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Physics, Chapter 11: Rotational Motion (The Dynamics of a Rigid Body)

Henry Semat
City College of New York

Robert Katz
University of Nebraska-Lincoln, rkatz2@unl.edu

11

Rotational Motion (The Dynamics of a Rigid Body)

11-1 Motion about a Fixed Axis

The motion of the flywheel of an engine and of a pulley on its axle are examples of an important type of motion of a rigid body, that of the motion of rotation about a *fixed axis*. Consider the motion of a uniform disk rotating about a fixed axis passing through its center of gravity C perpendicular to the face of the disk, as shown in Figure 11-1. The motion of this disk may be described in terms of the motions of each of its individual particles, but a better way to describe the motion is in terms of the *angle* through which the disk rotates. Calling two successive positions of a point in the plane of the disk P_1 and P_2, we find the angle of rotation by drawing radial lines from C to P_1 and to P_2. The angle θ between these two lines is the angle through which the disk has rotated; every point in the plane of the disk has rotated through the same angle θ in the same interval of time.

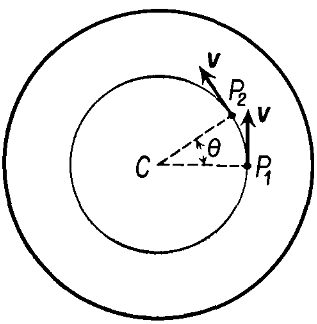


Fig. 11-1 Angle of rotation of a disk.

The angle θ is called the *angular displacement* of the body. Both the angle θ and the direction of the axis of rotation must be given in order to specify properly a rotational displacement.

In spite of the apparent similarity between the specification of a rotational displacement and a linear displacement, an arbitrary rotational displacement is *not* a vector quantity, for one cannot add rotational displacements in the same way that linear displacements are added. Let us imagine that a blackboard eraser has its length initially directed along the x axis, and that the top face of the eraser is initially perpendicular to the

y axis. If we rotate the eraser first about the y axis by 90°, then about the z axis by 90°, the eraser lies on its side. If the rotation is first performed about the z axis, then about the y axis, the eraser will stand on end. The resultant of these two operations depends on the order in which they are

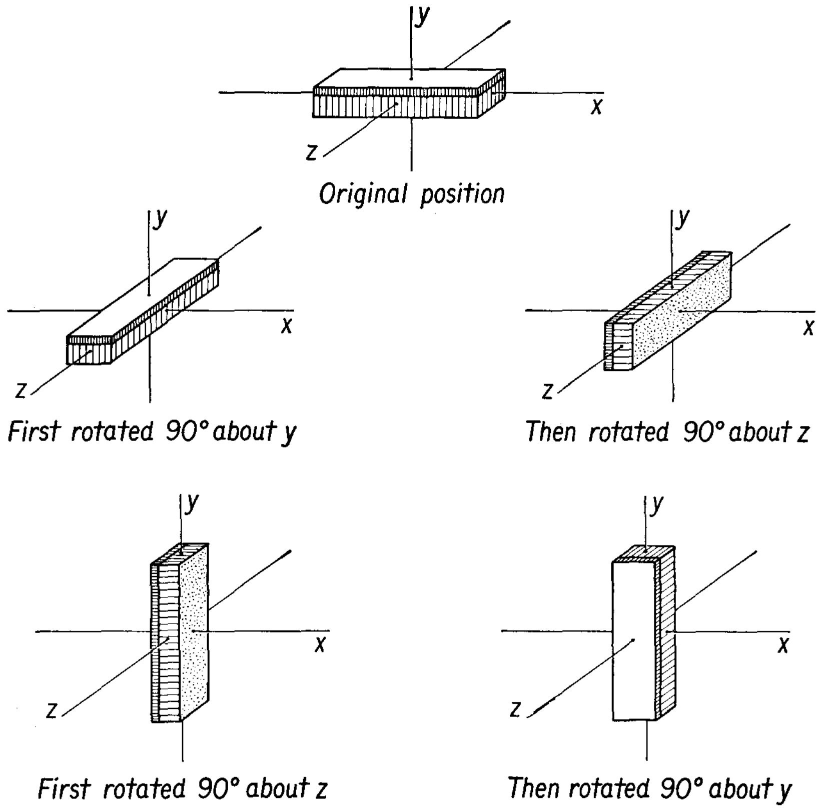


Fig. 11-2 The result of two finite rotations depends upon the order in which they are performed.

performed, as shown in Figure 11-2. As we have already seen, the resultant of two vectors, or of two linear displacements, does not depend on the order in which the sum is taken. Thus, although angular displacements involve both direction and magnitude, angular displacements of arbitrary magnitude cannot be true vectors.

If the rotational motion is restricted to rotation about a single *fixed* axis, it is possible to represent angular displacement as a vector quantity whose direction is parallel to that axis, in accordance with the right-hand rule previously given in the discussion of circular motion, for then the resultant of two angular displacements does not depend on the order of rotations.

When the angular displacement of a body is restricted to *infinitesimal rotations,* these infinitesimal rotations may be thought of as vector angular displacements, for it may be shown that the sum of two infinitesimal rotations does not depend upon the order in which these rotations are performed. For this reason angular velocity is a vector quantity, for it is the result of dividing an infinitesimal angular displacement, a vector, by time, a scalar.

11-2 Kinetic Energy of Rotation

A rigid body rotating with uniform angular speed ω about a fixed axis possesses kinetic energy of rotation. Its value may be calculated by summing up the individual kinetic energies of all the particles of which the body is composed. A particle of mass m_1 located at distance r_1 from the axis of rotation has kinetic energy given by $\frac{1}{2}m_1v_1^2$, where v_1 is the speed of the particle. There will be a similar term for each particle making up the body, so that we may write, for the total kinetic energy $\mathcal{E}_k$,

$$\mathcal{E}_k = \tfrac{1}{2}m_1v_1^2 + \tfrac{1}{2}m_2v_2^2 + \cdots + \tfrac{1}{2}m_nv_n^2,$$

so that

$$\mathcal{E}_k = \sum \tfrac{1}{2}m_iv_i^2.$$

Each particle of a rigid body rotates with uniform angular speed ω. Let us express the instantaneous linear speed of each particle in terms of the common angular speed. Remembering that $v = \omega r$, we substitute for v in the above equation to find

$$\mathcal{E}_k = \tfrac{1}{2}m_1r_1^2\omega^2 + \tfrac{1}{2}m_2r_2^2\omega^2 + \cdots + \tfrac{1}{2}m_nr_n^2\omega^2,$$

or

$$\mathcal{E}_k = \tfrac{1}{2}\omega^2(m_1r_1^2 + m_2r_2^2 + \cdots + m_nr_n^2).$$

Let us denote the factor in parentheses by the letter I; that is,

$$I = m_1r_1^2 + m_2r_2^2 + \cdots + m_nr_n^2,$$

or

$$\boxed{I = \sum m_ir_i^2,} \tag{11-1}$$

so that the kinetic energy of the rotating body may be written as

$$\boxed{\mathcal{E}_k = \tfrac{1}{2}I\omega^2.} \tag{11-2}$$

The factor I is called the *moment of inertia* of the rotating body with respect to the particular axis of rotation. The moment of inertia depends upon the manner in which the mass is distributed with respect to the axis.

Clearly, the moment of inertia will be greatest when the mass is farthest from the axis of rotation. In the motion of rotating systems, the moment of inertia plays a role analogous to that of the mass in translational systems or in linear motion. Unlike the mass, which is a constant for a particular body, the moment of inertia depends upon the location and direction of the axis of rotation as well as upon the way the mass is distributed.

11-3 Moments of Inertia of Simple Bodies

The moment of inertia of a system of particles is given by Equation (11-1) as

$$I = \sum m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + \cdots + m_n r_n^2.$$

Let us calculate the moment of inertia of several simple distributions of particles.

