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Abstract. In the 1970s, Feldman and Moore classified separably acting von Neumann algebras containing Cartan MASAs using measured equivalence relations and 2-cocycles on such equivalence relations. In this paper, we give a new classification in terms of extensions of inverse semigroups. Our approach is more algebraic in character and less point-based than that of Feldman-Moore. As an application, we give a restatement of the spectral theorem for bimodules in terms of subsets of inverse semigroups. We also show how our viewpoint leads naturally to a description of maximal subdiagonal algebras.

1. Introduction

Every abelian von Neumann algebra is isomorphic to $L^\infty(X, \mu)$ for a suitable measure space (X, μ). Because of this, the theory of von Neumann algebras is often described as "non-commutative integration." In a pair of landmark papers, Feldman and Moore [6, 7] pursued this analogy further. They showed that if $\mathcal{D} \simeq L^\infty(X, \mu)$ is a Cartan MASA in a separably acting von Neumann algebra $\mathcal{M}$, then there is a Borel equivalence relation $R \subseteq X \times X$ and a 2-cocycle c on R such that $\mathcal{M}$ is isomorphic to an algebra $M(R, c)$ consisting of certain measurable functions on R and $\mathcal{D}$ is isomorphic to the algebra $D(R, c)$ of functions supported on the diagonal $\{(x, x) : x \in R\}$ of R. The multiplication in $M(R, c)$ is essentially matrix multiplication twisted by the cocycle c. Feldman and Moore further show that the isomorphism classes of pairs $(\mathcal{M}, \mathcal{D})$ with $\mathcal{D}$ a Cartan MASA in a separably acting von Neumann algebra $\mathcal{M}$ is in bijective correspondence with the family of equivalence classes of pairs (R, c) where c is a 2-cocycle on the measured equivalence relation R. Twisting the multiplication by a cocycle originated in the work of Zeller-Meier for crossed products of von Neumann algebras [24, Section 8], which was itself an extension of the group-measure construction. The Cartan pairs of Feldman and Moore include these crossed products.

Feldman and Moore's work may be characterized as "point-based" in the sense that the basic objects used in their construction are functions determined up to null sets on appropriate measure spaces. As a result of the measure theory involved, the Feldman-Moore work is restricted to equivalence relations with countable equivalence classes and to von Neumann algebras with separable predual. Furthermore, their work demands considerable measure-theoretic prowess.

The goal of the present paper is to recast the Feldman-Moore work in algebraic terms. We bypass the measured equivalence relations used by Feldman and

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Moore and instead start with an axiomatization of the inverse semigroups which arise from measured equivalence relations. Here is a brief description of these inverse semigroups. Starting with a measured equivalence relation, Feldman and Moore consider the family $\mathcal{S}$ of all partial Borel isomorphisms $\phi : X \to X$ whose graph, $\mathrm{Graph}(\phi) := \{(\phi(x), x) : x \in X\}$, is a subset of R. With composition product, $\mathcal{S}$ becomes an inverse semigroup and the characteristic function of the set $\mathrm{Graph}(\phi)$ becomes a partial isometry in $M(R, c)$. The strong-$*$ closure $\mathcal{G}$ of the inverse semigroup generated by such isometries and $\mathbb{T}I$ is an inverse semigroup of partial isometries which generates $M(R, c)$. Further, $\mathcal{G}$ is an inverse semigroup extension of $\mathcal{S}$. We axiomatize the class of the inverse semigroups arising as partial Borel isomorphisms whose graph lies in a measured equivalence relation; we call members of this class of inverse semigroups *Cartan inverse monoids*.

Lausch [12] has developed a theory of extensions of inverse semigroups which parallels the theory of extensions of groups. In particular, Lausch shows that there is a natural notion of equivalence of extensions, and that up to equivalence, the family of extensions of a given inverse semigroup by an abelian inverse semigroup is parametrized by a 2-cohomology group. We replace the 2-cocycle on R appearing in the Feldman-Moore work with an extension of the Cartan inverse monoid $\mathcal{S}$ by the abelian inverse semigroup of partial isometries in the C^*-algebra generated by the idempotents of $\mathcal{S}$. From this data, we construct a Cartan MASA in a von Neumann algebra of the extension. This is accomplished in Theorem 5.12.

We show in Theorem 3.8 that any Cartan MASA $\mathcal{D}$ in a von Neumann algebra $\mathcal{M}$ determines an extension of the type mentioned in the previous paragraph. In combination, Theorems 3.8 and 5.12 show that these constructions are inverses of each other up to equivalence. Thus, we obtain the desired algebraic version of the Feldman-Moore work.

We note that our constructions apply to any pair $(\mathcal{M}, \mathcal{D})$ consisting of a Cartan MASA $\mathcal{D}$ in the von Neumann algebra $\mathcal{M}$. We require neither $\mathcal{M}$ to act separably, nor any hypothesis on Cartan inverse monoids which would correspond to countable equivalence classes of measured equivalence relations.

In constructing a Cartan pair from an extension, we build a representation of the Cartan inverse monoid analogous to the Stinespring representation of $\pi \circ E$, where $E : \mathcal{M} \to \mathcal{D}$ is the conditional expectation and π is a representation of $\mathcal{D}$ on the Hilbert space $\mathcal{H}$. Since the inverse semigroup has no innate linear structure (as $\mathcal{M}$ does) we use an operator-valued reproducing kernel Hilbert space approach. The construction of the corresponding reproducing kernel uses the order structure of $\mathcal{S}$ arising from the action of the idempotents of $\mathcal{S}$. This action should be viewed as the semigroup analogue of the bimodule action of $\mathcal{D}$ on $\mathcal{M}$.

An important application of the Feldman-Moore construction is to characterize the $\mathcal{D}$-bimodules of $\mathcal{M}$ in terms of suitable subsets of R. For Bures-closed $\mathcal{D}$-bimodules, such a characterization was obtained by Cameron, Pitts, and Zarikian [5, Theorem 2.5.1]. In Theorem 6.3 below, we reformulate this characterization in terms of subsets of $\mathcal{S}$, which we call spectral sets. As a result, we describe maximal subdiagonal algebras of $\mathcal{M}$ which contain $\mathcal{D}$ in terms of spectral sets. In particular, this provides a proof of [17, Theorem 3.5] that avoids the weak-$*$-closed Spectral Theorem for Bimodules [17, Theorem 2.5] whose proof unfortunately is incomplete.

2. Preliminaries

We begin with a discussion of the necessary ideas about Boolean algebras and inverse semigroups.

2.1. **Stone's representation theorem.** Let $\mathcal{L}$ be a Boolean algebra, and let $\widehat{\mathcal{L}}$ be the character space of $\mathcal{L}$, that is, the set of all lattice homomorphisms of $\mathcal{L}$ into the two element lattice $\{0,1\}$. For each $e \in \mathcal{E}$, let

$$G_e = \{\rho \in \widehat{L} \colon \rho(e) = 1\}.$$

Stone's representation theorem shows the sets $\{G_e \colon e \in \mathcal{L}\}$ form a basis for a compact Hausdorff topology on $\widehat{\mathcal{L}}$ ([20], or see e.g., [9]). In this topology, each set G_e is clopen. Thus Stone's theorem represents $\mathcal{L}$ as the algebra of clopen sets in $\widehat{\mathcal{L}}$. Equivalently, $\mathcal{L}$ can be viewed as the lattice of projections in $C(\widehat{\mathcal{L}})$.

We now show that $C(\widehat{\mathcal{L}})$ is the universal C^*-algebra of $\mathcal{L}$.

Definition 2.1. Let $\mathcal{L}$ be a Boolean algebra. A *representation* of $\mathcal{L}$ is a map $\pi : \mathcal{L} \to \operatorname{proj}(\mathcal{B})$ of $\mathcal{L}$ into the projection lattice of a C^*-algebra $\mathcal{B}$ such that for every $s, t \in \mathcal{L}$, $\pi(s \wedge t) = \pi(s)\pi(t)$.

Proposition 2.2. *Let $\mathcal{L}$ be a Boolean algebra with character space $\widehat{\mathcal{L}}$. For each $s \in \mathcal{L}$, let $\widehat{s} \in C(\widehat{\mathcal{L}})$ be the Gelfand transform, $\widehat{s}(\rho) = \rho(s)$. Then $C(\widehat{\mathcal{L}})$ has the following universal property: if $\mathcal{B}$ is a C^*-algebra and $\theta : \mathcal{L} \to \mathcal{B}$ is a representation such that $\theta(\mathcal{L})$ generates $\mathcal{B}$ as a C^*-algebra, then there exists a unique $*$-epimorphism $\alpha : C(\widehat{\mathcal{L}}) \to \mathcal{B}$ such that for every $s \in \mathcal{L}$,*

$$\theta(s) = \alpha(\widehat{s}).$$

Proof. By the definition of representation, $\theta(\mathcal{L})$ is a commuting family of projections, and, since $\theta(\mathcal{L})$ generates $\mathcal{B}$, $\mathcal{B}$ is abelian. For $\rho \in \widehat{\mathcal{B}}$, $\rho \circ \theta \in \widehat{\mathcal{L}}$. Moreover, the dual map $\theta^{\#} : \hat{\mathcal{B}} \to \hat{\mathcal{L}}$ given by $\widehat{\mathcal{B}} \ni \rho \mapsto \rho \circ \theta$ is continuous. Hence there is a $*$-homomorphism $\alpha : C(\hat{\mathcal{L}}) \to \mathcal{B}$ given by

$$\widehat{\alpha(f)} = f \circ \theta^{\#}.$$

For $s \in \mathcal{L}$ and $\rho \in \widehat{\mathcal{B}}$, we have

$$\widehat{\alpha(\widehat{s})}(\rho) = \widehat{s}(\rho \circ \theta) = \rho(\theta(s)) = \widehat{\theta(s)}(\rho),$$

so that $\theta(s) = \alpha(\widehat{s})$. Since $\theta(\mathcal{L})$ generates $\mathcal{B}$, the image of α is dense in $\mathcal{B}$, whence α is onto.

Suppose $\alpha_1 : C(\widehat{\mathcal{L}}) \to \mathcal{B}$ is another $*$-epimorphism of $C(\widehat{\mathcal{L}})$ onto $\mathcal{B}$ such that $\alpha_1(\widehat{s}) = \theta(s)$ for every $s \in \mathcal{L}$. Letting $\mathcal{A}$ be the $*$-algebra generated by $\{\widehat{s} : s \in \mathcal{L}\}$, we find $\mathcal{A}$ separates points of $\widehat{\mathcal{L}}$ and contains the constant functions. The Stone-Weierstrass Theorem shows $\mathcal{A}$ is dense in $C(\widehat{\mathcal{L}})$. Since $\alpha_1|_{\mathcal{A}} = \alpha|_{\mathcal{A}}$, we conclude that $\alpha_1 = \alpha$. ☐

2.2. **Inverse semigroups.** We discuss some results and definitions in the theory of inverse semigroups. For a comprehensive text on inverse semigroups, see Lawson [14].

A semigroup $\mathcal{S}$ is an *inverse semigroup* if there is a unique inverse operation on $\mathcal{S}$. That is, for every $s \in \mathcal{S}$ there is a unique element $s^\dagger$ in $\mathcal{S}$ satisfying

$$ss^\dagger s = s \text{ and } s^\dagger s s^\dagger = s^\dagger.$$

Two elements $s, t \in \mathcal{S}$ are *orthogonal* if $s^\dagger t = ts^\dagger = 0$. An inverse semigroup $\mathcal{S}$ is an *inverse monoid* if $\mathcal{S}$ has a multiplicative unit; we usually denote the unit with the symbol 1.

We denote the idempotents in $\mathcal{S}$ by $\mathcal{E}(\mathcal{S})$. The idempotents of an inverse semigroup form an abelian inverse subsemigroup. Further, $\mathcal{E}(\mathcal{S})$ determines the *natural partial order* on $\mathcal{S}$: given $s, t \in \mathcal{S}$, write $s \leq t$ if there is an idempotent $e \in \mathcal{S}$ such that

$$s = te.$$

We will often use the notation $(\mathcal{S}, \leq)$ when we "forget" the multiplication on $\mathcal{S}$ and simply consider $\mathcal{S}$ as a set with this natural partial order.

For $s, t \in \mathcal{S}$, we will use $s \wedge t$ for the greatest lower bound of $\{s, t\}$, if it exists. Likewise, $s \vee t$ will denote the least upper bound. In general inverse semigroups, $s \vee t$ and $s \wedge t$ need not exist. If for any $s, t \in \mathcal{S}$, $s \wedge t$ exists in $\mathcal{S}$, $(\mathcal{S}, \leq)$ is a *meet semilattice*.

Idempotents of the form $s^\dagger t \wedge 1$ are called *fixed point idempotents* by Leech [15]. When $(\mathcal{S}, \leq)$ is a meet semilattice, these are the idempotents which define the meet operation on $\mathcal{S}$.

Lemma 2.3 (Leech)**.** *Suppose $\mathcal{S}$ is an inverse monoid such that $(\mathcal{S}, \leq)$ is a meet semilattice. For any $s, t \in \mathcal{S}$, $s^\dagger t \wedge 1$ is the smallest idempotent e such that*

$$s \wedge t = se = te.$$

In particular, $(s \wedge t)^\dagger (s \wedge t) = s^\dagger t \wedge 1$.

An inverse semigroup $\mathcal{S}$ is *fundamental* if for $s, t \in \mathcal{S}$

$$ses^\dagger = tet^\dagger \text{ for all } e \in \mathcal{E}(\mathcal{S})$$

only when $s = t$. Equivalently, $\mathcal{S}$ is fundamental if the centralizer of $\mathcal{E}(\mathcal{S})$ in $\mathcal{S}$ is $\mathcal{E}(\mathcal{S})$. An inverse semigroup is *Clifford* if $s^\dagger s = ss^\dagger$ for all $s \in \mathcal{S}$. Fundamental and Clifford inverse semigroups play an important role in the theory of inverse semigroups. In fact, every inverse semigroup can be described as the extension of a Clifford inverse semigroup by a fundamental inverse semigroup. We explain these concepts now.

Let $\mathcal{S}$ and $\mathcal{P}$ be two inverse semigroups, and let $\pi \colon \mathcal{P} \to \mathcal{E}(\mathcal{S})$ be a surjective homomorphism. Suppose further that $\pi|_{\mathcal{E}(\mathcal{P})}$ is an isomorphism of $\mathcal{E}(\mathcal{P})$ and $\mathcal{E}(\mathcal{S})$. An inverse semigroup $\mathcal{G}$, together with a surjective homomorphism $q \colon \mathcal{G} \to \mathcal{S}$, is an *idempotent separating extension of $\mathcal{S}$ by $\mathcal{P}$* if there is an embedding ι of $\mathcal{P}$ into $\mathcal{G}$ such that

(a) $q(g) \in \mathcal{E}(\mathcal{S})$ if and only if $g = \iota(p)$ for some $p \in \mathcal{P}$; and
(b) $q \circ \iota = \pi$.

Unless explicitly stated to the contrary, all extensions considered in the sequel will be idempotent separating. Thus, we will use the phrase, 'extension of $\mathcal{S}$ by $\mathcal{P}$,' instead of 'idempotent separating extension of $\mathcal{S}$ by $\mathcal{P}$' when discussing extensions. Also, since $q \circ \iota = \pi$, we will typically suppress the map π and describe an extension of $\mathcal{S}$ by $\mathcal{P}$ using the diagram,

$$\mathcal{P} \overset{\iota}{\hookrightarrow} \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}.$$

The extension $\mathcal{P} \overset{\iota}{\hookrightarrow} \mathcal{G} \overset{q}{\twoheadrightarrow} \mathcal{S}$ is *a trivial extension* if there exists a semigroup homomorphism $j : \mathcal{S} \to \mathcal{G}$ such that $q \circ j = \mathrm{id}|_{\mathcal{S}}$.

We will sometimes identify $\mathcal{P}$ with $\iota(\mathcal{P})$, so that ι becomes the inclusion map. When this identification is made, we delete ι from the diagram of the extension and simply write

$$\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}.$$

We shall require a notion of equivalent extensions. The following definition is a modification of the definitions found in [12] and [14].

Definition 2.4. For $i = 1, 2$ let $\mathcal{S}_i$ and $\mathcal{P}_i$ be inverse semigroups, and suppose that $\tilde{\alpha} : \mathcal{S}_1 \to \mathcal{S}_2$ and $\underline{\alpha} : \mathcal{P}_1 \to \mathcal{P}_2$ are fixed isomorphisms of inverse semigroups. The extension

$$\mathcal{P}_1 \overset{\iota_1}{\hookrightarrow} \mathcal{G}_1 \xrightarrow{q_1} \mathcal{S}_1 \tag{2.1}$$

of $\mathcal{S}_1$ by $\mathcal{P}_1$ and the extension

$$\mathcal{P}_2 \overset{\iota_2}{\hookrightarrow} \mathcal{G}_2 \xrightarrow{q_2} \mathcal{S}_2 \tag{2.2}$$

of $\mathcal{S}_2$ by $\mathcal{P}_2$ are $(\underline{\alpha}, \tilde{\alpha})$-*equivalent* if there is an isomorphism $\alpha\colon \mathcal{G}_1 \to \mathcal{G}_2$ such that $q_2 \circ \alpha = \tilde{\alpha} \circ q_1$, and $\underline{\alpha} \circ \iota_2 = \iota_1 \circ \alpha$.

Notice that when the extensions (2.1) and (2.2) are $(\underline{\alpha}, \tilde{\alpha})$-equivalent, $\tilde{\alpha} \circ q_1 \circ \iota_1 = q_2 \circ \iota_2 \circ \underline{\alpha}$, that is,

$$\tilde{\alpha} \circ \pi_1 = \pi_2 \circ \underline{\alpha}.$$

Remark 2.5. Definition 2.4 differs slightly from that given in Lausch [12] and Lawson [14]. These authors assume that $\mathcal{P}_1 = \mathcal{P}_2$, $\mathcal{S}_1 = \mathcal{S}_2$, and both $\tilde{\alpha}$ and $\underline{\alpha}$ are the identity maps. While Definition 2.4 is essentially the same as that given by Lausch and Lawson, it enables us to streamline the statements of our main results.

In [12], Lausch also shows that equivalence classes of extensions of inverse semigroups may be parametrized by elements of a 2-cohomology group. Trivial extensions as defined above correspond to the neutral element of this cohomology group.

Another way to describe extensions of inverse semigroups is via congruences. Let $\mathcal{G}$ be an inverse semigroup. An equivalence relation R on $\mathcal{G}$ is a *congruence* if it behaves well under products, that is,

$$(v_1, v_2), (w_1, w_2) \in R \text{ implies } (v_1 w_1, v_2 w_2) \in R.$$

The quotient of $\mathcal{G}$ by R gives an inverse semigroup $\mathcal{S}$. Let $q : \mathcal{G} \to \mathcal{S}$ denote the quotient map. Let

$$\mathcal{P} = \{v \in \mathcal{G}\colon q(v) \in \mathcal{E}(\mathcal{S})\}. \tag{2.3}$$

Then $\mathcal{P}$ is a inverse semigroup, and $\mathcal{G}$ is a extension of $\mathcal{S}$ by $\mathcal{P}$.

The *Munn congruence* R_M on $\mathcal{G}$ is the congruence,

$$R_M := \{(v_1, v_2) \in \mathcal{G} \times \mathcal{G}\colon v_1 e v_1^\dagger = v_2 e v_2^\dagger \text{ for all } e \in \mathcal{E}(\mathcal{G})\}.$$

The Munn congruence is the maximal idempotent separating congruence on $\mathcal{G}$ and the quotient of $\mathcal{G}$ by R_M is a fundamental inverse semigroup $\mathcal{S}$. With $\mathcal{P}$ as in (2.3), $\mathcal{P}$ is a Clifford inverse semigroup, and $\mathcal{G}$ is an idempotent separating extension of $\mathcal{S}$ by $\mathcal{P}$.

We are interested in inverse monoids with a strong order structure. Parts (a–c) in the definition below may be found in Lawson [13].

Definition 2.6. An inverse monoid $\mathcal{S}$ with 0 is a *Boolean inverse monoid* if

(a) $(\mathcal{E}(\mathcal{S}), \leq)$ is a Boolean algebra;
(b) $(\mathcal{S}, \leq)$ is a meet semilattice;
(c) if $s, t \in \mathcal{S}$ are orthogonal, their join, $s \vee t$, exists in $\mathcal{S}$.

