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Course Portfolio for Math 872: Topology II

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Promoting graduate student engagement

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Abstract

In this course portfolio, I explore tactics for engaging a group of beginning mathematics graduate students taking the second semester of the first-year sequence in topology. Course objectives include computing algebraic invariants of topological spaces, understanding the Galois correspondence between covering spaces and fundamental groups, and using a range of invariants to distinguish spaces. My main instructional tools include a series of three “Activities” tailored to each student, a project involving computing invariants with the program SnapPy and using homology of covering spaces to distinguish 3-dimensional knots, and individual student board work during class periods. The activities served as benchmarks to gauge student understanding of key concepts; the first functioned as a pretest, the second was completed mid-semester, and the third served as a posttest. I have used the activities to analyze the progression of student understanding over the course of the semester, and in addition, I have examined student performance on midterm exams in order to determine the extent to which individual board work impacted learning. The portfolio also contains a number of examples of student work including the activities, projects, board work, and exam responses.

Keywords: Graduate education, mathematics, engagement, board work, computational project
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Chapter 1

Objectives of Course Portfolio

In a broad sense, I took part in this program in order to become more mindful of my teaching practices and more deliberate about the decisions I make in the classroom. I had taught Math 872, Topology II once previously, and two goals I had for my instruction in the spring semester of 2019 were to connect course concepts to their practical applications and to include more active practice during my lectures. By enumerating course objectives prior to the start of class, I was able to give the course a unified narrative from start to finish. Within the portfolio, I discuss the choices I made for the various course activities and project, and the nature of the in-class work at the chalkboard I used to engage students during our lecture periods.

The primary objective of this portfolio is to document the choices I made in order to accomplish my goals for the course and to analyze the effects of those choices on student learning. Additionally, I reflect on the course outcomes, including what went well and what did not. I also explore possible chances to improve these materials and methods the next time I teach this course.

A secondary goal of the portfolio is to be useful for instructors teaching introductory graduate courses. In mathematics in particular, many graduate courses continue to use a traditional lecture model in the classroom, and current trends in undergraduate education remain untapped in the graduate setting. While the traditional model certainly has many merits, alternative approaches are worth exploring, and I hope that the interested reader will be encouraged to do so.
Chapter 2

Benchmark Memo 1: Course Description

2.1 Course goals

The intended audience for this course is beginning students in the mathematics graduate program at UNL. It is the second course in a sequence of two “qual-leading” courses; students who take the course should be prepared to pass the qualifying examination in topology. The qualifying exam has a publicly available syllabus, set by members of the department’s Graduate Advisory Committee. The portion of the syllabus corresponding to this course reads as follows:

1. Fundamental groups: Fundamental group, induced homomorphism; free group, group presentation, Tietze’s theorem, amalgamated product of groups, Seifert-van Kampen Theorem; cell complex, presentation complex, classification of surfaces.

2. Covering spaces: Covering map, Lifting theorems; covering space group action; universal covering, Cayley complex; Galois Correspondence Theorem, deck transformation, normal covering; applications to group theory.

3. Homology: Simplicial homology, singular homology, induced homomorphism, homotopy invariance; exact sequence, long exact homology sequence, Mayer-Vietoris Theorem. Applications.

Since students may have several opportunities to take the qualifying exams, and topics tested can be drawn from anything on this list, it is important that they are familiar with each of these concepts if they are expected to be able to pass the qualifying exams. I see this as the primary goal of the course.

In addition, I developed my own set of goals that go beyond a list of topics: Defining
the fundamental group and homology and proving invariance, distinguishing topological spaces using the power of algebraic invariants, and understanding the basics of functoriality and categorification. The first goal relates to the important basic structures in the subject. The second goal is about applications of these structures. The third goal relates to seeing an underlying framework used to relate algebra to topology – these foundational ideas can be applied in a huge variety of contexts. My intention is that after taking my class, students will retain the basic definitions of fundamental group/homology, know how to compute these invariants, and be able to distinguish some canonical examples. If they continue to study topology, they should be to able recognize the same structures they see in Math 872 in more complicated situations.

The qualifying syllabus topics are typically structured in a way that reflects the course textbook, Allen Hatcher’s *Algebraic Topology*. The first section of this book defines the fundamental group, the second details covering space theory, and the third describes homology. This evolution can be seen as moving from the concrete to abstract; typically students grasp the idea of the fundamental group more clearly than the more nebulous idea of homology.

The general goal of the qual-leading courses is to give students exposure to a broad range of high-level mathematical ideas. My goals for this course fit into that framework; the fundamental group, covering spaces, and homology are the biggest elementary ideas from algebraic topology. The course syllabus is included in Appendix 6.1.

### 2.2 Course learning outcomes

At the end of this course, students will

1. Compute the fundamental group and homology groups of a given topological space using a variety of methods,

2. Connect the structures in covering space theory with the structures in group theory, and

3. Distinguish topological spaces using algebraic invariants.

### 2.3 Context

I chose this course because when I taught it (once) before, I struggled to balance details and minutiae with the big picture ideas. In a practical sense, I use the tools from Math 872 quite often in the course of my own research; however, I very rarely consider the foundational underpinnings that make these tools work. In this sense, it seems more important to stress application and computations. On the other hand, I would like
students to (at some level) see the beauty and elegance of the underlying mathematics. In total, balancing these two somewhat competing goals is difficult to achieve.

This difficulty was reflected in prior student evaluations following the spring semester of 2018. Some students complained that I spent too much time on proofs; others were upset that I did not go into great enough detail in other areas. This course is also more difficult in content than Math 871. Whereas Math 871 is a sort of gentle introduction to topology, the structure of Math 872 assumes that students (who may have just seen the Math 871 material for the first time) are fluent in the techniques of point-set topology. This rapid acceleration makes it easy to leave less advanced students behind, and I would like to make sure that those students still benefit from taking the class.

2.4 Enrollment/demographics

The course enrollment was seven students, most of whom are enrolled in the PhD program in mathematics. Six of the seven students had taken the Math 871, Topology I prerequisite course I taught in Fall 2018. The other student had taken the prerequisite course as an undergraduate. Five of the students were first-year graduate students and the two others were in their second year. In addition, three out of the seven students had received instruction that overlapped with some, but not all, of the course material at another institution. These differences in preparation added to the difficulty of teaching the course, since the repetition could bore some of the better prepared students. In their consent forms, two students requested that their names be included with their work in the Appendix, Chapter 6.
Chapter 3

Benchmark Memo 2: Teaching Decisions and Course Content

3.1 Teaching methods

The course met twice weekly for periods of 75 minutes, from 2:00pm until 3:15pm every Tuesday and Thursday, and I also held two hour-long office hours each week so that students could meet with me independently. The general method of content delivery was lecture. During lecture I typically alternated between presenting theory and examples in order to vary the type of content and to keep the students engaged. I often incorporated some element of active learning in each lecture; usually in the form of a short exercise or example for students to work out on their own. Additionally, I used the layout of the room to my advantage – there were seven sections of blackboard on the side and back walls of classroom, and there were seven students in the class; hence, on a number of occasions I asked students to go to the blackboard to work out a particular problem. During this time, I would walk around the classroom to answer specific questions or help an individual student with a hint.