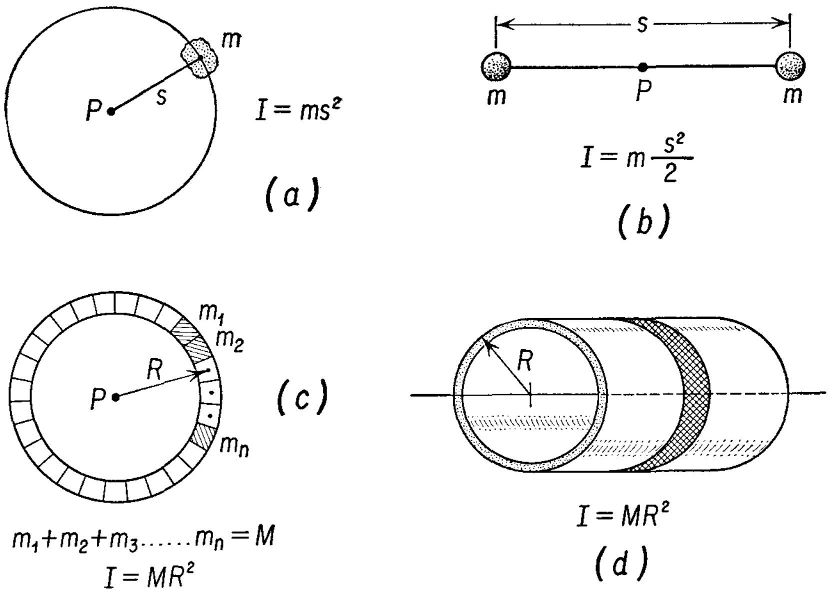


Fig. 11-3 Moments of inertia of some bodies of simple geometrical shapes. The axis is perpendicular to the paper and passes through P in (a), (b), and (c). In (d) the axis is the geometrical axis of the cylinder.

Consider a small stone of mass m attached to a long weightless string of length s, whose other end is fixed to a pivot P, as in Figure 11-3(a). Since there is only one mass to consider, the summation reduces to a single term, and the moment of inertia is given by $I = ms^2$.

A dumbbell, consisting of two equal masses m separated by a long weightless bar of length s free to rotate about its center of gravity at the point P midway between the two masses, as shown in Figure 11-3(b), has a moment of inertia given by

$$I = m\left(\frac{s}{2}\right)^2 + m\left(\frac{s}{2}\right)^2 = \frac{ms^2}{2}\cdot$$

A thin ring of mass M and mean radius R which is free to rotate about its center may be thought of as a collection of segments of mass m_1, m_2, m_3, and so on, as shown in Figure 11-3(c), each of which is located at a distance R from the axis of rotation. Applying Equation (11-1) to the ring, considered as a collection of particles, we find

$$\begin{aligned} I &= m_1R^2 + m_2R^2 + m_3R^2 + \cdots + m_nR^2 \\ &= (m_1 + m_2 + m_3 + \cdots + m_n)R^2, \end{aligned}$$

and since the summed mass of the segments is equal to the mass M of the ring, we find, for the moment of inertia of a hollow ring,

$$I = MR^2.$$

A hollow cylinder of mass M which is free to rotate about an axis through its center may be thought of as a stack of rings, as shown in Figure 11-3(d). From Equation (11-1) we see that the moment of inertia of a collection of matter about a given axis is simply the sum of the moments of inertia of each of the separate parts about the same axis. Thus the moment of inertia of a hollow cylinder of radius R about its axis is given by the same formula as the moment of inertia of a hollow ring, $I = MR^2$, where M now represents the mass of the cylinder.

A body which is composed of a distribution of matter rather than a collection of mass points must be imagined as segmented into small pieces approximating point masses. The moment of inertia is calculated by summing the quantity mr^2 over each of the imagined segments. Better approximations to the true moment of inertia of the body may be made by imagining the body to be broken up into finer and finer subdivisions. In the limit of an infinitesimally fine subdivision, the sum is replaced by an integral, and in the language of the calculus, if dm represents the mass of one element of the body of volume dv, located at a distance r from the axis of rotation, the moment of inertia of the body is given by

$$I = \int r^2\,dm. \tag{11-3a}$$

If the body is a homogeneous one of density ρ, the mass of a small element of volume dv is given by

$$dm = \rho\,dv,$$

and the moment of inertia may be written as

$$I = \int \rho r^2 \, dv. \tag{11-3b}$$

For bodies of complicated shape, the evaluation of the integral may be quite difficult, but for bodies of simple geometric shape, the evaluation of the integral is well within the reach of an introductory course in the calculus.

Illustrative Example. Calculate the moment of inertia of a rod of length L and cross-sectional area A about an axis perpendicular to the rod through one end, as shown in Figure 11-4. Suppose the density of the rod is ρ; the volume of an

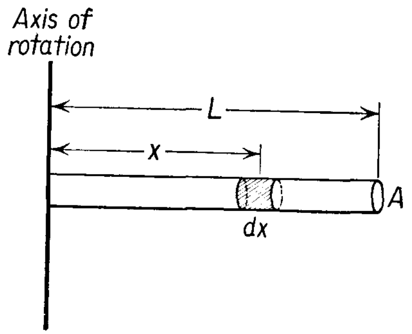


Fig. 11-4 Determining the moment of inertia of a rod.

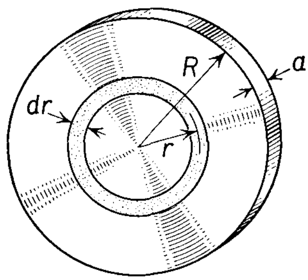


Fig. 11-5 Determining the moment of inertia of a solid disk.

element of length dx is given by $A\,dx$, and the mass dm of the element is $\rho A\,dx$. The moment of inertia of this element, located at a distance x from the axis of rotation, is $dI = \rho A x^2\,dx$, and the moment of inertia of the entire rod is obtained by summing, or integrating, the contributions to the moment of inertia of each element of the rod. Thus

$$\begin{aligned} I &= \int_0^L dI = \int_0^L \rho A x^2 \, dx = \left[\rho A \frac{x^3}{3}\right]_0^L \\ &= \rho A \frac{L^3}{3}. \end{aligned}$$

Remembering that AL is the volume of the rod, the mass of the rod M is given by ρAL. We write for the moment of inertia of the rod about a perpendicular axis through one end,

$$I = \frac{ML^2}{3}.$$

Illustrative Example. Calculate the moment of inertia of a flat solid disk of radius R and mass M about an axis through its center of mass and perpendicular to the plane of the disk. Let us suppose the disk to be of thickness a, and made of a homogeneous material of density ρ. To calculate the moment of inertia of the disk, we imagine the disk to be made up of a set of nested rings, as shown in Figure 11-5. The surface area of a ring of mean radius r and width dr is given by

$2\pi r\,dr$, and the mass dm of such a ring is the product of its volume by its density; that is, $dm = 2\pi r a \rho\,dr$. The moment of inertia of a ring is its mass times the square of its radius, and the contribution of the moment of inertia of any one such ring to the moment of inertia of the disk is

