Physics, Chapter 3: The Equilibrium of a Particle

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The Equilibrium of a Particle

3-1 Equilibrium

A particle which remains at rest or in uniform motion with respect to its frame of reference is said to be in *equilibrium* in that frame. Centuries ago it was recognized that the state of rest was a *natural* state of things, for it was observed that objects set in motion on the surface of the earth tended to come to rest. The maintenance of any horizontal motion on earth was thought to require the continued exercise of a force, hence to be a *violent* motion, while vertical motion like that of a falling body was thought to be *natural*. In heavenly bodies circular motion was thought to be natural. That uniform motion in a straight line was a universal equilibrium condition, a natural state of things, was not recognized until the work of Galileo (1564–1642) and Newton (1642–1727), which represented a very significant contribution to the study of mechanics and to our understanding of nature.

Newton summarized his conception of motion in three principles, which are today called *Newton's laws of motion,* the first of which may be stated as follows: *A body at rest will remain at rest, and a body in motion will continue in motion with constant speed in a straight line, as long as no net force acts upon the body.*

Although, as the result of much repetition, Newton's first law may today seem to be another trite statement, the result of simple common sense, it was indeed a very startling conception. No one of us has ever seen an object which moved with constant speed in a straight line for an infinite length of time either on the earth or in the heavens. Nevertheless the Newtonian formulation of the conditions of equilibrium has proved itself invaluable in our understanding of nature and is universally accepted as the basis for the formulation of an important division of mechanics. The experimental validity of the Newtonian formulation of equilibrium is reestablished each time a new structure is erected, each time an airplane flies.

3-2 Equilibrium of a Particle

According to Newton's first law, a particle is said to be in equilibrium if there is no net force acting on it. This does not mean that no forces act on the particle, but rather that the resultant of all the forces which do act on the particle is zero. The direction in which a force acts is an important fact needed for its specification. Force is, therefore, a vector quantity, and the resultant of the forces must be obtained by vector methods. If we label the forces acting on the particle by **A**, **B**, **C**, . . . , the condition for the equilibrium of a particle may be written in the form of an equation as

$$\boxed{\mathbf{R} = \mathbf{A} + \mathbf{B} + \mathbf{C} + \cdots = 0,} \tag{3-1}$$

where **R** is the resultant of the forces acting on the particle.

We have already seen that a vector equation may be resolved into three independent scalar equations, one for each of three mutually perpendicular directions, so that Equation (3-1) may be written as

$$R_x = 0 = A_x + B_x + C_x + \cdots , \tag{3-2a}$$

$$R_y = 0 = A_y + B_y + C_y + \cdots , \tag{3-2b}$$

$$R_z = 0 = A_z + B_z + C_z + \cdots , \tag{3-2c}$$

in which A_x, B_x, C_x, . . . are the x components of the forces **A**, **B**, **C**, . . . , respectively, and A_y, B_y, C_y, . . . and A_z, B_z, C_z, . . . are the y and z components of these forces, respectively.

We shall usually restrict our discussion to the equilibrium of bodies which are acted upon by forces all of which are in one plane. In such cases Equation (3-2c) is redundant, and the conditions for the equilibrium of a particle become

$$\boxed{\begin{aligned} R_x &= 0, && \text{(3-2a)} \\ R_y &= 0. && \text{(3-2b)} \end{aligned}}$$

The word *particle* has been used rather loosely to imply a small body on which the forces act *concurrently;* that is, all forces are directed toward a single point. At times the statements relating to the statics of a particle will be applied to larger bodies when these move in *translational* motion; that is, when there is no rotation of the body itself.

When we examine the equilibrium of a particle, we must be careful to isolate the particle in our minds and to examine the forces exerted on the particle by each object capable of exerting a force on it. We shall see that the conditions for the equilibrium of a particle will enable us to determine the forces exerted by many structural elements which make up a complicated assembly and thereby provide information essential to the design of complex structures.