One aspect of my lectures that I wish to highlight is that I intentionally limited the amount of detail present in my lecture notes. My notes usually consisted of a basic outline, so that I could work out the details for a given computation in real time. The purpose of this teaching method is to give students the opportunity to see the process of a proof or a computation rather than a polished, finished product. The trade-off, of course, was that I did not always present concepts perfectly; on several occasions, I made a misstep or error in a proof, and I would have to cross out several sentences on the board or return the following period to correct a mistake. Nonetheless, I believe this delivery method allows for appropriate pacing during lecture.
3.2 Course materials

The primary textbook for this course is traditionally *Algebraic Topology*, written by Allen Hatcher. The topics in the textbook align well with the syllabus for the qualifying exam, and so it is practical to continue to use this text in the near future. During the first two weeks of class, I lectured out of Chapter 1 of William Massey’s *Algebraic Topology: An Introduction*. In order to carry out interesting but tractable computations in algebraic topology, students should have a diverse collection of examples to draw from. Chapter 1 of Massey’s text introduces the classification of topological surfaces, a convenient set of examples to be used for computations throughout the course. Surfaces also necessitate concrete examples of structures, like triangulations and Euler characteristic, that come back more abstractly in the form of ∆-complex structures and (homological) Euler characteristic near the conclusion of the course. It is useful to build on these concrete structures to motivate the more abstract ideas in homology theory.

3.3 Outside activities used

During the course, I facilitated student learning in the following ways:

3.3.1 Weekly homework assignments

I asked students to complete weekly homework assignments consisting of four or five homework problems related to the concepts from that week’s lecture. Problems were designed so that students would have to work out calculations that were more detailed than those we had time to do during class. I also asked students to adapt proofs of theorems from class to other contexts, and sometimes I would leave small sections of proofs as homework exercises. In mathematics, some proof structures are best learned and understood when a student “discovers” the proof for themself. A secondary goal of the homework exercises was to give students practice writing detailed, structured proofs. Such skills are necessary to develop so that students can communicate more complicated mathematical ideas in research papers or dissertations.

3.3.2 Activities graded for participation

I asked students to complete three assignments, called “Activity 1,” “Activity 2,” and “Activity 3.” Each activity consisted of three pairs of topological spaces, and students were requested to respond to a simple question: Are these spaces different or the same? The activities were assigned at the beginning, middle, and end of the semester.
If a student answered one of the problems correctly, I replaced that problem with a new, more difficult one for their next activity. The activities were intended to be less rigorous and structured compared with the homework assignments. For this reason, I only graded the activities on completion, not on correctness. The activities are included in Appendix 6.2.

3.3.3 Course project

Near the end of the semester, I had students complete a course project using a well-known computer program, SnapPy, to distinguish topological knots. In general, the computations one can do by hand are not nearly involved enough to be relevant to a research mathematician. SnapPy is a program I frequently use in my research, and the goal of the project was for students to see how computer-aided calculations can be used in practice. Students responded to several theoretical questions, setting up the basis for the calculations, in addition to carrying out computations with SnapPy in order to distinguish mathematical knot of their own constructions. They turned in written responses to the theoretical questions and screenshots of the output of the program, along with some discussion of and conclusions about the output. The project is included in Appendix 6.4.

3.3.4 Exams

Students took a total of three unit exams, each of which covered roughly five weeks of material from lecture. The course is neatly divided into three units: Homotopy and the fundamental group, the theory of covering spaces, and homology. These were the topics for the first, second, and third exams, respectively. Each exam consisted of five questions, each worth 10 points, and students were asked to respond to four out of these five. Some questions contained multiple parts and all answers were expected to be written in well-justified complete sentences, mirroring the format of the qualifying exam a subset of the students will take. Occasionally, if a student performed poorly on an exam, I offered that student the option to write corrections on a separate sheet of paper to earn back a portion of the points they missed.

3.4 Rationale for teaching methods

Growing evidence, both from my personal experiences and from the educational literature, supports the use of active practice techniques in the classroom. Students best learn mathematics when they engage in that mathematics, and for this reason, I attempt to incorporate some form of active practice into each of my lectures, for any
course in the undergraduate or graduate mathematics curriculum. When students attempt to solve a given problem on a blackboard, something simple but useful happens: They get out of their seats, and they write their thoughts on a blackboard, where the rest of the class can see their work. Students benefit from seeing the examples worked out by their classmates, and they are held accountable for their own participation. Another reason I like to employ this technique is that I can ask students different questions that are tailored to their abilities and preparations. The students in the course had a wide range of backgrounds; thus, I could keep all students interested by giving the beginning students more foundational problems and giving the others more of a challenge.

As mentioned above, my lecture style is predicated on the belief that students will benefit from seeing me work out proofs and computations in real time, as opposed to copying or memorizing the steps in a clean, precise proof. With this practice, I draw from my own experience as a student: The algebraic topology course I took as a graduate student was taught in this manner, and it was one of the more useful courses I recall from graduate school. This process also elicits student participation; we established a number of results via proof by committee, and on occasion when I was able to get stuck on a proof, students could help me get back on track. I also believe that it is quite important to model failure as well as success in the classroom, because doing research-level mathematics often involves making mistakes, back-tracking, and being confused. A research article is merely a finished product; it does not tell the story of how the ideas came to be.

3.5 Rationale for outside activities

The purpose of the homework assignments was for students to interact with and to internalize fundamental concepts from lecture. As Paul Halmos wisely said: “The only way to learn mathematics is to do mathematics.” A secondary purpose is hone students’ proof-writing skills. The homework problems elevated the difficulty of in-class exercises. The exam problems were at a similar or slightly lower level due to time constraints. The format of the exams parallels the format of the qualifying exam, in which students are asked to respond to three out of four problems in algebraic topology. More generally, by allowing students to pick one problem to skip, I am able to write slightly more difficult problems, and students can take the exam with the knowledge that no single problem can make or break their exam score.

I designed Activities 1, 2, and 3 and the Project with a direct view of the course objectives: Students will learn how to distinguish topological spaces. The Activities were graded based on participation, not correctness, because I expected them not to know how to do the problems contained in Activity 1. Instead, I wanted them to think broadly about the same set of problems at three different periods during the semester,
so that they could observe the development of skill sets over time. The purpose of the project was for students to use their tools on complicated examples beyond what we had time to compute by hand, so that they could see firsthand how these tools are applied practically in low-dimensional topology.

3.6 Changes from the previous year

I taught this course for the first time in the spring semester of 2018. Teaching that section came with all of the usual challenges of teaching a course for the first time - I was not as familiar with the course material, and I did not fully grasp the narrative of the course. I incorporated active practice techniques in the form of “think-pair-share.” In the spring semester of 2019, the students cared less for this method. I struggled to get them to talk to each other, or someone would say the answer out loud before the other students had thought about the problem, and “think-pair-share” did not mesh well with the chemistry of this particular class. One possible explanation for why this method failed is the large range of ability levels and preparation of students in the class. Some knew the answers right away, while others took a bit longer. For this reason, incorporating individualized work at the blackboards more effectively engaged each student. I had not tried this technique in previous courses, but I was pleased with how well it was received by the students.

In spring 2019, having taught the course before, I had a better understanding of the scope of the content. I knew which topics to cover in greater depth and which topics to gloss over, and this made preparing lectures significantly easier. Additionally, my assignment of the Activities and the Project were new for spring 2019, motivated by my articulated course objectives.
Chapter 4

Benchmark Memo 3: Analysis of Student Learning

4.1 Analysis of Activities 1, 2, and 3

Students submitted three assignments called “Activity 1,” “Activity 2,” and “Activity 3,” at the beginning, middle, and end of the semester, respectively. Each activity included three problems, and each problem asked students to decide and justify whether two topological spaces were the same or different. If a student answered a problem correctly, it was replaced with a more difficult problem in the next activity; hence, after Activity 1, each student received a personalized Activity 2 and Activity 3 depending on their performance on the previous activity. The topological spaces included in Problems 1, 2, 3, 5, and 6 were different, while the spaces in Problems 4 and 7 were the same. The tables below shows the progress of each student, labeled A-G, on each problem.