$$dI = (2\pi r\,dr\,a\rho)r^2 = 2\pi a r^3 \rho\,dr.$$

The moment of inertia of the entire disk is found by adding the moments of

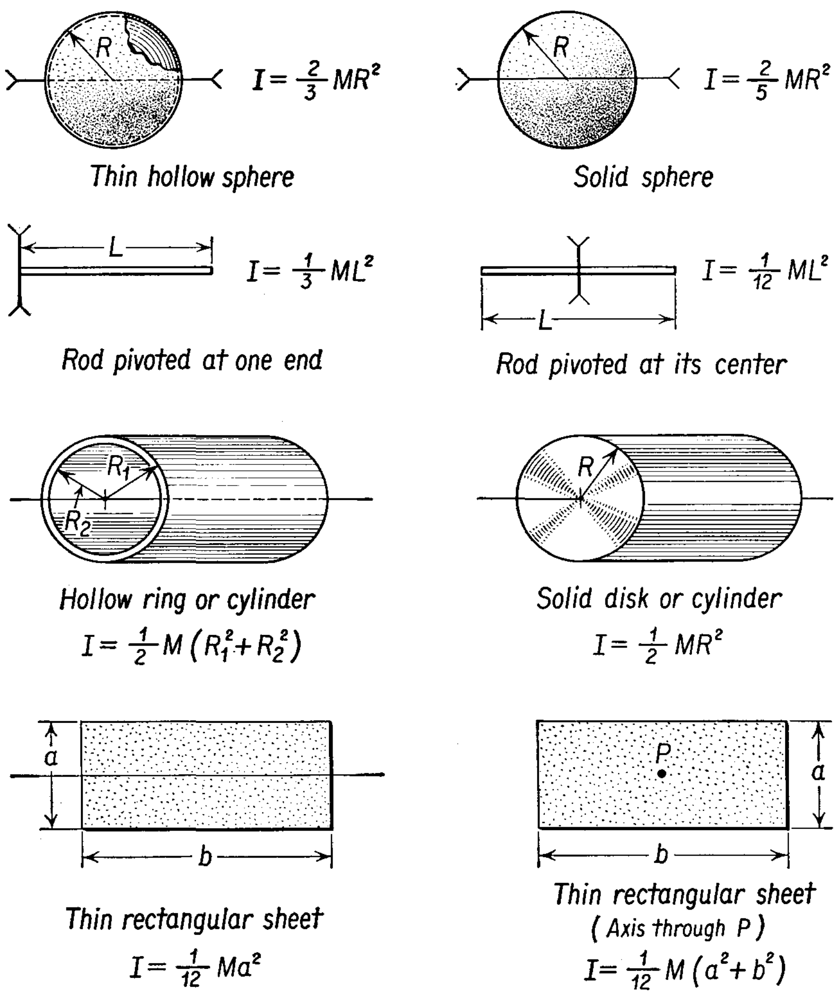


Fig. 11-6 Moments of inertia of several bodies of simple geometrical shapes.

inertia of all the rings which are imagined to constitute the disk. Thus

$$I = \int_0^R dI = \int_0^R 2\pi a\rho r^3\,dr = \left[2\pi a\rho\,\frac{r^4}{4}\right]_0^R$$
$$= \pi R^2 a\rho\,\frac{R^2}{2}\cdot$$

The factor $\pi R^2 a\rho$ may be recognized as the volume of the disk times density, or the mass M of the disk, and we have

$$I = \frac{MR^2}{2}\cdot$$

The moment of inertia of a solid disk about a central axis perpendicular to its face is half that of a hollow ring having the same mass and the same radius.

The moments of inertia of several bodies of simple geometric shapes are given in Figure 11-6. The moment of inertia is a useful and important concept in the study of the strength of materials, for it figures prominently in formulas for the strength of such members as angles and I beams.

The units for moment of inertia are those of mass multiplied by the square of a distance, for example, gm cm^2, or kgm m^2, or slugs ft^2.

11-4 The Parallel Axis Theorem

A theorem in mechanics which is very useful in the study of rotational motion is called the *parallel axis theorem* which states that *if the moment of inertia of a body about an axis through its center of mass is known, the moment of inertia of the body about any axis parallel to the first is given by the moment of inertia about the axis through the center of mass plus the product of the mass of the body by the square of the perpendicular distance between the two axes.* In the form of an equation we write

$$I = I_c + MR^2, \tag{11-4}$$

where I_c is the moment of inertia of the body about an axis through its center of mass, M is the mass of the body, and R is the perpendicular distance from the center of mass to the axis of rotation.

Illustrative Example. Find the moment of inertia of a hollow ring about an axis perpendicular to the plane of the ring which passes through a point on the circumference.

The moment of inertia of a hollow ring of mass M and radius R about an axis through its center of mass I_c perpendicular to the face of the ring has been shown to be equal to MR^2. The moment of inertia I of the ring about a parallel axis through its circumference is equal to

$$I = I_c + MR^2$$
$$= MR^2 + MR^2$$
$$= 2MR^2.$$

11-5 Torque and Angular Acceleration

In our discussion of the equilibrium of a rigid body, we found that, when the vector sum of all the torques acting on a body is zero, the body is in equilibrium as far as rotational motion is concerned. If an external torque acts on the body, it will acquire an angular acceleration $\boldsymbol{\alpha}$ given by

$$\boxed{\mathbf{G} = I\boldsymbol{\alpha},} \tag{11-5}$$

where **G** is the sum of all the external torques acting on the body about a fixed axis, and I is the moment of inertia of the body about the same axis.

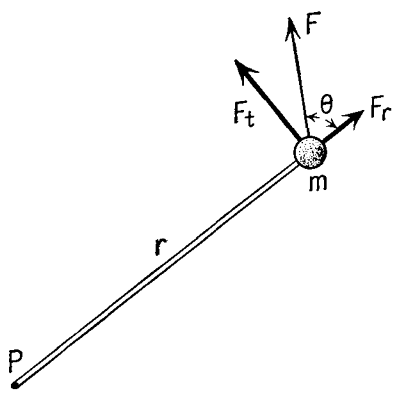


Fig. 11-7

Equation (11-5) may be derived from Newton's laws of motion and represents a special form of Newton's equation applied to rotational motion.

Suppose that a particle of mass m is constrained to move in a circular path by a rigid weightless rod of length r about a point P, as shown in Figure 11-7. An arbitrary force F can only cause it to move in a tangential direction, for motion in the radial direction is not permitted by the rod. If the angle between the force F and the rod is given by θ, we can resolve the force into a radial component F_r and a tangential component F_t. The radial component produces no torque about the axis through P, hence we need consider only the effect of the tangential component.

From Newton's second law we may write, for the tangential component,

$$F_t = ma_t,$$

where a_t is the tangential component of the acceleration of the particle. We have already seen that the tangential acceleration of a particle moving in circular motion may be related to its angular acceleration α through the equation

$$a_t = \alpha r,$$

and, substituting into the equation above, we find

$$F_t = m\alpha r.$$

From Figure 11-7, the value of the tangential component of F is given by

$$F_t = F \sin \theta.$$

Thus

$$F \sin \theta = m\alpha r,$$

and, multiplying both sides of the equation by r, we find

$$Fr \sin \theta = mr^2\alpha.$$

The quantity $Fr \sin \theta$ on the left-hand side of the equation is exactly the torque G exerted by the force F about the axis through P, while the quantity mr^2 is the moment of inertia I of the particle about the same axis. Thus

$$G = I\alpha,$$

and we have verified Equation (11-5) for the simple case of a mass particle constrained to rotate about a fixed axis. Note that the directions of the vector $\mathbf{G}$ and of the vector $\boldsymbol{\alpha}$ are both perpendicular to the plane of the paper, pointing outward, in accordance with the right-hand rule.

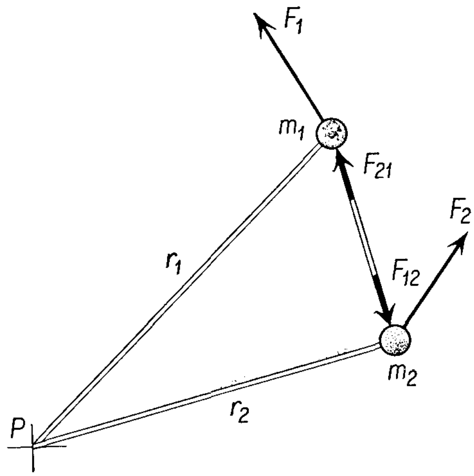


Fig. 11-8

Suppose we had a system of two particles m_1 and m_2 rigidly connected to each other and to the axis of rotation by a framework of weightless rods, as shown in Figure 11-8. The two particles and their framework are constrained to move in circular motion with a common angular velocity and common angular acceleration. Let the external force exerted on the particle of mass m_1 be F_1, the external force exerted on the second particle of mass m_2 be F_2, while the forces exerted *by* the second particle *on* the first is F_{21}, and *by* the first particle *on* the second is F_{12}. From Newton's third law these two forces must be equal and opposite to each other. Remembering that neither m_1 nor m_2 is free to move in the radial direction, we apply Newton's second law to the tangential motion of each particle:

$$(F_1)_t + (F_{21})_t = m_1(a_1)_t,$$

$$(F_2)_t + (F_{12})_t = m_2(a_2)_t.$$

Let us multiply the first of these equations by r_1 and the second by r_2. The product $(F_1)_t r_1$ is the torque G_1 of the force F_1 about the axis of rotation. Similarly, the product $(F_{21})_t r_1$ is the torque G_{21} of the internal force F_{21} about the axis of rotation, and we may write

$$G_1 + G_{21} = m_1(a_1)_t r_1 = m_1 r_1^2 \alpha,$$

$$G_2 + G_{12} = m_2(a_2)_t r_2 = m_2 r_2^2 \alpha.$$

Since F_{21} and F_{12} are directed along the same straight line in opposite directions, and since they have equal magnitudes, the torques G_{21} and G_{12} are equal and opposite:

$$G_{12} = -G_{21}.$$

Adding the two equations, we find

$$G_1 + G_2 = (m_1 r_1^2 + m_2 r_2^2)\alpha,$$

or

$$G = I\alpha.$$

Once again we have verified Equation (11-5), that the sum of the external torques acting on a rigid body is equal to the product of the moment of inertia by the angular acceleration. Following the same procedure, the equation may be shown to be true for an arbitrary number of mass particles connected by a rigid framework and hence for a continuous rigid body.

The equation $\mathbf{G} = I\boldsymbol{\alpha}$, is the rotational counterpart of the equation $\mathbf{F} = M\mathbf{a}$. The only assumption which has been introduced in the study of rotational motion is that the rotating body is a rigid one in which the force exerted by one part of the body on another lies along the line joining the two parts. We see also that the rotational equilibrium of a rigid body is a special case of Equation (11-5). A rigid body is in rotational equilibrium when its angular acceleration is zero, and hence when the sum of the external torques acting upon it is zero. This is exactly analogous to the translational equilibrium of a rigid body, which may be considered as a special case of the equation $\mathbf{F} = M\mathbf{a}$, for a body is in translational equilibrium when its acceleration is zero, and hence when the sum of the external forces is zero.

Illustrative Example. A disk 30 cm in diameter and having a mass of 900 gm is mounted so that it can rotate about a fixed axis passing through its center, as in Figure 11-9. A mass of 200 gm hangs from a string which is wound around the disk. Determine (a) the acceleration of the 200-gm mass, (b) the angular acceleration of the disk, and (c) the tension in the string.

To solve the problem we first examine the figure to establish a simple sign convention. The weight will be accelerated in the downward direction by the force of gravity, and at the same time the disk will be caused to rotate in a clockwise direction. Let us call the clockwise direction the positive direction of rotation, and the downward direction the positive direction of translation. From the nature of the force which can be exerted by a string, the upward force exerted by the string on the falling mass must be equal in magnitude to the downward force exerted on the disk by the string. We shall call the magnitude of this force S and indicate the directions on the diagram. From the nature of the constraints, we note that the only way the mass can receive a given downward displacement is for a length of string to unroll from the disk. If the radius of the disk is R, the disk must receive an angular displacement θ when the mass is displaced by a distance s such that $s = \theta R$. Note that the directions of the angular and linear displacements as well as their magnitudes are related by this equation, for a positive value of s implies a positive value of θ. Thus we have $v = \omega R$ and $a = \alpha R$, where v and a are the velocity and acceleration of the mass, positive downward, and ω and α are the angular velocity and angular acceleration of the disk, positive clockwise. With the relationships of constraint and the sign conventions established, we proceed to a formal solution of the problem.

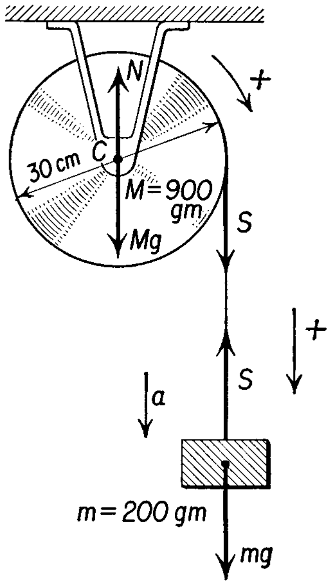


Fig. 11-9

Let us first consider the forces acting on the 200-gm mass. From Newton's second law we can write

$$mg - S = ma.$$

Now let us consider the rotation of the disk. The only force acting on the disk that exerts a torque about the axis of rotation through C is the pull of the cord S. If R is the radius of the disk and I is its moment of inertia, we find, on substituting in Equation (11-5),

$$G = SR = I\alpha,$$

and, since

$$a = \alpha R,$$

we have

$$SR = \frac{Ia}{R}\cdot$$

Substituting the value of S into the first equation, we find

$$mg - \frac{Ia}{R^2} = ma,$$

and, solving for a, we find

$$a = \frac{mg}{m + \frac{I}{R^2}}.$$

The moment of inertia of a uniform disk about an axis through C is $I = \frac{MR^2}{2}$,

so that
$$a = \frac{mg}{m + M/2}.$$

The numerical values are $m = 200$ gm, $M = 900$ gm, $R = 15$ cm, and $g = 980$ cm/sec^2, and we find

(a) $$a = 309 \frac{\text{cm}}{\text{sec}^2};$$

(b) $$\alpha = \frac{a}{R} = 20.5 \frac{\text{radians}}{\text{sec}^2};$$

(c) $$S = m(g - a)$$
$$= 200 \text{ gm} \times (980 - 309) \frac{\text{cm}}{\text{sec}^2},$$

so that
$$S = 134{,}800 \text{ dynes}.$$

The tension in the cord is less than the weight of the 200-gm mass hanging from its end.

11-6 Rotational Energy, Work, and Power

Whenever a rigid body is set into rotation about an axis, work is done by the torques acting on it to increase its kinetic energy of rotation. Suppose

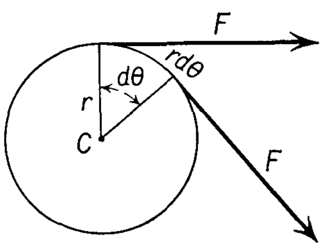


Fig. 11-10 The work done by a constant torque G when it acts through an angle $d\theta$ is $G\,d\theta$.

that a force F acts on the rim of a wheel of radius r and rotates the body through an angle $d\theta$, as shown in Figure 11-10. The displacement of the wheel is $r\,d\theta$, and the work done by this force is

$$d\mathscr{W} = Fr\,d\theta;$$

but
$$Fr = G;$$

therefore
$$d\mathscr{W} = G\,d\theta, \tag{11-6a}$$

or the work $d\mathcal{W}$ done by a torque G is equal to the product of the torque and the angle $d\theta$ through which it acts.

Just as in the case of the work done by a force, the work of an applied torque is done by the component of the torque parallel to the axis of rotation. In vector notation, if a torque **G** produces a rotation $d\boldsymbol{\theta}$, the work done is

$$\boxed{d\mathcal{W} = \mathbf{G} \cdot d\boldsymbol{\theta}.} \tag{11-6b}$$

If a constant torque acts on a rigid body which is rotating about a fixed axis, then, from the principle of conservation of mechanical energy, assuming no loss due to friction, the work done by the torque will produce a change in the kinetic energy of the body given by

$$\mathcal{W} = G\theta = \tfrac{1}{2}I\omega_f^2 - \tfrac{1}{2}I\omega_i^2, \tag{11-7}$$

in which ω_f is the final angular speed of the body, ω_i is the initial angular speed of the body, and θ is angular displacement through which the torque has acted.

Illustrative Example. The flywheel of a steam engine whose moment of inertia is 72 kg m² is given an angular speed of 150 rpm in 90 rev, starting from rest. Determine the torque, assuming it to be constant, which acted on the flywheel.

