Physics, Chapter 4: Statics of a Rigid Body

Henry Semat
City College of New York

Robert Katz
University of Nebraska-Lincoln, rkatz2@unl.edu

4

Statics of a Rigid Body

4-1 The Concept of a Rigid Body

In the preceding chapter we observed that a particle would remain in equilibrium, in a state of rest, or in a state of uniform motion in a straight line when the resultant of all the forces acting on it was equal to zero. This condition for equilibrium was extended to larger bodies under either of two possible conditions: If the forces acting on the body were concurrent, that is, if they were directed toward a single point, the body could be treated as if it were a particle; or if the body moved with uniform translational motion in which every particle of the body moved in the same fixed direction with uniform speed, the whole body could be treated as though it were a particle.

Many of the problems of the equilibrium of extended bodies do not fulfill these conditions. The forces acting on the body do not pass through a single point, and the motion of the body is not one of uniform translational motion but may include rotation as well. The motion of a body is often quite complicated, as in the case of a spiraling football. The ball is generally thrown so that it spins about its longer axis, but, in addition to its spinning motion, the axis of rotation itself rotates, and the ball has a general translational projectilelike motion superimposed upon the rotational motions.

We shall restrict ourselves to the study of rotation about a fixed axis and shall devote our attention first to the case of equilibrium of a body with respect to rotation about a fixed axis.

While all material bodies deform somewhat under the action of applied forces, it is convenient to think of them as nondeforming, or as rigid; we shall define a *rigid body as one in which all dimensions remain the same, regardless of the nature of the applied forces.* With this concept the statics of material bodies can be greatly simplified, for, instead of having to study the body as though it were a vast collection of particles to which the conditions of equilibrium must be applied to one particle at a time, the entire body may be treated as a single object, and its equilibrium may be studied through the introduction of a new concept called *torque*.

4-2 Moment of a Force; Torque

The effect of a force in producing rotation is determined by two factors, (a) the force itself and (b) the distance of the line of action of the force from some line considered as an *axis of rotation*. Suppose that a force **F** acts on a rigid body, as shown in Figure 4-1; its line of action is collinear with the

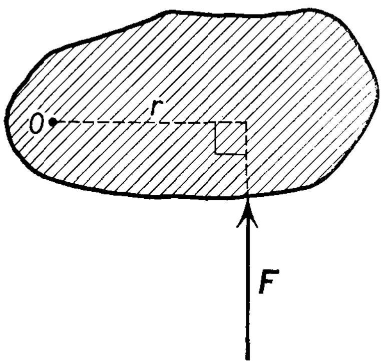

Fig. 4-1 Torque produced by a force **F** whose line of action is at a distance r from the axis through O perpendicular to the plane of the paper is $G = Fr$.

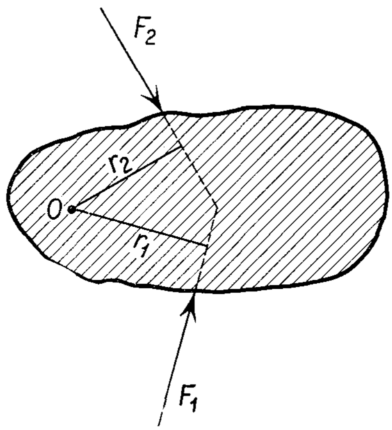

Fig. 4-2

vector **F**. Imagine an axis through point O perpendicular to the plane of the paper such that the distance from O to the line of action of the force **F** is r. The effect of the force in producing rotation about the axis through O, called the *moment of the force*, or the *torque*, is defined as the *product of the force and the perpendicular distance from the axis to the line of action of the force*. If G represents the magnitude of the torque, then

$$\boxed{G = Fr.} \tag{4-1}$$

As we view Figure 4-1, the torque will tend to produce a rotation of the body in a counterclockwise direction about an axis through O; the torque G is said to be in a counterclockwise direction. Figure 4-2 shows a rigid body acted upon by two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ at distances r_1 and r_2, respectively, from an axis through O perpendicular to the plane of the paper. The torque produced by $\mathbf{F}_1$ about O is F_1r_1 in a counterclockwise direction; the torque produced by $\mathbf{F}_2$ about O is F_2r_2 in a clockwise direction. By convention *a torque in a counterclockwise direction is usually called positive,* and one in a *clockwise direction is usually called negative.* Thus the total torque produced by these forces about O as an axis is

$$G = F_1r_1 - F_2r_2.$$

Whenever the torque produced by a force about a particular axis is to be determined, it is essential to find the *perpendicular distance* from the axis to the line of action of the force. In Figure 4-3 the force $\mathbf{F}$ is applied at the point E on the rim of a disk. To find the torque about an axis perpendicular

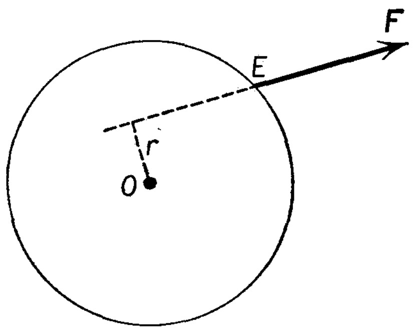

Fig. 4-3 Force $\mathbf{F}$ applied at point E produces torque $-Fr$ about an axis through O perpendicular to the plane of the paper.

to the plane of the paper through O at the center of the disk, it is necessary to extend the line of action of the force $\mathbf{F}$ as shown by the dotted line, and then to drop a perpendicular from O onto this line to obtain the perpendicular distance r. The torque of $\mathbf{F}$ about the axis through O is $-Fr$, the minus sign indicating that it acts in a clockwise direction.

The units used for expressing a torque must be those appropriate for the product of a force and a distance. Thus pound feet (lb ft), newton meters (nt m), and dyne centimeters (dyne cm) are the appropriate units of torque in the British gravitational, the mks, and the cgs systems of units, respectively.