The first activity was intended to be a pretest, assigned on the first day of class. Each student was assigned Problems 1, 2, and 3. The results are indicated in the table below, in which a blue square indicates a correct answer, and a red square indicates an incorrect answer. A “–” entry indicates that a student had not yet been assigned the given problem.

To my surprise, two of the students, Students A and B in the table, answered Problems 1 and 3 correctly. Initially, I had only planned for the activities to have three problems, but I decided to incorporate more problems as a result of Activity 1, designing the second activity to have two different forms, one with Problems 1, 2, and 3 for Students C-G, and another with Problems 2, 4, and 5 for Students A and B.
Activity 2 was assigned mid-semester on March 12th, 2019 and due on March 14th, 2019. The results are shown in the table below.

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At this point, I expected every student to be able to answer Problems 1 and 3 correctly, since these problems require the use of the fundamental group, a topic we had already covered in depth. Problem 2, on the other hand, requires the use of homology, a different tool we had not yet learned, and so I did not expect any students to get Problem 2 correct on Activity 2. Once again, I was surprised by Students A and B; in particular, Student A answered three problems, numbers 3, 4, and 5 correctly on Activity 2, whereas Student B correctly answered Problems 3 and 5 but faltered on Problem 4.

Activity 3 was assigned on April 15th and due on the first day of finals, April 29th. The results are indicated below.

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In Activity 3, Students B, C, and D answered one or more new problems correctly. At this point in the semester, I expected all students to be able to answer Problem 2 correctly. Notably, all five students correctly stated that the two spaces in question, $S^4$ and $S^2 \times S^2$ should be non-homeomorphic spaces (which was not the case in previous activities), yet each of the five explanations was not accurate, so the problem was marked incorrect. Student C’s work on Activity 3 can be seen in Appendix 6.3.1. All three problems have correct answers, but the first problem (Problem 2 from the table) has an incorrect explanation, while the others two explanations are mostly correct. A sample of Student E’s work on Activity 3 can be seen in Appendix 6.3.2; this student had a correct answer but an incorrect explanation.

Previously, some of these students believed that the spaces in Problem 2 should be the same; thus, this change certainly constitutes progress. It is also important to note that Problems 6 and 7 were exploratory in the sense that the spaces in Problem 6 could not be distinguished using tools from the class, and the spaces in Problem 7 are homeomorphic but it is also beyond the scope of the course to exhibit a homeomorphism. Nevertheless, Student B correctly guessed that the spaces in Problem 6 are different, noted that our tools could not tell them apart, and even guessed the tool (intersection forms) that would be able to distinguish them (see Appendix 6.3.3). This level of answer is as high as I could reasonably expect for any student in the course given what they learned over the course of the semester.

### 4.2 Analysis of the Project

The project required students to install the program SnapPy on their personal computers in order to compute homologies of 3-dimensional spaces related the topology of knotted loops. I frequently use this program in my own research, and I designed the project so that students could see how homology can be used in practice. The project included five problems, two of which were theoretical, and three of which were computational. The computational problems were straightforward to complete by following the step-by-step instructions in the project description, and in general students easily completed the computations. Problem 3 asked the students to design their own knots and use homology to distinguish the topological spaces determined by the knots. Problem 4 instead gave the students two knots which are indistinguishable by their torsion numbers, asking students to reflect on these computations. Problem 5 gives students a different invariant, hyperbolic volume, which SnapPy can compute and which does distinguish the two knots. The handout for the project is included in Appendix 6.4.

Students were quite creative in constructing their own knots for Problem 3, and all seven students were able to construct and distinguish their own constructed knots by following the commands given in the project description. Examples of student creativ-
ity is included in Appendix 6.5.1. Additionally, most students achieved a secondary
goal of the project, which is to recognize the limitations of invariants. Appendix 6.5.1
includes student work that recognizes this fact. Not every student, however, correctly
understood the goal of Problem 4. One student erroneously concluded that torsion
numbers should be the same for two knots with the same number of crossings, which
is incorrect. That work can be viewed in Appendix 6.5.2.

One student had a background in knot theory and decided to deviate from the instruc-
tions to make the assignment more challenging. Instead of entering the two knots in
Problem 4 by hand, they used a different problem to identify the knots, and then they
wrote a short script to carry out the requested computations. The student explained
all of this very clearly, and I was impressed by their finished work, which appears in
Appendix 6.5.3.

4.3 Analysis of board work in conjunction with Ex-
ams 2 and 3

Typically, I solicit informal feedback from students twice during a semester, roughly
four to five weeks in, and again just after the midpoint of the semester (Appendix 6.7).
Students filled out the first of these feedback forms on February 14th, 2019. When I
taught most of these students in the first semester, I often used a “think-pair-share”
activity to promote engagement and active practice. For this course, however, the
activity was not well-received, and students were clearly unenthusiastic about working
out short exercises with their peers during class time. Indeed, one student noted this
issue in their first review form, stating that the discussion “feels forced” (comment
included in Appendix 6.7.1).

After reconsidering my strategies for active practice during the class period, I decided
to incorporate individual board work as an alternative to “think-pair-share” activities
to see if this could stimulate student engagement. This type of activity was especially
useful in the second unit of course content, covering spaces, which is example-driven.
The first time I asked students to approach the board, I requested that they con-
struct an example of a covering space of a simple topological space, the wedge of two
circles.

The examples they constructed were varied, from the simple to complex, and the open-
ended nature of the exercise allowed for a range of ability levels: The students who were
seeing the topic for the first time could construct basic examples, and the students who
were reviewing a concept they already understood could find complicated and intricate
examples. Samples of the spaces students constructed appear in Appendix 6.8. In later
classes, board work activities built off of these examples: One day I asked students to
follow an algorithm to write down a presentation for a subgroup coming from their
particular covering space. As before, the complicated examples had more complicated presentations, whereas the simpler examples had simpler presentations, and the work was organically adapted to the ability level of each student. In a subsequent exercise at the board, students decided if their covering spaces satisfied a special type of property known as normality. Finally, I was able to draw on these specific examples to use in my own lecture, and since the students “owned” the examples, the material became more relatable.

As evidence that the board work exercises impacted student learning, consider the table below, which displays the results of the second exam, administered on March 25th, 2019. The exam consisted of five problems, of which students were asked to respond to four of their choosing. Each problem was graded out of 10 points. For the purpose of this table, a score between 8 and 10 is counted as “Correct,” and a score less than 8 is counted as “Incorrect.”

<table>
<thead>
<tr>
<th>Problem 1</th>
<th>Problem 2</th>
<th>Problem 3</th>
<th>Problem 4</th>
<th>Problem 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct</td>
<td>Correct</td>
<td>Correct</td>
<td>Correct</td>
<td>Correct</td>
</tr>
<tr>
<td>Skipped</td>
<td>Correct</td>
<td>Correct</td>
<td>Correct</td>
<td>Correct</td>
</tr>
<tr>
<td>Incorrect</td>
<td>Incorrect</td>
<td>Incorrect</td>
<td>Incorrect</td>
<td>Incorrect</td>
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I should note that in general, the second exam did not go well for most students. I scheduled the exam for the Monday after spring break, and when I started to grade the exam, it became clear that many were underprepared. After reviewing the seven exams, I decided to record grades only for those problems that I considered to be "Correct" with a score of eight or more out of ten. I returned the partially graded exams and asked students to make corrections on the other problems for partial credit.