The angle θ through which the torque acted is

$$\theta = 90 \times 2\pi = 180\pi \text{ radians.}$$

The final speed of the flywheel is

$$\omega_f = 150\,\frac{\text{rev}}{\text{min}} \times \frac{1\text{ min}}{60\text{ sec}} \times \frac{2\pi\text{ radians}}{1\text{ rev}}$$

$$= 5\pi\,\frac{\text{radians}}{\text{sec}}\cdot$$

Applying Equation (11-7) to the solution of the problem,

$$G \times 180\pi = \tfrac{1}{2} \times 72\text{ kg m}^2 \times 25\pi^2\,\frac{1}{\text{sec}^2},$$

$$G = 15.7\text{ nt m.}$$

If the constant torque G is applied for a time dt to the rotation about a fixed axis, then we may find the power $\mathcal{P}$ by dividing both sides of Equation (11-6b) by the time dt,

$$\mathcal{P} = \frac{d\mathcal{W}}{dt} = \mathbf{G} \cdot \frac{d\boldsymbol{\theta}}{dt}$$

$$\mathcal{P} = \mathbf{G} \cdot \boldsymbol{\omega}. \tag{11-8a}$$

When the torque is parallel to the axis of rotation,

$$\mathcal{P} = G\omega, \tag{11-8b}$$

for, by definition, the angular velocity is equal to the angular displacement divided by the time. Thus the power $\mathcal{P}$ expended by a constant torque G applied for a time t is equal to the product of the torque by the angular velocity. Equation (11-8b) is the rotational analogue of the equation $\mathcal{P} = Fv$.

Illustrative Example. A $\frac{1}{2}$-hp motor is designed to operate at a speed of 1,750 rpm. What is the torque which the shaft of the motor can exert when operating at the rated speed?

The power delivered by the motor is $\frac{1}{2} \times 550 \dfrac{\text{ft lb}}{\text{sec}}$, while its angular speed is $1{,}750 \times \dfrac{2\pi}{60} \dfrac{\text{radians}}{\text{sec}}$. Substituting in Equation (11-8b), we find

$$\frac{550}{2} \frac{\text{ft lb}}{\text{sec}} = G \times 1{,}750 \times \frac{2\pi}{60} \frac{\text{radians}}{\text{sec}},$$

$$G = 1.50 \text{ lb ft}.$$

11-7 Angular Momentum and Angular Impulse

A rigid body rotating with angular velocity $\boldsymbol{\omega}$ about a fixed axis has an angular momentum $\mathbf{p}_\theta$ about this axis given by

$$\mathbf{p}_\theta = I\boldsymbol{\omega}, \tag{11-9}$$

where I is the moment of inertia of the body about this axis. Note that since the angular velocity about a given axis is a vector quantity which lies parallel to the axis of rotation, in a direction given by the right-hand rule, and the moment of inertia about this axis is a scalar quantity, the *angular momentum is a vector quantity.* To change the angular momentum of a body, an external torque must be applied to it. Remembering that the instantaneous angular acceleration is given by the derivative of the angular velocity with respect to the time, we may write Equation (11-5) as

$$\mathbf{G} = I\frac{d\boldsymbol{\omega}}{dt} = \frac{d(I\boldsymbol{\omega})}{dt}.$$

Substituting $\mathbf{p}_\theta$ for $I\boldsymbol{\omega}$ from Equation (11-9), we have

$$\mathbf{G} = \frac{d}{dt}(\mathbf{p}_\theta), \tag{11-10}$$

or the torque acting on a rigid body is equal to the rate of change of the angular momentum. Although Equation (11-10) was here derived for a rigid body, it may be shown that a system of particles obeys the same rule;

that the rate of change of the total angular momentum of the system of particles is equal to the sum of the external torques acting on the system of particles. *In the absence of external torques, the angular momentum of a rigid body must be constant;* that is, there is no change in the angular momentum of a rigid body when the sum of the external torques is zero. This is known as the principle of *conservation of angular momentum,* and, like the principle of conservation of energy and the conservation of linear momentum, is one of the most important general principles of mechanics.

Just as in the case of linear motion, we may treat impulsive motion in the case of rotation by examining the incremental form of the equation relating the torque to the rate of change of angular motion. We may write

$$\mathbf{G} = \frac{\Delta \mathbf{p}_\theta}{\Delta t},$$

and, multiplying through the equation by the time interval Δt during which the torque $\mathbf{G}$ is applied, we find the angular impulse $\Delta \mathbf{J}_\theta$ to be

$$\Delta \mathbf{J}_\theta = \mathbf{G}\,\Delta t = \Delta \mathbf{p}_\theta. \tag{11-11}$$

Thus the change in angular momentum is equal to the angular impulse.

An example of angular-momentum changes due to an angular impulse is the operation of the clutch in an automobile where a rotating disk connected to the engine engages a second disk connected to the rear wheels.

Since the angular momentum is a vector quantity, a rigid body set spinning on its axis will maintain its direction of rotation as well as its angular speed, providing no external torque acts on it.

Examples of the operation of the principle of conservation of angular momentum are numerous, in everyday life as well as in astronomy and in atomic and nuclear physics. The force exerted by the sun and by other celestial bodies on the earth is directed through the center of the earth (to a good approximation) in accordance with Newton's law of universal gravitation. Since the axis of rotation of the earth passes through its center, these forces exert no torque on the earth about its axis of spin. Consequently, the angular momentum of the earth and the length of the day are constant.

Consider a stone attached to the end of a string being whirled in a horizontal circle. If the string is made to wind itself around a vertical stick, becoming shorter with each revolution, the stone is observed to whirl with increasing angular speed as the string winds itself up. As the string becomes shorter, the moment of inertia of the stone about its axis of rotation is decreased. Since the force exerted on the stone by the string is in the radial direction, there is no external torque exerted by the string on the stone. Its angular momentum remains constant, but the decrease in moment of inertia must be accompanied by an increase in its angular speed.

A cat manages to fall on its feet, a diver can land in the water headfirst, an ice skater can execute a pirouette on the toe of one skate, all through the action of the principle of conservation of angular momentum. The principle of conservation of angular momentum explains why the changing mass distribution of the earth, as the result of volcanoes, tides, and winds, affects the instantaneous speed of rotation of the earth on its axis. In atomic and nuclear physics the atom or nucleus is acted upon by external forces which act through the center of mass of the system. The angular momentum of an atom or nucleus about its center of mass is constant, and, in fact, the value of the angular momentum of an atom or a nucleus is one of the more important pieces of information which can be used to describe atomic or nuclear systems.

11-8 Rolling Motion

The motion of a wheel which is rolling along the ground, without slipping, can be considered in one of two ways: either as a rotation of the wheel about an axis through its center of gravity C and an additional translational motion of the entire wheel with the same velocity as the center of gravity,

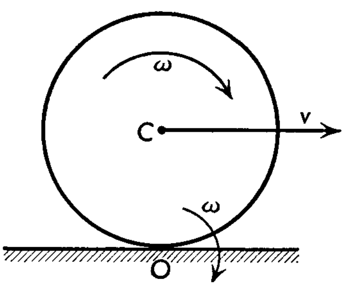


Fig. 11-11 Wheel rolling on the ground. ω is the angular velocity and v is the linear velocity of the center of gravity.

as shown in Figure 11-11, or as a rotation of the wheel about an *instantaneous axis* through the point of contact O between the wheel and the ground. A point in the body which is on the instantaneous axis is momentarily at rest. The instantaneous axis itself moves forward as the wheel moves forward, but it always remains parallel to itself and to the axis through the center of gravity. The angular velocity of the wheel about the instantaneous axis is the same as that about the axis through the center of gravity.

The general motion of a rigid body may be thought of as made up of two parts: one a motion of translation of the center of gravity, with the entire mass of the body acting as though it were concentrated there, and the other a motion of rotation of the body about an axis through the center of gravity. The angular velocity and angular acceleration are calculated by taking the torques and the moment of inertia about an axis through the center of gravity. The linear velocity and the linear acceleration of the center of gravity are then calculated by considering all the forces as though they acted through the center of gravity, and by applying Newton's second law of motion to this case.

Illustrative Example. A disk of radius R and mass M rolls without slipping down an inclined plane of height h. Discuss the motion of the disk.