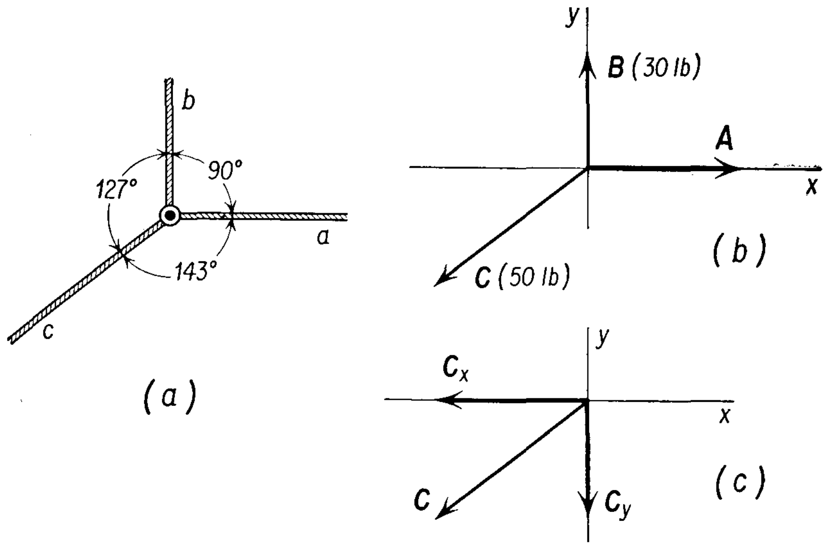

Fig. 3-1

Illustrative Example. Three strings are tied to a small ring (here considered to be a particle), as shown in Figure 3-1(a). The string b exerts a force of 30 lb upon the ring; the string c exerts a force of 50 lb upon the ring. What force must be exerted by the string a if the ring is to remain in equilibrium?

From experience we know that a taut string exerts a force directed along its own length. In Figure 3-1(b), the ring is placed at the origin of an x-y coordinate system, and the strings are replaced by the forces that they exert on the ring; the force of the string b is represented by the symbol **B**, and the force of string c is represented by the symbol **C**. While forces **B** and **C** are drawn to scale, the force **A** is not, since its value is not known.

To apply Equations (3-2) to the solution of the problem, we must resolve the force **C** into its x and y components C_x and C_y. From Figure 3-1(c) we see that

$$C_x = -C \cos 37° = -50 \text{ lb} \times 0.8 = -40 \text{ lb},$$

$$C_y = -C \cos 53° = -50 \text{ lb} \times 0.6 = -30 \text{ lb},$$

and, substituting numerical values into Equations (3-2), we have

$$R_x = A - 40 \text{ lb} = 0,$$

$$R_y = +30 \text{ lb} - 30 \text{ lb} = 0.$$

Thus the ring is in equilibrium in the y direction under the action of the applied forces and will be in equilibrium in the x direction if $R_x = 0$; that is, if $A = 40$ lb.

3-3 Newton's Third Law

Newton's third law of motion states that *whenever one body exerts a force on another, the second body exerts a force equal in magnitude and opposite in direction on the first body.* This law is sometimes called the *law of action and reaction.* In the solution of problems in statics, this concept is applied extensively. We focus our attention first on the equilibrium of one point in

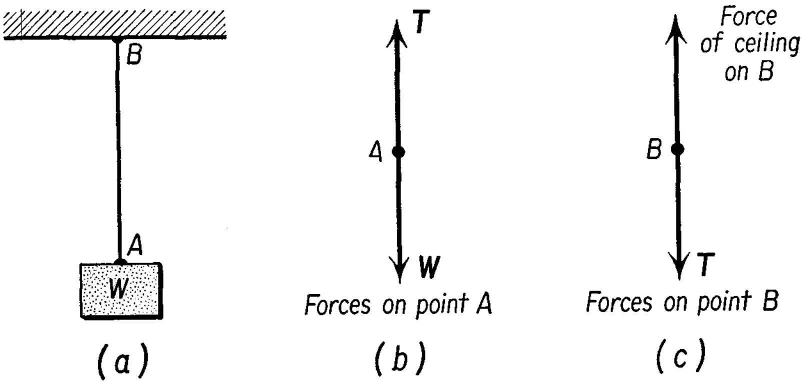

Fig. 3-2

the structure, then on a second, and a third, and so on, to find all the forces exerted on or by each member of the structure. If we pull against a rope, our hands exert a force on the rope, but the rope exerts an equal and opposite force on our hands. When a packing crate rests on the floor, the crate exerts a force on the floor and the floor exerts an equal and opposite force on the crate.