Appendices 6.9.1 and 6.9.2 contain responses for problems 1 and 2 from Exam 2 for students E and G. Note that Student E was one of the students whose problem 1 was not marked “Correct.” Nonetheless, this student had completely correct answers for parts (a) and (b) of problem 1 but did not answer part (c). Other than the unanswered part (c), both students E and G show mastery of the material in problem 1. Problem 2, on the other hand, is not correct for either student. Both students correctly identified that the problem should use the Seifert-Van Kampen Theorem, but otherwise the work is mostly incorrect.

These responses for problems 1 and 2 are representative of the entire class, and notably, problem 1 was closely related to board work, whereas problem 2 was not. Indeed, the second most successful problem, problem 5, was also connected to the board work we had done. I had expected problem 2 to be one of the easier problems on the exam, which was obviously not the case. Due to the apparent success of the board work, I decided to incorporate more of this type of instruction into the third unit, and problems 1, 2, 3, and 5 on the third exam were closely related to an activity the class had completed on the board. Below, a table indicates the results of Exam 3.

<table>
<thead>
<tr>
<th>Problem</th>
<th>Correct</th>
<th>Skipped</th>
<th>Incorrect</th>
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<tbody>
<tr>
<td>Problem 1</td>
<td>3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Problem 2</td>
<td>4</td>
<td>0</td>
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<td>Problem 3</td>
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<td>Problem 5</td>
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<td>0</td>
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Overall, there were more correct answers for Exam 3 than there were for Exam 2, 17 versus 13. In addition, three of the problems related to board work, problems 2, 3, and 5, went quite well, while students struggled with problem 1. Selected correct responses from Students E and G are included in Appendices 6.9.3 and 6.9.4. Comparing the results of Exam 2 and Exam 3, I conclude that the board work was a factor in improving
students’ examination scores and thus made a positive contribution to their learning. Other factors which may have also influenced exam scores are difficulty and timing of the exams, plus the additional motivation of students to do well after performing poorly on Exam 2. Three out of seven students indicated an appreciation for board work on my second informal feedback form (see Appendix 6.7.2), and on my formal evaluations at the end of the semester, one student said “Working on chalkboards was surprisingly helpful to my understanding of the course material.”
Chapter 5

Summary: Reflection on the Course

5.1 Successful course elements

5.1.1 Course objectives

At a basic level, setting and sticking to course objectives was a successful activity for me. By following the principles of backward design that I encountered in my readings for this program, I enumerated clear goals before I began to teach, and these goals served as guiding principles for almost all of my course activities. I felt that this foundation gave my course a unity that it had previously lacked when I first taught it in the spring of 2018.

5.1.2 Activities 1, 2, and 3

With the course objectives in mind, I incorporated two new course elements, the activities and project described and analyzed above. I would characterize the three activities as moderately successful; students were able to distinguish between a number of different spaces and some even surprised me by going above and beyond what I expected them to know and learn for the course. The student performance on the second problem, however, was at a level below what I expected (see Subsection 4.1). The primary purpose of the problem was for students use homology to distinguish spaces which could not be distinguished with the fundamental group, but for one of my chosen spaces, $S^2 \times S^2$, the homology calculation proved too difficult for many of the students. In the future, I will plan to replace this with a simpler space, $\mathbb{C}P^2$ (which, in fact, was Problem 5). Thus, since Problems 2 and 5 serve similar purposes, I would
eliminate Problem 2 entirely. I also enjoyed the conversational nature of student responses; grading these assignments on completion rather than correctness is a feature of the assignment that I would keep for future iterations.

5.1.3 Homology Project

The homology project was well-received and most (but not all) of the students followed instructions and turned in professional, polished projects. The majority of students were able to understand the objectives of the project, and I would certainly include an assignment like this in the future. Aspects of the project that I found particularly important were that it involved using computer calculations to find complicated homology groups, paralleling the work we were doing by hand in class, and in addition, it incorporated covering space theory, tying together multiple elements from the course curriculum.

5.1.4 Board work

At the beginning of the course, I planned to include active learning techniques, but my intended means of engagement was a “think-pair-share” activity that had worked well with most of the students the previous semester. However, the “think-pair-share” was not well-received or very useful, and so I had to change my approach mid-semester. Somewhat spontaneously, I decided to ask each student to construct their own examples of covering spaces on the seven blackboard sections at the side and back of the room (see the image above in Figure 4.1).

Students enjoyed the board work, both at the time and in formal and informal feedback. One important aspect of board work is that each student chose the level of difficulty of their constructed examples; thus, it succeeded in engaging the broad range of students making up the class. The impact of the board work was corroborated by student performances on Exams 2 and 3 (see Subsection 4.3).

5.2 Limitations of course elements

As discussed briefly above, one of the activity problems (Problem 2) was a bit too difficult; I overestimated how much material we could cover during the semester when I assigned this problem for the first activity. With the benefit of hindsight, I will modify the activities the next time I teach this course, so that the examples are demonstrating the same ideas, but the computations are slightly more tractable. I also had to adjust the activities mid-semester due to several students already having prerequisite
knowledge; this limited the use of the activities. However, now I can expect this in the future and modify the activities accordingly.

While I felt the homology project was useful, I also judged it to be slightly formulaic: Students simply had to follow instructions, and I wished that I could have incorporated more creative elements. In the future, I will also ask students to compute the fundamental groups of some spaces, and perhaps I will attempt to have the discussion of covering spaces be a guided activity instead of simply telling them how to use covering spaces. I mentioned above that most students seemed to take the activity seriously, but a couple did not. One student went so far as to answer the final questions in a much different way (see Appendix 6.5.3), and my guess is that student did not feel sufficiently challenged by the project as it was designed.

Finally, the board work exercises that went the best were constructing examples and performing computations of these examples; these activities were especially well-suited to the unit on covering space theory. When we entered the unit on homology, however, I felt that the board work was not always as engaging, especially when all students were working on the same problem. An obvious limitation of board work is that it takes away from lecture time, and we are not able to cover quite as much material when doing board work (although this could be viewed as a feature instead of a bug).

5.3 Future plans for the course

As with most aspects of teaching, iteration begets improvement. The course elements discussed in detail above were experimental in nature, and I had little idea of how well each would work. Now that I am more familiar with how students interact with these elements, I can tweak them to further improve student learning. I will certainly keep and possibly expand these elements in the future. For the activities, better planning of the exercises could make it possible for every student to answer the first four or five problems correctly, which would be a good goal the next time I teach this. Additionally, I will need to find some way get every student to take the activities seriously. While many students turned in high quality work, several students put much less effort into the activities than they did for the homework assignments, perhaps because the activities were only graded for completion. While I like this aspect of the grading, it might make sense for me to require the responses to be, for instance, typed, so that students take it more seriously. In the previous subsection, I talked about possible changes I will make to the homology project in order to make it more challenging and less formulaic.

I will never again give an exam the Monday after spring break. The second exam was a disaster, mostly because the students (and I) were not well-prepared for the exam. I had assumed that students would be too busy for an exam prior to spring break; while that may be true, the solution is not to have the exam directly following break.
5.4 Final thoughts

This course was a pleasure to teach, in large part because of my involvement in the Peer Review of Teaching program. While doing the work that went into this portfolio, I made similar considerations in the other course I taught this semester: Clearly identifying and stating my course objectives, and referring back to those objectives throughout the semester. After my experience preparing this portfolio, I will repeat this process for every course I teach in the future. It helps me get organized, it aids my designing of course materials, and it unifies content for students. In fact, I could do a better job of reminding students of these objectives at various points throughout the semester, and I could show them explicitly that they have met their learning goals.

