again, it follows that ψ is also rigid. □

The importance of the last two lemmas is that they immediately give the following theorem. In particular, invariants for regular isomorphisms of the towers (with rigid embeddings) are in fact invariant for the associated locally finite algebras.

Theorem 2.4. *Let $n \geq 3$. Suppose that*

$$A_1 \subseteq A_2 \subseteq \cdots \text{ and } A'_1 \subseteq A'_2 \subseteq \cdots$$

are towers of direct sums of $2n$-cycle algebras, where the inclusions are rigid. Then the locally finite algebras $\mathcal{A}_0 = \underrightarrow{\operatorname{alg\,lim}}\, A_k$ and $\mathcal{A}'_0 = \underrightarrow{\operatorname{alg\,lim}}\, A'_k$ are star *extendibly isomorphic by a map φ with inverse $\psi = \varphi^{-1}$ if and only there is commuting diagram*

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[dr, "\varphi_1"] & A_{m_1} \arrow[r] \arrow[dr, "\varphi_2"] & A_{m_2} \arrow[r] \arrow[dr, "\varphi_3"] & A_{m_3} \arrow[r] \arrow[dr, "\varphi_4"] & \cdots \mathcal{A}_0 \arrow[d, shift left, "\varphi{,}\psi"] \\
A_1' \arrow[r] & A_{n_1}' \arrow[r] \arrow[u, "\psi_1"'] & A_{n_2}' \arrow[r] \arrow[u, "\psi_2"'] & A_{n_3}' \arrow[r] \arrow[u, "\psi_3"'] & \cdots \mathcal{A}_0' \arrow[u, shift left]
\end{tikzcd}$$

where all the linking maps φ_k and ψ_j are rigid and $\varphi = \varinjlim \varphi_k$.

Proof. One direction is clear and for the other, assume that $\varphi : \mathcal{A} \to \mathcal{A}'$ is given. Then, since A_1 is finitely generated, there exists n_1 and m_1 so that

$$\varphi(A_1) \subseteq A_{n_1}' \text{ and } \psi(A_{n_1}') \subseteq A_{m_1}.$$

The restriction embeddings, $\varphi_1 : A_1 \to A_{n_1}'$ and $\psi_1 : A_{n_1}' \to A_{m_1}$, are star extendible and by hypothesis, have a rigid composition. Now Lemmas 2.1 and 2.3 apply, to show that φ_1 and ψ_1 are rigid. Continuing in this way, we obtain the required diagram. $\square$

3. Approximate Factorisations

To classify norm-closed direct limits, we need approximate versions of Lemmas 2.1 and 2.3. The latter will be used to define homology invariants for operator algebra direct limits. Readers interested only in the classification of algebraic direct limits may go to Section 4. For clarity and convenience, we have chosen not to determine the absolute dependence of δ upon ϵ and n in the proofs below.

Lemma 3.1. *Suppose $\epsilon > 0$ and $a = (a_{ij})$ is a block matrix in a $2n$-cycle algebra A, $n \geq 3$. There is constant $\delta > 0$ so that if a is δ-close to a partial isometry with initial and final projections in A, then each entry a_{ij} is ϵ-close to a partial isometry.*

The proof is routine.

Lemma 3.2. *Let $n \geq 3$, A_1, A_2, A_3 be $2n$-cycle algebras, and $\epsilon > 0$. Let $\varphi : A_1 \to A_2$ and $\psi : A_2 \to A_3$ be linear injections which are δ-close to star-extendible embeddings and are such that the composition $\psi \circ \varphi$ is δ-close to a rigid embedding, $i : A_1 \to A_3$. Then there is a sufficiently small δ, depending on ϵ and n, so that φ and ψ are ϵ-close to rigid embeddings.*

Proof. First observe that A_1 admits a direct sum decomposition $A_1 = A_1 \cap A_1^* + \mathrm{rad}\,(A_1)$ and $\varphi(a_1 + a_2) = \varphi_1(a_1) + \varphi_2(a_2)$ where φ_1 maps $A_1 \cap A_1^*$ to A_2 and φ_2 maps $\mathrm{rad}\,(A_1)$ to A_2. By standard finite-dimensional C*-algebra theory [1, Chapter III] φ_1 is close to a star-extendible injection, so we may replace it by a nearby C*-algebra homomorphism $\varphi_1' : A_1 \cap A_1^* \to A_2 \cap A_2^*$. Similarly, we may assume the restriction of ψ to $A_2 \cap A_2^*$ is a C*-algebra homomorphism.

