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Physics, Chapter 7: Work and Energy

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Work and Energy

7-1 Work Done by Forces

An extremely important concept that has been developed in physics is that of the *work done* on a body by the action of some external agent which exerts a force on this body and produces motion. For example, whenever someone lifts a body, he does work by exerting a force upward on it and moving it upward. Whenever a steam locomotive pulls a train, a series of processes takes place in the steam engine of the locomotive which enables it to exert a force on the train and move it in the direction of the force. The term *work*, as used in physics, is a technical term. Whenever work is done by an external agent on a body, *the work done is the product of the force which acts on the body and the distance through which the body moves while the force is acting on it, provided that the force and the distance through which the body moves are parallel to each other.*

In the initial development of the concept of work, we shall restrict our discussion to work done by a constant force. We shall later (Section 7-8) remove this restriction and treat the more general case of work done by a variable force.

If a constant force F acts on a body for a distance s in the direction of the force, then the work done $\mathscr{W}$ is, from the definition,

$$\boxed{\mathscr{W} = Fs.} \tag{7-1a}$$

If the force F and the distance s are not parallel, then only that component of the force which is in the direction of the motion does the work. For example, if a heavy block is to be moved, it may be more convenient to pull with a force F at some angle θ with respect to the ground, as shown in Figure 7-1. The component of F in the direction of the motion is $F \cos \theta$; if the block is moved through a distance s while this force is acting on it, the work done $\mathscr{W}$ is

$$\boxed{\mathscr{W} = Fs \cos \theta.} \tag{7-1b}$$

Work is a scalar quantity. The concept of physical work is often a confusing one, in part because of the way the word "work" is used in everyday language, and in part because there is no direct physiological analogue

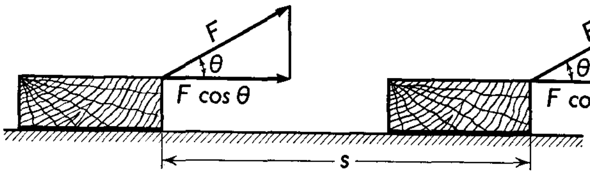

Fig. 7-1 Work done by a constant force F acting at an angle θ to the direction of its motion is $Fs \cos \theta$.

to the physical concept of work. In physics no force exerted, no matter how great nor for how long a time, generates any work unless there is a displacement. No force generates work unless the force has a component in the direction of the displacement. The centripetal force which a string

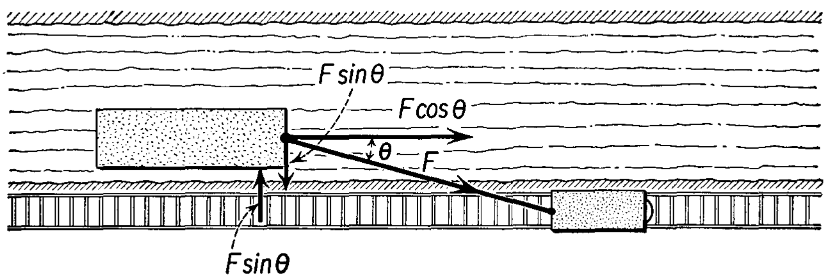

Fig. 7-2

exerts on a stone in uniform circular motion does no work upon the stone. *The sensation of tiredness has no direct relationship to physical work.* Suppose, for example, that you are called upon to support one end of a car while the driver changes a tire. When the operation is concluded you will have done no work, for, although you were called upon to exert a large force, there was no displacement of the car. Consider an even more unlikely situation. Suppose a barge is being towed through a canal by an engine located alongside the barge canal, as shown in Figure 7-2. If the force exerted by the engine is F, the component of the force parallel to the canal is $F \cos \theta$, while the component perpendicular to the canal is $F \sin \theta$. The perpendicular component tends to urge the barge against the side of the canal. You are called upon to exert a force against the side of the barge of magnitude $F \sin \theta$ to keep the barge from scraping along the walls of the canal, and you accompany the barge on its trip through the canal, continually exerting the required force. When the trip is completed you will have done no work on the barge, for the force exerted had no component in the direction of the displacement.

7-2 Units for Expressing Work

There are several different units that are used for expressing the work done. In every case the unit used must be equivalent to the product of a force by a distance. In the British gravitational system the unit used is the foot pound (ft lb), the product of the unit of distance by the appropriate unit of force. In the cgs system the analogous unit would be the dyne centimeter, but this unit has been given a special name, the *erg; one erg is the work done by a force of one dyne acting through a distance of one centimeter*, or

$$1 \text{ erg} = 1 \text{ dyne} \times 1 \text{ cm} = 1 \text{ dyne cm}.$$

In the mks system the unit of work is called the *joule. A joule is defined as the work done by a force of one newton acting through a distance of one meter;* that is,

$$1 \text{ joule} = 1 \text{ nt} \times 1 \text{ m} = 1 \text{ nt m}.$$

The relationship between the joule and the erg can be found readily from the facts that 1 nt = 100,000 dynes and 1 m = 100 cm. By the usual conversion procedure, we write

$$1 \text{ joule} = 1 \text{ nt m} \times \frac{100{,}000 \text{ dynes}}{1 \text{ nt}} \times \frac{100 \text{ cm}}{1 \text{ m}},$$

$$1 \text{ joule} = 10{,}000{,}000 \text{ dyne cm} = 10{,}000{,}000 \text{ ergs},$$

$$1 \text{ joule} = 10^7 \text{ ergs},$$

or

$$10^{-7} \text{ joule} = 1 \text{ erg}.$$

While both work and torque are compounded of the product of a force by a displacement and therefore have the same units, these are quite different things. Torque is a vector quantity, while work is a scalar quantity. Torque is the product of a force by a distance which is always measured in a direction perpendicular to the force, while work is the product of a force by a distance parallel to the direction of the force. While there need be no displacement to generate a torque, on the other hand, no work can be done without a displacement of the force.