In addition, we shall say $\mathcal{S}$ is a *locally complete Boolean inverse monoid* if $\mathcal{E}(\mathcal{S})$ is a complete Boolean algebra. Finally, $\mathcal{S}$ is a *complete Boolean inverse monoid* if $\mathcal{S}$ satisfies the additional condition,

(d) for every pairwise orthogonal family $S \subseteq \mathcal{S}$, $\bigvee_{s\in S} s$ exists in $\mathcal{S}$.

Remark 2.7. A complete Boolean inverse monoid is necessarily locally complete, see [9, Corollary 1, p. 46].

Example 2.8. At first glance, it may appear that local completeness for a Boolean inverse monoid $\mathcal{S}$ might imply that $\mathcal{S}$ is actually complete. Here is an example showing this is not the case. Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$, and let $\mathcal{D}$ be the set of all operators $T \in \mathcal{B}(\mathcal{H})$ for which each e_j is an eigenvector for T. Let $\mathcal{S}$ be the inverse semigroup generated by the projections in $\mathcal{D}$ and the rank-one partial isometries, $\{e_ie_j^*\}_{i,j\in\mathbb{N}}$. Then $\mathcal{E}(\mathcal{S})$ is a complete Boolean algebra, and $\{e_{j+1}e_j^* : j \in \mathbb{N}\}$ is a pairwise orthogonal family in $\mathcal{S}$, yet $\bigvee_{j=1}^{\infty} e_{j+1}e_j^* \notin \mathcal{S}$.

Our main application of Proposition 2.2 is when $\mathcal{S}$ is a Boolean inverse monoid and $\mathcal{L} = \mathcal{E}(\mathcal{S})$. For $i = 1, 2$, let $\mathcal{S}_i$ be Boolean inverse monoids and let $\mathcal{P}_i$ be the inverse semigroup of partial isometries in $\mathcal{D}_i := C(\widehat{\mathcal{E}(\mathcal{S}_i)})$. As in the proof of Proposition 2.2, any isomorphism θ of $\mathcal{E}(\mathcal{S}_1)$ onto $\mathcal{E}(\mathcal{S}_2)$ uniquely determines a homeomorphism $\theta^\#$ of $\widehat{\mathcal{E}(\mathcal{S}_2)}$ onto $\widehat{\mathcal{E}(\mathcal{S}_1)}$, which in turn gives a $*$-isomorphism, $\theta^{\#\#}$ of $\mathcal{D}_1$ onto $\mathcal{D}_2$. Define $\underline{\theta} := \theta^{\#\#}|_{\mathcal{P}_1}$. Clearly, $\underline{\theta}$ is an isomorphism of $\mathcal{P}_1$ onto $\mathcal{P}_2$.

The map $\underline{\theta}$ allows us to specialize Definition 2.4 for extensions of Boolean inverse monoids.

Definition 2.9. For $i = 1, 2$, let $\mathcal{S}_i$ be Boolean inverse monoids and $\mathcal{P}_i$ be the partial isometries in $C(\widehat{\mathcal{E}(\mathcal{S}_i)})$. The extensions

$$\mathcal{P}_1 \overset{\iota_1}{\hookrightarrow} \mathcal{G}_1 \xrightarrow{q_1} \mathcal{S}_1 \tag{2.4}$$

and

$$\mathcal{P}_2 \overset{\iota_2}{\hookrightarrow} \mathcal{G}_2 \xrightarrow{q_2} \mathcal{S}_2 \tag{2.5}$$

are *equivalent* if there are isomorphisms $\theta : \mathcal{S}_1 \to \mathcal{S}_2$ and $\alpha\colon \mathcal{G}_1 \to \mathcal{G}_2$ such that $q_2 \circ \alpha = \theta \circ q_1$, and $\iota_2 \circ \underline{\theta} = \alpha \circ \iota_1$. In other words, these extensions are equivalent if there is an isomorphism $\theta : \mathcal{S}_1 \to \mathcal{S}_2$ such that (2.4) is $(\underline{\theta}, \theta)$-equivalent to (2.5).

A *partial homeomorphism* of a topological space X is a homeomorphism between two open subsets of X. If s_1 and s_2 are partial homeomorphisms, their product s_1s_2 has domain $\operatorname{dom}(s_1) \cap \operatorname{range}(s_2)$ and for $x \in X$, $(s_1s_2)(x) = s_1(s_2(x))$. In the following proposition, whose proof is left to the reader, $\mathcal{O}$ denotes the family of clopen subsets of $\widehat{\mathcal{E}(\mathcal{S})}$ and $\mathrm{Inv}_{\mathcal{O}}$ will denote the inverse semigroup of all partial homeomorphisms of $\widehat{\mathcal{E}(\mathcal{S})}$ whose domains and ranges belong to $\mathcal{O}$.

Proposition 2.10. *Let $\mathcal{S}$ be a Boolean inverse monoid and $\mathcal{D} = C(\widehat{\mathcal{E}(\mathcal{S})})$. For $s \in \mathcal{S}$, the map $\mathcal{E}(\mathcal{S}) \ni e \mapsto s^\dagger es$ determines a partial homeomorphism β_s of $\widehat{\mathcal{E}(\mathcal{S})}$*

with

$$\mathrm{dom}(\beta_s) = \{\rho \in \widehat{\mathcal{E}(\mathcal{S})} : \rho(s^\dagger s) = 1\} \quad \textit{and} \quad \mathrm{range}(\beta_s) = \{\rho \in \widehat{\mathcal{E}(\mathcal{S})} : \rho(ss^\dagger) = 1\}$$

as follows: for $e \in \mathcal{E}(\mathcal{S})$ and $s \in \mathcal{S}$,

$$\beta_s(\rho)(e) = \rho(s^\dagger es).$$

The map $s \mapsto \beta_s$ is a one-to-one inverse semigroup homomorphism of $\mathcal{S}$ into the inverse semigroup $Inv_{\mathcal{O}}$. Moreover, β_s determines a partial action on $\mathcal{D}$: for $f \in \mathcal{D}$, define $s^\dagger fs \in \mathcal{D}$ by

$$(s^\dagger fs)(\rho) := f(\beta_s(\rho)).$$

In particular, when $e \in \mathcal{E}(\mathcal{S})$, $s^\dagger \chi_{G_e} s = \chi_{G_{s^\dagger es}}$.

Definition 2.11. We call an inverse semigroup $\mathcal{S}$ a *Cartan inverse monoid* if

(a) $\mathcal{S}$ is fundamental;
(b) $\mathcal{S}$ is a complete Boolean inverse monoid; and
(c) the character space $\widehat{\mathcal{E}(\mathcal{S})}$ of the complete Boolean lattice $\mathcal{E}(\mathcal{S})$ is a hyperstonean topological space.

The choice of name "Cartan" for these inverse monoids will become clear presently. For now we note that condition (c) in Definition 2.11 tells us that if $\mathcal{S}$ is a Cartan inverse monoid, then the lattice of idempotents $\mathcal{E}(\mathcal{S})$ is isomorphic to the lattice of projections in some abelian von Neumann algebra [21, Theorem III.1.18].

Remark 2.12. We emphasize that for two extensions of Cartan inverse monoids, equivalence is always to be taken in the sense of Definition 2.9.

Remark 2.13. Recall that a *pseudogroup* is an inverse semigroup $\mathcal{S}$ of partial homeomorphisms of a topological space X. By a theorem of V. Vagner [23] (or see [14, Section 5.2, Theorem 10]), an inverse semigroup $\mathcal{S}$ if fundamental if and only if $\mathcal{S}$ is isomorphic to a topologically complete pseudogroup $\mathcal{T}$ consisting of partial homeomorphisms of a T_0 space X; recall that $\mathcal{T}$ is *topologically complete* if the family $\{\mathrm{dom}(t) : t \in \mathcal{T}\}$ is a basis for the topology on X.

An application of Vagner's theorem yields a slightly different description of Cartan inverse monoids: $\mathcal{S}$ is a Cartan inverse monoid if and only if $\mathcal{S}$ is isomorphic to a pseudogroup $\mathcal{T}$ on a hyperstonean topological space X such that:

(a) $\{\mathrm{dom}(t) : t \in \mathcal{T}\} = \{E \subseteq X : E \text{ is clopen}\}$; and
(b) if $\{t_\alpha : \alpha \in \mathbb{A}\} \subseteq \mathcal{T}$ is such that the two families $\{\mathrm{dom}(t_\alpha) : \alpha \in \mathbb{A}\}$ and $\{\mathrm{range}(t_\alpha) : \alpha \in \mathbb{A}\}$ are each pairwise disjoint, then there exists $t \in \mathcal{T}$ such that for each $\alpha \in \mathbb{A}$, $t|_{\mathrm{dom}(t_\alpha)} = t_\alpha$.

Proposition 2.10 can be used to produce the isomorphism.

3. From Cartan MASAs to extensions of inverse semigroups

Our goal of this section is to show that every Cartan pair $(\mathcal{M}, \mathcal{D})$ uniquely determines an exact sequence of inverse semigroups. As we will see, these inverse semigroups will be Cartan inverse monoids. In Section 5 we show the converse: given an extension of a Cartan inverse monoid by a natural choice of inverse semigroup, we can construct a Cartan pair. Cartan inverse monoids will play a role analogous to measured equivalence relations of Feldman-Moore [6, 7].

Let $\mathcal{M}$ be a von Neumann algebra. Let $\mathcal{D}$ be a MASA (maximal abelian subalgebra) of $\mathcal{M}$. The *normalizers of* $\mathcal{D}$ *in* $\mathcal{M}$ are the elements $x \in \mathcal{M}$ such that

$$x\mathcal{D}x^* \subseteq \mathcal{D} \text{ and } x^*\mathcal{D}x \subseteq \mathcal{D}.$$

If a partial isometry $v \in \mathcal{M}$ is a normalizer, then we call v a *groupoid normalizer*. The collection of all groupoid normalizers of $\mathcal{D}$ in $\mathcal{M}$ is denoted by $\mathcal{GN}(\mathcal{M},\mathcal{D})$. It is not hard to show that $\mathcal{GN}(\mathcal{M},\mathcal{D})$ is an inverse semigroup with the adjoint serving as the inverse operation. The idempotents in the inverse semigroup $\mathcal{GN}(\mathcal{M},\mathcal{D})$ are the projections in $\mathcal{D}$.

Definition 3.1. A MASA $\mathcal{D}$ in the von Neumann algebra $\mathcal{M}$ is *Cartan* if

(a) there exists a faithful, normal conditional expectation E from $\mathcal{M}$ onto $\mathcal{D}$;
(b) the set of groupoid normalizers $\mathcal{GN}(\mathcal{M},\mathcal{D})$ spans a weak-$*$ dense subset of $\mathcal{M}$.

If $\mathcal{D}$ is a Cartan MASA in $\mathcal{M}$, we call the pair $(\mathcal{M},\mathcal{D})$ a *Cartan pair.*

Remark 3.2. A MASA $\mathcal{D}$ is usually defined to be Cartan if it satisfies condition (a) above, and if the unitary groupoid normalizers of $\mathcal{D}$ in $\mathcal{M}$ span a weak-$*$ dense subset of $\mathcal{M}$. This is equivalent to the definition given above. A proof of the equivalence can be found in [5, inclusion 2.8, p. 479].

Let $(\mathcal{M},\mathcal{D})$ be a Cartan pair. Let $\mathcal{G} = \mathcal{GN}(\mathcal{M},\mathcal{D})$ and let $\mathcal{P} = \mathcal{G}\cap\mathcal{D}$. Note that $\mathcal{P}$ is the set of all partial isometries in $\mathcal{D}$. Thus $\mathcal{P}$ and $\mathcal{G}$ are inverse semigroups with same set of idempotents, which is the set of projections in $\mathcal{D}$. That is,

$$\mathcal{E}(\mathcal{P}) = \mathcal{E}(\mathcal{G}) = \text{Proj}(\mathcal{D}).$$

Let R_M be the Munn congruence on $\mathcal{G}$, let $\mathcal{S}$ be the quotient of $\mathcal{G}$ by R_M, and let

$$q\colon \mathcal{G} \to \mathcal{S}$$

be the quotient map. It follows that $q|_{\mathcal{E}(G)}$ is a complete lattice isomorphism from the idempotents of $\mathcal{G}$ onto the idempotents of $\mathcal{S}$, so

$$\text{Proj}(\mathcal{D}) = \mathcal{E}(\mathcal{P}) = \mathcal{E}(\mathcal{G}) \simeq \mathcal{E}(\mathcal{S}).$$

Lemma 3.3. *Let $v \in \mathcal{G}$. Then $q(v) \in \mathcal{E}(\mathcal{S})$ if and only if $v \in \mathcal{P}$. Thus, $\mathcal{G}$ is an idempotent separating extension of $\mathcal{S}$ by $\mathcal{P}$.*

Proof. Suppose that $q(v) \in \mathcal{E}(\mathcal{S})$. This means that v is equivalent to an idempotent $e \in \mathcal{E}(\mathcal{G})$, that is, $(v,e) \in R_M$ for some $e \in \mathcal{E}(\mathcal{G})$. Since $vIv^\dagger = eIe^\dagger = e$, for any $f \in \mathcal{P}$ we have $vf = (vfv^\dagger)v = (efe)v = fev = fv$. It follows that v commutes with $\mathcal{D}$, and since $\mathcal{D}$ is a MASA in $\mathcal{M}$, we obtain $v \in \mathcal{P}$.

Conversely, if $v \in \mathcal{P}$, then $(v, vv^\dagger) \in R_M$. □

Definition 3.4. Let $(\mathcal{M},\mathcal{D})$ be a Cartan pair. We call the extension

$$\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$$

constructed above the *extension* for the Cartan pair $(\mathcal{M},\mathcal{D})$.

Proposition 3.5. *Let $\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$ be the extension for a Cartan pair $(\mathcal{M},\mathcal{D})$. Then $\mathcal{S}$ is a Cartan inverse monoid.*

Proof. By construction, $\mathcal{S}$ is a fundamental inverse semigroup. Since $\mathcal{E}(\mathcal{G})$ is the projection lattice of an abelian von Neumann algebra, it is a complete Boolean algebra and $\widehat{\mathcal{E}(\mathcal{G})}$ is hyperstonean. Since $\mathcal{E}(\mathcal{S})$ is isomorphic to $\mathcal{E}(\mathcal{G})$, $\widehat{\mathcal{E}(\mathcal{S})}$ is also hyperstonean.

We use [15, Theorem 1.9] to show that $\mathcal{S}$ is a meet semilattice. Indeed, given $s, t \in \mathcal{S}$, let

$$\begin{aligned} f &= \bigvee\{e \in \mathcal{E}(\mathcal{S})\colon e \leq st^\dagger\} \\ &= \bigvee\{e \in \mathcal{E}(\mathcal{S})\colon e \leq ts^\dagger\}. \end{aligned}$$

As $\mathcal{E}(\mathcal{S})$ is a complete lattice, we have that f exists in $\mathcal{E}(\mathcal{S})$. We then have $s \wedge t = ft = fs$, so $\mathcal{S}$ is a meet semi-lattice.

Finally, suppose that $S \subseteq \mathcal{S}$ is a pairwise orthogonal family. For each $s \in S$, let $v_s \in \mathcal{G}$ satisfy $q(v_s) = s$. Then $\{v_s : s \in S\}$ is a family in $\mathcal{G}$ such that for any $s, t \in S$ with $s \neq t$, $v_s^* v_t = v_s v_t^* = 0$. Then the range projections, $\{v_s v_s^* : s \in S\}$ are a pairwise orthogonal family; likewise, the source projections $\{v_s^* v_s : s \in S\}$ are pairwise orthogonal. Therefore, $\sum_{s\in S} v_s$ converges strongly in $\mathcal{M}$ to an element $w \in \mathcal{G}$. Put $r := q(w)$. Applying q to each side of the equality, $w(v_s^* v_s) = v_s$ yields $r \geq s$ for every $s \in S$. Notice also that $r^\dagger r = \bigvee\{s^\dagger s : s \in S\}$. Hence if $r' \in \mathcal{S}$ and $r' \geq s$ for every $s \in S$, then $r'^\dagger r' \geq r^\dagger r$. Then $r = r'(r^\dagger r)$, that is, $r' \geq r$. Thus, r is the least upper bound for S. It follows that $\mathcal{S}$ is a complete Boolean inverse monoid. This completes the proof. □

Our goal now is to show that Cartan pairs uniquely determine their extensions.

Definition 3.6. For $i = 1, 2$ let $(\mathcal{M}_i, \mathcal{D}_i)$ be Cartan pairs. An *isomorphism* from $(\mathcal{M}_1, \mathcal{D}_1)$ to $(\mathcal{M}_2, \mathcal{D}_2)$ is a $*$-isomorphism $\theta : \mathcal{M}_1 \to \mathcal{M}_2$ such that $\theta(\mathcal{D}_1) = \mathcal{D}_2$.

Remark 3.7. For $i = 1, 2$, let (X_i, μ_i) be probability spaces and suppose Γ_i are countable discrete groups acting freely and ergodically on (X_i, μ_i) so that each element of Γ_i is measure preserving. Put $\mathcal{M}_i = L^\infty(X_i) \rtimes \Gamma_i$ and $\mathcal{D}_i = L^\infty(X_i) \subseteq \mathcal{M}_i$. Then $(\mathcal{M}_i, \mathcal{D}_i)$ are Cartan pairs. In this context, equivalence in the sense of Definition 3.6 is often called orbit equivalence.

Theorem 3.8. *For $i = 1, 2$, suppose $(\mathcal{M}_i, \mathcal{D}_i)$ are Cartan pairs, with associated extensions*

$$\mathcal{P}_i \hookrightarrow \mathcal{G}_i \xrightarrow{q_i} \mathcal{S}_i.$$

Then $(\mathcal{M}_1, \mathcal{D}_1)$ and $(\mathcal{M}_2, \mathcal{D}_2)$ are isomorphic Cartan pairs if and only if their associated extensions are equivalent. Furthermore, when the extensions are equivalent and $(\mathcal{M}_i, \mathcal{D}_i)$ are in standard form, the isomorphism is implemented by a unitary operator.

Proof. An isomorphism of Cartan pairs restricts to an isomorphism of $\mathcal{GN}(\mathcal{M}_1, \mathcal{D}_1)$ onto $\mathcal{GN}(\mathcal{M}_2, \mathcal{D}_2)$. The fact that the extensions associated to $(\mathcal{M}_1, \mathcal{D}_1)$ and $(\mathcal{M}_2, \mathcal{D}_2)$ are equivalent follows easily from their construction.

Suppose now that the extensions are equivalent. Let $\alpha : \mathcal{G}_1 \to \mathcal{G}_2$ and $\tilde{\alpha} : \mathcal{S}_1 \to \mathcal{S}_2$ be inverse semigroup isomorphisms such that

$$\tilde{\alpha} \circ q_1 = q_2 \circ \alpha.$$

By examining the image of $\mathcal{E}(\mathcal{S}_1)$ under $\tilde{\alpha}$, we find that the isomorphism $\underline{\tilde{\alpha}}$ of $\mathcal{P}_1$ onto $\mathcal{P}_2$ induced by $\tilde{\alpha}$ is $\alpha|_{\mathcal{P}_1}$. Thus by Definition 2.9, $\alpha|_{\mathcal{P}_1}$ is the restriction of a $*$-isomorphism, again called α, of the von Neumann algebra $\mathcal{D}_1$ onto the von Neumann algebra $\mathcal{D}_2$.

Let $E_i : \mathcal{M}_i \to \mathcal{D}_i$ be the conditional expectations. We claim that

$$(\alpha \circ E_1)|_{\mathcal{G}_1} = E_2 \circ \alpha.$$

To see this, fix $v \in \mathcal{G}_1$, and let $\mathfrak{J} := \{d \in \mathcal{D}_1 : vd = dv \in \mathcal{D}_1\}$. Then $\mathfrak{J}$ is a weak-$*$ closed ideal of $\mathcal{D}_1$. Therefore, there exists a projection $e \in \mathcal{P}_1$ such that $\mathfrak{J} = e\mathcal{D}_1$. In fact,

$$e = \bigvee \{f \in \mathcal{E}(\mathcal{G}_1) : vf = fv \in \mathcal{P}_1\}.$$

Since $E_1(v^*)v$ and $vE_1(v^*)$ both commute with $\mathcal{D}$, they belong to $\mathcal{D}$; hence $E_1(v^*) \in \mathfrak{J}$. As $\mathfrak{J}$ is closed under the adjoint operation, $E_1(v) \in \mathfrak{J}$. Therefore, there exists $h \in \mathcal{D}_1$ such that $E_1(v) = eh$. It now follows that $E_1(v) = ve$. Since α is an isomorphism, we find $\alpha(e) = \bigvee \alpha(\mathfrak{J}) = \bigvee\{f_2 \in \mathcal{E}(\mathcal{G}_2) : \alpha(v)f_2 = f_2\alpha(v) \in \mathcal{P}_2\}$. Hence $E_2(\alpha(v)) = \alpha(v)\alpha(e)$ and the claim holds.