Personally, I like and thrive in the improvisational aspect of teaching. I make changes mid-lecture as a result of spontaneous ideas; I try to prove theorems from scratch so that students can see my thought process and can also witness my making mistakes, as a learning experience. I will continue to do these things in my courses. Nevertheless, I expect that the organizational framework provided by backward design and goal-setting will help me structure my future courses in a manner that will be most beneficial students, and this framework can coexist with the less well-planned aspects of my teaching.
Appendix

6.1 Syllabus

Topology II (872)  
Course Syllabus  
Spring Semester 2019

Prof. Alex Zupan  
Email: zupan@unl.edu  
Office: Avery Hall 308  
Office Hours: T 10:30am-11:30am, W 2:30pm-3:30pm  
Website: www.math.unl.edu/~azupan2  
TTh 2:00pm-3:15pm

Prerequisites: Math 871 and Math 417 or equivalent, or consent of the instructor.

Textbook: *Algebraic Topology*, by Allen Hatcher (Cambridge University Press). Hard copy recommended. Online version available at http://www.math.cornell.edu/~hatcher/AT/ATpage.html. We will primarily cover chapters 0-2. We will also cover Chapter 1 from *A Basic Course in Algebraic Topology* by William Massey.

Grading

Scheme: Your grade for this course will be determined by homework (40%), three activities (1% each), one project, to be turned in during dead week (6%), two midterm exams (17% each), and a final (17%). A final percentage of 90% or above will guarantee an A, 80% and above will guarantee a B, and 70% and above will guarantee a C. Depending on the performance of the class, it is possible these thresholds will be lower (but not higher).

Homework: Homework is worth 40% of your overall grade, and assignments will be posted on Canvas every Thursday. It is recommended that you work through every problem but only a proper subset will be required to be written up and handed in. Written homework will be due at 4pm on Thursdays, one week after the assignment has been posted. Late homework will not be accepted.

Peer Review of Teaching: This semester, I am conducting an inquiry into my teaching of this course. For this purpose, I will ask you to complete three short activities at the beginning, middle, and end of the semester (for a completion grade, each worth 1% of your final grade) and a brief project, to be turned in during dead week (work 6% of your final grade).

Exams: There will be two midterm exams, worth 17% each, and one final exam, worth 17%. The midterm exams will take place during the evenings, during a block of time that works for everyone, TBD. Our designated final exam time is **Monday, April 29th from 1pm-3pm**. Make-up exams will not be given except for school-sanctioned absences.

The Final Exam is 1pm-3pm Monday, April 29

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Expectations

Outside of class: You should expect to devote at least six to nine hours to this course outside of class every week. The best way to learn the course material is to interact with it every single day. The ideal student will read through the appropriate sections of the textbook before coming to class.

Academic Honesty: While you are encouraged to ask each other questions and to work together on the homework exercises, you are even more strongly encouraged to ponder and/or complete the homework problems independently before consulting your classmates. You will do yourself a disservice by copying without understanding the questions. The same policy applies to finding solutions online – to the extent that it is possible to regulate such behavior, solutions taken from an online source will be considered a violation of academic honesty. Any violation of university policies regarding academic integrity will be reported to the appropriate offices.

Technology: Please, no cell phones, tablets, or laptops in class unless you are using a device to take notes.

Other policies

ADA Notice: Students with disabilities are encouraged to contact the instructor for a confidential discussion of their individual needs for academic accommodation. It is the policy of the University of Nebraska-Lincoln to provide flexible and individualized accommodation to students with documented disabilities that may affect their ability to fully participate in course activities or to meet course requirements. To receive accommodation services, students must be registered with the Services for Students with Disabilities (SSD) office (http://www.unl.edu/ssd/home), 132 Canfield Administration, 472-3787 voice or TTY.

Course Evaluation: The Department of Mathematics Course Evaluation Form will be available through your Blackboard account during the last two weeks of class. You’ll get an email when the form becomes available. Evaluations are anonymous and instructors do not see any of the responses until after final grades have been submitted. Evaluations are important—the department uses evaluations to improve instruction. Please complete the evaluation and take the time to do so thoughtfully.

Department Grading Appeals Policy: The Department of Mathematics does not tolerate discrimination or harassment on the basis of race, gender, religion or sexual orientation. If you believe you have been subject to such discrimination or harassment in this or any math course, please contact the department. If, for this or any other reason, you believe that your grade was assigned incorrectly or capriciously, appeals should be made to (in order) the instructor, the department chair, the department grading appeals committee, and the college grading appeals committee.
6.2 Activities 1, 2, and 3

Math 872 Activity 1 1.8.2019

Answer the following questions, with justifications, to the best of your ability, on a separate sheet of paper. It is perfectly fine to write down intuitive thoughts not backed up with a proof. Please work by yourself and do not consult any outside sources. This activity is graded based on completion, not on correctness. The completed activity is due at the beginning of class on Thursday, January 10th.

Recall that $S^n$ is defined to be

$$S^n = \{ \bar{x} \in \mathbb{R}^{n+1} : ||\bar{x}|| = 1 \},$$

where $S^n$ has the subspace topology as a subset of the standard topology on $\mathbb{R}^{n+1}$.

1. Decide whether the spaces $S^4$ and $S^1 \times S^3$ are homeomorphic.

2. Decide whether the spaces $S^4$ and $S^2 \times S^2$ are homeomorphic.

3. Decide whether the spaces $S^2 \times S^2$ and $S^1 \times S^3$ are homeomorphic.
Answer the following questions, with justifications, to the best of your ability, on a separate sheet of paper. It is perfectly fine to write down intuitive thoughts not backed up with a proof. Please work by yourself and do not consult any outside sources. This activity is graded based on completion, not on correctness. The completed activity is due at the beginning of class on Thursday, March 14th.

Recall that $S^n$ is defined to be

$$S^n = \{ \vec{x} \in \mathbb{R}^{n+1} : ||\vec{x}|| = 1 \},$$

where $S^n$ has the subspace topology as a subset of the standard topology on $\mathbb{R}^{n+1}$.

Complex projective space, $\mathbb{CP}^n$, is defined to be the quotient space of $(\mathbb{C}^{n+1})^\times = \{ \vec{z} \in \mathbb{C}^{n+1} : \vec{z} \neq \vec{0} \}$ under the equivalence relation $\vec{z}_1 \sim \vec{z}_2$ if there exists $w \in \mathbb{C}$ such that $\vec{z}_2 = w \cdot \vec{z}_1$. The space $\mathbb{CP}^n$ is a $2n$-manifold.

1. Decide whether the spaces $S^4$ and $S^2 \times S^2$ are homeomorphic.

2. Decide whether the spaces $S^2$ and $\mathbb{CP}^1$ are homeomorphic.

3. Decide whether the spaces $S^4$ and $\mathbb{CP}^2$ are homeomorphic.
Answer the following questions, with justifications, to the best of your ability, on a separate sheet of paper. It is perfectly fine to write down intuitive thoughts not backed up with a proof. Please work by yourself and do not consult any outside sources. This activity is graded based on completion, not on correctness. The completed activity is due at 1pm on Monday, April 29th.

Recall that $S^n$ is defined to be

$$S^n = \{ \vec{x} \in \mathbb{R}^{n+1} : ||\vec{x}|| = 1 \},$$

where $S^n$ has the subspace topology as a subset of the standard topology on $\mathbb{R}^{n+1}$.

Complex projective space, $\mathbb{C}P^n$, is defined to be the quotient space of $(\mathbb{C}^{n+1})^* = \{ \vec{z} \in \mathbb{C}^{n+1} : \vec{z} \neq \vec{0} \}$ under the equivalence relation $\vec{z}_1 \sim \vec{z}_2$ if there exists $w \in \mathbb{C}$ such that $\vec{z}_2 = w \cdot \vec{z}_1$. The space $\mathbb{C}P^n$ is a $2n$-manifold.