Assume now that the sets of matrix units e_{ij}^k, $1 \le k \le 3$, have been chosen for A_1, A_2, and A_3 so that φ and ψ map matrix units of $A_1 \cap A_1^*$ and $A_2 \cap A_2^*$ to sums of matrix units. Since φ is close to star-extendible injection, $\varphi(e_{ij}^1)$ is close to $v_{ij}^1 = \varphi(e_{ii}^1)\varphi(e_{ij}^1)\varphi(e_{jj}^1)$. Let φ' be the linear map for which $\varphi'(e_{ij}^1) = v_{ij}^1$. Then φ' agrees with φ on $A_1 \cap A_1^*$ and φ' is a C_1-bimodule map, where C_1 is the diagonal masa spanned by $\{e_{ii}^1\}$ in A_1. Define ψ' similarly. Since φ and ψ are close to φ' and ψ', we need only prove the lemma for such maps.

Assume that $n = 3$; this simplifies the notation. The same argument applies for $n > 3$, with a smaller δ.

We can now write, as in the proof of Lemma 2.3,

$$\varphi(E_i) = \begin{bmatrix} a_i & & f_i \\ b_i & c_i & \\ & d_i & e_i \end{bmatrix}, \qquad \psi(F_i) = \begin{bmatrix} \alpha_i & & \lambda_i \\ \beta_i & \gamma_i & \\ & \delta_i & \epsilon_i \end{bmatrix},$$

for $i = 1, 2, \dots, 6$, where the E_i and F_i are rank-one six cycles in A_1 and A_2 respectively. These operators are close to partial isometries and, moreover, by assumption, φ and ψ are close to star-extendible homomorphisms. By Lemma 3.1, each of the entries of each $\varphi(E_i)$ and $\psi(F_i)$ is close to a partial isometry. Thus, we may assume that each of these entries is a partial isometry and that we still have φ, ψ close to star-extendible homomorphisms. Since $\beta_1\alpha_1^*$ has small norm (as $\psi(F_1)$ is close to a partial isometry) and $\psi \circ \varphi$ is close to a rigid map, it follows that the initial projection of $\alpha_1 \otimes a_1$ is close to the initial projection of $\beta_2 \otimes b_2$ and hence that the initial projection of a_1 is close to the initial projection of b_2.

Now redefine φ to obtain a new φ for which, in addition to our earlier assumptions, the initial and final projections of a_1, b_2, c_3, d_4, e_5 match up. Also, redefine f_6 as $a_1 b_2^* c_3 d_4^* e_5$, which is close to the original f_6 because φ is close to a star-extendible embedding. The map φ now has a summand, determined by the 6-cycle $a_1, \dots, f_6$, which is a rigid algebra homomorphism. Continuing in this way, we can construct a rigid algebra homomorphism which is close to the original φ. Performing similar constructions with ψ, it follows that ψ is also close to a rigid embedding. □

Theorem 3.3. *Let $n \ge 3$. Suppose that*

$$A_1 \subseteq A_2 \subseteq \cdots \text{ and } A_1' \subseteq A_2' \subseteq \cdots$$

are towers of direct sums of $2n$-cycle algebras, where the inclusions are rigid. Then the operator algebras $\mathcal{A} = \varinjlim A_k$ and $\mathcal{A}' = \varinjlim A_k'$ are star extendibly *isomorphic if and only if there is commuting diagram*

$$\begin{array}{ccccccccccc}
A_1 & \longrightarrow & A_{m_1} & \longrightarrow & A_{m_2} & \longrightarrow & A_{m_3} & \longrightarrow & A_{m_4} & \longrightarrow & \cdots \mathcal{A} \\
& \searrow \varphi_1 & \big\uparrow \psi_1 & \searrow \varphi_2 & \big\uparrow \psi_2 & \searrow \varphi_3 & \big\uparrow \psi_3 & \searrow \varphi_4 & \big\uparrow \psi_4 & \searrow \varphi_5 & \\
A'_1 & \longrightarrow & A'_{n_1} & \longrightarrow & A'_{n_2} & \longrightarrow & A'_{n_3} & \longrightarrow & A'_{n_4} & \longrightarrow & \cdots \mathcal{A}'
\end{array}$$

where all the linking maps φ_k and ψ_j are rigid.