Fix a faithful normal semi-finite weight ψ_1 on $\mathcal{D}_1$. Use α to move ψ_1 to a weight on $\mathcal{D}_2$, that is, $\psi_2 = \psi_1 \circ \alpha^{-1}$. Putting $\phi_i = \psi_i \circ E_i$, we see ϕ_i are faithful semi-finite normal weights on $\mathcal{M}_i$. Let $(\pi_i, \mathfrak{H}_i, \eta_i)$ be the associated semi-cyclic representations (the notation is as in [22]) and let $\mathfrak{n}_i := \{x \in \mathcal{M}_i : \phi_i(x^*x) < \infty\}$. By [5, Corollary 1.4.2], $\operatorname{span}(\eta_i(\mathcal{G}_i \cap \mathfrak{n}_i))$ is dense in $\mathfrak{H}_i$.

Let $n \in \mathbb{N}$ and suppose $v_1, \dots, v_n \in \mathcal{G}_1 \cap \mathfrak{n}_1$ and $c_1, \dots, c_n \in \mathbb{C}$. Since $(\alpha \circ E_1)|_{\mathcal{G}_1} = E_2 \circ \alpha$, it follows from the definition of ϕ_2 that $\alpha(v_j) \in \mathcal{G}_2 \cap \mathfrak{n}_2$ and

$$\begin{aligned}
\phi_2\left(\left(\sum_{i=1}^n c_i\alpha(v_i)\right)^* \left(\sum_{i=1}^n c_i\alpha(v_i)\right)\right) &= \sum_{i,j=1}^n \overline{c_i}c_j\phi_2(\alpha(v_i^*v_j)) \\
&= \sum_{i,j=1}^n \overline{c_i}c_j\phi_1(v_i^*v_j) \\
&= \phi_1\left(\left(\sum_{i=1}^n c_iv_i\right)^* \left(\sum_{i=1}^n c_iv_i\right)\right).
\end{aligned}$$

Hence the map

$$\eta_1\left(\sum_{i=1}^n c_iv_i\right) \mapsto \eta_2\left(\sum_{i=1}^n c_i\alpha(v_i)\right)$$

extends to a unitary operator $U : \mathfrak{H}_1 \to \mathfrak{H}_2$. It is routine to verify that for $v \in \mathcal{G}_1$, $U\pi_1(v) = \pi_2(\alpha(v))U$. Therefore the map $\theta : \mathcal{M}_1 \to \mathcal{M}_2$ given by $\theta(x) = \pi_2^{-1}(U\pi_1(x)U^*)$ is an isomorphism of $(\mathcal{M}_1, \mathcal{D}_1)$ onto $(\mathcal{M}_2, \mathcal{D}_2)$. $\square$

Let $\mathcal{M}$ be a von Neumann algebra, and let $\mathcal{D}$ be a MASA in $\mathcal{M}$. Even if $(\mathcal{M}, \mathcal{D})$ is not a Cartan pair, one can define $\mathcal{G}$, $\mathcal{P}$ and $\mathcal{S}$ as above to get an extension related to the pair $(\mathcal{M}, \mathcal{D})$. The inverse monoid $\mathcal{S}$ will again be a Cartan inverse monoid. However, if $\mathcal{D}$ is not a Cartan MASA in $\mathcal{M}$, the equivalence class of the extension

$$\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$$

may arise from a Cartan pair $(\mathcal{M}_1, \mathcal{D}_1)$ for which $\mathcal{M}$ and $\mathcal{M}_1$ are not isomorphic.

Proposition 3.9. *Let $\mathcal{M}$ be a von Neumann algebra and let $\mathcal{D}$ be a MASA in $\mathcal{M}$. Then the pair $(\mathcal{M}, \mathcal{D})$ determines an idempotent separating exact sequence of inverse semigroups*

$$\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S},$$

where $\mathcal{G} = \mathcal{GN}(\mathcal{M}, \mathcal{D})$*,* $\mathcal{P} = \mathcal{G} \cap \mathcal{D}$ *and* $\mathcal{S}$ *is a Cartan inverse monoid.*

4. Representing an extension

The goal of this section is to develop a representation for extensions of Boolean inverse monoids suitable for the construction of a Cartan pair from a given extension of a Cartan inverse monoid. Given an extension of a Boolean inverse monoid $\mathcal{S}$

$$\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$$

we will ultimately represent $\mathcal{G}$ by partial isometries acting on a Hilbert space. This will be achieved after several intermediate steps, each of which is interesting in its own right.

A Boolean inverse monoid $\mathcal{S}$ has sufficiently rich order structure to allow the construction of a representation theory of $\mathcal{S}$ as isometries on a right $\mathcal{D}$-module constructed from the order structure; as usual, $\mathcal{D} = C(\widehat{\mathcal{E}(\mathcal{S})})$. A key tool in moving from representations of $\mathcal{S}$ to representations of the extension $\mathcal{G}$ of $\mathcal{S}$ by $\mathcal{P}$ is the existence of an order preserving sections $j : \mathcal{S} \to \mathcal{G}$ which splits the exact sequence of ordered spaces, $(\mathcal{P}, \leq) \hookrightarrow (\mathcal{G}, \leq) \xrightarrow{q} (\mathcal{S}, \leq)$. Such sections are discussed in Subsection 4.1.

In Subsection 4.2 we construct the right Hilbert $\mathcal{D}$-module $\mathfrak{A}$ mentioned above. The module $\mathfrak{A}$ will have a reproducing kernel structure, with the lattice structure of $\mathcal{S}$ represented as $\mathcal{D}$-evaluation maps in $\mathfrak{A}$.

Finally, in Subsection 4.3, we represent $\mathcal{G}$ as partial isometries in the adjointable operators on $\mathfrak{A}$. The existence of the order preserving section plays an important role here.

We alert the reader that because we will we using the theory of right Hilbert modules, all inner products, either scalar-valued or $\mathcal{D}$-valued, will be conjugate linear in the first variable.

4.1. Order preserving sections.

Let $\mathcal{S}$ be a Boolean inverse monoid, let $\mathcal{P}$ be the partial isometries of $C(\widehat{\mathcal{E}(\mathcal{S})})$ and let $\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$ be an extension. Since q is onto, it has a *section*, that is, a map $j : \mathcal{S} \to \mathcal{G}$ such that $q \circ j = \mathrm{id}|_{\mathcal{S}}$. Notice that since $\mathcal{S}$ is fundamental, j is one-to-one. Our interest is in those sections which preserve order. When the extension is trivial, a splitting map may be taken to be a semigroup homomorphism, which is order preserving. The main result of this subsection is that when $\mathcal{S}$ is locally complete, then every extension of $\mathcal{S}$ by $\mathcal{P}$, trivial or not, has an order preserving section.

Definition 4.1. Let $\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$ be an extension. We will call a section $j : \mathcal{S} \to \mathcal{G}$ for q an *order preserving section for* q if

(a) $j(1) = 1$, and
(b) $j(s) \leq j(t)$ for every $s, t \in \mathcal{S}$ with $s \leq t$.

Lemma 4.2. *Let* $j : \mathcal{S} \to \mathcal{G}$ *be a section for* q*. The following statements are equivalent.*

(a) The map j *is an order preserving section for* q*.*
(b) For every $e, f \in \mathcal{E}(\mathcal{S})$ *and* $s \in \mathcal{S}$*,*

$$j(esf) = j(e)j(s)j(f).$$

(c) For every $s, t \in \mathcal{S}$*,* $j(s \wedge t) = j(s) \wedge j(t)$ *and* $j(1) = 1$*.*

Proof. $(a) \Rightarrow (b)$. Suppose j is an order preserving section. For any $e \in \mathcal{E}(\mathcal{S})$, $e \leq 1$, so $j(e) \leq 1$, hence $j(e) \in \mathcal{E}(\mathcal{G})$. Since $q|_{\mathcal{E}(\mathcal{G})}$ is an isomorphism of $\mathcal{E}(\mathcal{G})$ onto $\mathcal{E}(\mathcal{S})$ and $q \circ j = \mathrm{id}|_{\mathcal{S}}$, we obtain $j|_{\mathcal{E}(\mathcal{S})} = \left(q|_{\mathcal{E}(\mathcal{G})}\right)^{-1}$.

For $s \in \mathcal{S}$, it follows that

$$j(s)^\dagger j(s) = j(s^\dagger s),$$

because $q(j(s)^\dagger j(s)) = s^\dagger s = q(j(s^\dagger s))$.

Now suppose that $s \in \mathcal{S}$ and $e \in \mathcal{E}(\mathcal{S})$. Then $j(se) = j(se)\left(j(se)^\dagger j(se)\right) = j(se)j(s^\dagger s)j(e)$. Multiply both sides of this equality on the right by $j(e)$ to obtain $j(se) = j(se)j(e)$. Since $se \leq s$, the hypothesis on j gives

$$j(se) = j(se)j(e) \leq j(s)j(e).$$

Hence

$$j(se) = j(s)j(e)\left(j(se)^\dagger j(se)\right) = j(s)j(s^\dagger s)j(e) = j(s)j(s)^\dagger j(s)j(e) = j(s)j(e);$$

where the first equality follows from [14, p. 21, Lemma 6].

Similar considerations yield $j(es) = j(e)j(s)$ for every $e \in \mathcal{E}(\mathcal{S})$ and $s \in \mathcal{S}$. Thus $j(esf) = j(e)j(s)j(f)$.

$(b) \Rightarrow (c)$. Suppose $j(esf) = j(e)j(s)j(f)$ for all $e, f \in \mathcal{E}(\mathcal{S})$ and $s \in \mathcal{S}$. Then for any $e \in \mathcal{E}(\mathcal{S})$, $j(e) \in \mathcal{E}(\mathcal{G})$. Since j is a section for q, we find $j(1) = 1$. Notice also that $j|_{\mathcal{E}(\mathcal{S})}$ is an isomorphism of $(\mathcal{E}(\mathcal{S}), \leq)$ onto $(\mathcal{E}(\mathcal{G}), \leq)$. By Lemma 2.3, for $s, t \in \mathcal{S}$,

$$j(s)j(s^\dagger t \wedge 1) = j(s \wedge t) = j(t)j(t^\dagger s \wedge 1).$$

Therefore, $j(s \wedge t) \leq j(s) \wedge j(t)$. To obtain the reverse inequality, let $e \in \mathcal{E}(\mathcal{S})$ be the unique idempotent such that $j(e) = (j(s) \wedge j(t))^\dagger (j(s) \wedge j(t))$. Using [14, Lemma 6, p. 21], we find

$$j(se) = j(s) \wedge j(t) = j(te).$$

Since j is one-to-one, $te = se$. But then $se \leq s \wedge t$. Applying j to this inequality gives

$$j(s) \wedge j(t) = j(te) \leq j(s \wedge t),$$

and (c) follows.

$(c) \Rightarrow (a)$. Suppose $s, t \in \mathcal{S}$ with $s \leq t$. Then $s \wedge t = s$, so $j(s) = j(s \wedge t) = j(s) \wedge j(t) \leq j(t)$.

∎

It follows immediately that order preserving sections also preserve the inverse operation.

Corollary 4.3. *Let j be an order preserving section. Then for all $s \in \mathcal{S}$,*

$$j(s^\dagger) = j(s)^\dagger.$$

Proof. As

$$j(s)^\dagger j(s) = j(s^\dagger s),$$

it follows that

$$j(s^\dagger)^\dagger = j(s)(j(s)^\dagger j(s^\dagger)^\dagger) = j(s)(j(s^\dagger)j(s))^\dagger = j(s)j(s^\dagger s) = j(s).$$ ∎

Remark 4.4. Order preserving sections are implicit in the work of Feldman and Moore. Indeed if $\mathcal{S}$ is the Cartan inverse monoid consisting of all partial Borel isomorphisms ϕ of the standard Borel space $(X, \mathcal{B})$ whose graph, $\mathrm{Graph}(\phi)$, is contained in the measured equivalence relation R, then the map $\phi \mapsto \chi_{\mathrm{Graph}(\phi)}$ is an order preserving section of $\mathcal{S}$ into the inverse semigroup of groupoid normalizers of the Cartan pair $(M(R,c), D(R,c))$ constructed by Feldman and Moore.

Note that Lemma 4.2 and Corollary 4.3 hold for extensions of arbitrary Boolean inverse monoids. We do not know whether order preserving sections exist in general. However, Proposition 4.6 below shows that when $\mathcal{S}$ is a locally complete Boolean inverse monoid, such sections always exist.

Lemma 4.5. *Let $\mathcal{S}$ be an inverse monoid with 0 such that $(\mathcal{E}(\mathcal{S}), \leq)$ is a Boolean algebra. Let $s \in \mathcal{S}$, let $A_s := \{t \in \mathcal{S} : t \leq s\}$ and let $B_s := \{e \in \mathcal{E}(\mathcal{S}) : e \leq s^\dagger s\}$. Then $(A_s, \leq)$ and $(B_s, \leq)$ are Boolean algebras and the map τ_s given by $A_s \ni t \mapsto s^\dagger t$ is a complete order isomorphism of $(A_s, \leq)$ onto $(B_s, \leq)$.*

Proof. The proof is routine after observing that for $e_1, e_2 \in \mathcal{E}(\mathcal{S})$, $se_1 \wedge se_2 = s(e_1 \wedge e_2)$. ∎

We recall from Subsection 2.1, that for each $e \in \mathcal{E}(\mathcal{S})$, G_e is the clopen set in $\widehat{\mathcal{E}(\mathcal{S})}$ of characters supported on e.

Proposition 4.6. *Let $\mathcal{S}$ be a locally complete Boolean inverse monoid, and suppose $\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$ is an extension of $\mathcal{S}$ by $\mathcal{P}$. Then there exists an order preserving section $j : \mathcal{S} \to \mathcal{G}$ for q.*

Proof. We shall define j in steps, beginning with its definition on $\mathcal{E}(\mathcal{S})$. Recall that $q|_{\mathcal{E}(\mathcal{G})}$ is an isomorphism of $\mathcal{E}(\mathcal{G})$ onto $\mathcal{E}(\mathcal{S})$. Define j on $\mathcal{E}(\mathcal{S})$ by setting $j := (q|_{\mathcal{E}(\mathcal{G})})^{-1}$.

Next, choose a subset $\mathcal{B} \subseteq \mathcal{S}$ such that,

(a) $1 \in \mathcal{B}$;
(b) if $s_1, s_2 \in \mathcal{B}$ and $s_1 \neq s_2$, then $s_1 \wedge s_2 = 0$;
(c) $\mathcal{B}$ is maximal with respect to (b).

The second condition ensures that $\mathcal{B} \cap \mathcal{E}(\mathcal{S}) = \{1\}$.

Define j on the set $\mathcal{B}$ by choosing any function $j : \mathcal{B} \to \mathcal{G}$ such that $q \circ j = \mathrm{id}|_{\mathcal{B}}$ and $j(1) = j(1)$. For $s \in \mathcal{B}$, Let A_s and B_s be the sets as in Lemma 4.5. Lemma 4.5 shows that the map $A_s \ni t \mapsto s^\dagger t$ is an order isomorphism of $(A_s, \leq)$ onto $(B_s, \leq)$. Since $\mathcal{B}$ is pairwise meet orthogonal, the sets $\{A_s : s \in \mathcal{B}\}$ are pairwise disjoint. Hence we may extend j from $\mathcal{B}$ to $\bigcup_{s \in \mathcal{B}} A_s$ by defining $j(se) = j(s)j(e)$ when $s \in \mathcal{B}$ and $e \leq s^\dagger s$. By construction, j is order-preserving on $\bigcup_{s \in \mathcal{B}} A_s$.

We now wish to extend j to the remainder of $\mathcal{S}$. Let $L = \mathcal{S} \setminus \left(\bigcup_{s \in \mathcal{B}} A_s\right)$ and let $\phi : L \to \mathcal{G}$ be any map such that $q \circ \phi = \mathrm{id}_L$. We shall perturb ϕ so that it becomes order preserving and is compatible with the map j already defined on $\mathcal{S} \setminus L = \bigcup_{s \in \mathcal{B}} A_s$.

Fix $t \in L$ and put $w := \phi(t)$. By definition, $q(w) = t$. For each $s \in \mathcal{B}$, let $h_s := w^\dagger j(t \wedge s)$. Then $q(h_s) = t^\dagger (t \wedge s) = t^\dagger t \wedge t^\dagger s \in \mathcal{E}(\mathcal{S})$, whence $h_s \in \mathcal{P}$. Set

$$e_s := q(h_s^\dagger h_s).$$

Note that $e_s = (s \wedge t)^\dagger (s \wedge t)$. Then $\{e_s : s \in \mathcal{S}\}$ is a pairwise orthogonal subset of $\{e \in \mathcal{E}(\mathcal{S}) : e \leq t^\dagger t\}$. It follows that $\{G_{e_s} : s \in \mathcal{B}\}$ is a pairwise disjoint family of compact clopen subsets of $G_{t^\dagger t}$. Also, $G_{t^\dagger t}$ is a compact clopen subset of $\widehat{\mathcal{E}(\mathcal{S})}$. The maximality of $\mathcal{B}$ ensures that $H := \bigcup_{s\in\mathcal{B}} G_{e_s}$ is a dense open subset of $G_{t^\dagger t}$. We may thus uniquely define a continuous function $h : H \to \mathbb{C}$ such that $h|_{G_{e_s}} = h_s$. Since $\mathcal{S}$ is locally complete, $G_{t^\dagger t}$ is a Stonean space. By [21, Corollary III.1.8], h extends uniquely to a continuous complex valued function (again called h) on all of $G_{t^\dagger t}$. Extend h to all of $\widehat{\mathcal{E}(\mathcal{S})}$ by setting its values to be 0 on the complement of $G_{t^\dagger t}$. By construction, range$(h) \subseteq \mathbb{T} \cup \{0\}$, so $h \in \mathcal{P}$. Finally, set

$$j(t) = wh.$$

The construction shows that for $s \in \mathcal{B}$,

$$j(te_s) = j(t \wedge s) = j(t)j(e_s).$$

For $e \in \mathcal{E}(\mathcal{S})$ we have (using the facts that $\{r \in \mathcal{S} : r \le t\}$ is a complete Boolean algebra and $\bigvee\{e_s : s \in \mathcal{B}\} = t^\dagger t$),

$$j(te) = j\left(t \bigvee_{s\in\mathcal{B}} ee_s\right) = j\left(\bigvee_{s\in\mathcal{B}} te_se\right) = j\left(\bigvee_{s\in\mathcal{B}} (t\wedge s)e\right)$$

(now use Lemma 4.5 applied to the inverse monoid $\mathcal{G}$)

$$= \bigvee_{s\in\mathcal{B}} j(t\wedge s)j(e) = \bigvee_{s\in\mathcal{B}} j(t)j(e_s)j(e) = j(t)\left(\bigvee_{s\in\mathcal{B}} j(ee_s)\right) = j(t)j(e).$$

We have now defined j on L, so that it preserves order. Since j was defined earlier to be order preserving on $\mathcal{S} \setminus L$, we see that j is order preserving on all of $\mathcal{S}$. □

4.2. **A $\mathcal{D}$-valued reproducing kernel and a right Hilbert $\mathcal{D}$-module.** Let $\mathcal{S}$ be a Boolean inverse monoid and let $\mathcal{D} = C(\widehat{\mathcal{E}(\mathcal{S})})$. As always, we denote the partial isometries in $\mathcal{D}$ by $\mathcal{P}$. Let

$$\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$$

be an extension of $\mathcal{S}$ by $\mathcal{P}$. In this subsection, we will construct a $\mathcal{D}$-valued reproducing kernel $K : \mathcal{S}\times\mathcal{S} \to \mathcal{D}$ whose evaluation functionals $k_s(t) := K(t,s)$ represent the meet-lattice structure of $\mathcal{S}$ in the sense that the pointwise product of k_s with k_t satisfies $k_sk_t = k_{s\wedge t}$. The completion of span$\{k_s\}_{s\in\mathcal{S}}$ yields a $\mathcal{D}$-valued reproducing kernel right Hilbert $\mathcal{D}$-module, denoted $\mathfrak{A}$. There is an action of $\mathcal{S}$ on $\mathfrak{A}$ arising from the left action of $\mathcal{S}$ on itself: for $s,t \in \mathcal{S}$, $k_t \mapsto k_{st}$. We modify this action to produce a representation of the extension $\mathcal{G}$ on $\mathfrak{A}$ by partial isometries in the bounded adjointable maps $\mathcal{L}(\mathfrak{A})$. Finally, we obtain a class of representations of $\mathcal{G}$ on a Hilbert space using the interior tensor product $\mathfrak{A} \otimes_\pi \mathcal{H}$ where $(\pi, \mathcal{H})$ is a representation of $\mathcal{D}$ on the Hilbert space $\mathcal{H}$. When $\mathcal{S}$ is a Cartan inverse monoid and π is faithful, it is this representation of $\mathcal{G}$ that will generate a Cartan pair. We refer the reader to [3] for more on reproducing kernel Hilbert spaces.