Consider the forces associated with a hanging weight. If a body of weight W is supported by a rope fastened to the ceiling of a room, as in Figure 3-2(a), we can determine the tension in the rope by considering the equilibrium of a particle of the rope at point A. To do this we draw a second diagram called a *force diagram,* to distinguish it from the sketch illustrating the appearance of the system which we call the *space diagram.* The forces on A are the tension $\mathbf{T}$ in the rope and the weight of the body, as shown in the force diagram, Figure 3-2(b). Since the force of gravity is vertically downward, the rope must be in the vertical direction. The force $\mathbf{T}$ at A must be equal in magnitude to $\mathbf{W}$ but opposite in direction.

To determine the force that the ceiling must exert on the rope, we apply the results obtained from examining the equilibrium of point A to the examination of the equilibrium of an element of the rope at point B. The forces on this element are the tension in the rope $\mathbf{T}$ which at B is directed vertically downward, and the force of the ceiling of unknown magnitude and direction; see Figure 3-2(c). Since the element at B is in equilibrium, the sum of all forces acting on B must be zero, and the force exerted by the ceiling must be vertically upward, and in magnitude equal to T, hence equal to W.

3-4 Solution of Simple Problems in Statics

Illustrative Example. A *boom,* or a *strut,* whose weight can be neglected, is a typical example of a pinned beam; that is, the beam is connected to the other part of the structure by means of a strong pin or rod which passes freely through it. A beam at rest which is pinned at one end can exert a force either in tension or in compression (that is, a push or a pull), but this force too can only be exerted in a direction parallel to the beam, for if the beam is pinned and can rotate freely,

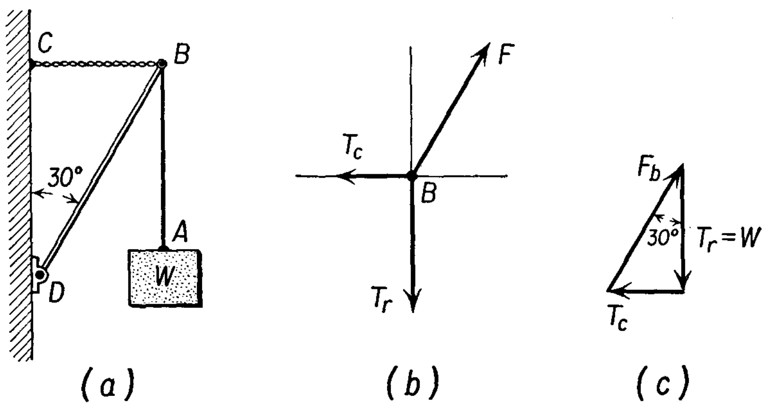

Fig. 3-3

any force perpendicular to its own length would cause it to rotate. Suppose a weight **W** of 60 lb hangs from a rope supported at the outer end of a boom of negligible weight, as in Figure 3-3(a). The boom is pinned at its lower end and is supported at its upper end by a horizontal chain fastened to the wall. To determine the forces exerted by each member, we first isolate a small segment of the rope near point A; as we have already seen, the tension in the rope $\mathbf{T}_r$ must be equal in magnitude to the weight W. To examine the equilibrium at point B, we draw a force diagram as in Figure 3-3(b), with the forces acting on the point B drawn as though they radiate out from a common origin. Since both the rope and chain can exert only tension forces along their own lengths, we label these forces $\mathbf{T}_r$ and $\mathbf{T}_c$ (for the tension in the chain), respectively, and draw them in appropriate directions. A pinned beam can exert a force only along its own length. The beam is in compression, and the force exerted by the beam must be a thrust away from the wall. The force exerted by the beam is **F**, as indicated in the figure. Since the point B is in equilibrium, the vector sum of all these forces acting on B must be zero.