Let $\overline{\mathbb{C}P^2}$ denote $\mathbb{C}P^2$ but with reversed orientation. If $X_1$ and $X_2$ are two 4-dimensional manifolds, we can construct a new 4-manifold $X_1 \# X_2$ by removing an open set homeomorphic to a 4-ball from each of $X_1$ and $X_2$, and identifying the $S^3$ boundaries via an orientation-reversing homeomorphism. This is called the connected sum of $X_1$ with $X_2$, akin to the way we can construct connected sums of surfaces by removing disks and gluing the $S^1$ boundaries.

1. Decide whether the spaces $S^2$ and $\mathbb{C}P^1$ are homeomorphic.

2. Decide whether the spaces $S^2 \times S^2$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ are homeomorphic.

3. Decide whether the spaces $(S^2 \times S^2) \# \overline{\mathbb{C}P^2}$ and $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# \overline{\mathbb{C}P^2}$ are homeomorphic.
6.3 Samples of student work from Activities

6.3.1 Student C, Activity 3

Problem 1

Decide whether the spaces \( S^4 \) and \( S^2 \times S^2 \) are homeomorphic.

*Proof.* Note that \( S^2 \times S^2 \) has no 4-dimensional holes. So \( H_4(S^2 \times S^2) = 0 \). We know that \( H_4(S^4) = \mathbb{Z} \) from class. Since homeomorphic spaces have identical homology groups, we conclude that \( S^4 \) and \( S^2 \times S^2 \) are not homeomorphic.

Problem 2

Decide whether the spaces \( S^2 \) and \( \mathbb{C}P^1 \) are homeomorphic.

*Proof.* We claim that the spaces \( S^2 \) and \( \mathbb{C}P^1 \) are homeomorphic: note that a cell complex structure for \( \mathbb{C}P^n \) is one each of the \( k \)-cells for \( k = 0, 2, 4, \ldots, 2k \). Thus in this case, the cell structure for \( \mathbb{C}P \) is one 0-cell and one 2-cell. There is only one way to “put these together,” which is by attaching a point to a sphere. In other words, \( \mathbb{C}P \) has a cell structure homeomorphic to \( S^2 \). Thus the spaces are homeomorphic.

Problem 3

Decide whether the spaces \( S^4 \) and \( \mathbb{C}P^2 \) are homeomorphic.

*Proof.* As seen in Problem 2, a cell-structure for \( \mathbb{C}P^2 \) is given by a 0-cell, a 2-cell, and a 4-cell. Thus the 2th homology group \( H_2(\mathbb{C}P^2) \) is not trivial, while \( H_2(S^4) = \mathbb{Z} \). Thus the two spaces cannot be homeomorphic.
2. I almost had myself convinced that $S^2$ and $CP^1$ are homeomorphic, but looking at the map I wanted to use now, I'm not even sure it's surjective. Since $CP^1$ is a 2-manifold, it should be able to draw it as the quotient of a disk which would hopefully be enlightening, but I'm not sure how to deal with starting out with 4 coordinates before taking the quotient.
6.3.3 Student B, Activity 3

Nick Meyer
- Activity III

②: My gut says (\(\mathbb{CP}^2 \# \mathbb{CP}^2, S^2 \times S^1\)) they are not homeomorphic. (for, if they were, the next exercise would be trivial).

Neither homology nor fundamental gp. can detect this though, for\n\[ H_r(\mathbb{CP}^2 \# \mathbb{CP}^2) \cong H_r(S^2 \times S^1) \ \forall r.\]

My intuition guesses that there's some diff. top trick to show this (intersection forms?)
Please put your completed project in my mailbox by 4pm on Friday, April 26th. For problems 3, 4, and 5, include screenshots of both your SnapPy Command Shell and your Plink Editor (if applicable).

1. Recall that a knot \( K_1 \subset S^3 \) is ambient isotopic to another knot \( K_2 \subset S^3 \) if there exists a map \( H: S^3 \times I \to S^3 \) such that

   (a) \( H(x, 0) = x \) for all \( x \in S^3 \);
   
   (b) for each fixed \( t \), the map \( h_t: S^3 \to S^3 \) given by \( h_t(x) = H(x, t) \) is a homeomorphism;
   
   (c) as sets of points in \( S^3 \), we have \( h_0(K_1) = K_2 \).

Prove that ambient isotopy defines an equivalence relation on knots in \( S^3 \).

2. Let \( K \) be a knot in \( S^3 \), and let \( N(K) \) be a neighborhood of \( K \) homeomorphic to an open solid torus, \( \text{int}(D^2) \times I \). The exterior \( E(K) \) is defined to be \( E(K) = S^3 - N(K) \). Use the Mayer-Vietoris sequence to prove that for any knot \( K \), we have \( H_1(E(K)) \cong \mathbb{Z} \).

3. Let \( k \in \mathbb{N} \). By problem 2 above, there is a natural homomorphism \( \varphi_k: \pi_1(E(K)) \to \mathbb{Z}_k \) given by the composition

\[
\pi_1(E(K)) \xrightarrow{\text{abelianization}} H_1(X) \cong \mathbb{Z} \xrightarrow{n \rightarrow n \mod k} \mathbb{Z}_k.
\]

Let \( Y_k \) denote the path-connected covering space associated with \( \ker(\varphi_k) \), which is guaranteed to exist by the Galois correspondence. Since \( H_1(Y_k) \) is a finitely generated abelian group, we have

\[
H_1(Y_k) \cong \mathbb{Z}^m \oplus \mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \cdots \oplus \mathbb{Z}_{m_i},
\]

where \( m_1 | m_2 | \cdots | m_i \). The numbers \( m_1, \ldots, m_i \) are called that \( k \text{th torsion numbers} \) of the knot \( K \).

Open SnapPy and type “M = Manifold()” into the Command Shell to bring up Plink Editor. Draw a knot \( K_1 \) with at least 8 crossings. Under the “Tools” menu select “Make alternating.”

Before (left) and after (right) the “Make alternating” command
Next, under “Tools” select “Send to SnapPy”. This command imports a $\Delta$-complex structure for $E(K_1)$ into SnapPy. Verify that the homology of $E(K_1)$ is $\mathbb{Z}$ by typing the command “M.homology()”.

Next, we will have SnapPy construct one of the covering spaces $Y_k$. Choose some $k \in \mathbb{N}$, and type “covers = M.covers(k, cover_type='cyclic’)”. This command creates a vector of all degree $k$ cyclic covers of $E(K_1)$ (this vector will only contain one element). Type the command “N = covers[0],” so that SnapPy assigns this cover to the manifold $N$. Finally, type “N.homology()” to compute the homology of $N$, which is homeomorphic to $Y_k$.

![SnapPy Command Shell]

An example with the trefoil and $k = 19$ and $k = 3$

Repeat this process for a second knot $K_2$. Find a value of $k$ such that the $k$th torsion numbers of $K_1$ and $K_2$ distinguish these knots. Conclude that $K_1$ and $K_2$ are not isotopic. (If this fails for some reason, keep choosing different knots until you can show they are different. Make sure that each knot has at least 8 crossings.)
4. Import these two knots into SnapPy using the “M = Manifold()” command and drawing each separately:

Images obtained from the fantastic website KnotInfo

Compute that $k$th torsion numbers for at least five different values of $k$. What did you find? What can you conclude?

5. SnapPy is quite good at computing a number of knot invariants beyond torsion numbers. One nice fact from the theory of 3-manifolds is that (most) knot exteriors admit a complete geometric structure, modeled local on 3-dimensional hyperbolic space, so that (almost) every knot has a well-defined hyperbolic volume. If $M$ is a knot exterior, “M.volume()” is the SnapPy command that computes hyperbolic volume – in fact, SnapPy is most well-known for computing hyperbolic invariants like this.