. We begin with the definition of the $\mathcal{D}$-valued reproducing kernel. By Proposition 4.6, there is an order preserving section $j : \mathcal{S} \to \mathcal{G}$, which we consider fixed throughout the remainder of this section.

Definition 4.7. Define $K : \mathcal{S}\times\mathcal{S} \to \mathcal{D}$ by

$$K(t,s) = j(s^\dagger t \wedge 1),$$

and for $s \in \mathcal{S}$, define $k_s : \mathcal{S} \to \mathcal{D}$ by

$$k_s(t) = K(t, s).$$

Remark 4.8. By Lemma 2.3, $K(s,t)$ is the source idempotent of $j(s \wedge t)$, that is, $K(s,t) = j((s \wedge t)^\dagger (s \wedge t))$. Notice also that K is symmetric, that is $K(s,t) = K(t,s)$ for all $s, t \in \mathcal{S}$. The function k_s should be thought of as the "s-th column" of the "matrix" $K(t,s)$.

We will show in Lemma 4.11 that K is positive in the sense that

$$0 \leq \sum_{i,j}^{n} \overline{c_i} c_j K(s_i, s_j)$$

for any finite collection of scalars $c_1, \ldots, c_n$ and $s_1, \ldots, s_n$ in $\mathcal{S}$. This positivity will allow us define a $\mathcal{D}$-valued inner product on $\operatorname{span}\{k_s\}_{s \in \mathcal{S}}$. The completion of this span, with respect to the norm from the inner-product, will be a right Hilbert $\mathcal{D}$-module (Proposition 4.13).

The following simple corollary to Lemma 2.3 is immediate, since j is an order preserving section.

Corollary 4.9. *For any $s, t \in \mathcal{S}$ and any $e \in \mathcal{E}(S)$ we have*

$$K(t, se) = K(te, s) = K(te, se) = K(t, s) j(e).$$

Thus

$$k_{se} = k_s j(e).$$

The significance of K is that an "integral" on $\mathcal{D} j(s^\dagger s)$, that is, a weight or state on $\mathcal{D}$ restricted to $\mathcal{D} j(s^\dagger s)$, may be translated in a consistent way using K to an "integral" on A_s. Remark 4.19 below explores this further in the context of the Feldman-Moore construction.

Corollary 4.9 shows how the map $s \mapsto k_s$ respects the order structure of $\mathcal{S}$. This is further cemented in the following lemma, where we show that the mapping $s \mapsto k_s$ is a meet-lattice representation of $\mathcal{S}$ as a family of functions from $\mathcal{S}$ into the lattice of projections of $\mathcal{D}$. Thus we are constructing a $\mathcal{D}$-module from the order structure of $\mathcal{S}$.

Lemma 4.10. *For $r, s, t \in \mathcal{S}$, we have*

$$(s^\dagger t \wedge 1)(r^\dagger t \wedge 1) = ((s \wedge r)^\dagger t) \wedge 1, \tag{4.1}$$

hence

$$K(t, r) K(t, s) = K(t, r \wedge s).$$

In particular,

$$k_r k_s = k_{r \wedge s}.$$

Proof. Take any $r, s, t \in \mathcal{S}$. Applying the isomorphism τ_t of $(A_t, \leq)$ onto $(B_t, \leq)$ of Lemma 4.5 and using Lemma 2.3, we obtain

$$\begin{aligned} \big(t^\dagger (s \wedge r)\big) \wedge 1 &= \tau_t(s \wedge r \wedge t) = \tau_t(s \wedge t) \wedge \tau_t(r \wedge t) \\ &= (t^\dagger s \wedge 1)(t^\dagger r \wedge 1). \end{aligned}$$

This equality is equivalent to (4.1). The remaining statements of the lemma follow. $\square$

We will now show that $K(t,s) = j(s^\dagger t \wedge 1)$ defines a positive $\mathcal{D}$-valued positive kernel. A key step in this will be to show that if ρ is a pure state on $\mathcal{D}$, then $\rho \circ K$ defines a positive semi-definite matrix on $\mathcal{S}$, in the sense of Moore as described in [3, p. 341]. That is, for each pure state ρ on $\mathcal{D}$, and for $s_1, \dots, s_n \in \mathcal{S}$ and $c_1, \dots, c_n \in \mathbb{C}$

$$0 \leq \sum_{i,j}^{n} \overline{c_i} c_j K_\rho(s_i, s_j),$$

where $K_\rho := \rho \circ K$. Thus, each pure state ρ will determine a reproducing kernel Hilbert space $\mathcal{H}_\rho$ of functions on $\mathcal{S}$, with kernel K_ρ and point-evaluation functions $(k_s)_\rho := \rho \circ k_s$.

Lemma 4.11. *Let $n \in \mathbb{N}$, let $c_1, \dots, c_n$ be complex numbers, and let $s_1, \dots, s_n \in \mathcal{S}$. Then, with respect to the positive cone in $\mathcal{D}$,*

$$0 \leq \sum_{i,j=1}^{n} \overline{c_i} c_j K(s_i, s_j).$$

Proof. Fix a pure state ρ on $\mathcal{D}$. Note that, as $\mathcal{D} = C(\widehat{\mathcal{E}(\mathcal{S})})$, we view ρ as being in $\widehat{\mathcal{E}(\mathcal{S})}$. With this identification, we have $\rho(K(s_i, s_j)) = \rho(s_j^\dagger s_i \wedge 1)$. Let $\mathbf{n} = \{1, \dots, n\}$ and

$$R = \{(i,j) \in \mathbf{n} \times \mathbf{n} : \rho(K(s_i, s_j)) = 1\}.$$

We shall show that R is a symmetric and transitive relation. Symmetry of R is immediate from the fact that $K(s_i, s_j) = K(s_j, s_i)$.

Suppose for some $i, j, k \in \mathbf{n}$, that $\rho(K(s_i, s_j)) = 1 = \rho(K(s_j, s_k))$. To show that R is transitive, we must show that $\rho(K(s_i, s_k)) = 1$. By Lemma 2.3 and Lemma 4.10, we have

$$\begin{aligned} K(s_i, s_j)K(s_j, s_k) = K(s_j, s_i)K(s_j, s_k) &= K(s_j, s_i \wedge s_k) \\ &\leq K(s_i, s_k). \end{aligned}$$

Applying ρ yields $1 = \rho(K(s_i, s_k))$. Thus $(i,k) \in R$, and R is transitive.

The symmetry and transitivity of R imply that if $(i,j) \in R$, then both (i,i) and (j,j) belong to R. Let

$$\mathbf{n}_1 := \{i \in \mathbf{n} : \rho(s_i^\dagger s_i) = 1\}.$$

Then $R \subseteq \mathbf{n}_1 \times \mathbf{n}_1$ and R is an equivalence relation on $\mathbf{n}_1$.

Write $\mathbf{n}_1 = \bigcup_{m=1}^r X_m$ as the disjoint union of the equivalence classes for R, and let $T(\rho) \in M_n(\mathbb{C})$ be the matrix whose ij-th entry is $\rho(K(s_j, s_i)) = \rho(j(s_i^\dagger s_j \wedge 1))$. Let $\{\xi_j\}_{j=1}^n$ be the standard basis for $\mathbb{C}^n$ and let $\zeta_m = \sum_{j \in X_m} \xi_j$. Then

$$T(\rho) = \sum_{m=1}^{r} \zeta_m \zeta_m^* \geq 0,$$

where $\zeta_m \zeta_m^*$ is the rank-one operator, $\xi \mapsto \langle \zeta_m, \xi \rangle \zeta_m$. Thus,

$$\rho\left(\sum_{i,j=1}^{n} \overline{c_i} c_j K(s_i, s_j)\right) = \left\langle T(\rho) \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}, \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \right\rangle \geq 0.$$

As this holds for every pure state ρ on $\mathcal{D}$, we find that $\sum_{i,j=1}^n \overline{c_i} c_j K(s_i, s_j) \geq 0$ in $\mathcal{D}$. ☐

Let

$$\mathfrak{A}_0 = \mathrm{span}\{k_s \colon s \in \mathcal{S}\}.$$

Our goal is show that there is a natural $\mathcal{D}$-valued inner product on $\mathfrak{A}_0$. Completing with respect to the norm induced by this inner product will yield a Banach space $\mathfrak{A}$ of functions from $\mathcal{S}$ into $\mathcal{D}$. For a function $u \in \mathfrak{A}$ and $d \in \mathcal{D}$, define ud to be the function from $\mathcal{S}$ into $\mathcal{D}$ given by

$$(ud)(s) := u(s)d.$$

We shall show that this makes $\mathfrak{A}$ into a right Hilbert $\mathcal{D}$-module.

In order to study the space $\mathfrak{A}_0$, we use the reproducing kernel Hilbert spaces generated by composing elements of $\mathfrak{A}_0$ with a given pure state ρ on $\mathcal{D}$.

For each pure state ρ on $\mathcal{D}$ and $f \in \mathfrak{A}_0$, write f_ρ for the function $\rho \circ f$ on $\mathcal{S}$. In particular, notice that for $s \in \mathcal{S}$, $(k_s)_\rho(t) = \rho(K(s,t))$. Likewise, set $K_\rho(t,s) := \rho(K(t,s))$. Finally, put

$$\mathfrak{A}_\rho := \mathrm{span}\{(k_s)_\rho : s \in \mathcal{S}\}.$$

It was shown in Lemma 4.11 that K_ρ is a positive matrix in the sense of [3]. Thus K_ρ determines a reproducing kernel Hilbert space $\mathcal{H}_\rho$, with $\mathfrak{A}_\rho$ dense in $\mathcal{H}_\rho$. For $u_\rho, v_\rho \in \mathfrak{A}_\rho$, we may find $n \in \mathbb{N}$, $c_i, d_i \in \mathbb{C}$, and $s_i, t_j \in \mathcal{S}$ so that $u_\rho = \sum_{i=1}^n c_i(k_{s_i})_\rho$ and $v_\rho = \sum_{i=1}^n d_i(k_{t_i})_\rho$. The formula,

$$\begin{aligned}\langle u_\rho, v_\rho\rangle_\rho := \left\langle \sum_{i=1}^n c_i(k_{s_i})_\rho, \sum_{j=1}^n d_i(k_{t_j})_\rho \right\rangle_\rho &= \sum_{i,j=1}^n \overline{c_i} d_j K_\rho(s_i,t_j) \\ &= \sum_{i,j=1}^n \overline{c_i} d_j \rho(K(s_i,t_j))\end{aligned}$$

gives a well-defined inner product on $\mathfrak{A}_\rho$ by the Moore-Aronszajn Theorem (see [3, paragraph (4), p. 344]). In particular, $\|u_\rho\| := \langle u_\rho, u_\rho\rangle_\rho$ is a norm on $\mathfrak{A}_\rho$.

We now are ready to define a $\mathcal{D}$-valued inner product on $\mathfrak{A}_0$. We wish the inner product to be conjugate linear in the first variable and to satisfy,

$$\langle k_s, k_t\rangle = k_t(s) = K(s,t).$$

As before, for $u, v \in \mathfrak{A}_0$, write u and v as linear combinations, $u = \sum_{i=1}^n c_i k_{s_i}$ and $v = \sum_{i=1}^n d_i k_{t_i}$ of elements from $\{k_s : s \in \mathcal{S}\}$. Then for a pure state ρ on $\mathcal{D}$,

$$\rho\left(\sum_{i,j=1}^n \overline{c_i} d_j K(s_i,t_j)\right) = \langle u_\rho, v_\rho\rangle_\rho .$$

Hence $\sum_{i,j=1}^n \overline{c_i} d_j K(s_i,t_j) \in \mathcal{D}$ depends only upon u and v and not on the choice of linear combinations from $\{k_s : s \in \mathcal{S}\}$ which represent them. Therefore, the following definition makes sense.

Definition 4.12. For $u = \sum_{i=1}^n c_i k_{s_i}$ and $v = \sum_{i=1}^n d_i k_{t_i}$ in $\mathfrak{A}_0$,

$$\langle u, v\rangle := \sum_{i,j=1}^n \overline{c_i} d_j K(s_i,t_j)$$

is a well-defined $\mathcal{D}$-valued inner product on $\mathfrak{A}_0$ which is sesquilinear in the first variable.

Finally, define $\mathfrak{A}$ to be the completion of $\mathfrak{A}_0$ relative to the norm,

$$\|u\| := \|\langle u, u\rangle\|_{\mathcal{D}}^{1/2}.$$

Proposition 4.13. *The space $\mathfrak{A}$ is a Banach space of functions from $\mathcal{S}$ to $\mathcal{D}$, satisfying*

$$u(s) = \langle k_s, u\rangle$$

for each $u \in \mathfrak{A}$ and $s \in \mathcal{S}$.

The right action of $\mathcal{D}$ on $\mathfrak{A}$ given by

$$(vh)(s) = v(s)h$$

for $h \in \mathcal{D}$, $v \in \mathfrak{A}$ and $s \in S$, makes $\mathfrak{A}$ into a right Hilbert $\mathcal{D}$-module.

Proof. It follows from Definition 4.12 that, for $u \in \mathfrak{A}_0$ and $s \in \mathcal{S}$, $u(s) = \langle k_s, u\rangle$. For any $u, v \in \mathfrak{A}_0$ and any pure state ρ on $\mathcal{D}$ we have

$$\rho(\langle u, v\rangle) = \langle u_\rho, v_\rho\rangle_\rho.$$

Hence

$$\langle u, v\rangle^* \langle u, v\rangle \leq \langle u, u\rangle\langle v, v\rangle$$

for any $u, v \in \mathfrak{A}_0$. For $s \in \mathcal{S}$, it now follows that the evaluation map $\varepsilon_s\colon \mathfrak{A}_0 \to \mathcal{D}$ given by

$$\mathfrak{A}_0 \ni u \mapsto u(s) \in \mathcal{D}$$

is continuous. Thus, the evaluation map ε_s extends to a continuous map on $\mathfrak{A}$. Therefore, $\mathfrak{A}$ is a Banach space of functions from $\mathcal{S}$ into $\mathcal{D}$, and for each $u \in \mathfrak{A}$ and $s \in \mathcal{S}$ we have $u(s) = \langle k_s, u\rangle$.

Write $\mathcal{D}_0$ for the linear span of the projections in $\mathcal{D}$. Note that, for $u, v \in \mathfrak{A}_0$ we have that $\langle u, v\rangle \in \mathcal{D}_0$. Further if $h \in \mathcal{D}_0$, then for any $v \in \mathfrak{A}_0$, $vh \in \mathfrak{A}_0$. To see this, write v and h as linear combinations, $v = \sum_{n=1}^N \alpha_n k_{t_n}$ and $h = \sum_{m=1}^N \beta_m j(e_m)$, where $\alpha_n, \beta_m \in \mathbb{C}$, $t_n \in \mathcal{S}$ and $e_m \in \mathcal{E}(\mathcal{S})$. Then, by Corollary 4.9, we have

$$vh = \sum_{n,m=1}^N \alpha_n\beta_m k_{t_n} j(e_m) = \sum_{n,m=1}^N \alpha_n\beta_m k_{t_n e_m} \in \mathfrak{A}_0.$$

A calculation, again using Corollary 4.9, shows that for $u \in \mathfrak{A}_0$, we have

$$\langle u, vh\rangle = \langle u, v\rangle\, h. \tag{4.2}$$

Continuity of multiplication by h then yields $vh \in \mathfrak{A}$ for every $v \in \mathfrak{A}$ and $h \in \mathcal{D}_0$. As the projections of $\mathcal{D}$ span a norm dense subset of $\mathcal{D}$, we then obtain $vh \in \mathfrak{A}$ for every $v \in \mathfrak{A}$ and $h \in \mathcal{D}$. Since equation (4.2) passes to the completions, the proof is complete. $\square$

4.3. **The construction of the representation.** Our goal in this subsection is to construct a representation of an extension $\mathcal{G}$ of a Boolean inverse monoid $\mathcal{S}$ by $\mathcal{P}$, where $\mathcal{P}$ is the semigroup of partial isometries in $\mathcal{D} = C(\widehat{\mathcal{E}(\mathcal{S})})$. We will represent $\mathcal{G}$ in the adjointable operators on $\mathfrak{A}$, where $\mathfrak{A}$ is the right Hilbert $\mathcal{D}$-module constructed in the previous subsection.

We fix an extension

$$\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}.$$

By Proposition 4.6, there is an order preserving section $j\colon \mathcal{S} \to \mathcal{G}$, such that $q \circ j = \mathrm{id}_{\mathcal{S}}$. Thus, while the extension need not be a trivial trivial extension of inverse semigroups, we do have a splitting at the level of partially ordered sets:

$$\begin{tikzcd} (\mathcal{P}, \leq) \arrow[r, hook] & (\mathcal{G}, \leq) \arrow[r, bend left, "q"] & (\mathcal{S}, \leq) \arrow[l, bend left, "j"] \end{tikzcd} .$$

A construction of Lausch [12] shows that, up to an equivalent extension, $\mathcal{G}$ can be viewed as the set

$$\{[s,p]\colon s \in \mathcal{S},\ p \in \mathcal{P},\ q(p^\dagger p) = s^\dagger s\}.$$

That is, every element $v = [s,p] \in \mathcal{G}$ consists of a function $p \in \mathcal{P}$ "supported" on an element $s \in \mathcal{S}$. The product on $\mathcal{G}$ is then determined by a cocycle function

$$\begin{aligned} \alpha\colon \mathcal{S} \times \mathcal{S} &\to \mathcal{P} \\ [s,t] &\mapsto j(st)^\dagger j(s) j(t). \end{aligned}$$

We do not wish to dwell on this viewpoint. It can be computationally cumbersome, and while it lies behind much of our work, most of it is unnecessary for our constructions. However, we will need to be able to describe certain elements of $\mathcal{G}$ in terms of their support in $\mathcal{S}$ and a function in $\mathcal{P}$. In order to do this, we construct a cocycle-like function, related to the cocycle α.

Definition 4.14. Define a cocycle-like function $\sigma\colon \mathcal{G} \times \mathcal{S} \to \mathcal{P}$ by

$$\sigma(v,s) = j(q(v)s)^\dagger v j(s) = j(s^\dagger q(v^\dagger)) v j(s).$$

Since

$$q(\sigma(v,s)) = s^\dagger q(v^\dagger v) s \in \mathcal{E}(\mathcal{S}),$$

$\sigma(v,s) \in \mathcal{P}$. Thus σ indeed maps $\mathcal{G} \times \mathcal{S}$ into $\mathcal{P}$.

Remark 4.15. Lausch's cocycle α can be recovered from σ as follows. For any $s,t \in \mathcal{S}$ we have

$$\sigma(j(s),t) = j(st)^\dagger j(s) j(t) = \alpha(s,t).$$

Also, observe that for all $v \in \mathcal{G}$ and $s \in \mathcal{S}$ we have

$$vj(s) = j(q(v)s)\sigma(v,s).$$

Thus elements of the form $vj(s)$ can be described in terms of an element $q(v)s \in \mathcal{S}$ and $\sigma(v,s) \in \mathcal{P}$. The left action of $\mathcal{G}$ on the set $j(\mathcal{S})$ will be used to construct the representation.

Lemma 4.16. *Let $N \in \mathbb{N}$ and let $s_1, \dots, s_N$ be non-zero elements of $\mathcal{S}$. There exists a finite set $A \subseteq \mathcal{S}$ with the following properties.*

- *(a) $0 \notin A$.*
- *(b) The elements of A are pairwise meet orthogonal.*
- *(c) Each $a \in A$ satisfies:*
  - *i) for $1 \leq n \leq N$, $a \wedge s_n \in \{a, 0\}$; and*
  - *ii) there exists $1 \leq n \leq N$ such that $a \wedge s_n = a$.*
- *(d) For each $1 \leq n \leq N$, $s_n = \bigvee \{a \in A : a \leq s_n\}$.*

Proof. Throughout the proof, when $s, t \in \mathcal{S}$, we use $s \setminus t$ to denote the element $s(s^\dagger s \wedge \neg(t^\dagger t))$; thus $s \setminus t$ is orthogonal to $s \wedge t$ and satisfies $(s \setminus t) \vee (s \wedge t) = s$.