There are three forces acting on the disk, its weight Mg, the normal force N perpendicular to the plane and passing through the center of the disk, and the

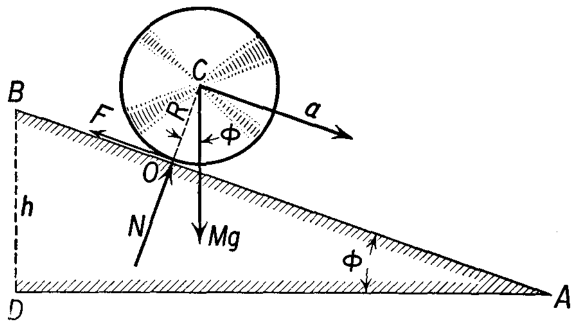


Fig. 11-12 Disk rolling down an inclined plane.

frictional force F acting parallel to the plane at the point of contact O, as illustrated in Figure 11-12. The instantaneous axis of rotation passes through point O and is perpendicular to the plane of the disk.

Let us determine the torques about the instantaneous axis through O. Since both F and N pass through O, they contribute nothing to the torque about the axis through O. The moment arm of the force Mg with respect to the axis through O is $R \sin \phi$, where ϕ is the angle of the inclined plane. Hence the torque equation becomes

$$MgR \sin \phi = I\alpha.$$

From the parallel axis theorem, the moment of inertia of the disk about an axis through O is

$$I = I_c + MR^2,$$

where I_c is the moment of inertia about the center of gravity. Solving the torque equation for α, we find

$$\alpha = \frac{MgR \sin \phi}{I},$$

and the acceleration of the center of gravity a is given by

$$a = \alpha R = \frac{MgR^2 \sin \phi}{I}.$$

It is interesting to compare the linear acceleration of the center of gravity of a solid disk of mass M with a hollow ring of the same mass. Note that the analysis thus far does not specify whether the disk is solid or hollow. For a solid disk we have seen that $I_c = \frac{1}{2}MR^2$, while for a hollow ring of the same mass and radius, $I_c = MR^2$, so that for a solid disk $I = \frac{3}{2}MR^2$, while for a hollow ring $I = 2MR^2$. Thus for a solid disk

$$a = \tfrac{2}{3}g \sin \phi,$$

while for a hollow ring

$$a = \tfrac{1}{2}g \sin \phi,$$

so that a solid disk will roll down an incline with greater acceleration than a hollow disk of the same radius. Note that neither the mass of the disk nor its radius appears in the expression for the acceleration. Thus all solid disks will roll down an inclined plane faster than all hollow rings. A similar relationship may be found for spheres.

It may be observed that the linear acceleration a of the center of gravity is less than the acceleration of a body which slides down a similar but frictionless inclined plane.

It is instructive to consider this problem from the energy point of view. If the disk rolls down the incline without slipping, there is no energy lost in doing work against the frictional force. If the disk starts at the top of the incline with zero kinetic energy, its total energy is its potential energy Mgh. At the bottom of the incline its energy is all kinetic. We may calculate the kinetic energy in two ways. First, let us consider that the motion of the disk at the bottom of the incline consists of rotation about the instantaneous point of contact. Let us consider the motion of a solid disk, whose moment of inertia about a point on its rim is $I = \frac{3}{2}MR^2$. The kinetic energy is all rotational about the point of contact, and we have

$$Mgh = \tfrac{1}{2}I\omega^2,$$

$$Mgh = \tfrac{3}{4}MR^2\omega^2,$$

$$\omega^2 = \frac{4gh}{3R^2},$$

$$\omega = \sqrt{\frac{4gh}{3R^2}},$$

where ω is the angular speed of the disk at the bottom of the incline.

Now let us consider the rolling disk, alternatively, as having its energy made up of two parts. The motion may be thought of as a translational motion of the center of gravity with kinetic energy $\frac{1}{2}Mv^2$, and a rotational motion about the center of gravity with kinetic energy $\frac{1}{2}I\omega^2$. The moment of inertia of a solid disk about its center of gravity is $I = \frac{1}{2}MR^2$. The total kinetic energy of the disk at the bottom of the incline is therefore given by

$$\mathcal{E}_k = \tfrac{1}{2}Mv^2 + \tfrac{1}{2}I\omega^2,$$

and, remembering that the linear velocity of the center of gravity is related to the angular velocity of the disk by

$$v = \omega R,$$

the kinetic energy becomes

$$\mathcal{E}_k = \tfrac{1}{2}M\omega^2R^2 + \tfrac{1}{2} \times \tfrac{1}{2}MR^2 \times \omega^2,$$

$$\mathcal{E}_k = \tfrac{3}{4}MR^2\omega^2,$$

exactly as before.

11-9 Rotational Motion and Linear Motion

In Section 6-6 we have already seen that many of the equations developed for linear motion could be transcribed to the problem of angular motion simply by replacing the symbols used to describe linear motion by a set of symbols appropriate to angular motion. From the present chapter it is possible to display some additional analogues. Thus in rotational motion the moment of inertia I plays a role analogous to that of the mass M in linear motion. The angular momentum $I\boldsymbol{\omega}$ plays a role analogous to that of the linear momentum $M\mathbf{v}$. The torque $\mathbf{G}$ and the force $\mathbf{F}$ play analogous roles. It is instructive to transcribe a number of the equations which have been thus far developed to display the similarity of the equations frequently used in mechanics for both linear and angular motions, as in Table 11-1. The formulas in the table have been somewhat simplified over the formulas developed in the text. In particular, the formulas are all indicated as relating scalar quantities, and the displacement and velocity are assumed parallel to the force in the equations for work and power.

Although there are analogous quantities used to describe linear motion and rotational motion, it must be recognized that these quantities represent quite different things. The mass is a constant quantity and is independent of the position of any coordinate frame, but this is distinctly not true of the moment of inertia. The moment of inertia is a different number for every different axis of rotation, whether the difference is due to position or to orientation. Similarly, it is difficult to conceive just how one might transport mechanical energy in the form of linear kinetic energy in a package, yet every engine has a flywheel which carries rotational kinetic energy. The behavior of a system having linear momentum in response to an applied force is very similar to the behavior of the system at rest to the same applied force. We shall see in the next section that the way a system having angular momentum responds to an applied torque is far more complex and more interesting.

11-10 The Gyroscope

We have stressed the fact that the angular momentum of a rigid body about an axis remains constant unless acted on by an external torque. In the

TABLE 11-1

Linear Motion	Angular Motion
$x = \bar{v}t$	$\theta = \bar{\omega}t$
$v = u + at$	$\omega = \omega_i + \alpha t$
$x = ut + \frac{1}{2}at^2$	$\theta = \omega_i t + \frac{1}{2}\alpha t^2$
$v^2 = u^2 + 2ax$	$\omega^2 = \omega_i^2 + 2\alpha\theta$
$\bar{v} = \dfrac{u + v}{2}$	$\bar{\omega} = \dfrac{\omega_i + \omega}{2}$
$F = ma$	$G = I\alpha$
$\mathscr{W} = Fs$	$\mathscr{W} = G\theta$
$\mathscr{P} = Fv$	$\mathscr{P} = G\omega$
$\mathscr{E}_k = \frac{1}{2}Mv^2$	$\mathscr{E}_k = \frac{1}{2}I\omega^2$
$p = mv$	$p_\theta = I\omega$
$F = \dfrac{dp}{dt}$	$G = \dfrac{dp_\theta}{dt}$
$J = \Delta p$	$J_\theta = \Delta p_\theta$
x . . . . corresponds to	θ
v	ω
a	α
F	G
M	I
p	p_θ
J	J_θ

previous sections we considered the change in angular momentum about the *same axis* as the original direction of the angular momentum. The direction of the torque vector was parallel to the direction of the angular momentum vector, and the change in the angular momentum vector was always an increase or a decrease in its length. Let us consider now the implications of a change in the *direction* of the angular-momentum vector.