Solving Equation (3-1) by the polygon method, we observe that the three force vectors, added in sequence head to tail, must form a closed triangle, for the resultant of the three forces, the vector drawn from the tail of the first to the head of the last, must be of length zero. The force vectors are parallel to their respective structural members; appropriate angles may therefore be identified on the force diagram from the space diagram; thus $\mathbf{F}$ makes an angle of 30° with the vertical force $\mathbf{T}_r$, and the vector $\mathbf{T}_r$ is perpendicular to $\mathbf{T}_c$. With this information the methods of trigonometry enable us to solve the problem. From Figure 3-3(c) we have

$$\frac{T_r}{F} = \cos 30°;$$

thus

$$F = \frac{T_r}{\cos 30°} = \frac{60\text{ lb}}{0.866} = 69.4\text{ lb}.$$

Furthermore,

$$\frac{T_c}{T_r} = \tan 30°;$$

thus

$$T_c = T_r \tan 30° = 60\text{ lb} \times 0.577 = 34.6\text{ lb}.$$

We have found the forces exerted by the three structural elements and have therefore obtained a complete solution to the problem. Since, by Newton's third law, the force exerted by each of these elements is equal and opposite to the force exerted on it, knowledge of the strength of materials enables the engineer to design a suitable structure.

Illustrative Example. A string is passed over two pulleys, and weights of 30 lb and 40 lb are hung from the ends. When a weight of 50 lb is hung on the string anywhere between the two pulleys, it is found that the angle made by the two parts of the string supporting the 50-lb weight is 90°, as shown in Figure 3-4(a), no matter where the weight is placed. The angle does not change when the pulleys are raised or lowered with respect to each other. Explain.

Examination of the equilibrium of the element of string at point A shows that the tension in the string at this point must be 30 lb. *The tension in the string is not altered when the string passes over a frictionless pulley.* A frictionless pulley is one in which the bearing of the pulley is perfectly smooth, although the surface of the pulley wheel may be quite rough. The tension in the string $\overline{BC}$ must be 30 lb. Similarly, the tension in the string $\overline{CD}$ must be 40 lb. At the point C there is an abrupt change in the direction of the string, as there must be when the tension is different in two parts of a string.

Consider the equilibrium of a particle at the point C. Three forces act on point C: namely, the two tensions in the strings and the weight of 50 lb. Since the point C is in equilibrium, the vector sum of these forces must equal zero. The force diagram is shown in Figure 3-4(b). The force polygon, drawn as Figure 3-4(c), is a 3–4–5 triangle, hence a right triangle, with the right angle between the 30-lb vector and the 40-lb vector. The directions $\overline{BC}$ and $\overline{CD}$ are parallel to the

30-lb and the 40-lb vectors, respectively. By a theorem in geometry which states that two angles whose sides are mutually perpendicular are either equal or supplementary, the angle BCD must be 90°.