4-3 Vector Representation of Torque

Only coplanar forces were considered in the above discussion; the axis about which the moments of the forces were taken was always at right angles to the plane containing the forces. In this simple case the direction of rotation, and hence the direction of the torque, was specified either as clockwise or counterclockwise. In the more general case where the forces are not coplanar and the axis of rotation may have any arbitrary direction, it is necessary to have a more general method of representing torque as a vector.

As we have already seen, conventional rectangular coordinate systems are *right-handed;* that is, if the fingers of the right hand are pointed in the direction of the x axis and the fingers are bent so that they point toward the direction of the y axis, the outstretched thumb will point in the direction of the z axis. The disposition of the fingers and thumb of the right hand are commonly used to represent vector quantities involving rotation. If the fingers of the right hand were used to grasp the disk illustrated in Figure 4-4, with the fingers pointing in the direction of the rotation which the force at A might produce, the extended thumb would point in the direction of the axis of rotation. To represent the torque produced by the force **F** at A by

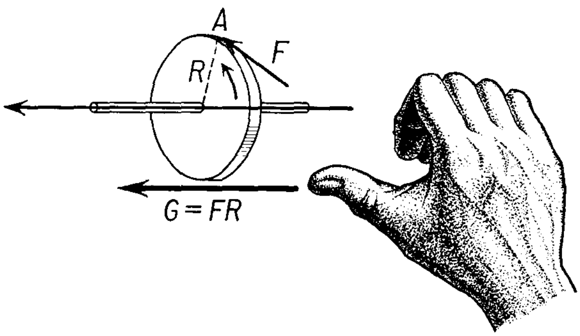

Fig. 4-4 The right hand rule: if the fingers of the right hand follow the direction of rotation, the thumb will point in the direction in which the arrow should be drawn along the axis of rotation.

a vector, we would draw a vector of magnitude given by $G = R \times F$ pointing along the line of the axis of rotation to the left. Conversely, if the torque vector were given as to the left, then, pointing the right thumb in the direction of the vector, the curled fingers of the right hand would point in the direction of rotation the torque would tend to produce.

4-4 Equilibrium of a Rigid Body

When a rigid body remains at rest under the action of a system of forces, the body is said to be in equilibrium. In addition, under certain special conditions a body may be in equilibrium even when it is in motion. For example, a rigid body is in equilibrium if it moves in such a way that every particle in the body moves with uniform speed in a straight line. Another type of equilibrium is that of a wheel rotating about its axis with uniform angular speed. For a rigid body to remain in equilibrium when acted upon by a set of forces, two conditions must be satisfied:

(a) *The vector sum of all the forces acting on the body must be zero.* This condition assures that there will be no change in the state of the translational motion. Writing the condition in the form of an equation, we have

$$\boxed{\mathbf{F}_1 + \mathbf{F}_2 + \cdots + \mathbf{F}_n = \sum_{i=1}^{n} \mathbf{F}_i = 0.} \tag{4-2}$$

We note that this is the same as the condition for the equilibrium of a particle.

(b) *The vector sum of all the torques acting on the body about any axis must be zero.* In dealing with two-dimensional problems, this is equivalent to saying that the sum of the clockwise torques about any axis must equal the sum of the counterclockwise torques about the same axis. Writing this condition in the form of an equation, we have

$$\boxed{\mathbf{G}_1 + \mathbf{G}_2 + \cdots + \mathbf{G}_n = \sum_{i=1}^{n} \mathbf{G}_i = 0.} \tag{4-3}$$

This condition on the torques, that the sum of the torques must equal zero, is a new condition for equilibrium applicable to a rigid body which was not pertinent to the equilibrium of a particle, for all the forces acting on a particle had to intersect in that particle. The forces acting on a rigid body do not generally act on a single point in the body and consequently will give rise to rotational motion unless Equation (4-3) is fulfilled.

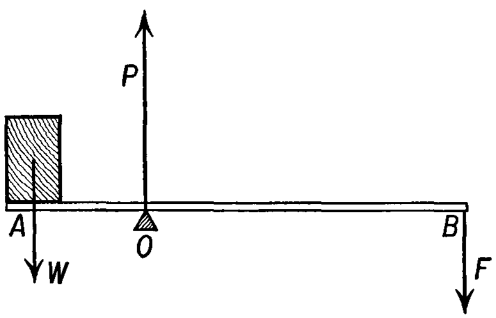

Fig. 4-5 Lever in equilibrium.

Illustrative Example. Let us analyze the forces associated with the operation of a lever. Essentially, a lever consists of a rigid bar AB, as in Figure 4-5, capable of rotating about a point of support O, called the *fulcrum*, which defines the axis of rotation. Suppose a weight $\mathbf{W}$ is placed at the end A and that some vertical force $\mathbf{F}$ is applied downward at the end B to keep the lever in equilibrium in a horizontal position. Applying Equation (4-2) to the equilibrium of the bar AB, since the forces **W** and **F** are both in the y direction, the only other possible force, the force exerted by the fulcrum at O, must also be in the y direction. Calling this force **P**, the vector equation for the forces must be

$$\mathbf{W} + \mathbf{F} + \mathbf{P} = 0,$$

and rewriting the equation with the symbols W, P, and F representing the magnitudes of the three forces, their directions being taken from the directions of the arrows on the figure, we have

$$-W + P - F = 0;$$

hence

$$P = W + F.$$

To apply the second condition for equilibrium, let us take moments of the forces about the point O with respect to an axis pointing normally out of the paper. If we consider O as the origin of a coordinate system with the positive x axis pointing toward the right to the point B, the positive y direction as the direction given by the vector **P**, then the positive z direction points normally out of the paper toward the reader, as given by the right-hand convention. The moment of **W** about O is $+W \times \overline{AO}$, since the rotation which would be generated by **W** would be counterclockwise, and the torque vector would point in the positive z direction. The moment of **F** about O is $-F \times \overline{OB}$, since this is clockwise; the moment of **P** about O is zero. All the torques are in the z direction, and we apply the conditions for equilibrium in the form of Equation (4-3)