We argue by induction on N. If $N = 1$, take $A = \{s_1\}$. Suppose now that $N > 1$ and the result holds whenever we are given non-zero elements $s_1, \dots, s_{N-1}$ of $\mathcal{S}$.

Let $s_1, \dots, s_N$ be non-zero elements of $\mathcal{S}$ and let A_{N-1} be the set constructed using the induction hypothesis applied to $s_1, \dots, s_{N-1}$. For each $b \in A_{N-1}$, let $C_b := \{b \wedge s_N, b \setminus s_N\} \setminus \{0\}$ and put $X := \bigcup_{b \in A_{N-1}} C_b$. Since the elements of A_{N-1} are pairwise meet disjoint, so are the elements of X. Let $t := \bigvee\{x \in X : x \leq s_N\}$ and let $r := s_N \setminus t$. Notice that $r \wedge x = 0$ for any $x \in X$. Finally, define

$$A := \begin{cases} \{r\} \cup X & \text{if } r \neq 0; \\ X & \text{if } r = 0. \end{cases}$$

Then A is pairwise meet orthogonal, and $0 \notin A$. By construction, we have $s_N = \bigvee\{a \in A : a \leq s_N\}$. Moreover, if $1 \leq n \leq N-1$ and $b \in A_{N-1}$ with $b \leq s_n$, then $b = \bigvee C_b$. Since A_{N-1} satisfies property (d), we obtain $s_n = \bigvee\{a \in A : a \leq s_n\}$. Thus A satisfies property (d) also.

Property (c) is equivalent to the statement that for $a \in A$,

$$\{0\} \neq \{a \wedge s_n : 1 \leq n \leq N\} \subseteq \{0, a\}.$$

For $a \in X$, this clearly holds. Suppose $r \neq 0$. Then $r \wedge s_N = r$, so $\{0\} \neq \{r \wedge s_n : 1 \leq n \leq N\}$. If $1 \leq n \leq N-1$, then $r \wedge s_n = r \wedge (\bigvee\{b \in X : b \leq s_n\}) = 0$. Hence $\{r \wedge s_n : 1 \leq n \leq N\} \subseteq \{r, 0\}$. Therefore A satisfies the requisite properties and we are done. $\square$

We now have all the ingredients we need to construct our representation of $\mathcal{G}$. We recall, that $\mathfrak{A}$ is the the right Hilbert $\mathcal{D}$-module constructed in Subsection 4.2.

Theorem 4.17. *For $v \in \mathcal{G}$ and $s \in \mathcal{S}$, the formula,*

$$\lambda(v)k_s := k_{q(v)s}\sigma(v, s)$$

determines a partial isometry $\lambda(v) \in \mathcal{L}(\mathfrak{A})$. Moreover, $\lambda : \mathcal{G} \to \mathcal{L}(\mathfrak{A})$ is a one-to-one representation of $\mathcal{G}$ as partial isometries in $\mathcal{L}(\mathfrak{A})$.

Proof. Fix $v \in \mathcal{G}$, and set $r := q(v)$. Given $s_1, \dots, s_N \in \mathcal{S}$, let A be the set constructed in Lemma 4.16. Choose $c_1, \dots, c_N \in \mathbb{C}$.

For $a \in A$ and $1 \leq m \leq N$, put

$$A_m := \{b \in A : b \leq s_m\} \quad \text{and} \quad c_a := \sum\{c_n : a \leq s_n\}.$$

Notice that the elements of A_m are pairwise orthogonal, and $\bigvee A_m = s_m$.

We first note that

$$\sum_{n=1}^{N} c_n k_{s_n} = \sum_{a \in A} c_a k_a. \tag{4.3}$$

To see this, for any $t \in \mathcal{S}$,

$$K(t, s_n) = \sum_{a \in A_n} K(t, a).$$

Thus,

$$\begin{aligned}\left(\sum_{n=1}^{N} c_n k_{s_n}\right)(t) &= \sum_{n=1}^{N} c_n K(t, s_n) = \sum_{n=1}^{N} c_n \sum_{a\in A_n} K(t,a) \\ &= \sum_{a\in A} c_a K(t,a) = \left(\sum_{a\in A} c_a k_a\right)(t).\end{aligned}$$

Secondly, we claim

$$\sum_{n=1}^{N} c_n k_{rs_n}\sigma(v, s_n) = \sum_{a\in A} c_a k_{ra}\sigma(v,a). \tag{4.4}$$

To see this, first notice that if $a \in A$ and $a \leq s_n$, then using the fact that j is order preserving we have

$$\sigma(v, s_n)K(t, ra) = \sigma(v, s_n)j(a^\dagger a)K(t, ra) = \sigma(v, a)K(t, ra).$$

Thus,

$$\begin{aligned}\left(\sum_{n=1}^{N} c_n k_{rs_n}\sigma(v, s_n)\right)(t) &= \sum_{n=1}^{N} c_n \sigma(v, s_n)K(t, rs_n) \\ &= \sum_{n=1}^{N} c_n \left(\sum_{a\in A_n} \sigma(v, s_n)K(t, ra)\right) \\ &= \sum_{n=1}^{N} c_n \left(\sum_{a\in A_n} \sigma(v, a)K(t, ra)\right) \\ &= \sum_{a\in A} c_a \sigma(v,a)K(t, ra) \\ &= \left(\sum_{a\in A} c_a k_a \sigma(v,a)\right)(t),\end{aligned}$$

as desired.

Notice that if $a, b \in A$ are distinct, then $\langle k_{ra}\sigma(v,a), k_{rb}\sigma(v,b)\rangle = 0 = \langle k_a, k_b\rangle$. Thus, using (4.4), then (4.3),

$$\begin{aligned}\left\langle \sum_{n=1}^{N} c_n k_{rs_n}\sigma(v,s_n), \sum_{n=1}^{N} c_n k_{rs_n}\sigma(v,s_n)\right\rangle &= \left\langle \sum_{a\in A} c_a k_{ra}\sigma(v,a), \sum_{a\in A} c_a k_{ra}\sigma(v,a)\right\rangle \\ &= \sum_{a\in A} |c_a|^2 j(a^\dagger r^\dagger ra) \\ &= \left\langle \sum_{a\in A} c_a k_{ra}, \sum_{a\in A} c_a k_{ra}\right\rangle \\ &\leq \sum_{a\in A} |c_a|^2 j(a^\dagger a) \\ &= \left\langle \sum_{a\in A} c_a k_a, \sum_{a\in A} c_a k_a\right\rangle \\ &= \left\langle \sum_{n=1}^{N} c_n k_{s_n}, \sum_{n=1}^{N} c_n k_{s_n}\right\rangle.\end{aligned}$$

Therefore,

$$\left\| \sum_{n=1}^{N} c_n \lambda(v) k_{s_n}\right\| \leq \left\| \sum_{n=1}^{n} c_n k_{s_n}\right\|.$$

It follows that we may extend $\lambda(v)$ linearly to a contractive operator from $\mathfrak{A}_0$ into $\mathfrak{A}$. Finally extend $\lambda(v)$ by continuity to a contraction in $\mathcal{B}(\mathfrak{A})$, the bounded operators on $\mathfrak{A}$.

We next show that $\lambda(v)$ is adjointable. Note that, for $s, t \in \mathcal{S}$, it follows from Lemma 2.3 and Corollary 4.9 that

$$K(t, rs) = K(r^\dagger t, s).$$

Further, setting $f = (s^\dagger r^\dagger t) \wedge 1 = j^{-1}(K(t, rs))$, it follows from Lemma 2.3 that

$$rs \wedge t = rsf = tf$$

and

$$s \wedge r^\dagger t = sf = r^\dagger tf.$$

Hence

$$\sigma(v,s)^\dagger K(rs,t) = \sigma(v^\dagger, t) K(s, r^\dagger t).$$

Therefore for any $s, t \in \mathcal{S}$,

$$\begin{aligned}\langle \lambda(v) k_s, k_t\rangle = \langle k_{rs}\sigma(v,s), k_t\rangle &= \sigma(v,s)^\dagger K(rs,t) \\ &= \sigma(v^\dagger, t) K(s, r^\dagger t) = \langle k_s, \lambda(v^\dagger) k_t\rangle.\end{aligned}$$

This equality implies that $\lambda(v)$ is adjointable and $\lambda(v)^* = \lambda(v^\dagger)$.

We now show that λ is a homomorphism. Suppose that $v_1, v_2 \in \mathcal{G}$ and $s \in \mathcal{S}$. Then

$$\begin{aligned}\lambda(v_1)(\lambda(v_2)k_s) &= \lambda(v_1)(k_{q(v_2)s}\sigma(v_2, s)) \\ &= (\lambda(v_1)k_{q(v_2)s})\sigma(v_2,s) \\ &= k_{q(v_1v_2)s}\ \sigma(v_1, q(v_2)s)\ \sigma(v_2, s).\end{aligned}$$

But

$$\begin{aligned}\sigma(v_1, q(v_2)s)\ \sigma(v_2, s) &= j(q(v_1)q(v_2)s))^\dagger v_1 j(q(v_2)s)\ j(q(v_2)s)^\dagger v_2 j(s)\\ &= j(q(v_1v_2)s))^\dagger v_1 j(q(v_2)s)\ j(s^\dagger q(v_2)^\dagger)v_2 j(s)\\ &= j(q(v_1v_2)s))^\dagger v_1(v_2 j(ss^\dagger)v_2^\dagger)v_2 j(s)\\ &= j(q(v_1v_2)s))^\dagger v_1 v_2 v_2^\dagger v_2 j(ss^\dagger)j(s)\\ &= j(q(v_1v_2)s))^\dagger v_1 v_2 j(s) = \sigma(v_1v_2, s).\end{aligned}$$

Hence $\lambda(v_1)\lambda(v_2)k_s = \lambda(v_1v_2)k_s$. As $\operatorname{span}\{k_s : s \in \mathcal{S}\}$ is dense in $\mathfrak{A}$, we conclude that $\lambda(v_1v_2) = \lambda(v_1)\lambda(v_2)$.

It follows that for every $e \in \mathcal{E}(\mathcal{G})$, $\lambda(e)$ is a projection. Furthermore, for $v \in \mathcal{G}$, $\lambda(v)$ is a partial isometry because $\lambda(v)^* = \lambda(v^\dagger)$.

It remains to show that λ is one-to-one. We first show that $\lambda|_{\mathcal{E}(\mathcal{G})}$ is one-to-one. So suppose $e, f \in \mathcal{E}(\mathcal{S})$ and $\lambda(j(e)) = \lambda(j(f))$. Then for every $s \in \mathcal{S}$, $k_{es}\sigma(j(e), s) = k_{fs}\sigma(j(f), s)$, whence $k_{es}j(s^\dagger es) = k_{fs}j(s^\dagger fs)$. Taking $s = 1$ gives $k_e j(e) = k_f j(f)$. Evaluating these functions at $t = 1$ gives $j(e) = j(f)$, so $\lambda|_{\mathcal{E}(\mathcal{G})}$ is one-to-one.

Now suppose $v_1, v_2 \in \mathcal{G}$ and $\lambda(v_1) = \lambda(v_2)$. Then

$$\lambda(v_1^\dagger v_1) = \lambda(v_1^\dagger v_2) = \lambda(v_1^\dagger v_2)^* = \lambda(v_2^\dagger v_1) = \lambda(v_2^\dagger v_2).$$

Likewise,

$$\lambda(v_1v_1^\dagger) = \lambda(v_1v_2^\dagger) = \lambda(v_2v_1^\dagger) = \lambda(v_2v_2^\dagger).$$

Hence $v_1^\dagger v_1 = v_2^\dagger v_2$ and $v_1v_1^\dagger = v_2v_2^\dagger$. For any $e \in \mathcal{E}(\mathcal{S})$, we have

$$\begin{aligned}\lambda(v_1 j(e)v_1^\dagger) &= \lambda(v_1v_1^\dagger v_1 j(e)v_1^\dagger v_1 v_1^\dagger) = \lambda(v_1 v_2^\dagger v_2 j(e) v_2^\dagger v_2 v_1^\dagger) = \lambda(v_2v_2^\dagger v_2 j(e)v_2^\dagger v_2 v_2^\dagger)\\ &= \lambda(v_2 j(e) v_2^\dagger).\end{aligned}$$

Hence $v_1 j(e) v_1^\dagger = v_2 j(e) v_2^\dagger$. Since this holds for every $e \in \mathcal{E}(\mathcal{S})$ and $\mathcal{S}$ is fundamental, we conclude that

$$q(v_1) = q(v_2).$$

Put $e := q(v_1^\dagger v_1)$ and $s := q(v_1)$. Since the functions $\lambda(v_1)k_e$ and $\lambda(v_2)k_e$ agree, we obtain, $k_s j(s)^\dagger v_1 = k_s j(s)^\dagger v_2$. Evaluating these functions at $t = s$ gives, $j(s)^\dagger v_1 = j(s)^\dagger v_2$. Now multiply each side of this equality on the left by $j(s)$ to obtain $v_1 = v_2$. □

We recall some facts about interior tensor products which may be found in [11, pages 38–44]. We will only need the interior tensor product of $\mathfrak{A}$ with a Hilbert space. If $\mathcal{H}$ is a Hilbert space and $\pi : \mathcal{D} \to \mathcal{H}$ is a $*$-representation, the balanced tensor product of $\mathfrak{A} \otimes_{\mathcal{D}} \mathcal{H}$ is the quotient of the algebraic tensor product of $\mathfrak{A}$ with $\mathcal{H}$ by the subspace generated by $\{ud \otimes \xi - u \otimes \pi(d)\xi : u \in \mathfrak{A}, d \in \mathcal{D}, \xi \in \mathcal{H}\}$. The balanced tensor product admits a semi-inner product given by

$$\langle u_1 \otimes \xi_1, u_2 \otimes \xi_2\rangle = \langle \xi_1, \pi(\langle u_1, u_2\rangle)\xi_2\rangle\,.$$

Let $N = \{x \in \mathfrak{A} \otimes_{\mathcal{D}} \mathcal{H} : \langle x, x\rangle = 0\}$. The completion of $(\mathfrak{A} \otimes_{\mathcal{D}} \mathcal{H})/N$ yields the *interior tensor product* of $\mathfrak{A}$ with $\mathcal{H}$, and is denoted $\mathfrak{A} \otimes_\pi \mathcal{H}$. Notice this is a Hilbert space.

Recall also that there is a $*$-representation $\pi_* : \mathcal{L}(\mathfrak{A}) \to \mathcal{B}(\mathfrak{A} \otimes_\pi \mathcal{H})$ given by

$$\pi_*(T)(u \otimes \xi) = (Tu) \otimes \xi. \tag{4.5}$$

This representation is strictly continuous on the unit ball of $\mathcal{L}(\mathfrak{A})$ and is faithful whenever π is faithful [11, p. 42]. The following is now immediate.

Corollary 4.18. *Let let $\pi : \mathcal{D} \to \mathcal{B}(\mathcal{H})$ be a $*$-representation of $\mathcal{D}$ on the Hilbert space $\mathcal{H}$. Then $\lambda_\pi := \pi_* \circ \lambda$ is a representation of $\mathcal{G}$ by partial isometries on $\mathfrak{A} \otimes_\pi \mathcal{H}$. If π is faithful, then λ_π is one-to-one.*

Remark 4.19. In this remark, we continue to outline a comparison of our constructions with those of Feldman and Moore. Full details are left to the interested reader.

Assume that $(X, \mathcal{B})$ is a standard Borel space, $R \subseteq X \times X$ is a Borel equivalence relation (with countable equivalence classes), μ is a quasi-invariant measure on X, $\mathcal{S}$ is the Cartan inverse monoid of all partial Borel isomorphisms on X whose graphs are contained in R, and ν is right-counting measure on R (see [6, Theorem 2]). Let c be a 2-cocycle on the equivalence relation R. As in [7, Section 2], we assume that c is normalized (and hence skew-symmetric) in the sense of [6, page 314]. Using the Feldman-Moore construction (c.f. [7, Section 2]), form the Cartan pair $(M(R,c), D(R,c))$. Recall that $M(R,c)$ consists of certain measurable functions on R and that $D(R,c)$ are those which are supported on the diagonal $\{(x,x) : x \in X\}$ of R. Note that $D(R,c) \simeq L^\infty(X,\mu)$.

As done in Section 3, let $\mathcal{G} = \mathcal{GN}(M(R,c), D(R,c))$ and let $\mathcal{P}$ be the partial isometries in $D(R,c)$. For $v \in \mathcal{G}$, the map

$$D(R,c)v^*v \ni dv^*v \mapsto vdv^* \in D(R,c)vv^*$$

is an isomorphism of $D(R,c)v^*v$ onto $D(R,c)vv^*$ and hence arises from a partial Borel isomorphism of X. This partial Borel isomorphism is $q(v)$. Finally, let $j : \mathcal{S} \to \mathcal{G}$ be given by $j(s) := \chi_{\mathrm{Graph}(s)}$. We have now explicitly described the the various components of the extension,

$$\begin{tikzcd} \mathcal{P} \arrow[r, hook] & \mathcal{G} \arrow[r, bend left, "q"] & \mathcal{S} \arrow[l, bend left, "j"] \end{tikzcd}$$

and the section j associated with a Cartan pair $(M(R,c), D(R,c))$ presented using the Feldman-Moore construction.

Next, we give a formula for the "cocycle-like" function of Definition 4.14 in terms of the Feldman-Moore data. For $v \in \mathcal{G}$, we know $h_v := j(q(v))^\dagger v \in D(R,c)$ and $v = j(q(v))h_v$. Using the fact that c is a normalized cocycle, for almost all $(x,y) \in R$ we obtain

$$v(x,y) = \chi_{\mathrm{Graph}(q(v))}(x,y) h_v(y,y).$$

Now for $s \in \mathcal{S}$, $\sigma(v,s) = j(q(v)s)^\dagger v j(s)$. A computation then shows that for $(x,y) \in R$,

$$(vj(s))(x,y) = \chi_{\mathrm{Graph}(q(v)s)}(x,y)\ h_v(s(y), s(y))\ c((q(v)s)(y), s(y), y),$$

and, (again using the fact that c is normalized)

$$\sigma(v,s)(x,y) = \chi_{\mathrm{Graph}(s^\dagger q(v^\dagger v)s)}(x,y)\ h_v(s(y), s(y))\ c((q(v)s)(y), s(y), y). \tag{4.6}$$

Let π be the representation of $D(R,c)$ on $\mathcal{H} := L^2(X,\mu)$ as multiplication operators: for $f \in D(R,c)$, $\xi \in L^2(X,\mu)$ and $x \in X$, $(\pi(f)\xi)(x) = f(x,x)\xi(x)$. Clearly π is a faithful, normal representation of $D(R,c)$.

Our next task is to observe that the representation λ_π of $\mathcal{G}$ on $\mathcal{B}(\mathfrak{A} \otimes_\pi \mathcal{H})$ is unitarily equivalent to the identity representation of $\mathcal{G}$ on $L^2(R, \nu)$.

For $s \in \mathcal{S}$ and $\xi \in L^2(X, \mu)$, let $F_{s,\xi}(x, y) := \xi(y)\chi_{\mathrm{Graph}(s)}(x, y)$. Then $F_{s,\xi} \in L^2(R, \nu)$. A computation (using Lemma 4.16 and similar to that in the first paragraph of Theorem 4.17) shows for $s_1, \dots, s_N \in \mathcal{S}$ and $\xi_1, \dots, \xi_N \in L^2(X, \mu)$,

$$\left\| \sum_{n=1}^{N} k_{s_n} \otimes \xi_n \right\|_{\mathfrak{A}\otimes_\pi \mathcal{H}} = \left\| \sum_{n=1}^{N} F_{s_n, \xi_n} \right\|_{L^2(R,\nu)}.$$

It follows there is an isometry $U \in \mathcal{B}(\mathfrak{A} \otimes_\pi \mathcal{H}, L^2(R, \nu))$ which satisfies $U(k_s \otimes \xi) = F_{s,\xi}$. In fact, U is a unitary operator.