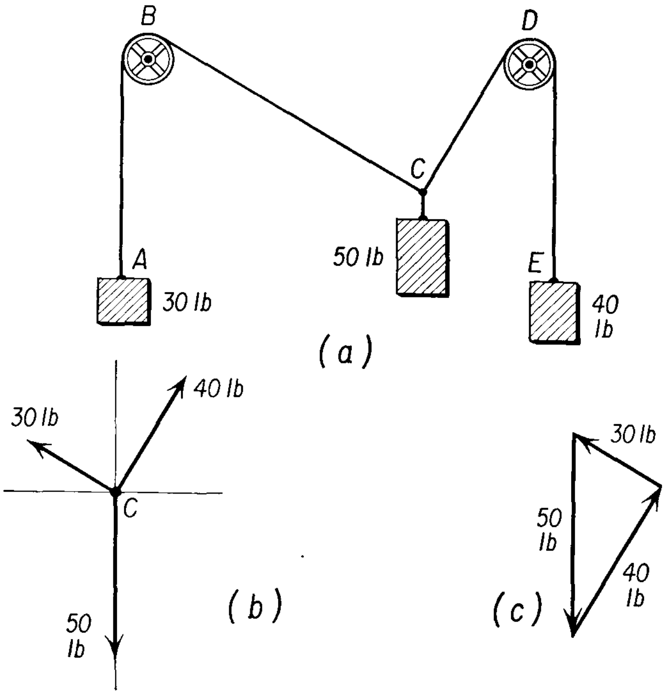

Fig. 3-4

We note again that the force diagram and the space diagram are not drawn to the same scale; distances on the force diagram do not necessarily transfer to the space diagram. Angles from the space diagram may be identified on the force diagram, because structural members often exert forces which bear a simple relationship to the directions of these members.

3-5 Frictional Forces

It is common experience that moving objects on the surface of the earth do not continue to move in a straight line with uniform speed, and that a force must be applied to maintain uniform motion. As part of our attempt to develop a consistent picture of nature, we have developed the concept of friction. The frictional force is conceived as a force that opposes the motion; this force must be equal in magnitude to the applied force required to keep a body sliding over a surface with uniform speed.

Friction between solid bodies can be classified into two types, *sliding friction* and *rolling friction.* Sliding friction occurs whenever the surface of one body slides over another; this kind of friction exists between the brake lining and the brake drum of the braking mechanism on the wheels of a car. Rolling friction exists between the wheels of a car and the road when no slippage occurs. It is common experience that considerably more effort is required to start a heavy object sliding across a floor than is needed to keep it moving once it has been started. We may thus distinguish between the force of *static* friction required to start an object in motion and the force of *kinetic* friction required to keep an object in motion, for they have different magnitudes. The results of many experiments show that, to a good approximation, the magnitude of the frictional force does not depend on the area of contact between the two surfaces but depends upon the nature of the surfaces and upon the force pressing them together. Calling the magnitude of the frictional force F_r and the magnitude of the normal force with which one surface presses against another N, we write

$$\boxed{F_r = fN,} \tag{3-3}$$

were f is a dimensionless constant of proportionality called the *coefficient of friction.* Some typical values of the coefficient of kinetic or sliding friction are shown in Table 3-1. While values in the table are all less than 1, there is no fundamental reason why the coefficient of friction should not have a value greater than 1. The coefficient of static friction is higher than the coefficient of kinetic friction.

TABLE 3-1 COEFFICIENTS OF SLIDING OR KINETIC FRICTION

Materials	
Wood on wood, dry	0.25–0.50
Metal on oak, dry	0.50
Leather on metal, dry	0.56
Metal on metal, dry	0.15–0.20
Steel on agate, dry	0.20
Masonry on clay, dry	0.51

The frictional force always acts in such a direction as to oppose the motion of one surface relative to another. When an object is in motion, the force of kinetic friction given by Equation (3-3) is always present. This is not true for static friction. The coefficient of static friction gives the maximum value of the frictional force—the force which must be applied to start the motion. As long as the object is at rest, the frictional force may take on any value (up to that maximum) necessary to fulfill the conditions of equilibrium (see the second *Illustrative Example* of Section 3-6 and Figure 3-6).

In screw-thread fastening devices, the effect of tightening a bolt against the work, or against a spring washer, is to increase the normal force with which the thread surfaces of the nut bear against the thread surfaces of the bolt. This increases the frictional force between the nut and bolt, as shown by Equation (3-3), and helps keep the nut from untwisting.

Illustrative Example. A steel block weighing 175 lb is pulled horizontally with uniform speed over another steel block. If the coefficient of kinetic friction between the two surfaces is 0.20, determine the force of friction between them.