Suppose that a bicycle wheel is mounted on an axle which protrudes beyond the end of the wheel, as shown in Figure 11-13 and that it is spinning in such a direction that its angular-momentum vector is nearly in the direction of the positive x axis. Let us rotate the axle through a positive angle $\Delta\theta$ in the x-y plane, as shown in Figure 11-14. Since the rotation is in a direction perpendicular to the direction of spin, the angular speed ω with which the wheel is spinning on its axle will not be altered, and the angular-momentum vector after the wheel has been rotated will be of the same length as before but will now point in a different direction. The change in the angular momentum $\Delta \mathbf{p}_\theta$ will be a vector directed from the head of the initial angular-momentum vector to the head of the final angular-momentum vector. Remembering that the torque is equal to the rate of change of the angular momentum,

$$G = \frac{d\mathbf{p}_\theta}{dt},$$

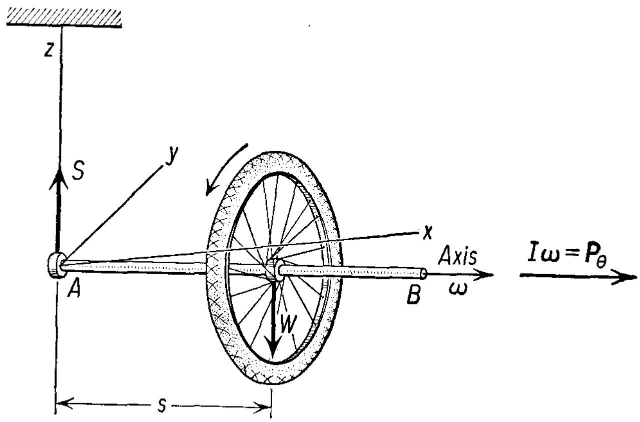


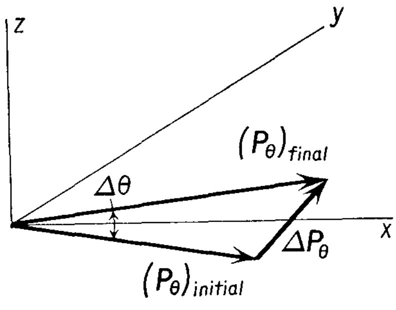
Fig. 11-13 Gyroscope.

which, for small increments in time Δt, may be written as

$$G = \frac{\Delta\mathbf{p}_\theta}{\Delta t}.$$

We see that the torque required to produce the rotation $\Delta\theta$ must be parallel to the change in the angular momentum and must be in the y direction.

Fig. 11-14



In other words, to produce a rotation of the bicycle wheel about the z axis requires that a torque be applied about the y axis. Such a torque would produce a rotation about the y axis *if the body were not spinning.* If the rotation $\Delta\theta$ is produced in a small time interval Δt, at a uniform rate Ω (capital omega), then

$$\Delta\theta = \Omega\,(\Delta t).$$

Referring again to Figure 11-14, for small angular displacement,

$$\Delta\theta = \frac{\Delta p_\theta}{p_\theta},$$

and, substituting from the above equation for $\Delta\theta$, we have

$$p_\theta\Omega\,(\Delta t) = \Delta p_\theta,$$

$$p_\theta\Omega = \frac{\Delta p_\theta}{\Delta t} = G,$$

$$I\omega\Omega = G. \tag{11-12a}$$

Thus the applied torque G is equal in magnitude to the product of the spin angular momentum by the angular speed with which the axis of rotation is itself rotated. A rotation of the spin axis is called *precession*. In this example the direction of spin is parallel to the x axis, the direction of the rotation of the axis is in the z direction, while the direction of the torque which must be applied to produce the rotation is in the y direction. We might write this in terms of the vector product

as

$$\mathbf{G} = \mathbf{\Omega} \times \mathbf{p}_\theta,$$

or

$$\mathbf{G} = I\mathbf{\Omega} \times \boldsymbol{\omega}, \tag{11-12b}$$

showing that the torque vector is perpendicular to both the spin vector $\boldsymbol{\omega}$ and the precession vector $\mathbf{\Omega}$.

Let us re-examine the problem by considering the effect of a torque on the bicycle wheel. Suppose that the left end of the axle of the bicycle wheel is supported by a string which is hung from the ceiling, as shown in Figure 11-13. Once the wheel has been set in motion, spinning about its axle, the forces acting on it are its weight W, acting through the center of gravity, and the tension in the string S. The torque produced by these forces about a horizontal axis through A perpendicular to the plane of the figure (in the positive y direction) is of magnitude Ws, where s is the distance from the center of gravity to A. This torque will produce a change of angular momentum per unit time in accordance with Equation (11-10). The vector representing the change of angular momentum per unit time will be parallel to the torque vector and hence in the positive y direction, directed into the plane of the paper.

Suppose we consider a very small time interval Δt during which this torque acts. The wheel must receive an increment of angular momentum $\Delta\mathbf{p}_\theta = \mathbf{G}(\Delta t)$, and in Figure 11-14 we add this increment of angular momentum to the initial angular momentum to find the final angular momentum of the wheel. We see that in the time Δt the final angular-momentum vector has been rotated by an angle $\Delta\theta$ about the vertical or z axis. The average angular speed with which the axis has been rotated $\Omega = \Delta\theta/\Delta t$ is the speed of *precession*. The forces S and W continue to act, and the rotational motion of the axle of the bicycle wheel will continue in a horizontal plane. Note that our first impression would be to assert that the wheel would tend to be twisted by the forces W and S so that the axle should tend to hang vertically. The rotating wheel is sometimes called a *gyroscope*, and the motion analyzed above is called *gyroscopic motion*.

Any rotating body can be considered as a gyroscope. When a torque acts to change the direction of spin, precessional motion will occur. The earth's axis of rotation precesses in the heavens in a circle of 23.5° radius with a period of 26,000 years. The torques which cause the axis of rotation to precess are due primarily to the gravitational attraction of the sun and moon on the earth's equatorial bulge. The motion of a top whose axis of spin is inclined to the vertical is a common example of precessional motion. When the top is not spinning, it is in unstable equilibrium when resting on its point. When the top is spinning, its most stable position is one in which its center of gravity is directly above the point of support, and when a spinning top is thrown, it climbs to this position in apparent disregard for equilibrium conditions; this is owing to the friction between the peg of the top and the surface. The propeller of an airplane acts like a gyroscope, and when the airplane turns the gyroscope will precess, unless the airplane has its propellers in pairs which rotate in opposite directions, so that the total angular momentum due to spin is zero.

Whenever any piece of rotating machinery is mounted on a moving platform, such as a ship or an airplane, the bearings of that machine must exert a torque on the shaft of the machine so that it will precess in the direction in which the platform is turning. The greater the rotational speed and the spin angular momentum, the greater is the torque required, according to Equations (11-12). For this reason, motors mounted on board ships or aircraft must have specially designed bearings, capable of withstanding far greater loads than would be required of the same appliance if the machine were used on a stationary platform.

If a gyroscope is mounted on earth so that its axis is parallel to the axis of rotation of the earth, that is, in a north-south direction, the rotating earth does not change the direction of the axis of rotation of the gyroscope in space, and there is no tendency for the gyroscope to precess. If the gyroscope is mounted with its axis in some other direction, the rotation of the earth will cause the gyroscope to precess, so that the direction of the true north can be recognized from the behavior of a rotating gyroscope. This is the basis of the gyrocompass.

Gyroscopic motion is the basis of the behavior of the bicycle. If a rider leans to the left, the front wheel of a bicycle will turn to the left as though to catch the rider in his fall. The discussion based on Figure 11-13 is suitable to a discussion of the behavior of a bicycle advancing in the $-y$ direction in which the rider leans to his left. From Figure 11-14 we see that the front wheel will turn into the rider's fall. A novice learning to ride must learn to let the bicycle do his thinking for him, while the skilled cyclist can ride without touching the handle bars by shifting his weight from side to side.

Problems

11-1. The flywheel of a gasoline engine is built in the form of a uniform disk of radius 1 ft and weighs 75 lb. The flywheel is rotating with an angular speed of 3,300 rpm. Determine the kinetic energy of the flywheel.