A computation using (4.6) shows that for $v \in \mathcal{G}$, $s \in \mathcal{S}$ and $\xi \in L^2(X, \mu)$,

$$vF_{s,\xi} = F_{q(v)s,\pi(\sigma(v,s))\xi}.$$

Hence

$$\begin{aligned} U\lambda_\pi(v)(k_s \otimes \xi) &= U(k_{q(v)s}\sigma(v, x) \otimes \xi) = U(k_{q(v)s} \otimes \pi(\sigma(v, x))\xi) \\ &= F_{q(v)s,\pi(\sigma(v,s))\xi} = vF_{s,\xi}, \end{aligned}$$

so that $U\lambda_\pi(v)U^* = v$, so that λ_π is unitarily equivalent to the identity representation, as desired.

Remark 4.19 shows that our construction of the representation λ_π includes the Feldman-Moore construction as a special case. Of course, we have yet to show that the von Neumann algebra generated by $\lambda_\pi(\mathcal{P})$ is a Cartan MASA in the von Neumann algebra generated by $\lambda_\pi(\mathcal{G})$. We do this in the next section.

5. The Cartan pair associated to an extension

In this section we construct a Cartan pair from an extension. We will show in Theorem 5.12 that the extension associated to this Cartan pair is equivalent to the original extension. Thus, Theorem 3.8 and Theorem 5.12 show that there is a one-to-one correspondence between equivalence classes of Cartan pairs and equivalence classes of extensions of Cartan inverse monoids.

Let $\mathcal{S}$ be a Cartan inverse monoid, and let $\mathcal{P}$ be the partial isometries in $\mathcal{D} := C(\widehat{\mathcal{E}(\mathcal{S})})$. Because $\widehat{\mathcal{E}(\mathcal{S})}$ is assumed to be a hyperstonean space, $\mathcal{D}$ is $*$-isomorphic to an abelian von Neumann algebra. In the sequel, we assume that $\mathcal{D}$ is an abelian von Neumann algebra. Let

$$\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$$

be an extension, and fix an order preserving section $j \colon \mathcal{S} \to \mathcal{G}$.

We denote by $\mathfrak{A}$ the right Hilbert $\mathcal{D}$-module constructed in Subsection 4.2. Let π be a faithful, normal representation of $\mathcal{D}$, and let

$$\lambda_\pi \colon \mathcal{G} \to \mathcal{B}(\mathfrak{A} \otimes_\pi \mathcal{H})$$

be the representation of $\mathcal{G}$ by partial isometries, as constructed in Theorem 4.17 and Corollary 4.18.

Definition 5.1. Let

$$\mathcal{M}_q = (\lambda_\pi(\mathcal{G}))'' \text{ and } \mathcal{D}_q = (\lambda_\pi(\mathcal{E}(\mathcal{G}))''.$$

Our goal in this section is to show that $(\mathcal{M}_q, \mathcal{D}_q)$ is a Cartan pair. The definition of $\mathcal{M}_q$ and $\mathcal{D}_q$ depends upon the choice of π and, because $\lambda : \mathcal{G} \to \mathcal{L}(\mathfrak{A})$ depends on the choice of j, $\mathcal{M}_q$ and $\mathcal{D}_q$ also depend on j. However, we shall see in Theorem 5.12 that the *isomorphism class* of $(\mathcal{M}_q, \mathcal{D}_q)$ depends only on the extension $\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$ and not upon π or j. We begin by constructing the conditional expectation.

5.1. **A conditional expectation.** In this subsection we construct the faithful, normal conditional expectation from $\mathcal{M}_q$ onto $\mathcal{D}_q$. This expectation will be constructed from the natural map from $\mathcal{S}$ onto $\mathcal{E}(\mathcal{S})$: the map

$$s \mapsto s \wedge 1.$$

This is an idempotent map from $\mathcal{S}$ onto $\mathcal{E}(\mathcal{S})$, which is the identity on $\mathcal{E}(\mathcal{S})$.

This idempotent map induces an idempotent mapping from $\mathcal{G}$ to $\mathcal{P}$, which will be the identity on $\mathcal{P}$. We call this map Δ, and define it by setting

$$\Delta(v) := vj(q(v) \wedge 1),$$

for all $v \in \mathcal{G}$. First note that

$$q(\Delta(v)) = q(v)(q(v) \wedge 1) = q(v) \wedge 1 \in \mathcal{E}(\mathcal{S}),$$

thus $\Delta(v) \in \mathcal{P}$ for all $v \in \mathcal{G}$. Further, if $v \in \mathcal{P}$ then $q(v) \in \mathcal{E}(\mathcal{S})$, thus

$$\Delta(v) = vj(q(v) \wedge 1) = vj(q(v)) = v.$$

Our goal now is to show that, given $v \in \mathcal{G}$, the formula,

$$E(\lambda_\pi(v)) := \lambda_\pi(\Delta(v))$$

extends to a faithful conditional expectation $E : \mathcal{M}_q \to \mathcal{D}_q$. It will take a bit more machinery before we can do this.

Let

$$\mathfrak{B} = \overline{\operatorname{span}}\{k_e \colon e \in \mathcal{E}(\mathcal{S})\} \subseteq \mathfrak{A}.$$

Note that $\mathfrak{B}$ is a right Hilbert $\mathcal{D}$-submodule of $\mathfrak{A}$. Proposition 2.2 shows that $\lambda|_{\mathcal{E}(\mathcal{G})}$ extends to a $*$-monomorphism $\alpha_\ell : \mathcal{D} \to \mathcal{L}(\mathfrak{A})$. For any $e, f \in \mathcal{E}(\mathcal{S})$,

$$\alpha_\ell(j(f))k_e = k_{fe}\sigma(j(e), f) = k_{fe} = k_{ef} = k_e j(f),$$

and for $s \in \mathcal{S}$,

$$\alpha_\ell(j(f))k_s = k_s\ j(s^\dagger f s).$$

It follows that, for any $\xi \in \mathfrak{B}$, $d \in \mathcal{D}$ and $s \in \mathcal{S}$,

$$\alpha_\ell(d)\xi = \xi d \quad \text{and} \quad \alpha_\ell(d)k_s = k_s j(s^\dagger d s).$$

That is, the representation $\alpha_\ell(\cdot)$, restricted to $\mathfrak{B}$, is given by the right module action of $\mathcal{D}$ on $\mathfrak{B}$.

Proposition 5.2. *For $s \in \mathcal{S}$, the map $k_s \mapsto k_{s\wedge 1}$ uniquely determines a projection $P \in \mathcal{L}(\mathfrak{A})$ with range $\mathfrak{B}$. Moreover, for each $v \in \mathcal{G}$,*

$$P\lambda(v)P = \lambda(\Delta(v))P.$$

Proof. Let $N \in \mathbb{N}$, let $c_1, \ldots, c_N \in \mathbb{C}$ and $s_1, \ldots, s_N \in \mathcal{S}$. Put $u = \sum_{n=1}^N c_n k_{s_n}$ and $v = \sum_{n=1}^N c_n k_{(s_n \wedge 1)}$. We claim that, as elements of $\mathcal{D}$,

$$\langle v, v \rangle \leq \langle u, u \rangle .$$

Indeed, let $B \in M_N(\mathcal{D})$ be the matrix whose mn-th entry is $j((s_m \wedge 1)(s_n \wedge 1))$, let $A \in M_N(\mathcal{D})$ be the matrix whose mn-th entry is $j(s_n^\dagger s_m \wedge 1) = K(s_m, s_n)$, and let $D \in M_N(\mathcal{D})$ be the diagonal matrix whose n-th diagonal entry is $s_n \wedge 1$. Lemma 4.11 shows that $A \geq 0$, and it is clear that D is a projection. Corollary 4.9 implies that

$$B = AD = DA.$$

In particular, $0 \leq B \leq A$, so that if $C \in M_{N1}(\mathcal{D})$ is the column matrix whose $n1$-th entry is $c_n I$, we obtain

$$\langle v, v \rangle = C^* BC \leq C^* AC = \langle u, u \rangle ,$$

as claimed.

It follows that $k_s \mapsto k_{(s \wedge 1)}$ extends linearly to contraction P on $\mathfrak{A}$. Let $s, t \in \mathcal{S}$ and put $e = s \wedge t \wedge 1$. By Lemma 4.9, $e = e^\dagger e = t^\dagger (s \wedge 1) \wedge 1 = s^\dagger (t \wedge 1) \wedge 1$. Hence,

$$\begin{aligned} \langle P k_s, k_t \rangle &= \langle k_{s \wedge 1}, k_t \rangle = k_t(s \wedge 1) = j(t^\dagger (s \wedge 1) \wedge 1) \\ &= j(s^\dagger (t \wedge 1) \wedge 1) = k_s(t \wedge 1) = \langle k_s, k_{t \wedge 1} \rangle = \langle k_s, P k_t \rangle. \end{aligned}$$

It follows that P is adjointable. As P is idempotent, P is a projection in $\mathcal{L}(\mathfrak{A})$. Obviously, $\operatorname{range}(P) = \mathfrak{B}$.

Let $s \in \mathcal{S}$ and $v \in \mathcal{G}$. Set $r = q(v)$. Then,

$$\begin{aligned} P\lambda(v) P k_s &= P k_{r(s \wedge 1)} \sigma(v, s \wedge 1) = P k_{r(s \wedge 1)} j(s \wedge 1) j(r)^\dagger v = P k_r j(s \wedge 1) j(r)^\dagger v \\ &= k_{r \wedge 1} j(s \wedge 1) j(r)^\dagger v = k_1 j(r \wedge 1) j(s \wedge 1) j(r)^\dagger v = k_{s \wedge 1} j(r \wedge 1) v. \end{aligned}$$

On the other hand,

$$\begin{aligned} \lambda(\Delta(v)) P k_s &= \lambda(v j(r \wedge 1)) k_{s \wedge 1} = \lambda(v) \lambda(j(r \wedge 1)) k_{s \wedge 1} = \lambda(v) k_{s \wedge 1} j(r \wedge 1) \\ &= k_{r(s \wedge 1)} \sigma(v, s \wedge 1) j(r \wedge 1) = k_{r(s \wedge 1)} j(s \wedge 1) j(r)^\dagger v j(r \wedge 1) \\ &= k_{r(s \wedge 1)} j(s \wedge 1) v j(r \wedge 1) = k_{r(s \wedge 1)} v j(r \wedge 1) = k_{(s \wedge 1)} j(r \wedge 1) v. \end{aligned}$$

Thus $P\lambda(v) P k_s = \lambda(\Delta(v)) P k_s$. As this holds for every $s \in \mathcal{S}$, it follows that $P\lambda(v) P = \lambda(\Delta(v)) P$. ☐

Lemma 5.3. *Define $V : \mathcal{H} \to \mathfrak{A} \otimes_\pi \mathcal{H}$ by $V\xi := k_1 \otimes \xi$. Then V is an isometry for which the following properties hold:*

(a) for every $s \in \mathcal{S}$ and $\xi \in \mathcal{H}$, $V^*(k_s \otimes \xi) = \pi(j(s \wedge 1))\xi$;
(b) $VV^* = \pi_*(P)$;
(c) for every $v \in \mathcal{G}$, $V^* \lambda_\pi(v) V = \pi(\Delta(v))$.

Proof. That V is an isometry follows from the fact that $\langle k_1, k_1 \rangle = I \in \mathcal{D}$. Indeed, for $\xi \in \mathcal{H}$, we have

$$\langle V\xi, V\xi \rangle = \langle k_1 \otimes \xi, k_1 \otimes \xi \rangle = \langle \xi, \pi(\langle k_1, k_1 \rangle)\xi \rangle = \langle \xi, \xi \rangle .$$

Notice that for $s \in \mathcal{S}$ and $\xi, \eta \in \mathcal{H}$,

$$\langle V\xi, k_s \otimes \eta \rangle = \langle k_1 \otimes \xi, k_s \otimes \eta \rangle = \langle \xi, \pi(k_s(1))\eta \rangle = \left\langle \xi, \pi(j(s^\dagger \wedge 1))\eta \right\rangle .$$

Since $s^\dagger \wedge 1 = s \wedge 1$, we find that $V^*(k_s \otimes \eta) = \pi(j(s \wedge 1))\eta$. Hence $VV^*(k_s \otimes \eta) = k_1 \otimes \pi(j(s \wedge 1))\eta = k_{s \wedge 1} \otimes \eta = \pi_*(P)(k_s \otimes \eta)$. So $VV^* = \pi_*(P)$.

By Proposition 5.2 we have

$$P\lambda(v)P = \lambda(\Delta(v))P.$$

Applying π_* to each side of this equality yields

$$\pi_*(P)\lambda_\pi(v)\pi_*(P) = \pi_*(\lambda(\Delta(v)))\pi_*(P) = \pi_*(\alpha_\ell(\Delta(v)))\pi_*(P).$$

A calculation gives $\pi_*(\alpha_\ell(\Delta(v)))\pi_*(P) = V\pi(\Delta(v))V^*$, so that

$$\pi_*(P)\lambda_\pi(v)\pi_*(P) = V\pi(\Delta(v))V^*.$$

Part (c) now follows from parts (a) and (b). ◻

We will now show that $\mathcal{D}$ and $\mathcal{D}_q$ are isomorphic. Thus, an expectation onto a faithful image of $\mathcal{D}$ will give rise to an expectation onto $\mathcal{D}_q$. We begin with two lemmas.

Lemma 5.4. *For $d \in \mathcal{D}$ the map $U : \mathcal{D} \to \mathfrak{B}$ given by $Ud = \alpha_\ell(d)k_1$ is an isometry of $\mathcal{D}$ onto $\mathfrak{B}$. Furthermore, the map $d \mapsto U^*P\alpha_\ell(d)U$ is the regular representation of $\mathcal{D}$ onto itself.*

Proof. For any $d \in \mathcal{D}$ we have

$$\langle Ud, Ud\rangle = \langle \alpha_\ell(d)k_1, \alpha_\ell(d)k_1\rangle = \langle k_1 d, k_1 d\rangle = d^*d.$$

Thus, U is an isometry.

To prove the remainder of the Lemma, we note that, for any $d, h \in \mathcal{D}$, we have

$$U^*P\alpha_\ell(d)Uh = U^*P\alpha_\ell(d)\alpha_\ell(h)k_1 = U^*\alpha_\ell(dh)k_1 = dh. \qquad \square$$

Lemma 5.5. *For every $d \in \mathcal{D}$,*

$$V\pi(d)V^* = \pi_*(\alpha_\ell(d))\pi_*(P).$$

Proof. This is a simple calculation. For $d \in \mathcal{D}$, $s \in \mathcal{S}$ and $\xi \in \mathcal{H}$,

$$\begin{aligned} V\pi(d)V^*(k_s \otimes \xi) &= V\pi(dj(s\wedge 1))\xi = k_1 \otimes \pi(dj(s\wedge 1))\xi \\ &= k_1 d \otimes \pi(j(s\wedge 1))\xi = \alpha_\ell(d)k_1 \otimes \pi(j(s\wedge 1))\xi \\ &= \pi_*(\alpha_\ell(d))\,(k_1 \otimes \pi(j(s\wedge 1))\xi) = \pi_*(\alpha_\ell(d))(k_{s\wedge 1} \otimes \xi) \\ &= \pi_*(\alpha_\ell(d))\pi_*(P)(k_s \otimes \xi). \end{aligned} \qquad \square$$

Proposition 5.6. *The image of $\mathcal{D}$ under $\pi_* \circ \alpha_\ell$ is a von Neumann algebra and the map $\Phi : \pi(\mathcal{D}) \to (\pi_* \circ \alpha_\ell)(\mathcal{D})$ given by $\Phi(\pi(d)) = \pi_*(\alpha_\ell(d))$ is an isomorphism of $\pi(\mathcal{D})$ onto $\mathcal{D}_q$.*

Proof. Clearly Φ is a $*$-homomorphism. Lemma 5.4 and Lemma 5.5 show that Φ is an isomorphism of C^*-algebras. To see that $\pi_*(\alpha_\ell(\mathcal{D}))$ is a von Neumann algebra, it suffices to show that $\pi_*(\alpha_\ell(\mathcal{D}))$ is strongly closed.

For $s \in \mathcal{S}$, the map $\mathcal{D} \ni d \mapsto s^\dagger ds \in \mathcal{D}j(s^\dagger s)$ is a $*$-homomorphism of the von Neumann algebra $\mathcal{D}$ onto the von Neumann algebra $\mathcal{D}j(s^\dagger s)$ and hence is normal. Also for $s \in \mathcal{S}$, $d \in \mathcal{D}$ and $\xi \in \mathcal{H}$,

$$\pi_*(\alpha_\ell(d))(k_s \otimes \xi) = k_s \otimes \pi(s^\dagger ds)\xi, \tag{5.1}$$

since

$$\alpha_\ell(d)k_s = k_s\ (s^\dagger ds).$$

Let $\mathcal{N}$ denote the strong closure of $\pi_*(\alpha_\ell(\mathcal{D}))$ and fix $x \in \mathcal{N}$. Kaplansky's density theorem ensures that there exists a net $d_i \in \mathcal{D}$ such that $\|d_i\| \le \|x\|$ and

$\pi_*(\alpha_\ell(d_i))$ converges strongly to x. Equation (5.1) applied with $s=1$ implies that $\pi(d_i)$ is a strongly Cauchy net and hence converges strongly. Thus, d_i converges σ-strongly to an element $d \in \mathcal{D}$. But another application of equation (5.1) shows that $\pi_*(\alpha_\ell(d_i))u \to \pi_*(\alpha_\ell(d))u$ for every $u \in \operatorname{span}\{k_s \otimes \xi : s \in \mathcal{S} \text{ and } \xi \in \mathcal{H}\}$. Since (d_i) is a bounded net, we obtain the strong convergence of $\pi_*(\alpha_\ell(d_i))$ to $\pi_*(\alpha_\ell(d))$. Hence $x \in \pi_*(\alpha_\ell(\mathcal{D}))$ as desired.

Finally, for every $e \in \mathcal{E}(\mathcal{S})$, a calculation gives $\pi_*(\alpha_\ell(j(e))) = \lambda_\pi(j(e))$. Thus $\pi_*(\alpha_\ell(\mathcal{D})) = \lambda_\pi(\mathcal{E}(\mathcal{G}))'' = \mathcal{D}_q$. ☐

We are at last ready to define the conditional expectation E from $\mathcal{M}_q$ onto $\mathcal{D}_q$. Recall that for $v \in \mathcal{G}$, $V^*\lambda_\pi(v)V = \pi(\Delta(v)) \in \pi(\mathcal{D})$. Thus, Proposition 5.6 shows that the following definition of E carries $\mathcal{M}_q$ into $\mathcal{D}_q$.

Definition 5.7. Define the conditional expectation $E\colon \mathcal{M}_q \to \mathcal{D}_q$ by

$$E(x) = \Phi(V^*xV).$$

By construction, E is normal, idempotent and $E|_{\mathcal{D}_q} = \mathrm{id}_{\mathcal{D}_q}$. Thus, E is indeed a normal conditional expectation. We conclude this subsection by recording some facts about E that will be useful.

Lemma 5.8. *For any $v \in \mathcal{G}$ and $x \in \mathcal{M}_q$ we have*

$$E(\lambda_\pi(v)) = \lambda_\pi(\Delta(v)),$$

and

$$E(\lambda_\pi(v)^* x \lambda_\pi(v)) = \lambda_\pi(v)^* E(x) \lambda_\pi(v).$$

Proof. The first part follows from the definition of E. For the second, we will show for $v, w \in \mathcal{G}$ we have

$$\Delta(w^\dagger v w) = w^\dagger \Delta(v) w.$$

The result will then follow from the normality of E. Take $v, w \in \mathcal{G}$. Setting $r := q(w)$, we have,

$$\begin{aligned}
\Delta(w^\dagger v w) &= w^\dagger v w \left(j(r^\dagger q(v) r \wedge 1)\right) \\
&= w^\dagger v \left(w j(r^\dagger q(v) r \wedge 1) w^\dagger\right) w \\
&= w^\dagger v \left(j(r r^\dagger q(v) r r^\dagger \wedge r r^\dagger)\right) w \\
&= w^\dagger v \left(j((q(v) \wedge 1) r r^\dagger)\right) w \\
&= w^\dagger v j(q(v) \wedge 1) w = w^\dagger \Delta(v) w.
\end{aligned}$$ ☐

5.2. **The Cartan pair.** Our next goal is to show that $(\mathcal{M}_q, \mathcal{D}_q)$ is a Cartan pair. That $\mathcal{D}_q$ is regular in $\mathcal{M}_q$ is straightforward. Much less straightforward is showing that $\mathcal{D}_q$ is a MASA in $\mathcal{M}_q$ and that E is faithful. The normality of E and a result of Kovács-Szűcs [10, Proposition 1], imply that if $\mathcal{D}_q$ is a MASA in $\mathcal{M}_q$, then E is faithful. On the other hand, as we shall see below, the fact that $\mathcal{D}_q$ is a MASA follows from faithfulness of E and the fact that the set of normalizing partial isometries span a weak-$*$ dense subset of $\mathcal{D}_q$. Thus there is a *"Which comes first, $\mathcal{D}_q$ is a MASA or E is faithful?"* problem. It may be possible to give a direct proof that $\mathcal{D}_q$ is a MASA, but we will proceed by showing that E is faithful.