$$\sum G = W \times \overline{AO} - F \times \overline{OB} = 0,$$

from which

$$W \times \overline{AO} = F \times \overline{OB},$$

so that

$$\frac{W}{F} = \frac{\overline{OB}}{\overline{AO}}.$$

The distances $\overline{AO}$ and $\overline{OB}$ are called the *lever arms* of the respective forces **W** and **F**. Thus, in the case of a lever, W and F are in the inverse ratio of their lever arms. By placing the fulcrum closer to **W**, we shall now need a smaller force **F** to lift **W**. The fulcrum may be placed at any point along the bar, and the positions of **W** and **F** may be moved around to get almost any desired result consistent with the approximation that the bar remains a rigid body. Many common tools are applications of the principle of the lever, as may be seen from an analysis of the use of the shovel, crowbar, tongs, wrench, tweezers, pliers, scissors, chain tightener, nail puller, and nutcracker.

Illustrative Example. A strong steel bar 5 ft long is supported at its two ends A and B, as shown in Figure 4-6. A weight of 160 lb is placed 2 ft from end A. Neglecting the weight of the bar, determine the forces exerted by the supports.

The forces acting on the steel bar are shown in Figure 4-6. The forces exerted by the supports are shown as $\mathbf{F}_A$ and $\mathbf{F}_B$. From the first condition of equilibrium, we get

$$F_A + F_B - 160\text{ lb} = 0.$$

In applying the second condition for equilibrium, we are at liberty to choose any axis of rotation. Let us choose an axis through the point A directed normally out of the paper. Following the previous example, we call this the positive z direction. The sum of the moments of all the forces about A is zero, yielding

$$F_A \times 0 - 160\text{ lb} \times 2\text{ ft} + F_B \times 5\text{ ft} = 0,$$

from which

$$F_B = 64\text{ lb}.$$

Substituting this back into the first equation gives us

$$F_A = 96\text{ lb}.$$

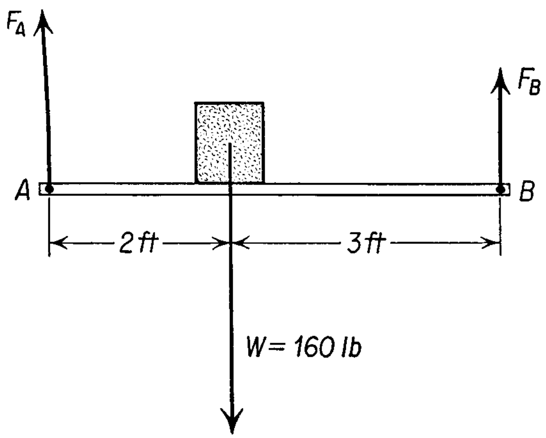

Fig. 4-6

This example really represents the solution of a great many problems in statics. If the line AB represents a simple bridge, then F_A and F_B represent the forces exerted by the bridge piers, and we have solved the problem of the loads borne by piers under one particular load distribution. If the line AB represents the bed of a truck, as it well might with the substitution of somewhat different numbers for the distance and weight, then **W** might represent the weight of the engine, and the two forces might represent the load borne by the front and rear tires.

Illustrative Example. A rod 8 ft long, and considered to be weightless, is pinned to a wall at one end, as shown in Figure 4-7(a). To support the rod horizontally a cord 10 ft long is fastened to the outer end of the rod and to the wall a distance of 6 ft above the pin. A 64-lb weight W is hung from the rod a distance of 3 ft from the pinned end. Find the tension in the cord and the force exerted by the pin on the rod.

We observe that we are here concerned with the equilibrium of a rigid body, namely the rod. From the dimensions given, the space figure is a 3–4–5 right triangle, and the angle ACD is 37°. Let us isolate the rod AC and label all the forces acting on it as shown in Figure 4-7(b). Since we know neither the magnitude nor the direction of the force exerted by the pin at A, we label the components of this force $\mathbf{A}_x$ and $\mathbf{A}_y$, and draw them in the directions we expect these forces to act. Although we know the direction of the tension in the cord, it is more convenient to work in terms of the components of the tension $\mathbf{T}_x$ and $\mathbf{T}_y$. The forces on the rod are then A_x, A_y, W, T_x, and T_y, where these symbols in italics once again represent the magnitudes of the forces, the directions being

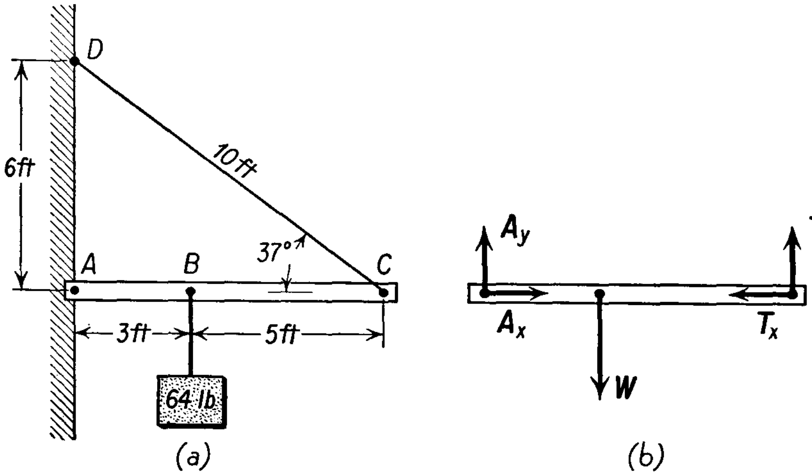

Fig. 4-7

given in the diagram. Following such a procedure, if one of the forces proves to have a negative value on solution of the problem, the direction of the particular force will be opposite to that shown in the figure.