The force pressing the two surfaces together is the weight of the upper block; hence $N = 175$ lb. From Equation (3-3), the force of friction is

$$F_r = fN,$$

$$F_r = 0.2 \times 175\text{ lb} = 35\text{ lb}.$$

3-6 Body on an Inclined Plane

When a body rests upon an inclined plane, it is sometimes convenient to resolve the forces acting on the body into components parallel and perpendicular to the plane. The force exerted by a plane surface may similarly be resolved into a component parallel to the plane, which is called the frictional force, and a component perpendicular to the plane, called the normal force. In describing a surface by the word *smooth,* we imply that the surface is frictionless and is capable of exerting only a force normal to itself. When it is called a *rough* surface, we imply that it is capable of exerting a force in any outward direction, and the component of that force parallel to the surface is the frictional force.

Illustrative Example. A weight of 80 lb rests on a smooth plane which is inclined at an angle of 37° with the horizontal, as shown in Figure 3-5(a). What is the magnitude of the horizontal force **F** which will keep the block from sliding down the plane?

The forces acting on the body are shown in Figure 3-5(b); they are the horizontal forces **F**, the force of gravity **W**, and the force of the smooth plane **N**, which must be normal to the plane. The vector sum of **W**, **F**, and **N** is zero, hence these vectors must form a closed triangle, as shown in Figure 3-5(c). The angle between **W** and **N** in Figure 3-5(c) is equal to the angle made by the inclined plane with the horizontal and is 37°. Since a 37° right triangle is approximately a 3–4–5 right triangle, and $W = 80$ lb, we see that $F = 60$ lb and $N = 100$ lb. Thus a horizontal force of 60 lb will keep a body of weight 80 lb from sliding down a smooth inclined plane. According to Newton's first law, such a force will also keep the body sliding up or down the plane at uniform speed once the body has achieved that speed, for under the application of a 60-lb horizontal force the vector sum of the applied forces is equal to zero.

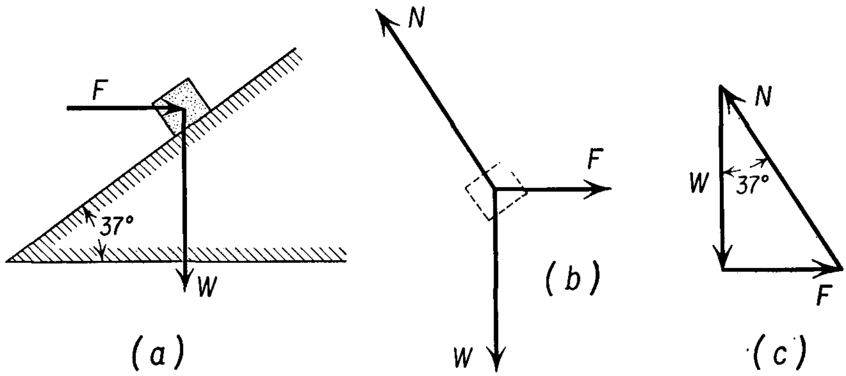

Fig. 3-5

Illustrative Example. A steel block weighing 100 lb rests upon a plank which is inclined at an angle θ of 30° with the horizontal, as shown in Figure 3-6(a). The coefficient of static friction is 0.8. What is the frictional force between the block and the plank?