Illustrative Example. Referring to Figure 7-1, suppose that the body is pulled along a level floor by a rope making an angle of 30° with the floor. If the body is moved a distance of 15 m, and if the force F is 40 nt, the work done is

$$\mathscr{W} = Fs\cos\theta = 40 \text{ nt} \times 15 \text{ m} \times \cos 30°$$

$$= 40 \times 15 \times 0.866 = 520 \text{ nt m},$$

or

$$\mathscr{W} = 520 \text{ joules}.$$

7-3 The Scalar Product

We have seen in the preceding paragraphs that work is a scalar quantity, yet it is composed of the product of two vector quantities—the force and the displacement. Such operations occur quite often in physics and have

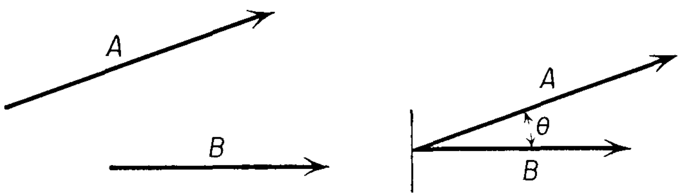

Fig. 7-3 The scalar or dot product of two vectors **A** and **B** is given by $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$, where θ is the angle between the two vectors when they are drawn from a common origin.

been given a special name, the *scalar product* of two vectors. If we have two vector quantities, such as the vectors **A** and **B** in Figure 7-3, *we define the scalar product of these two vectors as the product of the magnitude of* **A** *by the magnitude of* **B** *by the cosine of the angle between them.* The scalar product of the two vectors **A** and **B** is sometimes called the *dot product* because it is represented by writing **A·B**. Thus we have

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta, \tag{7-2}$$

where θ is the angle between the vectors **A** and **B** when they are drawn from a common origin. The result of multiplying two vectors by the operation called the scalar product is a scalar quantity; that is, it has magnitude only; there is no direction to be associated with the scalar product.

We see that the definition of the scalar product is perfectly adapted for the representation of work, for work has been defined as a scalar quantity, the result of the multiplication of a vector, force, by another vector, displacement. If a force **F** applied to a body produces a displacement **s**, the work done is the scalar product of the force and the displacement for

$$\mathscr{W} = \mathbf{F}\cdot\mathbf{s} = Fs\cos\theta. \tag{7-3a}$$

In an incremental displacement $\Delta\mathbf{s}$, the incremental work is

$$\Delta\mathscr{W} = \mathbf{F}\cdot\Delta\mathbf{s} = F\,\Delta s\cos\theta, \tag{7-3b}$$

and in the limit of small displacement,

$$d\mathscr{W} = \mathbf{F}\cdot d\mathbf{s} = F\,ds\cos\theta. \tag{7-3c}$$

In these formulas boldface type has been used to represent vector quantities, while italics have been used to represent the magnitudes of these quantities, or to represent scalar quantities. The great advantage of the representation of work as the dot product is that there is never any ambiguity as to which angle is referred to as θ in the formula $\mathscr{W} = Fs\cos\theta$. From the definition of the dot product, the angle θ is always the angle between the vectors $\mathbf{F}$ and $\mathbf{s}$ when these two vectors are drawn from a common origin. There is the further advantage in the representation $\mathscr{W} = \mathbf{F}\cdot\mathbf{s}$ in that there will be no confusion about the fact that work is a scalar quantity, for the result of the operation called the dot product is always a scalar quantity.

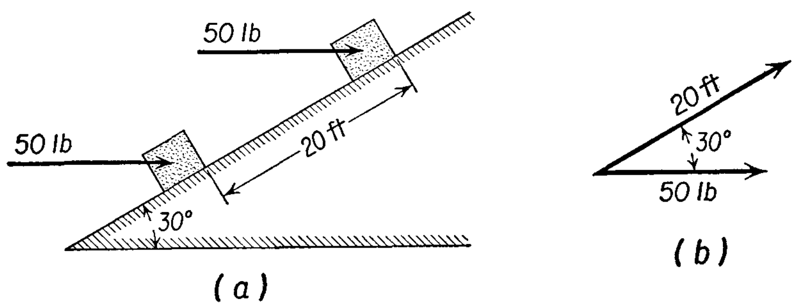

Fig. 7-4

Illustrative Example. A body is pushed up a 30° inclined plane a distance of 20 ft by a horizontal force of 50 lb, as shown in Figure 7-4(a). Find the work done on the body. In Figure 7-4(b) the force vector and the displacement vector are shown drawn from a common origin. The angle θ appropriate for use in Equation (7-1b) or (7-3a) is 30°. Substituting into the equation, we find

$$\begin{aligned}\mathscr{W} &= \mathbf{F}\cdot\mathbf{s} = Fs\cos\theta\\ &= 50\text{ lb} \times 20\text{ ft} \times \cos 30° = 50 \times 20 \times 0.866\text{ ft lb}\\ &= 866\text{ ft lb}.\end{aligned}$$

When the displacement $\mathbf{s}$ is in the same direction as the applied constant force **F**, the angle θ is zero. The work $\mathscr{W}$ done is

$$\mathscr{W} = \mathbf{F}\cdot\mathbf{s} = Fs\cos 0^\circ = Fs.$$

Since F and s represent the magnitudes of the two vectors, they are both positive quantities. Thus $\mathscr{W}$ is positive and represents the work done *by* the agency applying the force on the body which has been displaced.

In many cases the displacement is opposite in direction to the applied force, as, for example, when a moving body is slowed down by the action of an external force. When the displacement vector **s** is opposite to the applied force **F**, the angle θ between the two vectors is 180°. Since the cos 180° $= -1$, we find that the work done *by* the applied force is negative, or

$$\mathscr{W} = -Fs.$$

We interpret this result by saying that negative work has been done by the agency applying the force to the body which has been displaced. Alternatively, we may say that positive work has been done by the body which has been displaced upon the agency exerting the force. Thus, when a baseball is caught by a fielder, the ball does work upon the fielder.

The discussion of Figure 7-2 illustrates a case in which the applied force is perpendicular to the displacement. Here the angle θ is 90°, and the work done is zero.

7-4 The Vector Product

The extension of the concept of multiplication to vector quantities requires some additional consideration. By analogy with arithmetic, it is clear that the resulting product should involve the product of the magnitudes of the two vectors, but the question remains as to what to do about the directions; and should the resulting quantity be a vector or a scalar? In the preceding section we have seen the virtue of one type of product, the scalar product, in which the product of two vector quantities is a scalar quantity which has been so defined as to be ideally suited to represent work.

A second product, called the *vector product*, has been defined so as to be ideally suited to represent torque. Given two vectors **A** and **B**, as in Figure 7-5, we define their vector product as a vector **C** which is perpendicular to the plane formed by **A** and **B** whose magnitude is given by

$$C = AB\sin\theta, \tag{7-4}$$

and whose direction is given by the right-hand rule, or by the direction of advance of a right-handed screw which is made to rotate from the direction of **A** into the direction of **B**. The vector product is usually represented by the symbol × and is therefore called the *cross product*, to distinguish it from the scalar product or dot product.