We apply the component form of Equation (4-2) for the translational equilibrium of a rigid body:

$$\sum F_x = A_x - T_x = 0; \tag{a}$$

$$\sum F_y = A_y - W + T_y = 0. \tag{b}$$

Since T_x and T_y are components of a force $\mathbf{T}$, we may write

$$\frac{T_y}{T_x} = \tan 37° = \tfrac{3}{4}. \tag{c}$$

At this stage we have three equations in four unknowns, A_x, A_y, T_x, and T_y, and we need an additional relationship among these quantities to obtain a solution to the problem.

The second condition for equilibrium, Equation (4-3), provides the necessary relationship. Once again the positive z direction is taken as pointing out of the paper. The axis of rotation will be taken in the z direction, and the location of the axis of rotation will be chosen through the pin at A. The line of action of the forces $\mathbf{A}_x$, $\mathbf{A}_y$, and $\mathbf{T}_x$, all pass through the point A; hence these forces produce zero torque about an axis through A. It was for this reason that the point A was chosen as the location of the axis of rotation, and not because the pin was located at A. The point C would have been an equally good choice for the location of the axis of rotation.

Substituting in Equation (4-2) for the torques about an axis through A, we obtain

$$\sum G_A = 0 = A_y \times 0 \text{ ft} + A_x \times 0 \text{ ft} - 64 \text{ lb} \times 3 \text{ ft} + T_x \times 0 \text{ ft} + T_y \times 8 \text{ ft},$$

from which

$$64 \times 3 \text{ lb ft} = 8 \times T_y \text{ ft};$$

hence

$$T_y = 24 \text{ lb.} \tag{d}$$

With this result the entire problem is reduced to algebra. From Equation (c) we get

$$T_x = \frac{T_y}{\tan 37°} = \frac{24 \text{ lb}}{0.75} = 32 \text{ lb.} \tag{e}$$

From Equations (b) and (d) we find

$$A_y - 64 \text{ lb} + 24 \text{ lb} = 0,$$

so that

$$A_y = 40 \text{ lb.}$$

From Equations (a) and (e) we find that

$$A_x = T_x = 32 \text{ lb.}$$

Hence the tension in the rope T is of magnitude

$$T = (T_x^2 + T_y^2)^{1/2} = [(32)^2 + (24)^2]^{1/2} = 40 \text{ lb.}$$

The direction of $\mathbf{T}$ is known from the statement of the problem. The magnitude of the force on the pin A is given as

$$A = (A_x^2 + A_y^2)^{1/2} = [(32)^2 + (40)^2]^{1/2} = 51.2 \text{ lb};$$

the direction of the force $\mathbf{A}$ can be expressed in terms of the angle θ that it makes with the rod considered as the x axis; thus

$$\theta = \text{arc tan} \frac{A_y}{A_x} = \text{arc tan} \frac{40}{32} = 51.4°.$$

4-5 Center of Gravity

In all our previous discussions in which it was necessary to consider the weight of a body, we represented it by a single force $\mathbf{W}$ downward. Actually, the earth exerts a force of attraction on each particle of the body; the weight of the body is the resultant of all the forces which act on all the particles of the body. We ask whether it is possible to think of an extended distribution of matter as though all its weight were concentrated at a single point in space. A plumb bob, a weight hung on the end of a string, represents an approximation of a particle. When a plumb bob is suspended, the weight hangs directly beneath the point of support. From an experimental viewpoint, if there is a single point associated with an extended object where all the weight appears to be concentrated, this point should always come to rest beneath the point of support, no matter how the object is suspended. If an extended object is suspended from first one, then another, of several different points of support, the vertical lines through these points always intersect in a single point called the *center of gravity*. A single upward force of magnitude equal to the weight of the body will be sufficient to produce equilibrium if this force is applied at the center of gravity, regardless of the orientation of the body.

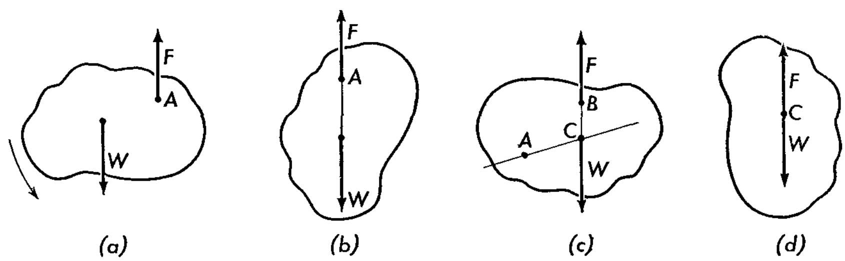

Fig. 4-8 Method of determining the position of the center of gravity of a body.

Suppose the body shown in Figure 4-8(a) is supported by a vertical force **F** at A, equal in magnitude to the weight of the body **W**, shown acting through the center of gravity. Considering an axis of rotation through A, the force **W** generates a torque which tends to rotate the body in the counterclockwise direction. The sum of the torques is not zero, and the body is not in equilibrium. Only when the center of gravity lies directly beneath the point of support, as in Figure 4-8(b), are the two conditions for equilibrium fulfilled. If the body is now supported at some other point B, the body will once again come to equilibrium, with its center of gravity beneath the point of support. The vertical line drawn through A when the body was in the position given in Figure 4-8(b) and the vertical line drawn through the second point of support B shown in Figure 4-8(c) intersect in the center of gravity C. Finally, when the body is supported at its center of gravity, the resultant of the force of support **F** and the force of gravity **W** is zero and therefore generates no torque about any point of support or about any other possible axes of rotation. Hence the body is in equilibrium in any orientation when it is supported at the center of gravity. The center of gravity is the balance point of the body.

If a body is homogeneous, that is, made of the same material throughout, and of simple geometric shape, such as a rectangular stick or a disk, a square plate or a sphere, the center of gravity lies at the geometrical center of the body. The center of gravity need not always lie at a place where any of the matter of the body is located. For example, the center of gravity of a hollow ball lies at the center of the ball, and the center of gravity of a bottle lies somewhere within the bottle. Nevertheless, the location of the center of gravity is rigidly fixed to the body and cannot be moved without altering the body to which it is "attached."