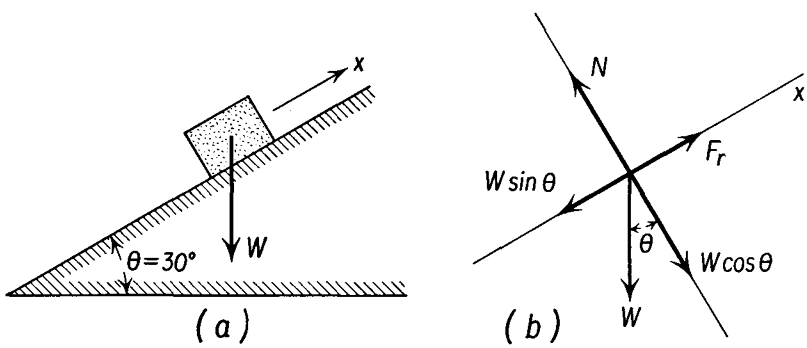

Fig. 3-6

It is convenient to choose the direction of the x axis as parallel to the plank and the direction of the y axis as perpendicular to the plank. The forces acting on the block are shown in Figure 3-6(b); they are the gravitational force $\mathbf{W} = 100$ lb acting vertically downward, the force exerted by the plank on the block which is resolved into the force $\mathbf{N}$ normal to the plank, and the frictional force $\mathbf{F}_r$ parallel to the plank. We choose the direction of $\mathbf{F}_r$ in order to oppose the tendency of the block to slide down the plank. The weight of the block is also resolved into x and y components; the y component is $W \cos \theta = -86.6$ lb, and the x component is $W \sin \theta = -50$ lb. Applying Equation (3-2b) for equilibrium in the y

direction, we have

$$N - 86.6 \text{ lb} = 0,$$

$$N = 86.6 \text{ lb}.$$

From Equation (3-3) the maximum value of the frictional force is

$$F_r \text{ (max)} = 0.8 \times 86.6 = 69.3 \text{ lb}.$$

But from Equation (3-2a) the conditions for the equilibrium of the block require the magnitude of the frictional force given by

$$F_r - 50 \text{ lb} = 0,$$

so that

$$F_r = 50 \text{ lb}.$$

Hence the frictional force is 50 lb, which is less than the maximum value obtained from the coefficient of static friction.

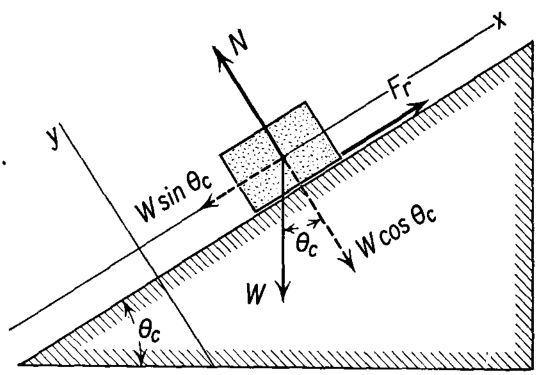

Fig. 3-7 Angle of repose on a rough inclined plane.

If the coefficient of kinetic friction between a block of weight W and a plane is given by f, it is interesting to consider the angle of inclination of the plane with the horizontal at which the block will continue to slide down the plane with uniform speed. Let us call this angle θ_c.

Once started in motion down the plane, the block will slide down with uniform speed if the sum of the forces parallel to the plane is equal to zero. Referring to Figure 3-7, the magnitude of the y component of the weight of the block is given by $W \cos \theta_c$, and this is equal to the magnitude of the normal force N. The frictional force is given by

$$F_r = fN = fW \cos \theta_c.$$

Since the sum of the x components of the forces must be equal to zero, we have

$$-W \sin \theta_c + F_r = 0,$$

and, substituting for F_r, we get

$$W \sin \theta_c = fW \cos \theta_c,$$

so that

$$f = \tan \theta_c. \tag{3-4}$$

The coefficient of sliding friction is given by the tangent of the angle at which the block, if started, will slide down the plane with uniform speed. At any angle slightly less than $\theta_c = \text{arc} \tan f$, the block will come to rest. The angle θ_c is called the *angle of repose.* Considerations similar to these illustrate why a pile of coal has uniformly sloping sides, and why some materials will stand in steeper piles than others. A knowledge of the *angle of repose,* the angle of elevation of the surface of the pile of granular materials, is of practical value in the design of appropriate storage bins.

Problems

3-1. A body weighing 15 lb hangs from one end of a vertical cord. What is the tension in the cord?

3-2. A body weighing 35 lb is hung from the center of a cord. The angle between the two parts of the cord is 120°. Determine the tension in the cord.

3-3. A body weighing 120 lb hangs from a cord which is attached to the ceiling. A horizontal force pushes the body out so that the cord makes an angle of 30° with the vertical. Determine (a) the magnitude of the horizontal force and (b) of the tension in the cord.