Proof. Let $\Phi : \mathcal{A} \to \mathcal{A}'$ be a star-extendible isomorphism. Then the restriction $\varphi_0 = \Phi|_{A_1}$ is close to a linear injection $\varphi : A_1 \to A'_k$, for suitably large k. Similarly, $\psi_0 = \Phi^{-1}|_{A'_k}$ is close to a linear injection $\psi : A'_k \to A_l$ for suitably large l, and the composition, $\psi \circ \varphi$ is close to the given rigid embedding $A_1 \to A_l$.

We cannot immediately apply Lemma 3.2 as stated, since we only know that φ is to close to a star-extendible embedding from A_1 to $\mathcal{A}'$, and not one from A_1 to A'_k. However, the same proof applies to this case, so we can apply the lemma. Thus, φ and ψ are close to rigid embeddings, φ_1 and ψ_1 say. Since $\psi_1 \circ \varphi_1$ is close to the given rigid embedding, it follows that they have the same induced maps on K_0 and H_1 and so by Lemma 4.2 below are inner conjugate. Adjusting φ_1 by a unitary in A_{n_1}, we obtain a commuting triangle. Continuing in the usual way, we can build the required diagram. □

The argument above shows more than we have stated, namely that each star-extendible isomorphism Φ from $\mathcal{A}$ to $\mathcal{A}'$ determines a star-extendible isomorphism $\{\varphi_k, \psi_k\}$ for the towers, which is unique up to an approximately inner automorphism, that is, to a pointwise limit of unitary automorphisms. Finally, if we define $H_1\mathcal{A}$ to be the limit group $H_1\mathcal{A}_0$ for the locally finite algebras $\mathcal{A}_0$, then we immediately have the following corollary.

Corollary 3.4. *The abelian group $H_1\mathcal{A}$ is well-defined and is an invariant for star-extendible isomorphisms.*

4. Invariants and Classification

We now develop the invariants and prove the main theorem.

To each rigid embedding between $2m$-cycle algebras we may associate an ordered $2m$-tuple, $(r_1, \dots, r_{2m})$, which we call the *multiplicity signature*, which is the set of multiplicities of the $2m$ classes of multiplicity one embeddings in the direct sum decomposition of φ. This signature depends on the identification of the reduced digraphs of A_1 and A_2 with D_{2m} and the labeling of the automorphisms of D_{2m}. Plainly, embeddings φ, φ' are conjugate (inner unitarily equivalent) if and only if they have the same multiplicity signature.

For the scaled K_0-group classification of limits of finite-dimensional semisimple algebras, the cornerstone lemma is that two embeddings between such algebras are conjugate if and only if they induce the same K_0-group homomorphism. Not surprisingly, the K_0-group homomorphism does not determine the conjugacy class of embeddings between $2m$-cycle algebras (see also the discussion following Lemma 4.2 below). However, if $H_1\varphi$ denotes the group homomorphism $\mathbb{Z} \to \mathbb{Z}$ given by multiplication by

$$r_1 - r_2 + r_3 - \cdots - r_{2m}$$

then, as we see below, $K_0\varphi$ and $H_1\varphi$ together do determine the multiplicity signature of φ and hence the conjugacy class of φ.

It is natural then to seek a complete classification of the locally finite algebras of Theorem 2.4 in terms of K_0A and the abelian group

$$H_1A := \varinjlim (\mathbb{Z}, H_1\alpha_i)$$

That this group is indeed an invariant for star-extendible isomorphism follows from Theorem 2.4.

It has already been made clear, in the consideration of 4-cycle algebras in Donsig and Power [4] and in Power [9], that, beyond the scaled ordered group K_0A and the abelian group H_1A, it is necessary to consider a number of other invariants. For some subfamilies of algebras, these additional invariants simplify or vanish but in the most general case, the appropriate invariant is a joint scale, ΣA in $K_0A \oplus H_1A$, which embodies these additional invariants.