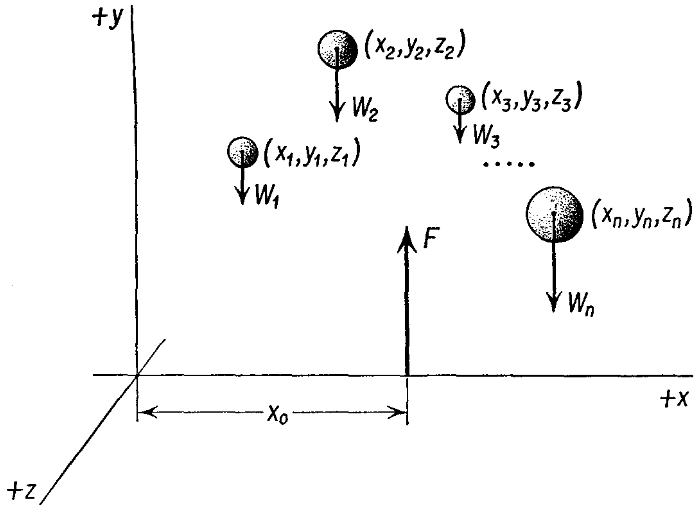

Fig. 4-9 The single force **F** acting through the center of gravity of the system of particles will support the system in equilibrium.

The location of the center of gravity of a distribution of particles may easily be calculated from the conditions of equilibrium for a rigid body. Consider a collection of n particles, each of which has weight W_i where $i = 1, 2, 3, \ldots n$, and is located at coordinates (x_i, y_i, z_i), as shown in Figure 4-9. To find the coordinates of the center of gravity, we imagine that these weights are attached to a rigid weightless framework, and we seek the location of a single force **F** which will support the system in equilibrium. The equilibrium for translational motion will be assured if **F** satisfies the first condition for equilibrium. Thus, summing the forces as shown in the figure,

$$\sum F_y = +F - W_1 - W_2 - W_3 - \cdots - W_n = 0,$$

from which

$$F = W_1 + W_2 + W_3 + \cdots + W_n = \sum W_i.$$

To satisfy the second condition for equilibrium, the sum of the torques acting on the system about any axis must be zero. We choose an axis of rotation directed along the z axis, passing through the origin. Each of the forces $\mathbf{W}_i$ is acting in the $-y$ direction, while the force $\mathbf{F}$ is acting in the $+y$ direction through an unknown point whose coordinates may be taken as (x_0, y_0, z_0). The moment arm of the force $\mathbf{F}$ about the chosen axis is given by x_0, while the moment arm of a force $\mathbf{W}_i$ is given by its x coordinate x_i. Applying Equation (4-3) for determining the z components of the torque, we find

$$\sum G_z = +Fx_0 - W_1x_1 - W_2x_2 - W_3x_3 - \cdots - W_nx_n = 0;$$

thus
$$x_0 = \frac{W_1x_1 + W_2x_2 + W_3x_3 + \cdots + W_nx_n}{F},$$

so that
$$\boxed{x_0 = \frac{\sum W_ix_i}{\sum W_i}.} \tag{4-4a}$$

By reorienting the system so that the x axis is vertically upward, we can find the y coordinate of the center of gravity

$$\boxed{y_0 = \frac{\sum W_iy_i}{\sum W_i},} \tag{4-4b}$$

and in one additional reorientation we obtain

$$\boxed{z_0 = \frac{\sum W_iz_i}{\sum W_i}.} \tag{4-4c}$$

A distribution of matter not made up of point particles can be imagined to be divided into pieces of simple geometric shapes. Each of these may be replaced by a point particle of the same weight located at its center of gravity, and the location of the center of gravity of the body may then be calculated from Equations (4-4).

Illustrative Example. Find the location of the center of gravity of a carpenter's square made of sheet steel. The body dimensions of the rule are 24 in. × 2 in., and the dimensions of the tongue are 16 in. × $1\frac{1}{2}$ in. The square, laid onto a coordinate system, is illustrated in Figure 4-10. Suppose the square is made of material weighing σ (sigma) lb/in.^{2}. We divide the square up into two simple rectangles—a body section of dimensions 24 in. × 2 in.^{2} and a tongue section of dimensions 14 in. × $1\frac{1}{2}$ in.^{2}, as shown in the figure. The center of gravity of each of these sections is located at the center of that section. Thus we may imagine the body section whose cross-sectional area is 48 in.^{2} to be replaced by a particle weighing 48σ lb located at the point whose (x, y, z) coordinates are given by (12, 1, 0). Similarly, the tongue section may be replaced by a particle

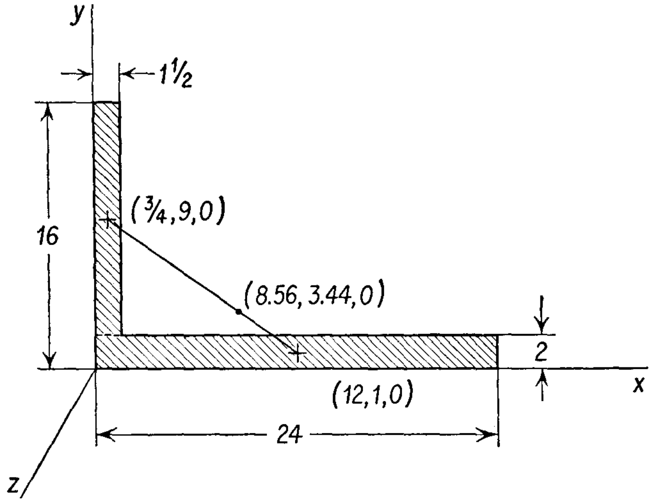

Fig. 4-10

weighing 21σ lb located at a point whose coordinates are $(\frac{3}{4}, 9, 0)$. For the case of two point particles, Equations (4-4) reduce to

$$
\begin{aligned}
x_0 &= \frac{W_1 x_1 + W_2 x_2}{W_1 + W_2} \\
&= \frac{48\sigma \text{ lb} \times 12 \text{ in.} + 21\sigma \text{ lb} \times \frac{3}{4} \text{ in.}}{48\sigma \text{ lb} + 21\sigma \text{ lb}} \\
&= 8.56 \text{ in.}, \\
y_0 &= \frac{W_1 y_1 + W_2 y_2}{W_1 + W_2} \\
&= \frac{48\sigma \text{ lb} \times 1 \text{ in.} + 21\sigma \text{ lb} \times 9 \text{ in.}}{48\sigma \text{ lb} + 21\sigma \text{ lb}} \\
&= 3.44 \text{ in.},
\end{aligned}
$$

and, since the figure may be thought to be in the x-y plane,

$$z_0 = 0.$$

Thus the coordinates of the center of gravity have been obtained. As shown in the figure, the center of gravity of the system lies along the line joining the centers of gravity of the base and the tongue of the square.