Suppose a force **F** acts on a body at a point P, as illustrated in Figure 7-6. The vector directed from the axis of rotation to the point P is **r**. Then

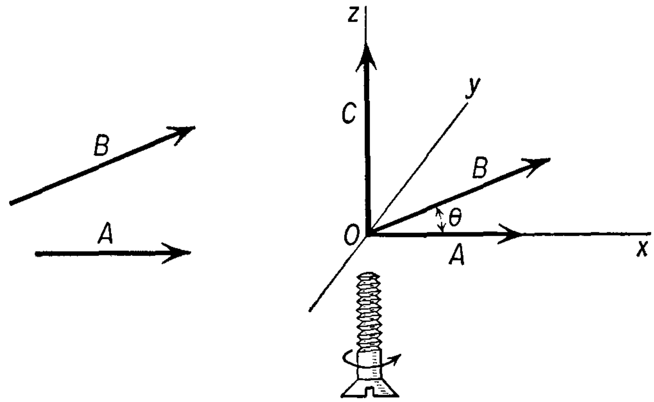

Fig. 7-5 The vector product, or cross product, of two vectors **A** and **B** is the vector **C**, written as $\mathbf{A} \times \mathbf{B} = \mathbf{C}$. The magnitude of **C** is given by $C = AB \sin \theta$, and the direction of **C** is perpendicular to the plane formed by **A** and **B** pointing in the direction of advance of a right-hand screw turned so as to advance from **A** to **B**. As in the scalar product, the angle θ is the angle between **A** and **B** when these are drawn from a common origin.

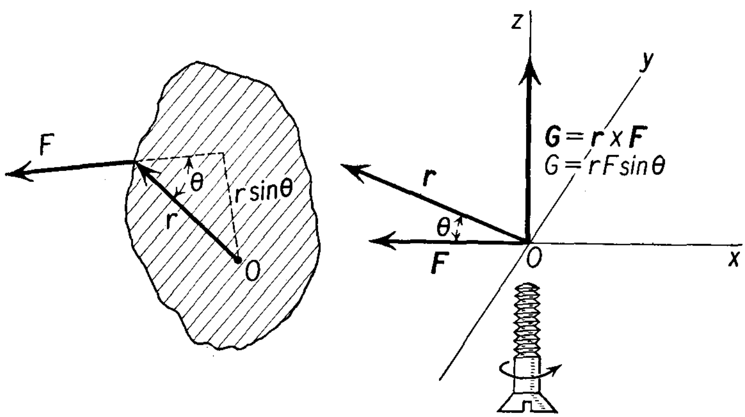

Fig. 7-6 The torque of the force **F** about an axis through O is given by $\mathbf{G} = r \times \mathbf{F}$.

the torque **G** developed by **F** about the axis of rotation is given by

$$\mathbf{G} = \mathbf{r} \times \mathbf{F}, \tag{7-5}$$

and the magnitude of **G** is G, given by

$$G = rF \sin \theta, \tag{7-6}$$

which is precisely the product of the force by the perpendicular distance between the line of action of the force and the axis of rotation, the moment arm. The direction of the torque vector is given by the same right-hand rule in the cross product as the direction previously given for the determination of the torque.

Again the great virtue of the cross-product notation for the representation of torque lies in the fact that the one rule for the determination of the cross product is adequate for many laws of physics. The same cross product will reappear in the discussion of angular momentum, in the relationship between electricity and magnetism, and in other places in physics. The cross product serves to simplify the notation and the formulas which must be learned.

We must note that the order of the factors which appear in the cross product is of some importance. While the magnitude of $\mathbf{A}\times\mathbf{B}$ is the same as the magnitude of $\mathbf{B}\times\mathbf{A}$, the two vectors are opposite to each other in direction, for a right-handed screw which rotates from **A** to **B** advances in the opposite direction from one which advances from **B** into **A**, and we write

$$\mathbf{A}\times\mathbf{B} = -\mathbf{B}\times\mathbf{A}. \tag{7-7}$$

7-5 Work and Energy

An important question which arises from this discussion concerns the result of the work done by the various forces which act on different bodies. In some cases the results are immediately obvious. For example, the work done by a force which accelerates a body produces a change in its speed; the work done in lifting a body produces an increase in the height of the body with respect to its former position; the work done against frictional forces produces an increase in the temperature of one or more of the bodies involved. In other cases the results may not be so obvious. Some bodies may become charged electrically; others may become magnetized. These changes will be discussed at the appropriate places in the text. One general conclusion can be drawn here; that is, that whenever work is done, some change is produced in the body or system of bodies on which the forces acted. To describe these changes, another technical term is used. We say that the work done produces *a change in the energy of the body or system of bodies.* In the first case above, the energy of motion or the *kinetic energy* of the body is changed; in the second case the positional energy or *potential energy* of the body is increased. In each case *the change in energy is defined as equal to the work done on the body or system of bodies.* From this, it follows that the units used in expressing the energy of a system are the same as the units of work. We shall see that it is possible for a body to gain energy as a result of work done upon it, and, conversely, a body may lose energy by doing work upon a second object.

7-6 Kinetic Energy

Suppose that a constant force F acts on a body of mass m for a distance s in the direction of F, as shown in Figure 7-7. The work done on the body by the force F is

$$\mathscr{W} = Fs.$$

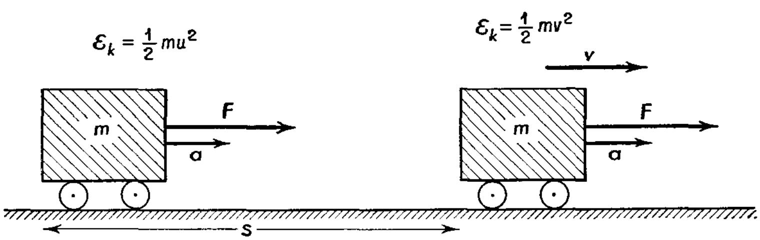

Fig. 7-7 The work done by a force in accelerating a body increases its kinetic energy.