Proposition 5.9. *The conditional expectation E is faithful.*

Proof. Let $\mathcal{C}$ denote the center of $\mathcal{M}_q$. We claim that $E|_{\mathcal{C}}$ is faithful. Let $x \in \mathcal{C}$ and suppose $E(x^*x) = 0$. The definition of E from Definition 5.7 and Proposition 5.6 show that $xV = 0$. Notice that $\sigma(j(s), 1) = j(s^\dagger s)$ so that $\lambda(j(s))k_1 = k_s j(s^\dagger s) = k_s$ (see Corollary 4.9). Hence for $s \in \mathcal{S}$ and $\xi \in \mathcal{H}$,

$$x(k_s \otimes \xi) = x\lambda(j(s))(k_1 \otimes \xi) = \lambda(j(s))x(k_1 \otimes \xi) = \lambda(j(s))xV\xi = 0.$$

Since the span of such vectors is a dense subspace of $\mathfrak{H}$, we conclude that $x = 0$.

Let $\mathfrak{J} := \{x \in \mathcal{M}_q : E(x^*x) = 0\}$. Then $\mathfrak{J}$ is a left ideal of $\mathcal{M}_q$. Lemma 5.8 implies that for $x \in \mathfrak{J}$ and $v \in \mathcal{G}$, $x\lambda_\pi(v) \in \mathfrak{J}$. It now follows that $\mathfrak{J}$ is a two-sided ideal of $\mathcal{M}_q$ as well. Since $\mathfrak{J}$ is weak-$*$-closed, by [21, Proposition II.3.12], there is a projection $Q \in \mathcal{C}$ such that $\mathfrak{J} = Q\mathcal{M}_q$. As $Q \in \mathfrak{J}$ and $E|_{\mathcal{C}}$ is faithful, we obtain $Q = 0$. Thus $\mathfrak{J} = (0)$, that is, E is faithful. $\square$

Proposition 5.10. *The subalgebra $\mathcal{D}_q$ is a MASA in $\mathcal{M}_q$.*

Proof. The proof has several preliminary steps. Let $\mathcal{D}_q^c$ be the relative commutant of $\mathcal{D}_q$ in $\mathcal{M}_q$.

Step 1: We first show $\lambda_\pi(\mathcal{G}) \cap \mathcal{D}_q^c \subseteq \mathcal{D}_q$. To see this, suppose $v \in \mathcal{G}$ and $\lambda_\pi(v) \in \mathcal{D}_q^c$. In particular, $\lambda_\pi(v)$ commutes with every element of $\lambda_\pi(\mathcal{E}(\mathcal{G}))$. Since λ_π is one-to-one, v commutes with every element of $\mathcal{E}(\mathcal{G})$. Since $\mathcal{S}$ is a fundamental inverse monoid, it follows that $v \in \mathcal{P}$. Therefore $\lambda_\pi(v) \in \mathcal{D}_q$.

Step 2: Next, we claim that if $x \in \mathcal{D}_q^c$, then for every $v \in \lambda_\pi(\mathcal{G})$, $vE(v^*x) \in \mathcal{D}_q$.[1] Given such x and v, we have, for each $d \in \mathcal{D}_q$,

$$\begin{aligned} xd - dx = 0, &\quad \text{so} \\ v^*xd - v^*dvv^*x = 0. &\quad \text{Apply } E \text{ to obtain} \\ E(v^*x)d - v^*dvE(v^*x) = 0; &\quad \text{multiplying on the left by } v \text{ yields} \\ vE(v^*x)d - dvE(v^*x) = 0. & \end{aligned}$$

Thus, $vE(v^*x) \in \mathcal{D}_q^c$. Let $E(v^*x) = u|E(v^*x)|$ be the polar decomposition of $E(v^*x)$. Then u is a partial isometry in $\mathcal{D}_q$, so $u \in \lambda_\pi(\mathcal{P})$. Also, $vu|E(v^*x)|$ is the polar decomposition of $vE(v^*x)$. As $vE(v^*x) \in \mathcal{D}_q^c$, we conclude that $vu \in \lambda_\pi(\mathcal{G}) \cap \mathcal{D}_q^c$, so by Step 1, $vu \in \mathcal{D}_q$. But $|E(v^*x)| \in \mathcal{D}_q$, so $vE(v^*x) \in \mathcal{D}_q$.

Step 3: For every $v \in \lambda_\pi(\mathcal{G})$, $v - E(v) \in \lambda_\pi(\mathcal{G})$. To see this, observe that since $v^*E(v) \in \mathcal{D}_q$, we have $v^*E(v) = E(v^*E(v)) = E(v^*)E(v)$. As $E(v) \in \lambda_\pi(\mathcal{P})$, we have $I - E(v)^*E(v) \in \lambda_\pi(\mathcal{P})$. Hence $\lambda_\pi(\mathcal{G}) \ni v(I - E(v^*)E(v)) = v - E(v)$, as desired.

With these preliminaries completed, we now prove the proposition. Let $x \in \mathcal{D}_q^c$. If $w \in \lambda_\pi(\mathcal{G})$ and $E(w) = 0$, by Step 2, we have

$$wE(w^*x) = E(wE(w^*x)) = E(w)E(w^*x) = 0.$$

Multiplying on the left by w^* shows that $E(w^*x) = 0$ whenever $w \in \lambda_\pi(\mathcal{G}) \cap \ker E$.

By Step 3, we obtain for every $v \in \lambda_\pi(\mathcal{G})$,

$$E(v^*x) = E((v^* - E(v^*))x) + E(E(v^*)x) = E(v^*)E(x).$$

[1] In order to be consistent with previous notation, we should start with $w \in \mathcal{G}$ and prove $\lambda_\pi(w)E(\lambda_\pi(w)^*x) \in \mathcal{D}_q$. But it is notationally cleaner to write $v := \lambda_\pi(w)$ instead. We will continue to do this when there is little danger of confusion.

Since $\mathcal{M}_q$ is the weak-$*$-closed linear span of $\lambda_\pi(\mathcal{G})$ and E is normal, we conclude that for every $x \in \mathcal{D}_q^c$,

$$E(x^*x) = E(x^*)E(x). \tag{5.2}$$

Replacing x by $x - E(x)$ in (5.2) shows that for every $x \in \mathcal{D}_q^c$,

$$E((x - E(x))^*(x - E(x))) = 0.$$

By faithfulness of E, $x = E(x) \in \mathcal{D}_q$ for every $x \in \mathcal{D}_q^c$. This completes the proof. ☐

We are now ready to show that $(\mathcal{M}_q, \mathcal{D}_q)$ is a Cartan pair.

Theorem 5.11. *The pair $(\mathcal{M}_q, \mathcal{D}_q)$ is a Cartan pair.*

Proof. By Proposition 5.10, $\mathcal{D}_q$ is a MASA in $\mathcal{M}_q$. By Proposition 5.9, there is a faithful conditional expectation from $\mathcal{M}_q$ onto $\mathcal{D}_q$. Finally, as

$$\lambda_\pi(\mathcal{G}) \subseteq \mathcal{GN}(\mathcal{M}_q, \mathcal{D}_q)$$

and the span of $\lambda_\pi(\mathcal{G})$ is weak-$*$ dense in $\mathcal{M}_q$ it follows that $\mathcal{GN}(\mathcal{M}_q, \mathcal{D}_q)$ spans a weak-$*$ dense subset of $\mathcal{M}_q$. ☐

We showed in Proposition 3.5 and Theorem 3.8 that a Cartan pair uniquely determines an extension by a Cartan inverse monoid. To complete our circle of ideas, we now want to show that the extension for $(\mathcal{M}_q, \mathcal{D}_q)$ is equivalent to the extension

$$\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}.$$

from which $(\mathcal{M}_q, \mathcal{D}_q)$ was constructed.

Theorem 5.12. *The extension associated to the Cartan pair $(\mathcal{M}_q, \mathcal{D}_q)$ is equivalent to the extension*

$$\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$$

from which $(\mathcal{M}_q, \mathcal{D}_q)$ was constructed.

Moreover, the isomorphism class of $(\mathcal{M}_q, \mathcal{D}_q)$ depends only upon the equivalence class of the extension (and not on the choice of representation π or section j).

Proof. Let R_M and $R_{M,\pi}$ be the Munn congruences for $\mathcal{G}$ and $\lambda_\pi(\mathcal{G})$ respectively. Because λ_π is an isomorphism of $\mathcal{G}$ onto $\lambda_\pi(\mathcal{G})$, (v, w) belongs to R_M if and only if $(\lambda_\pi(v), \lambda_\pi(w))$ belongs to $R_{M,\pi}$. Let $q_\pi : \lambda_\pi(\mathcal{G}) \to \lambda_\pi(\mathcal{G})/R_{M,\pi}$ be the quotient map. Then the map $\widetilde{\lambda}_\pi := q_\pi \circ \lambda_\pi \circ j$ is an isomorphism of S onto $\lambda_\pi(\mathcal{G})/R_{M,\pi}$ such that $\widetilde{\lambda}_\pi \circ q = q_\pi \circ \lambda_\pi$. It is now clear that the extensions

$$\mathcal{P} \hookrightarrow \mathcal{G} \xrightarrow{q} \mathcal{S}$$

and

$$\lambda_\pi(\mathcal{P}) \hookrightarrow \lambda_\pi(\mathcal{G}) \xrightarrow{q_\pi} \widetilde{\lambda}_\pi(\mathcal{S})$$

are equivalent.

Our next task is to show that

$$\lambda_\pi(\mathcal{G}) = \mathcal{GN}(\mathcal{M}_q, \mathcal{D}_q). \tag{5.3}$$

It will then follow immediately that $\lambda_\pi(\mathcal{P}) \hookrightarrow \lambda_\pi(\mathcal{G}) \xrightarrow{q_\pi} \widetilde{\lambda}_\pi(\mathcal{S})$ is the extension associated to $(\mathcal{M}_q, \mathcal{D}_q)$.

Claim 1: If $u \in \mathcal{GN}(\mathcal{M}_q, \mathcal{D}_q)$, then $uE(u^*)$ is a projection in $\mathcal{D}_q$, and

$$uE(u^*) = E(uE(u^*)) = E(u)E(u^*). \tag{5.4}$$

To see this, suppose $d \in \mathcal{D}_q$. Then

$$uE(u^*)d = uE(u^*d) = uE(u^*duu^*) = uu^*duE(u^*) = duE(u^*).$$

Since $\mathcal{D}_q$ is a MASA in $\mathcal{M}_q$, $uE(u^*) \in \mathcal{D}_q$. Next,

$$uE(u^*)uE(u^*) = uE(u^*uE(u^*)) = uu^*uE((E(u^*)) = uE(u^*),$$

so $uE(u^*)$ is a projection in $\mathcal{D}_q$. The equality (5.4) is now obvious.

By construction, $\lambda_\pi(\mathcal{G}) \subseteq \mathcal{GN}(\mathcal{M}_q, \mathcal{D}_q)$. To establish the reverse inclusion, fix $v \in \mathcal{GN}(\mathcal{M}_q, \mathcal{D}_q)$; without loss of generality, assume $v \neq 0$.

Claim 2: There exists $p \in \lambda_\pi(\mathcal{E}(\mathcal{G}))$ such that: a) $vp \in \lambda_\pi(\mathcal{G})$, b) $p \leq v^*v$, and c) $vp \neq 0$. Since $\lambda_\pi(\mathcal{G})'' = \mathcal{M}_q$, it follows (as in the proof of [5, Proposition 1.3.4]) that there exists $w \in \lambda_\pi(\mathcal{G})$ such that $wE(w^*v) \neq 0$. Let $p = v^*wE(w^*v)$. By Claim 1, $p \in \mathcal{D}_q$ is a projection. It is evident that $p \leq v^*v$. Moreover, (5.4) implies that $E(v^*w)E(w^*v) = |wE(w^*v))|^2 = p$, so $E(w^*v)$ is a partial isometry in $\mathcal{D}_q$, so $wE(w^*v) \in \lambda_\pi(\mathcal{G})$. Since $E(w^*v) = w^*v(v^*wE(w^*v))$, we obtain,

$$0 \neq wE(w^*v) = w(w^*v(v^*wE(w^*v))) = vv^*wE(w^*v) = vp.$$

Thus Claim 2 holds.

Now let $\mathcal{F} \subseteq \{p \in \mathcal{D}_q : p \text{ is a projection and } p \leq v^*v\}$ be a maximal pairwise orthogonal family of projections such that for each $p \in \mathcal{F}$, $0 \neq vp \in \lambda_\pi(\mathcal{G})$. Set $Q := \bigvee \mathcal{F}$. The maximality of $\mathcal{F}$ implies that $Q = v^*v$. Indeed, if $Q \neq v^*v$, then $Q_1 := v^*v - Q$ is a projection in $\mathcal{D}_q$ and applying Claim 2 to vQ_1 yields a projection $v^*v \geq p \in \mathcal{D}_q$ such that $0 \neq vp \in \lambda_\pi(\mathcal{G})$ which is orthogonal to every element of $\mathcal{F}$.

For each $p \in \mathcal{F}$, set

$$w_p := \lambda_\pi^{-1}(vp), \quad s_p = q(w_p), \quad h_p = j(s_p)^\dagger w_p \quad \text{and} \quad e_p = s_p^\dagger s_p.$$

Then

$$h_p \in \mathcal{P}, \quad vp = \lambda_\pi(w_p) \quad \text{and} \quad p = \lambda_\pi(j(e_p)).$$

Also, $\{s_p : p \in \mathcal{F}\}$ is a pairwise orthogonal family in $\mathcal{S}$ and hence the sum $\sum_{p\in\mathcal{F}} h_p$ converges weak-$*$ in $\mathcal{D}$. Let

$$s = \bigvee_{p\in\mathcal{F}} s_p, \quad e := \bigvee_{p\in\mathcal{F}} e_p, \quad \text{and} \quad h = \sum_{p\in\mathcal{F}} h_p.$$

Thus, $h \in \mathcal{P}$ and $h^\dagger h = j(s^\dagger s)$. Now set

$$w := j(s)h \in \mathcal{G}.$$

We claim that $\lambda_\pi(w) = v$. Observe that $v^*v = \lambda_\pi(w^*w)$. Also, for $p \in \mathcal{F}$, $se_p = s_p$, so

$$\lambda_\pi(w)p = \lambda_\pi(wj(e_p)) = \lambda_\pi(j(s)h_p) = \lambda_\pi(j(s_p)h_p) = \lambda_\pi(j(s_p)j(s_p)^\dagger w_p) = vp.$$

Therefore,

$$\lambda_\pi(w) = \lambda_\pi(w)Q = vQ = v.$$

Hence $v \in \lambda_\pi(\mathcal{G})$. Therefore

$$\lambda_\pi(\mathcal{P}) \hookrightarrow \lambda_\pi(\mathcal{G}) \xrightarrow{q_\pi} \widetilde{\lambda}_\pi(\mathcal{S})$$

is the extension for $(\mathcal{M}_q, \mathcal{D}_q)$.

Suppose that π' is a faithful normal representation of $\mathcal{D}$ and that and $j' : \mathcal{S} \to \mathcal{G}$ is an order preserving section for q. Let $(\mathcal{M}_q', \mathcal{D}_q')$ be the Cartan pair constructed using π' and j' as in Theorem 5.11. Then the previous paragraphs show that the extensions associated to $(\mathcal{M}_q, \mathcal{D}_q)$ and $(\mathcal{M}_q', \mathcal{D}_q')$ are equivalent extensions. By

Theorem 5.12, $(\mathcal{M}_q, \mathcal{D}_q)$ and $(\mathcal{M}'_q, \mathcal{D}'_q)$ are isomorphic Cartan pairs. The proof is now complete. $\square$

6. The Spectral Theorem for Bimodules and Subdiagonal Algebras

In this section, we provide two illustrations of how our viewpoint may be used to reformulate and address the validity of a pair of important assertions found of Muhly, Saito and Solel [17].

Muhly, Saito and Solel studied the weak-$*$-closed $\mathcal{D}$-bimodules in a Cartan pair $(\mathcal{M}, \mathcal{D})$ as they relate to the underlying equivalence relation R from the Feldman-Moore construction. Roughly speaking, they claimed in [17, Theorem 2.5] that if $B \subseteq \mathcal{M}$ is a weak-$*$-closed $\mathcal{D}$-bimodule in $\mathcal{M}$, then there is a Borel subset $A \subseteq R$ such that B consists of all operators in $\mathcal{M}$ whose "matrices" are supported in B. This statement is commonly known as the Spectral Theorem for Bimodules. It has been known for some time that there is a gap in the proof of [17, Theorem 2.5], see e.g. [1]. When the equivalence relation R is hyperfinite, the result was shown to hold by Fulman [8, Theorem 15.18]. When $\mathcal{M}$ is a hyperfinite factor, R is hyperfinite, .

An alternate approach to the Spectral Theorem for Bimodules was given by Cameron, Pitts, and Zarikian in [5]. Rather than characterizing weak-$*$-closed $\mathcal{D}$-bimodules, Cameron, Pitts and Zarikian show that the lattice of Bures-closed $\mathcal{D}$-bimodules is isomorphic to the lattice of projections in a certain abelian von Neumann algebra $\mathcal{Z}$ associated to the pair $(\mathcal{M}, \mathcal{D})$, see [5, Theorem 2.5.8]. Moreover, the work in [5] shows that the Spectral Theorem for Bimodules holds if and only if every weak-$*$-closed $\mathcal{D}$-bimodule in $\mathcal{M}$ is closed in the Bures topology. The approach in [5] does not rely on the Feldman-Moore construction.

Our first goal, accomplished in the first subsection, is to give a description of the Bures-closed $\mathcal{D}$-bimodules in a Cartan pair $(\mathcal{M}, \mathcal{D})$ in terms of certain subsets of $\mathcal{S}$, see Theorem 6.3. This description of the bimodules in $\mathcal{M}$ is a direct analogue of the spectral assertion for bimodules of Muhly, Saito and Solel. The advantage of the description given in Theorem 6.3 over that in [5] is that Bures-closed bimodules of $\mathcal{M}$ are parametrized in terms of data directly obtained from the associated extension, so there is no need to consider the projection lattice of $\mathcal{Z}$. In Corollary 6.4, we use Aoi's Theorem to refine this result to parametrize the von Neumann algebras between $\mathcal{M}$ and $\mathcal{D}$. In the second subsection, we use our work to give a description of the maximal subdiagonal algebras of $\mathcal{M}$ which contain $\mathcal{D}$, see Theorem 6.10 below. Theorem 6.10 provides a proof of Muhly, Saito, and Solel's main representation theorem, [17, Theorem 3.5], which avoids the (as yet) unproven weak-$*$ version of the Spectral Theorem for Bimodules.

6.1. $\mathcal{D}$-Bimodules and Spectral Sets.

Definition 6.1 ([4]). The *Bures topology* on $\mathcal{M}$ is the locally convex topology generated by the family of seminorms

$$\{T \mapsto \sqrt{\tau(E(T^*T))} : \ \tau \in (\mathcal{D}_*)^+\}.$$

We define the following subsets of $\mathcal{S}$.

Definition 6.2. A subset A of a Cartan inverse monoid $\mathcal{S}$ is a *spectral set* if

(a) $s \in A$ and $t \leq s$ implies $t \in A$; and
(b) $\{s_i\}_{i \in I}$ is a pairwise orthogonal family in A, then $\bigvee_{i \in I} s_i \in A$.

Given two spectral sets $A_1, A_2 \subseteq \mathcal{S}$, define their *join span*, denoted $A_1 \veebar A_2$, to be the set of all elements of $\mathcal{S}$ which can be written as the join of two orthogonal elements, one from A_1 and the other from A_2, that is,

$$A_1 \veebar A_2 := \{s \in \mathcal{S} : \text{ there exists } s_i \in A_i \text{ such that } s_1 s_2^\dagger = s_1^\dagger s_2 = 0 \text{ and } s = s_1 \vee s_2\}.$$

It is not hard to see that $A_1 \veebar A_2$ is the smallest spectral set containing $A_1 \cup A_2$. Thus the spectral sets in $\mathcal{S}$ form a lattice, with join given by $\veebar$ and meet given by intersection $\cap$. We aim to show the existence of a lattice isomorphism between the spectral sets in $\mathcal{S}$ and the Bures-closed $\mathcal{D}$-bimodules in $\mathcal{M}$.