***Definition* 4.1.** Let A be a locally finite algebra, as in Theorem 2.4. Then the *joint scale*, ΣA, is the subset of $K_0A \oplus H_1A$ given by the elements of the form

$$(K_0\varphi(e_{11} \oplus e_{22}), H_1\varphi(g))$$

where $\varphi : A(D_{2m}) \to A$ is a rigid embedding and g is a fixed generator for $H_1(A(D_{2m})) = \mathbb{Z}$. Furthermore, we define the unital joint scale to be the subset arising from unital embeddings φ.

It is possible to give an intrinsic formulation of H_1A for a digraph algebra A and hence an intrinsic formulation of $H_1\varphi$, although we do not do so here.

Lemma 4.2. *If A is $2m$-cycle algebra and φ, ψ are rigid embeddings from $A(D_{2m})$ to A with*

$$(K_0\varphi(e_{11} \oplus e_{22}), H_1\varphi(g)) = (K_0\psi(e_{11} \oplus e_{22}), H_1\psi(g)),$$

then φ and ψ are conjugate.

This lemma has essentially the same proof as the analogous 4-cycle result [4, Lemma 11.4]; see also [8, Proposition 3.4].

It is instructive to note the possible variation in $H_1\varphi$ once $K_0\varphi$ is specified. If φ has multiplicity signature $(r_1, \ldots, r_{2m})$, and $K_0\varphi = K_0\psi$, then for suitable integers k, ψ has multiplicity signature

$$(r_1 + k, r_2 - k, r_3 + k, \ldots, r_{2m} - k)$$

and $H_1\psi = H_1\varphi + [2mk]$. The range for k is $-M_2 \le k \le M_1$ where M_1 and M_2 are the maximum and minimum values determining nonnegative signature. The set

$$\mathrm{hr}\,(\varphi) = \{H_1\psi : \psi \text{ rigid and } K_0\psi = K_0\varphi\}$$

is called the *homology range* of φ and is realized as an interval in the mod $2m$ congruence class $M_1 + 2m\mathbb{Z}$. These remarks already suggest that there can be congruence class obstructions to the lifting of K_0H_1 group isomorphisms

$$\gamma_0 \oplus \gamma_1 : K_0A \oplus H_1A \to K_0A' \oplus H_1A'$$

to algebra isomorphisms. (For further discussions in the case of 4-cycle algebras see [4].)

***Definition* 4.3.** Let $n \ge 3$ and suppose that

$$A_1 \subseteq A_2 \subseteq \cdots \text{ and } A_1' \subseteq A_2' \subseteq \cdots$$

are towers of direct sums of $2n$-cycle algebras, where the inclusions are rigid. Letting $\mathcal{A} = \varinjlim A_k$ and $\mathcal{A}' = \varinjlim A_k'$, we say that a scaled, ordered group isomorphism

$$\gamma_0 : K_0\mathcal{A} \to K_0\mathcal{A}'$$

is *of rigid type* if there is some rigid isomorphism $\Gamma : \mathcal{A} \to \mathcal{A}'$ so that $K_0\Gamma = \gamma_0$. Equivalently γ_0 is induced by a commuting diagram of rigid embeddings (as in Theorem 2.4).

At the finite-dimensional level, a K_0-group homomorphism, of 6-cycle algebras is of rigid type if it can be realized as a sum of two integral matrices of the form:

$$\begin{bmatrix} a & b & c & 0 & 0 & 0 \\ c & a & b & 0 & 0 & 0 \\ b & c & a & 0 & 0 & 0 \\ 0 & 0 & 0 & a & b & c \\ 0 & 0 & 0 & c & a & b \\ 0 & 0 & 0 & b & c & a \end{bmatrix} + \begin{bmatrix} d & e & f & 0 & 0 & 0 \\ e & f & d & 0 & 0 & 0 \\ f & d & e & 0 & 0 & 0 \\ 0 & 0 & 0 & e & f & d \\ 0 & 0 & 0 & f & d & e \\ 0 & 0 & 0 & d & e & f \end{bmatrix}$$

Similar sums describe the inclusions for larger $2n$-cycle algebras.

We can now obtain the main theorem of the paper, which (but for the case of even 4-cycle systems) completes the classification of operator algebras begun in Power [7, Chapter 11] and continued in Donsig and Power [4] and in Power [9].