Under the action of a constant force, the body will receive an acceleration a given by

$$a = \frac{F}{m},$$

and the velocity of the body will be increased from its initial value u to some final value v, given by Equation (2-27c) as

$$v^2 = u^2 + 2as,$$

and, substituting F/m for a, we find

$$v^2 = u^2 + 2\frac{Fs}{m}\cdot$$

We now multiply the equation above by $\frac{m}{2}$, substitute the value $\mathscr{W}$ for the product Fs, and transpose to find

$$\boxed{\mathscr{W} = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mu^2.} \tag{7-8a}$$

We call the quantity $\frac{1}{2}mv^2$ the final *kinetic energy* of the body and $\frac{1}{2}mu^2$ its initial kinetic energy, and we say that the work done on the body by the force F acting through a distance s has produced a change in the kinetic energy of the body.

In general, the kinetic energy $\mathscr{E}_k$ of a body of mass m moving with speed v is given by

$$\boxed{\mathscr{E}_k = \tfrac{1}{2}mv^2.} \tag{7-8b}$$

Just as work is a scalar quantity, so is kinetic energy a scalar quantity.

7-7 Potential Energy in a Uniform Gravitational Field

When a mass m is placed in a uniform gravitational field in which the intensity of the gravitational field is g, the gravitational force is given by mg, the weight of the body. If such a body is lifted through a height h, the work done by the agency which lifts it is mgh (see Figure 7-8). We say that the work done has increased the *potential energy* $\mathscr{E}_p$ of the body, and we write as a defining equation

$$\boxed{\mathscr{E}_p = mgh.} \tag{7-9}$$

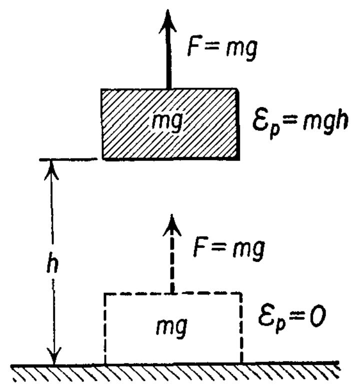

Fig. 7-8 The work done in lifting a body increases its potential energy.

The position at which the potential energy is zero is quite arbitrary and, in fact, makes no difference in the consideration of a particular problem. We shall always be interested in the change in potential energy associated with a change in position, and shall not attempt to ascribe a meaning to the value of the potential energy itself. For this reason it is often convenient to take the initial position of the body as the position of zero potential energy.

The potential energy in the earth's gravitational field is sometimes called the *gravitational potential energy*, to distinguish it from other forms of potential energy such as the energy of a stretched spring or the energy of an electric charge in an electric field, which will be discussed in the later chapters of this book.

7-8 Potential Energy of a Spring

Bodies which are acted on by external forces generally undergo changes in size or shape. A helical spring is an important example of such a body.

If a spring is fixed at one end and a force F is applied to the other end, an extension s is produced which is proportional to F and is given by the equation

$$F = ks, \tag{7-10}$$

where k is called the *constant of the spring;* if the force is measured in pounds and the extension is measured in feet, the constant k is given in pounds per foot. Each helical spring has a constant k associated with it, whose value depends upon the material from which it is made, the wire diameter, the number of turns per unit length, and the diameter of the spring itself. The linear relationship between applied force and displacement of a helical spring is the basis of the uniform scale of a spring balance.

To stretch a spring by an incremental amount Δs when it has already been stretched by an amount s requires that an amount of work $\Delta \mathscr{W}$ be done by the agency doing the stretching, which is given by

$$\Delta \mathscr{W} = F \, \Delta s = ks \, \Delta s.$$

The total work done in stretching the spring from the position of zero extension to the maximum extension s_m is given as the sum of the work done in the incremental extensions from the initial to the final positions. Letting the incremental displacement become infinitesimal, as the number of displacements increases appropriately, we write, in the language of the calculus,

$$\mathscr{W} = \int_0^{\mathscr{W}} d\mathscr{W} = \int_0^{s_m} ks \, ds = \tfrac{1}{2}ks_m^2.$$

The same result may be attained by observing that since the force on the spring varies at a constant rate, the average value of the force exerted by the spring, $\bar{F}$, is half the sum of the initial and final forces. In the form of an equation,

$$\bar{F} = \frac{ks_m}{2},$$

and the work done in stretching the spring is the product of the average force by the total displacement, or

$$\mathscr{W} = \bar{F}s_m = \tfrac{1}{2}ks_m^2 \, .$$

Calling the elastic energy in a stretched spring $\mathscr{E}_e$, and dropping the subscript m, we get

$$\mathscr{E}_e = \tfrac{1}{2}ks^2. \tag{7-11}$$

The energy of a stretched spring is associated with position or deformation rather than with motion and is therefore a form of potential energy.

7-9 Conservation of Energy

In the absence of dissipative forces, such as friction, the total mechanical energy of a system is constant. This principle, known as the *principle of conservation of mechanical energy,* is of great usefulness in the solution of many problems in mechanics. Initially discovered in mechanics, the principle of energy conservation has since been extended to include heat energy, radiant energy, electrical energy, chemical energy, and the energy associated with transformations of mass, so that it has become one of the most fundamental and unifying principles of physics, and indeed, of all science. It was Albert Einstein who first showed, from his work in the theory of relativity, that mass and energy were equivalent and could be interchanged in accordance with the formula

$$\mathcal{E} \equiv m. \tag{7-12a}$$

If the equivalence between mass and energy is to be expressed in systems of units in which mass and energy units are defined separately, it is necessary to apply a conversion factor to convert units of mass to units of energy. In the cgs and mks systems we write

$$\mathcal{E} = mc^2, \tag{7-12b}$$

where c is the velocity of light, 3×10^{10} cm/sec or 3×10^8 m/sec. In the cgs system of units, $\mathcal{E}$ is expressed in ergs, m in grams, and c, the velocity of light, in centimeters per second. In the mks system $\mathcal{E}$ is expressed in joules, m in kilograms, and c in meters per second.

Einstein's mass-energy relationship has been widely verified experimentally. The nuclear reactor is an example of the practical application of this relationship, in which energy in the form of mass is converted to energy in the form of heat.

In more general and extended form, the principle of conservation of energy states that *energy may be transformed from one type to another without loss,* and that *in a closed system the total amount of energy remains constant.*

One of the chief contributions of physics in the service of society has been in identifying and defining the concept of energy. This remarkable idea, and its subsequent theoretical and experimental development, has resulted in mechanical devices for the utilization of fuels such as coal and petroleum, of the mechanical energy of rain, and of the conversion of mass to other forms of energy by nuclear processes. As a result, each generation in this last century has lived in a substantially different way from that of its parents.