Compute the hyperbolic volumes of the knots $K_1$ and $K_2$ from the previous problem. What did you find? What can you conclude?
6.5 Samples of student work from Homology Project

6.5.1 Student E, Homology Project

3. We draw $K_1$ and $K_2$ in Plink:

and compute the 6th torsion numbers of both $K_1$ and $K_2$.

The 6th torsion numbers of $K_1$ are then: 11, while the 6th torsion numbers of $K_2$ are: 152 and 35416. So $E(K_1)$ and $E(K_2)$ are not homeomorphic. Then $K_1$ and $K_2$ are not ambient isotopic.
6.5.2 Student F, Homology Project

The First Knot is given within the variable N while the second knot is given in the variable G. These were calculated for $k = 1, 2, 3, 4, 5$ the homologies for these values of k were all the same. Thus we can conclude that there is evidence that homology is an alternating crossing invariant. By this I mean that if the crossing is alternating and the number of crossings in the knots is the same then their nth homologies will be the same. However there are restrictions upon needing to have crossing numbers greater than 8 to ensure these properties hold. Additionally this is just conjecture as we have not strictly proven anything about this.
6.5.3 Student B, Homology Project

MATH872: Homology Project
Nick Meyer

Problem 4: Import these two knots into SnapPy using the $M = \text{Manifold()}$ command and drawing each separately:

(a) $K_3 = 9_{28}$
(b) $K_4 = 9_{29}$

Figure 2: Diagrams of Selected Knots

Compute that $k$th torsion numbers for at least five different values of $k$. What did you find? What can you conclude?

Solution. First, as a note, to figure out which knots these were, I computed DT codes, used the Knot Atlas Mathematica Package to compute Gauss Codes, and then used Knotilus to view the resulting knots and compare to the images provided. The first knot listed in this section was the knot $K_3 = 9_{28}$, and the second was $K_4 = 9_{29}$.

To complete the exercise, I used the following code.

```python
K3 = Manifold('9_28')
K4 = Manifold('9_29')
ks = list(range(3, 12))
K3hom = {k: K3.covers(k, cover_type = 'cyclic')[0].homology() for k in ks}
K4hom = {k: K4.covers(k, cover_type = 'cyclic')[0].homology() for k in ks}
print('Do 9_28 and 9_29 have same torsion numbers for various small n?'
print('Yes!' if K3hom == K4hom else 'No!')
```

The result of the execution was the following.

```
Do 9_28 and 9_29 have same torsion numbers for various small n?:
Yes!
```

Let me explain how this code works. The first three lines are standard setup, defining the knots $K_3$ and $K_4$, as well as creating the (ordered) list $ks = \{3, 4, 5, \ldots, 11\}$. The next two lines work in the same way, I will explain the first. The dictionary (a list of values indexed by some set of keys) $K3hom$ is created with keys $k \in ks$ and values being the homology of the $k$-fold cyclic cover of $K3$. Two dictionaries are equal if and only if they have the same keys and for each key, the values stored in each dictionary under that key are equal. Hence the conclusion line (the last line of code) will print $\text{Yes!}$ if $K3$ and $K4$ have the same homology groups for each $k \in ks$ and will print $\text{No!}$ otherwise.

Since the code printed $\text{Yes!}$, we see that $K3$ and $K4$ have the same homology groups for various small $k$. However, from this, we cannot include anything about the isotopy classes of the knots, since invariants being equal does not imply that the knots are isotopic.
6.6 Feedback form

Math 872 Feedback 3.28.2019

What are several things you like about the way this class is run?

What kinds of things would you like to see changed?

Any additional comments?
6.7 Selected feedback

6.7.1 First feedback

—I like the idea of us discussing things amongst ourselves, but this semester it feels forced and too short a period of time (please forgive the lack of good grammar in this comment.)

6.7.2 Second feedback

More board work as it requires a slightly higher level of preparedness from the students and forces more careful thought on the material being presented.

doing things at the board is useful but terrifying

I liked going to boards. I'd think of more activities that could be like this
6.8 Selected student board work
6.9 Selected Exam 2 and 3 responses

6.9.1 Student E, Exam 2

Completely explain all of your answers. You may use facts from class or from the textbooks but you must cite what you are using.

1. Let $X = S^1 \lor S^1 \lor S^1$, with base point $x_0$ the wedge point, shown below.

(a) Construct a 3-sheeted path-connected covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$

(b) Suppose $\pi_1(X, x_0) = \langle a, b, c \rangle$. Using your answer to part (a), determine a presentation for $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

We fix a basepoint $\tilde{x}_0$ and choose a spanning tree of the covering space:

Then $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ is generated by:

$\langle a, bc, cac^{-1}, cbcc^{-1}, c^3ac^{-2}, c^3, c^5b \rangle$.

So $p_*(\pi_1(\tilde{X}, \tilde{x}_0)) = \langle a, bc, cac^{-1}, cbcc^{-1}, c^3ac^{-2}, c^3, c^5b \rangle$. 
(c) Is \( p_*(\pi_1(\tilde{X}_0, \tilde{x}_0)) \) from part (b) normal in \( \pi_1(X, x_0) \)? Explain why or why not.

2. Let \( Z \) be the quotient space obtained by identifying three Möbius bands along their common boundary curve. Determine a presentation for \( \pi_1(Z) \).

First we will identify two Möbius bands. We divide the space into two open sets \( A \) and \( B \), with each consisting of one Möbius band and some small band of points from the other band. Then each of \( A \) and \( B \) has the same fundamental group as a Möbius band and \( A \cap B \) is also a band (but it goes around \( A \) and/or \( B \) twice).
6.9.2 Student G, Exam 2

1. Let $X = S^1 \vee S^1 \vee S^1$, with base point $x_0$ the wedge point, shown below.

(a) Construct a 3-sheeted path-connected covering space $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$

(b) Suppose $\pi_1(X, x_0) = \langle a, b, c \rangle$. Using your answer to part (a), determine a presentation for $p_* (\pi_1(\tilde{X}, \tilde{x}_0))$.

(c) Is $p_* (\pi_1(\tilde{X}_0, \tilde{x}_0))$ from part (b) normal in $\pi_1(X, x_0)$? Explain why or why not.

Yes. For any vertex in $\tilde{X}$, there exists a deck transf. with $\tilde{x}_0$ being sent to that point. (Rotate 0°, 120°, or 240°). So therefore $p_* (\pi_1(\tilde{X}_0, \tilde{x}_0))$ is normal by a theorem or something we proved.
2. Let $Z$ be the quotient space obtained by identifying three Möbius bands along their common boundary curve. Determine a presentation for $\pi_1(Z)$.

So $Z = A \cup B \cup C$, where $A$, $B$, and $C$ are the first, second, and third Möbius bands, respectively. Notice that $A$, $B$, and $C$ are path-connected and connected and that $A \cap B$ and $(A \cap B) \cap C$ are also path-connected and connected; these intersections are nonempty and are the common boundary curve. Then by SVK,

$$\pi_1(Z) = \frac{\left(\pi_1(A) \times \pi_1(B)\right) \ast \pi_1(C)}{N}$$

where $N$ is the normal subgroup generated by $i_{A*}(e) \cdot i_{B*}(e)^{-1}$ and $M$ the normal subgroup generated by $i_{A \cap B*}(d) \cdot i_{C*}(d)^{-1}$ for $e \in \pi_1(A \cap B)$ and $d \in \pi_1((A \cap B) \cap C)$.