Theorem 4.4. *Let $n \geq 3$ and suppose that*

$$A_1 \subseteq A_2 \subseteq \cdots \text{ and } A_1' \subseteq A_2' \subseteq \cdots$$

are towers of direct sums of $2n$-cycle algebras, where the inclusions are rigid. Then the following conditions are equivalent:

(a) *There is a commuting diagram*

$$\begin{array}{ccccccccc} A_1 & \longrightarrow & A_{m_1} & \longrightarrow & A_{m_2} & \longrightarrow & A_{m_3} & \longrightarrow & \cdots \mathcal{A} \\ & \searrow \varphi_1 & \uparrow \psi_1 & \searrow \varphi_2 & \uparrow \psi_2 & \searrow \varphi_3 & \uparrow \psi_3 & \searrow \varphi_4 & \updownarrow \varphi,\psi \\ A_1' & \longrightarrow & A_{n_1}' & \longrightarrow & A_{n_2}' & \longrightarrow & A_{n_3}' & \longrightarrow & \cdots \mathcal{A}' \end{array}$$

where all the linking maps φ_i and ψ_j are rigid.

(b) *The locally finite algebras $\mathcal{A}_0 = \operatorname{alg}\varinjlim A_k$ and $\mathcal{A}_0' = \operatorname{alg}\varinjlim A_k'$ are star-extendibly isomorphic.*

(c) *The operator algebras $\mathcal{A} = \varinjlim A_k$ and $\mathcal{A}' = \varinjlim A_k'$ are star extendibly isomorphic.*

(d) *There is an abelian group isomorphism*

$$\gamma_0 \oplus \gamma_1 : K_0\mathcal{A} \oplus H_1\mathcal{A} \to K_0\mathcal{A}' \oplus H_1\mathcal{A}',$$

where γ_0 is a scaled, ordered group isomorphism of rigid type and $\gamma_0 \oplus \gamma_1$ preserves the joint scale. Also, in the unital case it suffices to preserve the unital joint scale.

Proof. From Theorems 2.4 and 3.3, we know that the first three conditions are equivalent. Clearly, the first implies the fourth, so it remains only to show that the fourth implies the first. The argument follows a familar scheme, seen for example in [4, Theorem 11.5] and [9, Theorem 5.2], but for completeness, we outline the argument.

Consider a multiplicity one rigid embedding $\varphi : A(D_{2n}) \to A_1$, which determines an element $([\varphi(e_{11} \oplus e_{22})], \delta)$ of the joint scale of A_1. By condition (d), there is some rigid embedding $\theta : A(D_{2n}) \to A_k'$, for some k, so that

$$\gamma_0 \oplus \gamma_1([\varphi(e_{11} \oplus e_{22})], \delta) = ([\theta(e_{11} \oplus e_{22})], H_1\theta(g)).$$

Letting $\eta = \theta \circ \varphi^{-1}$, we obtain a rigid embedding $\eta : \varphi(A(D_{2n})) \to A_k'$. We may assume that matrix units have been chosen for A_k' so that η maps matrix units to sums of matrix units.

We claim that η has an extension $A_1 \to A_k'$ (after possibly increasing k), which we also denote η. To see this first observe that since γ_0 is a scaled group isomorphism there is a C*-algebra homomorphism

$$\eta_0 : A_1 \cap A_1^* \to A_k' \cap (A_k')^*$$

so that $K_0\eta_0$ is equal to γ_0 restricted to K_0A_1 and, moreover, η_0 maps matrix units to sums of matrix units. Since each matrix unit $e \in A_1$ admits a unique factorization $e = e_1 f e_2$ where $e_1, e_2 \in A_1 \cap A_1^*$ and f is in $\varphi(A(D_{2n}))$ it follows that there is a unique star extendible extension of η and η_0 to a rigid embedding $A_1 \to A'_k$. Although η depends on φ and η_0 the inner conjugacy class of η is determined.

Next, consider a multiplicity one rigid embedding $\psi : A(D_{2n}) \to A'_k$ which induces an element of the joint scale $([\psi(e_{11} \oplus e_{22})], H_1\psi(g))$. We assume for convenience that matrix units have been chosen for A'_k so that ψ maps matrix units to sums of matrix units. As with η, there is some rigid embedding $\zeta : \psi(A(D_{2n})) \to A_l$, for some l, so that

$$\gamma_0^{-1} \oplus \gamma_1^{-1}([\psi(e_{11} \oplus e_{22})], H_1\psi(g)) = ([\zeta(e_{11} \oplus e_{22})], H_1\zeta(g)).$$

We may assume that matrix units have been chosen for A_l so that ζ maps matrix units to sums of matrix units.