7-10 Kinetic and Potential Energy Transformations

Suppose that a ball of mass m is thrown vertically upward with an initial velocity u. Since the acceleration of the ball is equal to g, directed vertically downward, we may calculate from Equation (2-27c) that the ball will rise to a height h given by the equation

$$u^2 = 2gh.$$

Consider the application of the principle of the conservation of mechanical energy to the same problem. Since no mechanical energy is transformed into heat by frictional processes, the total energy that the ball has initially will also be the total energy at every point in its path. The kinetic energy of the ball at the instant it is thrown is $\frac{1}{2}mu^2$, and its potential energy is zero. At the highest point h in its path, its kinetic energy is zero and its potential energy is mgh. Equating the initial total energy to the final total energy, we find

$$\tfrac{1}{2}mu^2 = mgh,$$

from which

$$u^2 = 2gh.$$

The speed of the ball at any known height above the initial point can be found in a similar manner.

The motion of a simple pendulum provides another interesting example of the transformation of energy and the usefulness of the energy-conservation principle in the solution of problems in mechanics. A simple pendulum consists of a small ball of mass m attached to one end of a string of negligible weight and of length L. The other end of the string is attached to some fixed point O, as shown in Figure 7-9. When at rest, the string hangs vertically with the ball at its lowest position C. Let us call the potential energy of the pendulum zero when it is in this position. When the ball is drawn aside from its lowest position and released, it moves in the arc of a circle of radius L. Suppose the ball is moved to position A, at a height h above point C. The only work which is done on the ball is that done against the force of gravity in lifting the ball; it increases the potential energy by the amount mgh. As the ball swings from point A to point C some of its energy is transformed from potential to kinetic energy until, at C, all of it is kinetic energy. As it moves from C to B, some of its kinetic energy is now transformed into potential energy, and, at B, all of it is potential energy again. The pendulum may be viewed as a device for the continuous exchange of energy between these two forms.

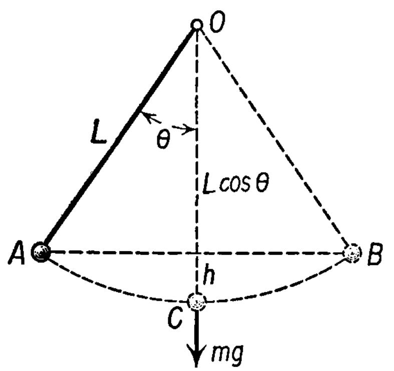

Fig. 7-9 Motion of a simple pendulum.

In the simple pendulum the ball moves with variable acceleration. The tension in the rope is not constant, and the methods of the preceding chapters are not adequate to analyze the motion of the pendulum. This analysis may be undertaken by the application of the energy principle.

Illustrative Example. A simple pendulum of length 10 cm with a pendulum bob of mass 5 gm is drawn aside so that the string makes an angle of 30° with the vertical. Find the linear speed of the pendulum bob when the pendulum has swung back so that its deflection is only 15°.

At any angle of deflection θ, the height h of the pendulum bob above its initial position C (Figure 7-9) is given by

$$h = L - L\cos\theta.$$

The potential energy of the pendulum, when $\theta = 30°$, is equal to $(\mathcal{E}_p)_i$ such that

$$(\mathcal{E}_p)_i = 5\text{ gm} \times g \times (10\text{ cm} - 10\text{ cm} \times \cos 30°).$$

The kinetic energy at this position is zero.
The potential energy of the pendulum, when $\theta = 15°$, is equal to $(\mathcal{E}_p)_f$ such that

$$(\mathcal{E}_p)_f = 5\text{ gm} \times g \times (10\text{ cm} - 10\text{ cm} \times \cos 15°).$$

The kinetic energy $\mathcal{E}_k$ at $\theta = 15°$, when the velocity of the bob is v, is

$$\mathcal{E}_k = \tfrac{1}{2}mv^2 = \tfrac{1}{2} \times 5\text{ gm} \times v^2.$$

Equating the total energy at $\theta = 30°$ to the total energy at $\theta = 15°$, we find

$$5 \times 10 \times 980 \times 0.134\text{ erg} = \tfrac{1}{2} \times 5\text{ gm} \times v^2 + 5 \times 10 \times 980 \times 0.034\text{ erg},$$

$$5 \times 10 \times 980 \times 0.10\text{ erg} = \tfrac{5}{2}\text{ gm} \times v^2,$$

so that

$$v = 44.3\,\frac{\text{cm}}{\text{sec}}\cdot$$

A mass which is oscillating at the end of a spring provides another illustration of an exchange between potential and kinetic energy. In this this case the energy is shared three ways. The mass may have kinetic energy and gravitational potential energy, while the spring itself has elastic energy associated with its stretch and compression. Here too, the force acting on the mass is not constant, and the methods developed in earlier chapters for treating motion resulting from the action of a constant force are inadequate to the problem. The principle of conservation of energy once again enables us to find a solution.

Illustrative Example. A 6-lb weight, when placed on a vertical spring, stretches it 2 in., at which point it is in equilibrium. The weight is then pulled down an additional 4 in. and released. (a) Determine the energy of the vibrating system. (b) What is the maximum upward displacement of the weight? (c) With what speed is the weight moving when it is 1 in. above the equilibrium position?

(a) The spring constant k may be determined from Equation (7-10). From the statement of the problem, we note that a force of 6 lb stretches the spring

2 in. or $\frac{1}{6}$ ft, and we find that

$$k = \frac{6\text{ lb}}{2\text{ in.}} = 3\,\frac{\text{lb}}{\text{in.}} = 36\,\frac{\text{lb}}{\text{ft}}\cdot$$

Let us take the zero level of energy as the neutral position of the spring, as shown in Figure 7-10. The vertical displacement of the weight will be called h, taken