We may represent the procedure for finding the center of gravity of an extended body in the form of an integral by replacing the summation signs in Equations (4-4) by integral signs. Thus we have

$$x_0 = \frac{\int x\,dw}{\int dw} = \frac{\int x\,dw}{W}, \tag{4-5a}$$

$$y_0 = \frac{\int y\,dw}{W}, \tag{4-5b}$$

$$z_0 = \frac{\int z\,dw}{W}, \tag{4-5c}$$

where dw is the weight of a small volume element of the body located at coordinates x, y, z, and the total weight of the body is represented by W.

4-6 Discussion and Further Examples

The problems of statics vary greatly in difficulty, but if they are soluble at all they are soluble by the methods and principles developed in this chapter. The two fundamental principles which govern the equilibrium of a rigid body, and which govern the equilibrium of a particle in the limiting case that the rigid body is composed of a single particle, are: *The vector sum of all the forces acting on the body must be zero. The vector sum of all the torques about any axis acting on the body must be zero.* Written in equation form, these two statements are

$$\boxed{\begin{aligned}\textstyle\sum F &= 0; \quad &(4\text{-}6a)\\ \textstyle\sum G &= 0. \quad &(4\text{-}6b)\end{aligned}}$$

These two equations, in extremely concise form, represent our entire knowledge of the forces exerted by and on structural elements and form the analytical foundation upon which all structures are built. While, in general, equilibrium is interpreted to mean a state of rest with respect to the earth, it must be recognized that rest and uniform motion in a straight line are equivalent conditions, according to Newton's first law of motion. Thus it is that the very same equations apply to the equilibrium of a structure moving with uniform speed, and the analytic procedures which apply to the construction of a crane or a bridge may also be used in the design of an airplane.

Illustrative Example. A wagon wheel 26 in. in diameter and weighing 10 lb rests against a square curb 8 in. high, as shown in Figure 4-11. What horizontal force applied to the axle is necessary to push the wheel over the curb?

The wagon wheel will start to rise when the supporting force exerted by the roadway on the wheel is zero. At that time the forces acting on the wheel, as shown in Figure 4-11(b), are the unknown horizontal force **H**, the force of gravity **W** acting at the center of gravity of the wheel, and the force of the curb **P** against

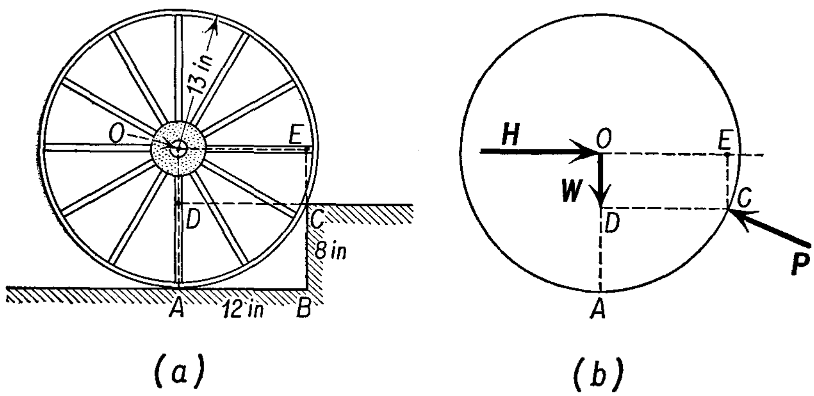

Fig. 4-11

the wheel. Let us choose an axis of rotation normal to the plane of the paper at the curb C. The moment arm of the force **W** is the distance $\overline{DC}$, 12 in. The moment arm of the force **H** is $\overline{EC} = \overline{OD} = 5$ in. Applying the torque condition for equilibrium, we know that the sum of the torques G_C about an axis normal to the plane of the paper through C is equal to zero; or

$$\sum G_C = 0 = W \times \overline{DC} - H \times \overline{EC},$$

or

$$10 \text{ lb} \times 12 \text{ in.} - H \times 5 \text{ in.} = 0,$$

so that

$$H = 24 \text{ lb}.$$

Illustrative Example. A ladder 26 ft long and weighing 30 lb leans against a smooth wall 24 ft from the ground and rests on a rough floor 10 ft from the wall. A man weighing 200 lb climbs 20 ft up the ladder before the ladder starts to slip [see Figure 4-12(a)]. (a) Find the forces exerted on the ladder by the floor and the wall. (b) What is the coefficient of static friction between the ladder and the floor?