(I don't remember what the fundamental group of a Möbius band is isomorphic to, so I didn't simplify.)
6.9.3 Student E, Exam 3

2. Recall that a retract of $X$ onto a subspace $A$ is a map $r : X \to A$ such that $r|_A = \text{Id}_A$.

(a) Prove that if there exists a retract $r : X \to A$, then the map $i_* : H_n(A) \to H_n(X)$ induced by the inclusion $i : A \to X$ is injective for all $n$.

First, consider $r : A \to A$. For any $a \in A$, $r(a) = r(r(a)) = r(a) = a$. So $r = \text{Id}_A$. Then, for any $n$, $(r)_* : H_n(A) \to H_n(X)$ induces $i_* : H_n(A) \to H_n(X)$. But we also know that $(r)_* = r_* \circ i_* = \text{Id}_n(A)$, so $r_* \circ i_* = \text{Id}_n(A)$. Thus, $r_* \circ i_*$ is injective and it must be the case that $i_*$ is injective.

(b) Prove that if there exists a retract $r : X \to A$, then $H_n(X, A) \cong H_n(X)/i_*H_n(A)$ for all $n$.

We know there is an exact sequence

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{\partial} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \xrightarrow{i_*} \cdots$$

From part (a), each $i_*$ is injective and so has $\ker i_* \cong 0$. Then, since the sequence is exact, each $\partial$ has $\ker \partial \cong 0$ and we have that

$$0 \to H_n(A) \xrightarrow{i_*} H_n(X) \to H_n(X, A) \to 0$$

is a short exact sequence. So $H_n(X, A) \cong H_n(X)/i_*H_n(A)$. 

\[\frac{5}{5}\]
3. Let $X$ be the quotient space obtained by gluing the boundary of a Möbius band $M$ the equator $c$ of a 2-sphere $S^2$. Use the Mayer-Vietoris sequence to compute the reduced homology groups $\tilde{H}_n(X)$ for $n \geq 1$. (You may state the homology groups $\tilde{H}_n(S^2)$ and $\tilde{H}_n(S^1)$ without proof.)

\[ \text{need } A, B \text{ so that } X = \text{int}(A) \]

Let $A = M$, and $B = S^2$, so that $A \cap B \cong S^1$, the equator of $B$. We also have $A \cup B = X$, so using the Mayer-Vietoris sequence,

\[ \ldots \rightarrow \tilde{H}_n(A \cap B) \xrightarrow{\partial_n} \tilde{H}_n(A) \otimes \tilde{H}_n(B) \rightarrow \tilde{H}_n(X) \xrightarrow{\partial_{n+1}} \tilde{H}_{n-1}(A \cap B) \rightarrow \ldots \]

is exact.

We know $\tilde{H}_n(A) \cong \tilde{H}_n(S^1)$, so for $n \geq 3$, $\tilde{H}_n(A) \cong 0$, $\tilde{H}_n(B) \cong 0$, and $\tilde{H}_{n-1}(A \cap B) \cong 0$. Then the sequence

\[ \ldots \rightarrow 0 \rightarrow \tilde{H}_n(X) \rightarrow 0 \rightarrow \ldots \]

is exact, so $\tilde{H}_n(X) \cong 0$ for $n \geq 3$.

For $n = 2$, we look at

\[ \tilde{H}_2(M) \rightarrow \tilde{H}_2(A) \otimes \tilde{H}_2(B) \rightarrow \tilde{H}_2(X) \xrightarrow{\partial_3} \tilde{H}_1(A \cap B) \otimes \tilde{H}_1(B), \]

so $0 \rightarrow 0 \rightarrow \tilde{H}_2(X) \rightarrow 0 \rightarrow \tilde{H}_1(A \cap B) \otimes \tilde{H}_1(B)$

is exact.

Suppose $[c]$ generates $\tilde{H}_1(A \cap B)$, and $[a]$ generates $\tilde{H}_1(A)$. Then we can compute $\partial_3([c])$. Since $A$ is a Möbius band, with boundary $c$, $\partial_3([c]) = 2[a] \neq 0$. Since $\tilde{H}_1(A \cap B)$ is generated by only one element, we conclude that $\text{ker}(\partial_3) \cong 0$. Then $\text{im} \partial_3 \cong 0$, and we have that

\[ 0 \rightarrow \tilde{H}_2(X) \xrightarrow{\partial_3} 0 \]

is exact. Also $\tilde{H}_2(X) \cong \mathbb{Z}$.

Next, $\tilde{H}_1(A \cap B) \rightarrow \tilde{H}_1(A) \otimes \tilde{H}_1(B) \rightarrow \tilde{H}_1(X) \rightarrow \tilde{H}_0(A \cap B)$ is exact,

and we already know $\partial$ is the zero map, so

\[ 0 \rightarrow \tilde{H}_1(X) \rightarrow \tilde{H}_0(A \cap B) \rightarrow \mathbb{Z} \rightarrow 0 \rightarrow 0 \]

is exact.

So $\tilde{H}_1(X) \cong \tilde{H}_0(A \cap B) \cong \mathbb{Z}$.

QED
6.9.4 Student G, Exam 3

5. Suppose that \( \{A_n\}, \{B_n\}, \{C_n\} \) are chain complexes, where (abusing notation, as in class) we let \( \partial \) denote the boundary map for each chain complex. Suppose further that \( i : A_n \to B_n \) and \( j : B_n \to C_n \) are chain maps inducing a short exact sequence of chain complexes. Let \( i_* : H_n(A) \to H_n(B) \) and \( j_* : H_n(B) \to H_n(C) \) be the induced maps on homology, and let \( \partial : H_n(C) \to H_{n-1}(A) \) be the induced map defined in class, where for each \( [c] \in H_n(C) \), there exists \( b \in B_n \) such that \( j(b) = c \), there exists \( a \in A_{n-1} \) such that \( i(a) = \partial_n(b) \), and \( \partial([c]) \) is defined as \([a] \in H_{n-1}(A)\).

(a) Prove that \( \text{Im } i_* \subset \text{Ker } j_* \).

\[
\begin{array}{c}
0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0
\end{array}
\]

\( \text{Let } [b] \in \text{Im } i_* \). So \( [b] = i_*([a]) \) for some \( [a] \in H_n(A) \). Then since

\( 0 \to A \xrightarrow{i} B \xrightarrow{j} C \to 0 \) is a SES,

\( j \circ i = 0 \), meaning that

\[
\begin{align*}
\partial_n([b]) &= j(\partial_n([a])) \\
&= \partial_n([a]) \\
&= [0].
\end{align*}
\]

Thus, \( [b] \in \text{Ker } j_* \), and so \( \text{Im } i_* \subset \text{Ker } j_* \). \( \Box \)

(b) Prove that \( \text{Ker } i_* \subset \text{Im } \partial \).

\[
\begin{array}{c}
a \xrightarrow{i} \partial_n(a) \xrightarrow{} 0
\end{array}
\]

Let \( [c] \in \text{Ker } i_* \). So \( [c] \in H_n(A) \) with \( i_*([c]) = 0 \). So there exists \( a \in A_n \) with \( i_*([a]) = [0] \). Now, there also exists \( b \in B_n \) with \( \partial_n(b) = 0 \).

Then consider \( [c] \in H_n(C) \) where \( c \) is \( j(b) \). Now, we know that there exists \( b \in B_n \) with \( j(b) = c \) by definition of \( C \). And \( a \in A_n \) with \( i(a) = \partial_n(b) \). So \( \partial([c]) = [a'] = [a] \).

Thus, \( [a'] \in \text{Im } \partial \), and so \( \text{Ker } i_* \subset \text{Im } \partial \). \( \Box \)