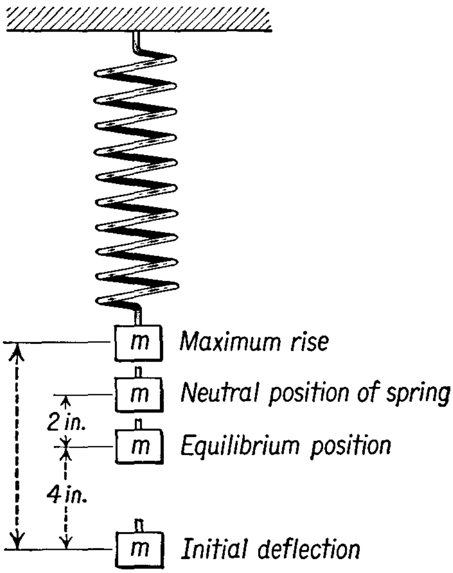

Fig. 7-10 The motion of an oscillating spring.

negative downward from the neutral position and positive upward from the neutral position. The energy $\mathcal{E}$ of the system at any position is given by

$$\mathcal{E} = \tfrac{1}{2}mv^2 + mgh + \tfrac{1}{2}kh^2. \tag{a}$$

We may evaluate $\mathcal{E}$ at the position of maximum displacement where $v = 0$. Noting that $mg = 6$ lb, $h = -\frac{1}{2}$ ft, and $k = 36\,\frac{\text{lb}}{\text{ft}}$, we find that

$$\begin{aligned}\mathcal{E} &= 0 + 6\text{ lb} \times (-\tfrac{1}{2}\text{ ft}) + \tfrac{1}{2} \times 36\,\frac{\text{lb}}{\text{ft}} \times (-\tfrac{1}{2}\text{ ft})^2 \\ &= -3\text{ ft lb} + 4.5\text{ ft lb},\end{aligned}$$

so that $\qquad \mathcal{E} = +1.5$ ft lb.

(b) Since the system is acted on by no frictional forces, the total mechanical energy of the system must be constant. At the extremes of the motion, the weight is instantaneously at rest, and we may find these extremes by finding the values of h at which $v = 0$. Substituting these values in Equation (a), we get

$$1.5\text{ ft lb} = 0 + 6\text{ lb} \times h + \tfrac{1}{2} \times 36\,\frac{\text{lb}}{\text{ft}} \times h^2.$$

Omitting dimensions to simplify the expression, and transposing,

we get $$18h^2 + 6h - 1.5 = 0;$$

and, applying the quadratic formula, we find

$$h = \tfrac{1}{6}, \quad \text{and} \quad h = -\tfrac{1}{2}$$

for the two solutions; that is, the weight oscillates from a point 2 in. above the undisplaced spring position to a point 6 in. below the undisplaced spring position, or 4 in. above and below the equilibrium position of the hanging weight.

(c) When the weight is 1 in. above its equilibrium position, it is 1 in. below the neutral position of the spring. To find the speed v with which the weight is moving at this point, we again apply the condition for conservation of mechanical energy, as given by Equation (a). The total energy $\mathcal{E}$ of the system is still 1.5 ft lb, but now some of the energy is kinetic; the value of h is now $-\frac{1}{12}$ ft. The equation now becomes

$$1.5 \text{ ft lb} = \tfrac{1}{2} \times \tfrac{6}{32} \text{ slug} \times v^2 - 6 \text{ lb} \times (\tfrac{1}{12} \text{ ft}) + \tfrac{1}{2} \times 36 \frac{\text{lb}}{\text{ft}} \times (\tfrac{1}{12} \text{ ft})^2,$$

from which $$v^2 = 20 \frac{\text{ft}^2}{\text{sec}^2},$$

yielding $$v = \pm 4.47 \frac{\text{ft}}{\text{sec}};$$

that is, the weight may be rising or falling with a speed of 4.47 ft/sec when the weight is 1 in. above its equilibrium position.

7-11 Power

In many cases the time in which a given amount of work is done is of great importance. The term *average power* is defined as the average rate of doing work, or *the work done divided by the time during which the work is done.*

$$\overline{\mathcal{P}} = \frac{\mathcal{W}}{t}, \tag{7-13}$$

in which $\overline{\mathcal{P}}$ represents the average power and t the time in which the work $\mathcal{W}$ is done. We define the instantaneous power $\mathcal{P}$ as

$$\mathcal{P} = \lim_{\Delta t \to 0} \frac{\Delta \mathcal{W}}{\Delta t} = \frac{d\mathcal{W}}{dt}. \tag{7-14}$$

The units of power are the units of work divided by the time. Thus, in the British gravitational system, the unit of power is foot pounds per second; in the cgs system power is expressed in ergs per second; and in the mks system power is expressed in joules per second. Only in the mks system of units is the unit of power given a special name, the *watt.* By definition *one watt equals one joule per second.* Note that although watts and kilowatts (abbreviated kw; equalling 1,000 watts) are widely used in connection with electrical apparatus, there is nothing particularly electrical about the definition of the watt. The watt is a unit of power which has been adopted by manufacturers of electrical equipment.

Another unit of power which is widely used in engineering but which is not an appropriate unit in any of our systems of units is the *horsepower* (hp) which owes its use to tradition rather than to any currently logical reason. When steam engines were first introduced, they were used to replace horses. Careful measurements of the rate at which a horse could do mechanical work were made by James Watt (1736–1819) who concluded that an average draft horse could exert a force of 150 lb while walking at a rate of 2.5 mi/hr. Thus 550 ft lb/sec or 33,000 ft lb/min were defined as the horsepower. These definitions are in current use today. The relationship between the foot pound per second, the watt, and the horsepower are

$$746 \text{ watts} = 1 \text{ hp} = 550 \frac{\text{ft lb}}{\text{sec}} \cdot$$

Such units as horsepower hour and kilowatt hour, often found in technical use, are units of power times units of time and therefore represent work.

We can derive an equation which is often quite useful, relating the power expended, the applied force, and the velocity. If the applied force and the displacement are in the same direction, we have seen that

$$d\mathscr{W} = F\,ds,$$

and, dividing by the time interval dt, we find

$$\mathscr{P} = \frac{d\mathscr{W}}{dt} = F\frac{ds}{dt},$$

so that

$$\mathscr{P} = Fv. \tag{7-15}$$

The power delivered to a body is the product of the force acting on it and the velocity of the body, when the force and the velocity are in the same direction. If we wish to take into account the possibility of different directions for the force and velocity, we perform the same operation of division by time on the vector form of the equation defining work. From Equations (7-3)

$$d\mathscr{W} = \mathbf{F}\cdot d\mathbf{s} = F\,ds\cos\theta,$$

and, dividing *by dt*,

$$\mathscr{P} = \mathbf{F}\cdot\mathbf{v} = Fv\cos\theta. \tag{7-16}$$

Illustrative Example. An engine is delivering 1,200 hp to an airplane in level flight at a uniform speed of 300 mi/hr. Determine the total of all the resistive forces (drag) acting on the airplane.