We begin by isolating the ladder and labeling the forces acting on it, as shown in Figure 4-12(b). The unknown force exerted by the floor at the point a is called **A**, with components $\mathbf{A}_x$ and $\mathbf{A}_y$. The entire weight of the ladder **W** of 30 lb acts vertically downward through its center of gravity located at the middle of the ladder. The weight of the man **M** of 200 lb acts vertically downward through a point 20 ft up the ladder. The smooth wall exerts a force **B** which must be perpendicular to the wall. Once again, italic symbols represent the magnitudes

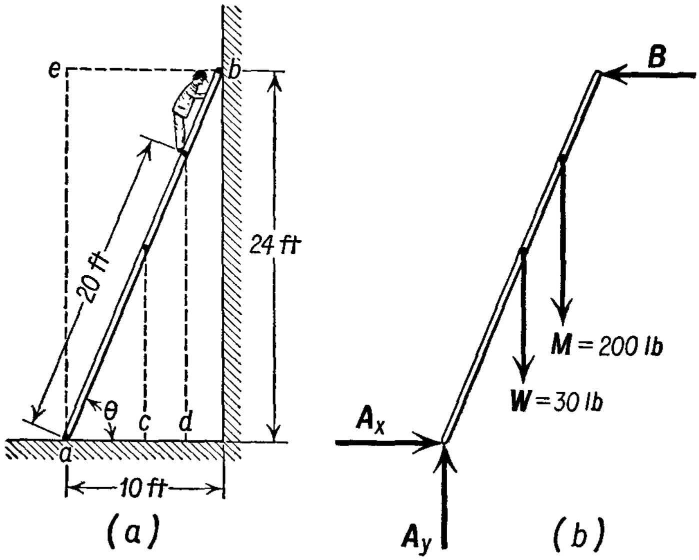

Fig. 4-12

of the forces, with directions given by the directions of the arrows. From the condition for equilibrium for the x components of the forces acting on the ladder, we have

$$\sum F_x = A_x - B = 0,$$

while for the y components we have

$$\sum F_y = A_y - 30\text{ lb} - 200\text{ lb} = 0,$$

so that

$$A_y = 230\text{ lb}.$$

Applying the conditions that the sum of the torques on the ladder must be zero, we choose an axis perpendicular to the plane of the paper through any convenient point such as a and get

$$\sum G_a = 0 = A_x \times 0 + A_y \times 0 - W \times \overline{ac} - M \times \overline{ad} + B \times \overline{ea}.$$

Substituting numerical values, we obtain

$$-30\text{ lb} \times 5\text{ ft} - 200\text{ lb} \times \tfrac{100}{13}\text{ ft} + B \times 24\text{ ft} = 0,$$

from which

$$B = \frac{150\text{ lb ft} + 1{,}540\text{ lb ft}}{24\text{ ft}} = 70.4\text{ lb};$$

and since $A_x = B$, from a preceding equation,

$$A_x = 70.4\text{ lb}.$$

The coefficient of static friction has been defined from the equation

$$F_r = fN.$$

In this example the force A_x is the frictional force, and A_y is the normal force, so that the coefficient of friction is equal to

$$f = \frac{A_x}{A_y} = \frac{70.4 \text{ lb}}{230 \text{ lb}} = 0.31.$$

Note that the coefficient of static friction was obtained from an analysis of the forces on the ladder when the ladder was on the point of slipping, when the force of static friction was at its maximum value.

Problems

4-1. Determine the torque produced by a force of 6 lb acting horizontally on the top of a bicycle wheel 24 in. in diameter with respect to an axis through its axle.

4-2. A torque of 5 ft lb is required to swing open a door which is 30 in. wide. What is the least force that must be exerted to open the door if it is applied (a) at a distance of 30 in. from the line of hinges and (b) at a distance of 24 in. from this line?

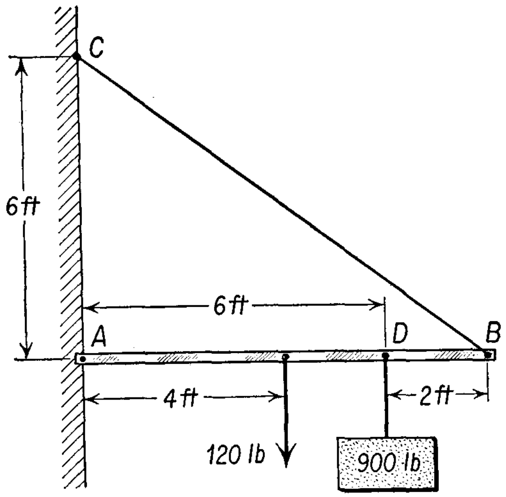

Fig. 4-13

4-3. A uniform horizontal bar AB, 8 ft long and weighing 120 lb, is pinned to the wall at A, while a steel cable 10 ft long extends out from a point C on the wall and is fastened to the bar at the point B, as shown in Figure 4-13. This bar supports a weight of 900 lb at a point D, 6 ft from the wall. Determine (a) the tension in the cable, (b) the vertical component and (c) the horizontal component of the force at A.

4-4. A man carries a bar 6 ft long which has two loads, one of 40 lb and the other of 60 lb, hung from its ends. At which point should the man hold the bar to keep it horizontal? Neglect the weight of the bar.

4-5. If the bar in Problem 4-4 is uniform and weighs 20 lb, determine the point at which the man should hold the bar to keep it horizontal.

4-6. A load of 180 lb is hung from a bar 10 ft long at a point 6 ft from one end. Two men carry this bar in a horizontal position. How big a force does each man exert, assuming that the bar is supported at its ends?

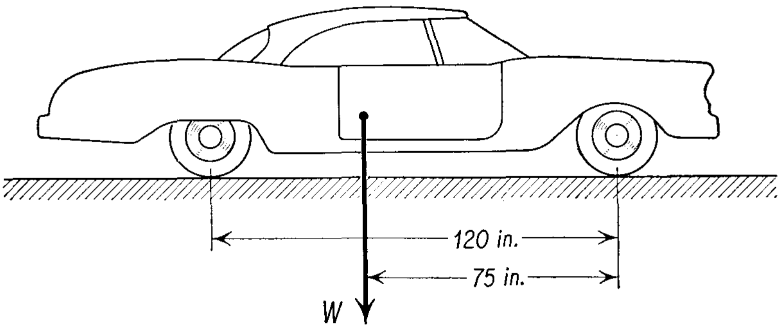

Fig. 4-14

4-7. A car weighing 3,200 lb has a wheel base of 120 in., and its center of gravity is 75 in. from the front wheels (see Figure 4-14). Determine the force (a) that the two front wheels exert on the ground and (b) that the two rear wheels exert on the ground.