3-4. A rope 20 ft long has its ends fastened to the tops of two poles 16 ft apart. A weight of 240 lb hangs 8 ft from one end of the rope. Determine the tension in each section of the rope.

3-5. In order to pull a car out of a rut, a man ties a rope around a tree and attaches the other end to the front bumper of the car. The man then pulls on the middle of the rope in a direction at right angles to the line from the tree to the car. (a) Determine the tension in the rope if the man exerts a force of 80 lb when the angle between the two parts of the rope is 160°. (b) What force does the rope exert on the car?

3-6. A box weighing 70 lb is held up by two ropes, one of which makes an angle of 30° with the vertical while the other makes an angle of 60° with the vertical. Find the tension in each rope.

3-7. A boom in the form of a uniform pole, whose weight may be neglected, is hinged at its lower end. The boom is held at an angle of 60° with the ground by means of a horizontal cable attached to its upper end. (a) Determine the tension in the cable when a load of 1,000 lb is attached to the upper end. (b) Determine the thrust exerted by the boom.

3-8. One end A of a rigid, light, horizontal bar is attached to a wall, while the other end C is supported by a rope which is attached to a point D on the wall directly above A, as shown in Figure 3-8. The length of the bar AC is 12 ft, and the length of the rope CD is 13 ft. Determine (a) the tension in the rope and (b) the thrust exerted by the bar when a weight of 3,000 lb is hung from C.

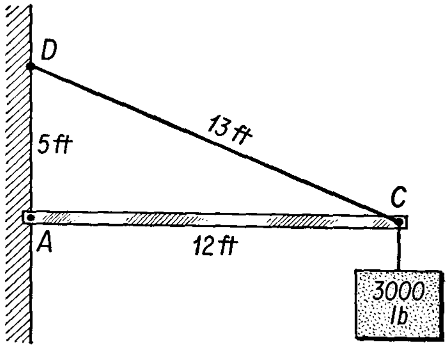

Fig. 3-8

3-9. A car weighing 3,500 lb is on a hill which rises 5 ft for every 100 ft of length. Determine the component of the weight which acts parallel to the hill.

3-10. A crate weighing 150 lb is held on a smooth inclined plane by means of a rope tied to this crate and to the top of the plane. If the inclination of the plane to the horizontal is 30°, (a) determine the tension in the rope and (b) the push of the plane against the crate. (c) What will be the tension in the rope if it is used to pull the crate up the plane at a uniform rate? (d) Determine the tension in the rope if the crate is allowed to slide down the plane at a uniform rate.

3-11. A barrel weighing 120 lb is held on a smooth inclined plane by means of a force applied horizontally. The inclination of the plane is 37°. Determine (a) the magnitude of the horizontal force and (b) the push of the plane.

3-12. A block weighing 2 lb is dragged along a rough, level floor at uniform speed by a rope which makes an angle of 30° with the floor. If the coefficient of kinetic friction between the floor and the block is 0.3, find the tension in the rope.

3-13. In Problem 3-10 the coefficient of static friction between the plane and the crate is 0.2 and the coefficient of kinetic friction is 0.1. Recalculate your answers to parts (a), (b), (c), and (d).

3-14. (a) A block weighing 50 lb rests on a horizontal plane. Find the frictional force between the block and the plane. The coefficient of static sliding friction is 0.8. (b) What is the frictional force between the block and the plane when the plane is inclined at an angle of 30° with the horizontal?

3-15. A box weighing 100 lb is pushed at constant speed, up a rough plane inclined at an angle of 37° with the horizontal by a steady horizontal force of 85 lb. (a) Find the frictional force and (b) the coefficient of kinetic sliding friction between the block and the plane. (c) Find the horizontal force which must be applied to lower the block down the plane.

3-16. A steeple jack sits in a chair which is fastened to a long rope. The rope is passed over a pulley fixed at the top of the steeple and hangs down within reach of the steeple jack. If the steeple jack and chair weigh 150 lb, with what force must he pull on the free end of the rope to raise himself at a steady rate? Neglect the weight of the rope.