An airplane flying with uniform velocity is in equilibrium. Hence the force supplied by the airplane engine and propeller must be equal and opposite to all the resisting forces acting on the airplane. This force can be found by means of Equation (7-15). The force supplied by the airplane is in the same direction as the velocity. The units supplied in the problem must first be converted to the system of units in which we will work, the British gravitational system.

$$\mathcal{P} = 1{,}200 \text{ hp} = 1{,}200 \text{ hp} \times \frac{550 \text{ ft lb/sec}}{1 \text{ hp}},$$

$$\mathcal{P} = 1{,}200 \times 550 \frac{\text{ft lb}}{\text{sec}}.$$

$$v = 300 \frac{\text{mi}}{\text{hr}} = 300 \frac{\text{mi}}{\text{hr}} \times \frac{1 \text{ hr}}{3{,}600 \text{ sec}} \times \frac{5{,}280 \text{ ft}}{1 \text{ mi}},$$

$$v = 440 \frac{\text{ft}}{\text{sec}}.$$

From Equation (7-15)

$$F = \frac{\mathcal{P}}{v} = \frac{1{,}200 \times 550 \text{ ft lb/sec}}{440 \text{ ft/sec}} = 1{,}500 \text{ lb}.$$

7-12 Simple Machines

In common terminology the words "engine" and "machine" are often used interchangeably. For present purposes we shall define an engine as a device which converts other forms of energy into mechanical energy, and a machine as a device which transmits mechanical energy. Neither an engine nor a machine is capable of delivering more energy in an interval of time than it consumes in that same interval of time, for this would be a violation of the principle of conservation of energy. In general, any device always delivers less energy than is supplied to it. We find it convenient to define the mechanical efficiency of a device as

$$e = \frac{\text{work delivered by the device}}{\text{work supplied to the device}}. \tag{7-17}$$

For an ideal frictionless machine the efficiency $e = 1$.

Most complicated machines can be considered to be made up of a combination of simple elements, called *simple machines*, such as the inclined plane, the lever, the pulley system, and the screw and nut. There are usually two reasons for using a simple machine: one is that the magnitude of the force which can be applied without the aid of a machine is insufficient to move the required load, and the other is that the direction of the applied force is not appropriate to the desired result. Input quantities will be distinguished by the subscript i, and output quantities by the subscript o. If a force F_i is applied to the machine through a distance s_i, the input

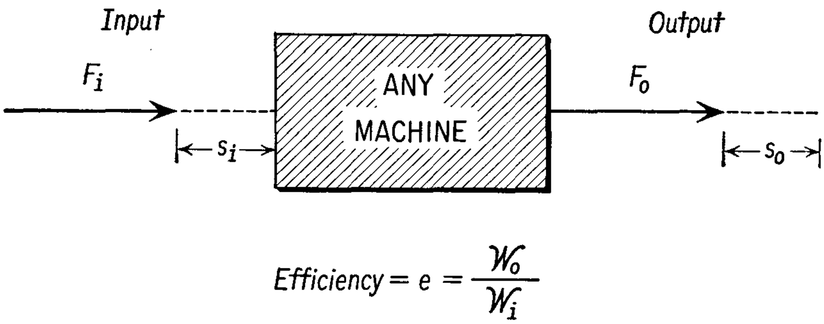

Fig. 7-11 Simple machines.

work will be W_i; the machine will perform an amount of work W_o by applying a force F_o through a distance s_o, as indicated in Figure 7-11.

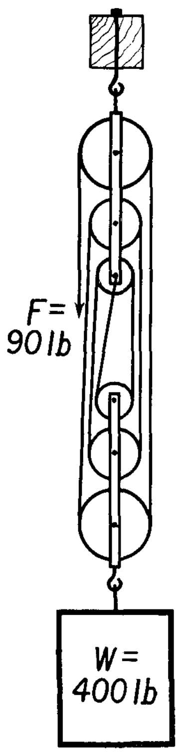

Fig. 7-12 Pulley system.

Illustrative Example. The pulley system of Figure 7-12 is used to hoist a safe weighing 400 lb by the application of a 90-lb force to the free end of the rope. Determine the efficiency of the pulley system.

To determine the efficiency of the system, we note that when the free end of the rope moves a distance $6h$, the weight moves a distance h. The work done on the machine is

$$W_i = 90 \text{ lb} \times 6h = 540h \text{ lb}.$$

The work done by the machine is

$$W_o = 400 \text{ lb} \times h = 400h \text{ lb}.$$

The efficiency of the machine is

$$e = \frac{W_o}{W_i} = \frac{400h}{540h} = 0.74.$$

Problems

7-1. A trunk weighing 150 lb is pulled across a floor for a distance of 12 ft by a horizontal force of 50 lb. (a) How much work is done? (b) If the trunk was pulled across the floor at uniform speed, what is the coefficient of kinetic friction between the trunk and the floor?

7-2. A man pulls a sled by means of a cord attached to it, exerting a force of 16 lb at an angle of 60° with the horizontal. How much work is done in pulling this sled for a distance of 250 ft?

7-3. (a) Calculate the work done in lifting a body whose weight is 140 lb through a height of 8 ft. (b) How much is the increase in its potential energy?

7-4. Determine the kinetic energy of an airplane whose weight is 30 tons if it is moving with a speed of 250 mi/hr.

7-5. A body weighing 100 lb is pushed up a rough inclined plane by a force of 75 lb acting parallel to the plane. The plane is inclined at an angle of 30° with the horizontal and is 24 ft long. (a) How much work is done in moving the body to the top of the inclined plane? (b) What is its potential energy when at the top of the plane? (c) How much work was done against friction?

7-6. A body weighing 75 lb slides down an inclined plane 16 ft high and 80 ft long. It reaches the bottom of the incline with a speed of 24 ft/sec. (a) What is its potential energy at the top of the inclined plane? (b) How much kinetic energy does it possess when it reaches the bottom of the plane? (c) Determine the force of friction between the body and the plane.

7-7. A box weighing 150 lb slides down an incline 20 ft long from the second floor of a building to the first floor 12 ft below. The frictional force exerted on the box by the incline is 48 lb. (a) How much potential energy does the box lose in sliding down? (b) How much energy is used up in moving the box against the frictional force? (c) How much kinetic energy does the box have when it gets to the bottom? (d) What is the coefficient of kinetic friction between the box and the plane?