For any weak-$*$-closed bimodule $B \subseteq \mathcal{M}$, let

$$\mathcal{GN}(B, \mathcal{D}) := B \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}).$$

It is shown in [5, Proposition 2.5.3] that

$$\overline{\operatorname{span}}^{\text{w-}*}(\mathcal{GN}(B, \mathcal{D})) \subseteq B \subseteq \overline{\operatorname{span}}^{\text{Bures}}(\mathcal{GN}(B, \mathcal{D})).$$

Also, if B is a Bures-closed $\mathcal{D}$-bimodule, then $B = \overline{\operatorname{span}}^{\text{Bures}}(\mathcal{GN}(B, \mathcal{D}))$ [5, Theorem 2.5.1].

For a Bures-closed $\mathcal{D}$-bimodule $B \subseteq \mathcal{M}$, define $\Theta(B) \subseteq \mathcal{S}$ by

$$\Theta(B) = q(\mathcal{GN}(B, \mathcal{D})).$$

Further, define a map Ψ from the collection of spectral sets in $\mathcal{S}$ to Bures-closed $\mathcal{D}$-bimodules in $\mathcal{M}$ by

$$\Psi(A) = \overline{\operatorname{span}}^{\text{Bures}}(j(A)),$$

which is necessarily a Bures-closed $\mathcal{D}$-bimodule.

The following is a restatement of [5, Theorem 2.5.8] in terms of spectral sets which is in the same spirit as the original assertion of [17, Theorem 2.5].

Theorem 6.3 (Spectral Theorem for Bimodules)**.** *There is a lattice isomorphism of the lattice of Bures-closed $\mathcal{D}$-bimodules onto the lattice of spectral sets in $\mathcal{S}$.*

Proof. Let B be a Bures-closed $\mathcal{D}$-bimodule in $\mathcal{M}$ and let $A := \Theta(B)$. We will first show that A is a spectral set in $\mathcal{S}$. Since B is a $\mathcal{D}$-bimodule, if $s \in A$ and $t \leq s$, then $t \in A$. Next, suppose that $\{s_i\}_{i \in I}$ is a pairwise orthogonal family in A and let $s = \bigvee s_i$. For $i \neq k$, the orthogonality of s_i and s_k implies that $j(s_i)$ and $j(s_k)$ are partial isometries with orthogonal initial spaces and orthogonal range spaces. Therefore, the sum $\sum_{i \in I} j(s_i)$ converges strong-$*$ to an element $v \in \mathcal{GN}(\mathcal{M}, \mathcal{D})$. As the Bures topology is weaker than the strong-$*$ topology, $v \in \mathcal{GN}(B, \mathcal{D})$. For every $i \in I$, $q(vj(s_i^\dagger s_i)) = s_i$, and it follows that $q(v) = s$. Thus $j(s) \in B$, and hence $s \in A$. Therefore $A = \Theta(B)$ is a spectral set.

We now prove that $A = \Theta(\Psi(A))$. Clearly, $A \subseteq \Theta(\Psi(A))$. If $A \neq \Theta(\Psi(A))$, then there exists $t \in \Theta(\Psi(A))$ such that $t \wedge s = 0$ for all $s \in A$. Thus, suppose $t \in \mathcal{S}$ and $t \wedge s = 0$ for all $s \in A$. Then $E(j(t)^* j(s)) = 0$ for all $s \in A$. It follows from Corollary 2.3.2 and Lemma 1.4.6 of [5], that t is not in the Bures-closed bimodule generated by $j(A)$. Hence $A = \Theta(\Psi(A)$.

That $\Psi(\Theta(B)) = B$ follows from the fact that B is generated as a $\mathcal{D}$-bimodule by $B \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Finally, the order preserving properties follow by the definitions of Θ and Ψ. $\square$

Recall that a sub-inverse monoid $\mathcal{T} \subseteq \mathcal{S}$ is *full* if $\mathcal{E}(\mathcal{T}) = \mathcal{E}(\mathcal{S})$. Let

$$W := \{\mathcal{N} \subseteq \mathcal{M} : \mathcal{N} \text{ is a von Neumann algebra and } \mathcal{D} \subseteq \mathcal{N} \subseteq \mathcal{M}\}$$

and let

$$T := \{\mathcal{T} \subseteq \mathcal{S} : \mathcal{T} \text{ is a full Cartan inverse submonoid}\}.$$

It follows by Aoi's Theorem [1] that if $\mathcal{M}$ has a separable predual, then for any $\mathcal{N} \in W$, $(\mathcal{N}, \mathcal{D})$ forms a Cartan pair. Cameron, Pitts and Zarikian give an alternative proof of Aoi's Theorem [5, Theorem 2.5.9]. Their approach shows that every von Neumann algebra $\mathcal{N}$ with $\mathcal{D} \subseteq \mathcal{N} \subseteq \mathcal{M}$ is Bures-closed and does not require that $\mathcal{M}$ has a separable predual. We note that, while Aoi's original approach relied on the Feldman-Moore construction of Cartan pairs, the proof in [5] is independent of the work of Feldman and Moore. The following corollary to Theorem 6.3 is immediate.

Corollary 6.4. *The map $\Theta|_W$ is a bijection of W onto S and $\Theta_W^{-1} = \Psi|_S$.*

6.2. **Subdiagonal Algebras.** If $\mathcal{N}$ is a von Neumann algebra such that $\mathcal{D} \subseteq \mathcal{N} \subseteq \mathcal{M}$, [5, Theorem 2.5.9] shows that $\mathcal{N}$ is Bures-closed and there exists a unique Bures-continuous, faithful conditional expectation $\Phi_{\mathcal{N}} : \mathcal{M} \to \mathcal{N}$.

We record the following two lemmas. We first show that under certain circumstances, the Bures closure of an algebra is again an algebra. Then we show that given a von Neumann algebra $\mathcal{D} \subseteq \mathcal{N} \subseteq \mathcal{M}$, the conditional expectation $\Phi_{\mathcal{N}}$ is multiplicative on certain subalgebras of $\mathcal{M}$.

Lemma 6.5. *Suppose $\mathcal{A}$ is a weak-$*$-closed subalgebra of $\mathcal{M}$ containing $\mathcal{D}$, and let $\mathcal{N} := \mathcal{A} \cap \mathcal{A}^*$. Then the Bures closure of $\mathcal{A}$ is a subalgebra of $\mathcal{M}$ and $\mathcal{N} = \overline{\mathcal{A}}^{Bures} \cap (\overline{\mathcal{A}}^{Bures})^*$.*

Proof. Let $\mathcal{B}$ be the Bures closure of $\mathcal{A}$ and choose $X \in \mathcal{B}$. By [5, Theorem 2.5.1], there exists a net $X_\lambda \in \operatorname{span} \mathcal{GN}(\mathcal{A}, \mathcal{D})$ such that Bures-$\lim X_\lambda = X$. Let $v \in \mathcal{GN}(\mathcal{A}, \mathcal{D})$. Since $E(v^*(X_\lambda - X)^*(X_\lambda - X)v) = v^* E((X_\lambda - X)^*(X_\lambda - X))v$, it follows that Bures-$\lim(X_\lambda v) = Xv$. Thus $Xv \in \mathcal{B}$.

Now suppose that $Y \in \mathcal{B}$. We may write $Y =$ Bures-$\lim Y_\lambda$, where $Y_\lambda \in \operatorname{span}(\mathcal{GN}(\mathcal{A}, \mathcal{D}))$. Then $XY_\lambda \in \mathcal{B}$ for every λ. Moreover, the estimate,

$$E((X(Y - Y_\lambda))^*(X(Y - Y_\lambda))) \leq \|X\|^2 E((Y - Y_\lambda)^*(Y - Y_\lambda))$$

implies that XY_λ Bures converges to XY, so $XY \in \mathcal{B}$. Thus $\mathcal{B}$ is an algebra.

By [5, Theorem 2.5.1], $\mathcal{A} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) = \mathcal{B} \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})$. Therefore, $\mathcal{A} \cap \mathcal{A}^* \cap \mathcal{GN}(\mathcal{M}, \mathcal{D}) = \mathcal{B} \cap \mathcal{B}^* \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})$. But $\mathcal{N}$ is the Bures closure of $\mathcal{A} \cap \mathcal{A}^* \cap \mathcal{GN}(\mathcal{M}, \mathcal{D})$, so $\mathcal{A} \cap \mathcal{A}^* = \mathcal{B} \cap \mathcal{B}^*$. ☐

Lemma 6.6. *Suppose $\mathcal{A}$ is a Bures-closed subalgebra of $\mathcal{M}$ containing $\mathcal{D}$, and let $\mathcal{N} := \mathcal{A} \cap \mathcal{A}^*$. Then for $X, Y \in \mathcal{A}$, $\Phi_{\mathcal{N}}(XY) = \Phi_{\mathcal{N}}(X)\Phi_{\mathcal{N}}(Y)$*

Proof. Let $\mathcal{J} := \ker(\Phi_{\mathcal{N}}|_{\mathcal{A}})$. We shall show that $\mathcal{J} \cap \mathcal{GN}(\mathcal{A}, \mathcal{D})$ is a semigroup.

Suppose first that $u, v \in \mathcal{GN}(\mathcal{A}, \mathcal{D})$, that $0 \in \{\Phi_{\mathcal{N}}(u), \Phi_{\mathcal{N}}(v)\}$, and $uv \in \mathcal{N}$. We claim that $uv = 0$. To see this, suppose that $\Phi_{\mathcal{N}}(u) = 0$. As $\mathcal{N}$ is closed under adjoints, $\mathcal{A} \ni v(v^*u^*) = (vv^*)u^* \in \mathcal{A}^*$, so $vv^*u^* \in \mathcal{N}$. Hence

$$vv^*u^* = \Phi_{\mathcal{N}}(vv^*u^*) = vv^*\Phi_{\mathcal{N}}(u)^* = 0.$$

It follows that $v^*u^* = uv = 0$. A similar argument shows that $uv = 0$ under the assumption that $\Phi_{\mathcal{N}}(v) = 0$, so the claim holds.

Now let $u, v \in \mathcal{J} \cap \mathcal{GN}(\mathcal{A}, \mathcal{D})$. By [5, Lemma 2.3.1(a)], there exists $p \in \operatorname{proj}(\mathcal{D})$ such that $uvp = \Phi_{\mathcal{N}}(uv)$. The claim applied to u and vp shows that $uvp = 0$, so $\mathcal{J} \cap \mathcal{GN}(\mathcal{A}, \mathcal{D})$ is a semigroup.

Let $\mathcal{A}_0 = \operatorname{span} \mathcal{GN}(\mathcal{A}, \mathcal{D})$. For $i = 1, 2$, let $X_i \in \mathcal{A}_0$. Then $\Phi_{\mathcal{N}}(X_i) \in \mathcal{A}_0$. Write $X_i = \Phi_{\mathcal{N}}(X_i) + Y_i$ where $Y_i = X_i - \Phi_{\mathcal{N}}(X_i) \in \operatorname{span}(\mathcal{J} \cap \mathcal{GN}(\mathcal{A}, \mathcal{D}))$. Since $\mathcal{J} \cap \mathcal{GN}(\mathcal{A}, \mathcal{D})$ is a semigroup, $\operatorname{span}(\mathcal{J} \cap \mathcal{GN}(\mathcal{A}, \mathcal{D}))$ is an algebra. Then

$$\begin{aligned}\Phi_{\mathcal{N}}(X_1X_2) &= \Phi_{\mathcal{N}}(X_1)\Phi_{\mathcal{N}}(X_2) + \Phi_{\mathcal{N}}(\Phi_{\mathcal{N}}(X_1)Y_2 + Y_2\Phi_{\mathcal{N}}(X_2)) + \Phi_N(Y_1Y_2)\\ &= \Phi_{\mathcal{N}}(X_1)\Phi_{\mathcal{N}}(X_2).\end{aligned}$$

As $\Phi_{\mathcal{N}}$ is Bures continuous, the previous equality also holds for $X_i \in \overline{\mathcal{A}_0}^{\text{Bures}}$, and we are done. ∎

Definition 6.7. Let $\mathcal{A}$ be a weak-$*$-closed subalgebra of $\mathcal{M}$ such that $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{M}$, and put $\mathcal{N} = \mathcal{A} \cap \mathcal{A}^*$. Then

(a) $\mathcal{A}$ is *subdiagonal* if $\mathcal{A} + \mathcal{A}^*$ is weak-$*$ dense in $\mathcal{M}$ and $\Phi_{\mathcal{N}}|_{\mathcal{A}}$ is multiplicative;
(b) $\mathcal{A}$ is *maximal subdiagonal* if there is no subdiagonal subalgebra $\mathcal{B}$ of $\mathcal{M}$ with $\mathcal{B} \cap \mathcal{B}^* = \mathcal{A} \cap \mathcal{A}^*$ which properly contains $\mathcal{A}$;
(c) $\mathcal{A}$ is *triangular* if $\mathcal{A}$ is subdiagonal and $\mathcal{A} \cap \mathcal{A}^* = \mathcal{D}$; and
(d) $\mathcal{A}$ is *maximal triangular* if there is no triangular subalgebra $\mathcal{B}$ of $\mathcal{M}$ with $\mathcal{B} \cap \mathcal{B}^* = \mathcal{D}$ which properly contains $\mathcal{A}$.

The following is an immediate consequence of Lemma 6.5 and Lemma 6.6.

Corollary 6.8. *If $\mathcal{A}$ is a subdiagonal subalgebra of $\mathcal{M}$ containing $\mathcal{D}$, then $\overline{\mathcal{A}}^{Bures}$ is a subdiagonal algebra with $\mathcal{A} \cap \mathcal{A}^* = \overline{\mathcal{A}}^{Bures} \cap \left(\overline{\mathcal{A}}^{Bures}\right)^*$. In particular, every maximal subdiagonal algebra $\mathcal{A}$ with $\mathcal{D} \subseteq \mathcal{A} \subseteq \mathcal{M}$ is Bures-closed.*

Muhly, Saito, and Solel assert that any subdiagonal algebra containing $\mathcal{D}$ is maximal subdiagonal. As their proof ([17, p. 263]) depends on the spectral theorem for weak-$*$-closed bimodules, their assertion remains open. However, because maximal subdiagonal algebras are Bures-closed, it is possible to modify their ideas to give descriptions of the maximal subdiagonal and maximal triangular subalgebras of $\mathcal{M}$ which contain $\mathcal{D}$. To do this, some notation is helpful. A submonoid of $\mathcal{S}$ which is also a spectral set is a *spectral monoid*. Let

$$\textsc{msd}(\mathcal{S}) := \{A \subseteq \mathcal{S} : A \text{ is a spectral monoid containing } \mathcal{E}(\mathcal{S}) \text{ and } A \veebar A^\dagger = \mathcal{S}\}$$

and

$$\textsc{mtr}(\mathcal{S}) := \{A \in \textsc{msd}(\mathcal{S}) : A \cap A^\dagger = \mathcal{E}(\mathcal{S})\}.$$

Remark 6.9. The sets $\textsc{mtr}(\mathcal{S})$ and $\textsc{msd}(\mathcal{S})$ correspond to the sets $\mathfrak{P}$ and $\mathfrak{P}'$ of [17, p. 258 and 262], respectively.

Theorem 6.10. *The restriction of Ψ to $\textsc{msd}(\mathcal{S})$ gives a bijection of $\textsc{msd}(\mathcal{S})$ onto the the set of all maximal subdiagonal algebras in $\mathcal{M}$ containing $\mathcal{D}$. In addition, the restriction of Ψ to $\textsc{mtr}(\mathcal{S})$ is a bijection of $\textsc{mtr}(\mathcal{S})$ onto the set of all weak-$*$-closed maximal triangular subalgebras of $\mathcal{M}$ containing $\mathcal{D}$.*

Proof. Let $A \in \textsc{msd}(\mathcal{S})$. Since $A \veebar A^\dagger = \mathcal{S}$, $\mathcal{GN}(\mathcal{M}, \mathcal{D}) \subseteq \Psi(A) + \Psi(A)^*$, so $\Psi(A) + \Psi(A)^*$ is weak-$*$ dense in $\mathcal{M}$. Thus, Lemma 6.5 and Lemma 6.6 shows that $\Psi(\mathcal{A})$ is a Bures-closed subdiagonal algebra.

Suppose $\mathcal{B} \subseteq \mathcal{M}$ is a subdiagonal algebra with $\mathcal{B} \cap \mathcal{B}^* = \Psi(A) \cap \Psi(\mathcal{A})^*$ and $\Psi(A) \subseteq \mathcal{B}$. If $u \in \mathcal{GN}(\mathcal{B}, \mathcal{D})$, then we may find orthogonal elements $s_1 \in A$ and $s_2 \in A^\dagger$ such that $q(u) = s_1 \vee s_2$. Then $u = w_1 + w_2$, where $w_i = uj(s_i^\dagger s_i)$. As

$\mathcal{D} \subseteq \mathcal{B}$, $w_2 \in \mathcal{B}$. On the other hand, $q(w_2)^\dagger = s_2^\dagger \in A$, so $w_2^* \in \Psi(A)^* \subseteq \mathcal{B}^*$, hence $w_2 \in \mathcal{B} \cap \mathcal{B}^* \subseteq \Psi(A)$. As $w_1 \in \Psi(A)$, we obtain $u \in \Psi(A)$. Therefore, $\mathcal{GN}(\mathcal{B}, \mathcal{D}) \subseteq \Psi(A)$. We then obtain $\mathcal{B} \subseteq \overline{\operatorname{span}}^{\text{Bures}}(\mathcal{GN}(\mathcal{B}, \mathcal{D})) \subseteq \Psi(A)$. Thus $\Psi(A) = \mathcal{B}$, so $\Psi(A)$ is maximal subdiagonal.

On the other hand, suppose $\mathcal{A} \subseteq \mathcal{M}$ is a maximal subdiagonal algebra containing $\mathcal{D}$. Set $\mathcal{N} := \mathcal{A} \cap \mathcal{A}^*$ and let $A := \Theta(\mathcal{A})$. Since $\mathcal{D} \subseteq \mathcal{A}$, $\mathcal{E}(\mathcal{S}) \subseteq A$; moreover, A is a monoid because q is a homomorphism and $\mathcal{GN}(\mathcal{A}, \mathcal{D})$ is a monoid. We need to show that $\mathcal{S} = A \veebar A^\dagger$.

To do this, let $s \in \mathcal{S}$ and set $v = j(s)$. Using [5, Lemma 2.3.1(a)] twice, there exist projections $p_+, p_- \in \operatorname{proj}(\mathcal{D})$ such that:

i) $vp_+ \in \mathcal{A}$ and $vp_+^\perp$ is $\mathcal{D}$-orthogonal to $\mathcal{A}$; and
ii) $vp_+^\perp p_- \in \mathcal{A}^*$ and $vp_+^\perp p_-^\perp$ is $\mathcal{D}$-orthogonal to $\mathcal{A}^*$.

Then $vp_+^\perp p_-^\perp$ is $\mathcal{D}$-orthogonal to $\mathcal{A} + \mathcal{A}^*$ and hence $\mathcal{D}$-orthogonal to $\mathcal{M}$. Therefore, $vp_+^\perp p_-^\perp = 0$, so that $vp_+^\perp = vp_+^\perp p_- \in \mathcal{A}^*$. Then $s = q(v) = q(vp_+) \vee q(vp_+^\perp) \in A \veebar A^\dagger$. Thus, $\Theta(\mathcal{A}) \in \textsc{msd}(\mathcal{S})$.

By Theorem 6.3, the restriction of Ψ to the class of maximal subdiagonal algebras containing $\mathcal{D}$ is a bijection onto $\textsc{msd}(\mathcal{S})$.

It is easy to see that for any maximal triangular algebra $\mathcal{A}$ containing $\mathcal{D}$, $\Psi(\mathcal{A}) \in \textsc{mtr}(\mathcal{S})$ and that if $A \in \textsc{mtr}(\mathcal{S})$, then $\Theta(A)$ is a maximal triangular algebra. Thus the restriction of Ψ to the class of maximal triangular algebras containing $\mathcal{D}$ is a bijection onto $\textsc{mtr}(\mathcal{S})$. ☐

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