11-2. A small copper disk of 15 cm radius and 350 gm mass is rotating with an angular speed of 12 radians/sec about an axis through its center. Determine (a) the kinetic energy of the disk and (b) its angular momentum.

11-3. A pulley 6 in. in diameter is mounted so that it can rotate about a fixed axis through its center. The pulley weighs 12 lb and has a moment of inertia of 0.02 slug ft^2. A constant force of 3 lb is applied to the rim of the pulley by means of a cord wrapped around it. Determine (a) the angular acceleration of the pulley and (b) the angular speed it has at the end of 10 sec, assuming that the pulley was initially at rest.

11-4. Find the moment of inertia of a dumbbell consisting of two spheres of radius 10 cm connected by a cylindrical rod 1 cm in radius and 50 cm long about an axis through the center of gravity perpendicular to the rod. The dumbbell is made of iron of density 7.8 gm/cm^3.

11-5. Prove the parallel axis theorem for the case of two equal point masses.

11-6. A wheel in the form of a uniform disk of mass 900 gm and radius 8 cm is mounted so that it can rotate about a fixed horizontal axis passing through its center. A cord is wrapped around the circumference of the wheel, and a mass of 50 gm is attached to its free end. (a) Determine the angular acceleration of the wheel when it is released. (b) Determine the linear acceleration of the 50-gm mass. (c) Determine the tension in the cord. (d) Determine the angular velocity of the wheel at the end of 5 sec. (e) Determine the kinetic energy of the entire system when the mass has fallen through a distance of 10 cm.

11-7. A wheel having a radius of 6 cm is mounted so that it can rotate about a fixed horizontal axis passing through its center. A cord wrapped around the circumference of the wheel has a mass of 250 gm attached to its free end. When allowed to fall, the mass takes 5 sec to fall a distance of 100 cm. Determine (a) the angular acceleration of the wheel and (b) its moment of inertia.

11-8. A rod 50 cm long and weighing 5 lb is pivoted at one end. The rod is raised to a horizontal position and released. (a) What is the angular velocity of the rod when it is at an angle of 45° with the horizontal. (b) What is the angular acceleration of the rod at the same deflection?

11-9. A solid cylinder 2 ft in diameter and weighing 64 lb starts at the top of a rough plane 24 ft long and inclined at an angle of 30° with the horizontal and rolls down without slipping. (a) How much energy did the cylinder have at the top of the hill? (b) How much energy will it have at the bottom of the hill? (c) Determine its angular velocity at the bottom of the hill.

11-10. Show that Kepler's law of areas (Section 6-14) is consistent with the law of conservation of angular momentum, for circular orbits.

11-11. A steel hoop rolls without sliding down a plane inclined at an angle of 30° with the horizontal. The mass of the hoop is 600 gm, and its radius is 8 cm. Determine (a) its moment of inertia about its instantaneous axis of rotation, (b) its angular acceleration, (c) the force of friction between the hoop and the plane, and (d) the linear velocity of the center of the hoop when the hoop has rolled 1 m down the incline.

11-12. A gasoline engine develops 75 hp when turning at 3,300 rpm. Determine the torque delivered by this engine to the drive shaft.

11-13. A solid spherical ball of radius 1 ft is rolled toward a curb of height 4 in. What must be the speed of the center of gravity of the ball if the ball is to jump the curb?

11-14. A boy weighing 100 lb stands at the center of a brass turntable 6 in. thick and 10 ft in diameter. The turntable is rotated with an angular speed of 1 radian/sec. The boy walks out along a radius to a point 5 ft from the center of the turntable. What is the angular speed with which the disk is now rotating? The density of brass is 8.6 gm/cm^3.

11-15. A solid disk having a mass of 1 kg and a radius of 2 cm is wrapped with string. The free end of the string is supported from a point in the ceiling, and the disk is released. Determine the angular speed of the disk when the center of gravity of the disk has fallen 1 m.

11-16. A top having a moment of inertia of 5,000 gm cm^2 is spinning at a speed of 25 rev/sec at an angle of 30° with the vertical. The top has a mass of 500 gm, and its center of gravity is 4 cm from its point. The spin is counterclockwise, as seen from above. (a) What is the angular velocity of precession of the top axis? (b) As seen from above, is it clockwise or counterclockwise?

11-17. A unicycle has a wheel 36 in. in diameter and a mass of 1 slug. Neglect the weight of the seat and frame. A man weighing 150 lb sits on the unicycle, so that his center of gravity is 4 ft from the bottom of the wheel, and pedals the unicycle until it is moving with a speed of 20 ft/sec. The man leans to his right so that man and cycle make an angle of 1° with the vertical. What is now the direction and angular speed of precession of the axle of the unicycle?

11-18. A satellite of mass 10 kg is launched at the equator due north with a velocity of 10^6 m/hr. The satellite is to fly along the surface of a sphere of radius 6.5×10^6 m concentric with the earth. (The earth's radius is approximately 6.4×10^6 m.) When the satellite reaches 37° north latitude, what are the components of its velocity with respect to the earth (a) in the north-south direction, and (b) in the east-west direction? Assume that the satellite experiences no drag. Take into account the necessity to conserve angular momentum.

11-19. When a car is going forward, the engine and flywheel are rotating counterclockwise as viewed by the driver. In which direction will the car tend to go if the front wheels are suddenly lifted by a bump in the road?

11-20. Discuss the motion of an airplane whose propellers suddenly stop rotating from the point of view of the conservation of angular momentum.

11-21. A man sits on a piano stool holding a spinning bicycle wheel in his hands. The axis of the bicycle wheel is directed vertically. He turns the wheel end for end while remaining seated, and finds that he begins to rotate in the same direction that the wheel was spinning originally. Why? What happens if he once again reverses the wheel?

11-22. A man sits on a piano stool which is at rest. The man and stool have a weight of 200 lb, and may be approximated by a vertical cylinder of radius 1 ft. The man catches a 5-oz baseball moving horizontally with a speed of 80 ft/sec at a distance of 1 ft from the axis of rotation of the stool. (a) What is the angular momentum of the baseball with respect to the axis of the stool at the instant it is caught? (b) What is the angular velocity of the system consisting of the man, stool, and ball after the ball is caught?

11-23. Show that the moment of inertia of a thin rectangular sheet of sides a and b is $I = \frac{1}{12} Ma^2$, when the axis of rotation lies in the plane of the sheet, through the center of mass, and is parallel to the side b, as in Figure 11-6. [HINT: Start with the expression for the moment of inertia of a rod and integrate.]

11-24. Find the moment of inertia of a thin flat sheet cut in the form of the quadrant of a circle of radius R with respect to an axis normal to the plane of the quadrant and passing through the center of the circle.

11-25. A 2-kg mass is suspended from a string which is wound over the axle of a wheel. It is observed that the mass has a downward acceleration of 2 m/sec^2. The radius of the axle is 0.05 m and the radius of the wheel is 1.5 m. At $t = 0$ the system is at rest. (a) What is the angular acceleration of the wheel? (b) What is the angular velocity of the wheel at $t = 5$ sec? (c) What is the radial acceleration of a point on the rim of the wheel at $t = 5$ sec? (d) What is the moment of inertia of the wheel and axle?

11-26. A solid sphere of radius 10 cm and mass 250 gms rolls without slipping down an inclined plane which makes an angle of 37° with the horizontal. (a) What is the linear acceleration of the center of mass of the sphere parallel to the plane? (b) What is the angular acceleration of the sphere about an axis through its center of mass? (c) What is the angular velocity of the sphere about an axis through its center of mass when the sphere has rolled a distance of 5 m down the incline (measured along the incline)?

11-27. An electric motor which turns at a speed of 3600 rpm has an armature of mass 10 kg. The armature may be approximated by a solid cylinder of radius 5 cm. The motor is mounted on an airplane which turns to the right through a 90° arc in 15 sec. Assume that the airplane turns without banking. What is the magnitude and direction of the torque exerted on the spinning armature by the bearings if (a) the axis of the motor is mounted vertically? (b) the motor is mounted horizontally with its axis parallel to the wings? (c) the motor is horizontal with its axis perpendicular to the wings?