7-8. A ball is thrown upward at an angle of 60° with the horizontal at a speed of 50 ft/sec. (a) From energy considerations, find the speed of the ball when it reaches the top of its path. (b) How high will the ball go?

7-9. A simple pendulum consists of a thin string of negligible mass with a steel ball of 450 gm mass attached to one end. The distance from the point of support to the center of the ball is 100 cm. The ball is pulled aside until the string makes an angle of 37° with the vertical. (a) How much potential energy does the pendulum have in this position? (b) With what velocity will the ball reach the lowest position after it is released? (c) How fast will the ball be moving when the string makes an angle of 10° with the vertical?

7-10. A helical spring hangs vertically with its lower end at $y = 0$. When a 45-lb weight is attached to it and lowered gently, the spring is stretched 1.5 in. The spring is pulled down an additional 4.0 in. and is released. (a) Determine the constant of the spring in lb/ft. (b) Determine the total energy of the system, taking the zero level of energy at $y = 0$. (c) Determine the y coordinate of the highest position the body will attain after being released. (d) Determine the speed of the body when $y = -2$ in.

7-11. What power must be delivered to a car which is moving at a speed of 45 mi/hr if the sum of all the resisting forces acting on the car is 180 lb?

7-12. How much power must a man weighing 160 lb develop if he runs up a flight of stairs 9 ft high in 5 sec?

7-13. The engines of a fighter plane deliver 2,000 hp to keep the plane in level flight with a constant velocity of 400 mi/hr. The plane weighs 5 tons. (a) Determine the sum of all the forces opposing the motion of the plane. (b) Assuming that these drag forces remain constant, what will be the speed of the plane when it climbs at an angle of 5° with the horizontal? (c) What will be the speed of the plane when it descends at an angle of 5° with the horizontal?

7-14. A body weighing 96 lb drops from a height of 4 ft above the top of a spring and compresses it. If the constant of the spring is 12 lb/in., determine the decrease in length of the spring.

7-15. When a body is attached to a tension spring and gently lowered to its equilibrium position, the spring is stretched by an amount s. If the same body is attached to the spring and permitted to drop, show that the maximum deflection of the spring is $2s$.

7-16. The pulley system sketched in Figure 7-13 consists of an upper fixed block containing two pulley wheels and a lower movable block also containing two pulley wheels. (a) A body of weight W is attached to the movable block. How big a force F would have to be applied to lift the weight uniformly it there were no friction? (b) In one such pulley system a force of 80 lb was needed to lift a 240-lb weight. What was the efficiency of this simple machine?

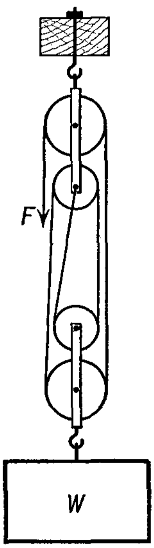

Fig. 7-13 Pulley system.

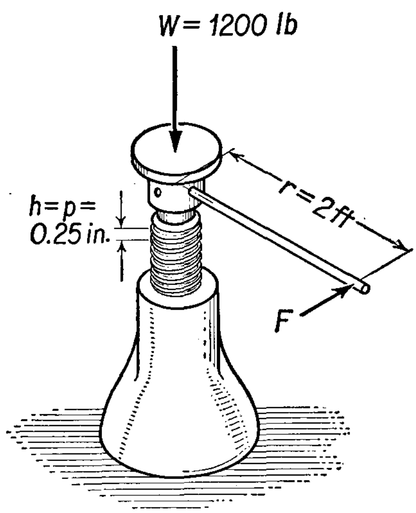

Fig. 7-14 Jack screw.

7-17. The screw thread of an automobile jack has a pitch of 0.25 in. (the pitch is the distance the screw moves forward in one complete rotation) and is operated by a lever 2 ft long, as shown in Figure 7-14. If the efficiency of the jack is 0.30, determine the force that must be applied to the end of the lever to lift a load of 1,200 lb.

7-18. The hammer of a pile driver weighs 1,200 lb and falls through a height of 6 ft to drive a pile into the ground. (a) How much energy does the hammer have when it strikes the pile? (b) If the pile is driven a distance of 6 in., determine the average resisting force acting on the pile.

7-19. A chemical balance customarily weighs to a sensitivity of about 0.1 mg. How much energy, in ergs, would be liberated in a chemical reaction if the change in mass were just detectable?

7-20. Two vectors **A** and **B** are drawn in the xy plane radiating out from the origin with the heads of the two vectors lying at the points whose coordinates are $(x, y) = (4, 0)$ and $(5, 12)$, respectively. Find (a) the scalar product $\mathbf{A}\cdot\mathbf{B}$ and (b) the vector product $\mathbf{A}\times\mathbf{B}$ between the two vectors.

7-21. Two vectors are drawn in the x-y plane. The tail of vector **A** lies at point (6, 0) while its head lies at point (9, 4). The head of vector **B** is at the origin, while its tail is at point (0, 6). Find (a) the dot product $\mathbf{A}\cdot\mathbf{B}$, (b) the cross product $\mathbf{A}\times\mathbf{B}$, and (c) the cross product $\mathbf{B}\times\mathbf{A}$ between the two vectors.

7-22. The output of an electric motor is 5 hp. Determine the velocity with which a load of 400 lb can be lifted?

7-23. An automobile weighing 2,500 lb is driven by an engine which develops 50 hp. On level ground the automobile has a maximum speed of 75 mi/hr. What is the greatest speed with which the automobile can climb a 10 per cent grade? (A 10 per cent grade is one which rises 1 ft in 10 ft along the incline.)

7-24. Starting with the definitions of the watt and the horsepower, show that 746 watts = 1 horsepower.

7-25. A ball weighing 4 ounces is at rest on the floor of a train which is moving at 60 mi/hr. What is the kinetic energy of the ball (a) as determined by an observer on the train? (b) By an observer on the ground? (c) Explain the discrepancy between the two cases.

7-26. The altitude at which the potential energy is zero may be set arbitrarily by choosing some level, such as sea level, as the level of zero altitude. In what way may we establish an arbitrary zero of kinetic energy? (HINT: Are the laws of mechanics equally valid in two coordinate frames moving at uniform speed with respect to each other?)