As with η, ζ has a star extendible extension $A'_k \to A_l$ (after possibly increasing l), which we also denote ζ. Since $\zeta \circ \eta$ and the inclusion map from A_1 to A_l induce the same $K_0 \oplus H_1$ map, we may replace ζ by an inner conjugate map so that the algebra maps are equal. Continuing in this way, we can construct the required diagram. $\square$

For an elementary illustration of the theorem we now consider unital limit algebras for which $A \cap A^*$ is a direct sum of two UHF C*-algebras. We say such algebras are *of matroid type*. For a fixed positive integer d consider the stationary systems of the form

$$A(D_6) \longrightarrow A(D_6) \otimes M_{3d} \longrightarrow A(D_6) \otimes M_{(3d)^2} \longrightarrow \cdots$$

where the k-th embedding α_k satisfies

$$K_0(\alpha_k) = d \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}, \; H_1(\alpha_k) = [s].$$

There are $d+1$ possible values of s, namely the integers that lie in the homology range

$$S = \{-3d, -3d+6, \ldots, 3d-6, 3d\}$$

and so there are $d+1$ associated operator algebras $A_s, s \in S$.

The group $K_0(A_s)$ identifies naturally with the subgroup of $\mathbb{Q} \oplus \mathbb{Q}$ corresponding to the generalised integer $3^\infty d^\infty$ with order unit $1 \oplus 1$ whilst $H_1(A_s)$

identifies naturally with the subgroup determined by s^∞. In the exceptional case $s = 0$, $H_1(A_s) = 0$ and so the joint scale then reduces to the usual scale in $K_0(A_s)$.

In general therefore, the unital part of the joint scale is the subset of

$$\mathbb{Z}\left[\frac{1}{3^\infty d^\infty}\right] \oplus \mathbb{Z}\left[\frac{1}{3^\infty d^\infty}\right] \oplus \mathbb{Z}\left[\frac{1}{s^\infty}\right]$$

arising from unital rigid embeddings $\varphi : A(D_6) \to A_s$. In the case of the extreme values of s the algebras A_{3d}, A_{-3d}, are isomorphic and the unital joint scale coincides with the subset of elements $1/3 \oplus 1/3 \oplus h$ where for some positive integer m one has $h = k/(3d)^m$ where $k \equiv (3d)^m \bmod 6$ and $-(3d)^m \leq k \leq (3d)^m$. In particular the homology component of the unital joint scale is a symmetric set in a symmetric finite interval of $H_1 A_s$. In contrast, in the nonextreme case, $s \neq 3d, -3d$, the unital joint scale is identified with the set of elements for which h merely satisfies the congruence restriction. (In our simple example the congruence restriction has no consequence if d is even whilst if d is odd then the homology part of the unital joint scale corresponds to odd numerators.) Indeed, for such an element h in $\mathbb{Z}[1/s^\infty]$ we may choose t large enough so that

$$h = \frac{k}{(3s)^m} = \frac{k'}{(3s)^t}$$

where $k' \leq (3d)^t$. Then there is a unital embedding

$$\psi : A(D_6) \to A(D_6) \otimes M_{(3d)^t}$$

with $H(\psi) = [k']$, and so ψ determines the element $1/3 \oplus 1/3 \oplus h$.

The H_1 group is already a distinguishing invariant for A_{s_1}, A_{s_2} if s_1 and s_2 have different prime divisors. On the other hand if s_1 and s_2 are nonextreme and have the same prime divisors then one can verify that the unital joint scales coincide and so, by the theorem, A_{s_1}, A_{s_2} are isomorphic.

It follows from similar observations that if A, A' are unital stationary limits of $2n$-cycle algebras, of nonextreme type, then A, A' are star extendibly isomorphic if and only if $C^*(A)$ and $C^*(A')$ are isomorphic and $H_1(A)$ and $H_1(A')$ are isomorphic abelian groups. In fact the same conclusion is true for the more general limits of "homologically limited" systems of matroid type, that is, those for which the natural scale of H_1 is not a finite interval but, as above, coincides with $H_1(A)$.

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