4-8. A car weighing 3,600 lb has a wheel base of 125 in., and its center of gravity is 80 in. from the front wheels. Two passengers sit in the front seat. If their combined weight is 400 lb and if their center of gravity is at a point 60 in. from the front wheels, determine the shift in the center of gravity produced by the passengers.

4-9. A boom in the form of a uniform pole weighing 400 lb is hinged at the lower end. The boom is held at an angle of 60° with the ground by means of a horizontal cable attached to its upper end. (a) Determine the tension in the cable when there is no load on the boom. (b) Determine the tension in the cable when a load of 1,000 lb is attached to the upper end of the boom.

4-10. A door 8 ft high and 3 ft wide weighs 80 lb, and its center of gravity is at its geometrical center. The door is supported by hinges 1 ft from top and bottom, each hinge carrying half the weight. Determine the horizontal component of the force exerted by each hinge on the door.

4-11. A uniform ladder 25 ft long rests against a smooth vertical wall. The ladder weighs 30 lb. The lower end of the ladder is 15 ft from the wall. A man weighing 150 lb climbs up the ladder until he is 20 ft from the base of the ladder, at which point the ladder starts to slip. What is the coefficient of friction between the ladder and the floor?

4-12. Two rods, each of length 10 ft and weight 5 lb, are joined to make a 30° V. Find the center of gravity of the V.

4-13. Find the center of gravity of a collection of weights located at the corners of an equilateral triangle, each side of length a. The three weights are 1, 2, and 3 lb, respectively. Place the x-axis along the line joining the 1 and 3 lb weights with the origin at the 1 lb weight.

4-14. A card table is made of 4 straight legs of dimensions 1 in. $\times$ 1 in. $\times$ 24 in., each weighing 1 lb, which are fastened to the corners of a square table top 30 in. on an edge by 1 in. thick. The table top weighs 5 lb. Find the center of gravity of the table.

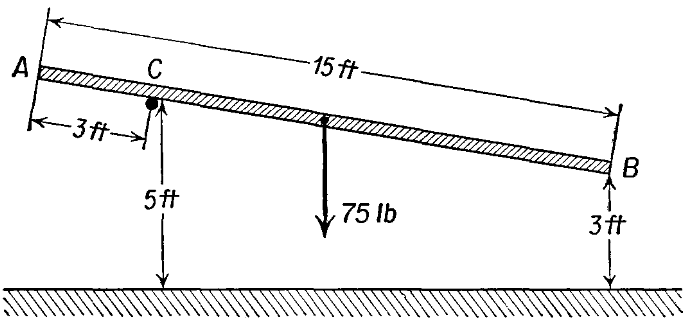

Fig. 4-15

4-15. A uniform beam 15 ft long weighing 75 lb is supported 3 ft from its upper end A by a smooth cylindrical rail which is 5 ft from the ground, as shown in Figure 4-15. What force must be exerted at the lower end B of the beam, located 3 ft from the ground, in order to support the beam?

4-16. A chain 5 ft long is placed on a horizontal table so that part of it hangs over the edge. If it starts to slip when 2 ft of chain hang over the side, find the coefficient of starting friction between the chain and the table.

4-17. Find the location of the center of gravity of a square sheet of metal of edge 4 in. which has had a smaller square of edge 1 in. cut out of one corner.

4-18. A uniform ladder 20 ft long and weighing 35 lb rests against a smooth wall at an angle of 30° to the wall. A 200-lb man stands 15 ft up the ladder. If the coefficient of friction between the floor and the ladder is 0.1, what additional horizontal force must be exerted at the base of the ladder to keep it from slipping?

4-19. Show that the center of gravity of a thin uniform board cut in the form of an isosceles triangle of altitude h is at a point $\frac{2}{3}h$ from the vertex on the perpendicular bisector of the base. [HINT: Choose a set of x-y coordinate axes with the origin at the vertex and the x axis along the perpendicular bisector, as shown in Figure 4-16. Take an element of the board formed by two lines a distance dx apart parallel to the base. The area of this element is $2y\,dx$, and its weight is $dw = \sigma \cdot 2y\,dx$, where σ is the weight per unit area. Then apply Equation (4-5a). Note that

$$y = \frac{b}{a}x$$

where $2b$ is the width of the base.]

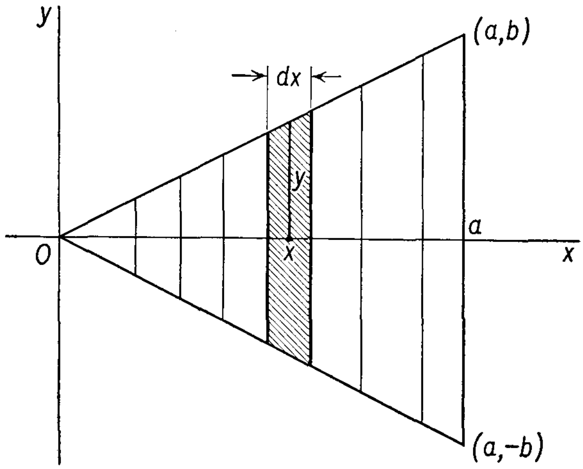

Fig. 4-16

4-20. Find the center of gravity of a thin board cut in the form of a 3–4–5 right triangle. [HINT: Apply the result of Problem 4-19.]

4-21. An irregular slab of material is pivoted at one corner by a horizontal pin, and is supported by a vertical force of 80 lb located 10 ft to the right of the pin. The slab weighs 200 lb. (a) How far to the right of the pivot is the center of gravity located? (b) What is the force on the object due to the pivot?

4-22. Show that if the resultant of a set of concurrent forces is zero, the sum of the moments of these forces about any axis is zero.

4-23. Using the second condition for the equilibrium of a body, show that when a body is in equilibrium under the action of three nonparallel forces, these forces must pass through a single point; that is, the forces are concurrent.