Physics, Chapter 12: Periodic Motion

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12

Periodic Motion

12-1 Introduction

One of the more important problems in mechanics is the study of periodic motions, that is, *motions which repeat themselves in regular intervals of time, called the period.* An example of periodic motion which we have already encountered is uniform circular motion, in which the velocity and acceleration of the body at a given angular position were always the same. If a particle was found at a given position at a time t, we could be sure that it would return to that position at time $t + T$ later, where T was the period of the rotational motion. A body in uniform circular motion moves under the influence of a varying force, the centripetal force, which, though constant in magnitude, varies in direction. In general, an object moving in periodic motion must move under the influence of a varying force which is directed to some equilibrium position or to some neutral position. In uniform circular motion this position is the center of the circle.

There are many other types of periodic motion. Among these are the *oscillations* of a weight attached to a vertical spring, the motion of a simple pendulum, and the oscillations of the balance wheel of a watch. A glass marble bouncing up and down on a hard steel anvil represents such a motion; see Figure 12-1(a). If the collision between the marble and the anvil is perfectly elastic, the marble rises after each collision to the height h_0 from which it was dropped. The motion is perfectly repetitive, as shown in Figure 12-1(b), where the height h of the marble has been plotted as a function of time. We see that the graph is really a succession of parabolas displaced along the time axis by the period T. In actual practice it is not possible to achieve a perfectly elastic collision between the marble and the anvil. The marble loses a fraction of its energy on each collision, and the motion is not perfectly periodic, as shown by the dotted lines in Figure 12-1(b). The interval between successive impacts becomes shorter and shorter, and the motion is known as an *aperiodic* motion, that is, nearly but not quite periodic.

Other examples of oscillating motion include the movement of a piston in a gasoline engine, the vibrations of a string of a musical instrument, of the wingtips of an airplane in flight, and of a building or a suspension bridge

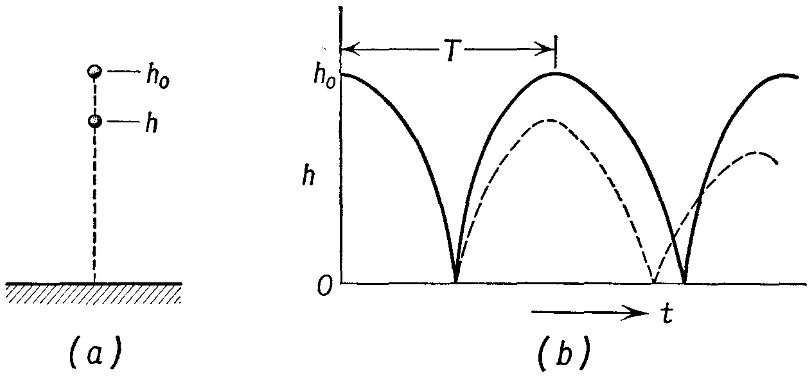

Fig. 12-1 (a) Ball bouncing off an anvil. (b) Periodic motion (solid line) of a perfectly elastic marble bouncing on a rigid anvil, and the aperiodic motion (dotted line) of a glass marble on a steel anvil.

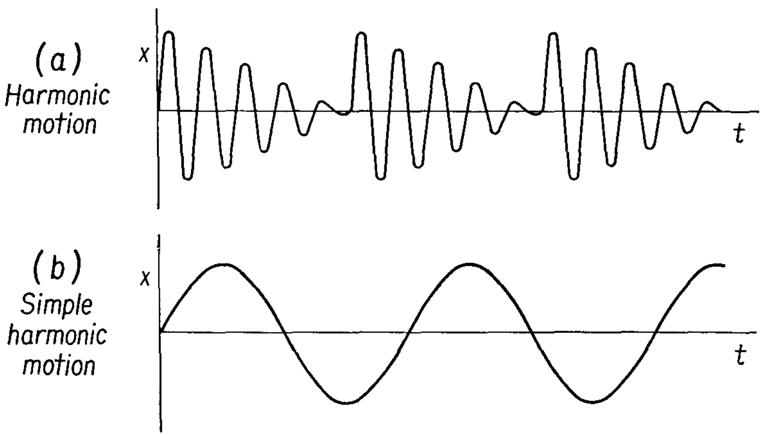

Fig. 12-2 Harmonic motion. (a) Graph of a series of damped oscillations. (b) Graph of simple harmonic motion.

in a high wind, and the bobbing of a ship or of a bell buoy. Many of these motions can be described by the use of a combination of sine or cosine functions of time and are consequently called *harmonic* motions. The simplest of these motions follows a single sine or cosine curve and is called *simple harmonic motion.* Examples of harmonic motion and simple harmonic motion are shown in Figure 12-2.

12-2 Elasticity

In an earlier discussion of collision problems, we used the word *elastic* to describe a process in which mechanical energy was conserved. In the present section the word "elastic" will be used to describe a different but related property of matter. When a force is applied to a solid body made of any one of a great variety of materials, the shape of the body is altered in a nonpermanent way. When the force is removed, the body returns to its original shape, as in the example of a coil spring. In general, the body deforms until the restoring force exerted by the body is equal and opposite to the applied force. Experiments conducted by Robert Hooke (1635–1703) showed that the nonpermanent deformation of many bodies was directly proportional to the force which created the deformation. The name *Hooke's law* is applied to all cases where *the restoring force exerted by a deformed object is proportional to the deformation.* Many materials follow Hooke's law over a limited range of deformation, and one finds that, over this range of applicability of Hooke's law, an object returns to its initial configuration without permanent deformation when the applied force is released. Such materials are called *elastic.* In neither sense of the uses to which we have put the word "elastic" is rubber a perfectly elastic material, for the deformation of a rubber band is not proportional to the applied force, nor is the impact between a rubber ball and an anvil a collision in which mechanical energy is conserved.

We have already seen that a helical spring stretches in proportion to the applied force, but we need not restrict ourselves to linear motion. In a clock spring, or a rod clamped at one end, the angular displacement is proportional to the applied torque tending to twist the rod or the spring.

To describe the elastic restoring force exerted by a stretched spring or by a beam which is deflected from its equilibrium position, we may write

$$\boxed{\mathbf{F} = -k\mathbf{x},} \tag{12-1}$$

which states that the force $\mathbf{F}$ exerted by the spring or the beam is proportional to the displacement $\mathbf{x}$ and is in a direction opposite to that displacement, hence the minus sign. The spring constant k is a constant of proportionality which has the dimensions of force per unit length.

To describe the elastic restoring torque of a clock spring or a twisted shaft, we may write

$$\boxed{\mathbf{G} = -\kappa\boldsymbol{\theta},} \tag{12-2}$$

where **G** is the restoring torque exerted by the spring or shaft when it has been twisted from its equilibrium position through an angle $\boldsymbol{\theta}$. We represent $\boldsymbol{\theta}$ as a vector quantity in this case, for the axis of rotation is fixed in space. The torque exerted by the spring is opposite in direction to the displacement. The constant κ (kappa) is a constant of proportionality which has the dimensions of torque per unit angular displacement.

When an object obeying Hooke's law, as represented in Equations (12-1) and (12-2), *is displaced from its equilibrium position and released, the subsequent motion is simple harmonic.* An elastic system in which the restoring force is directly proportional to the displacement is said to obey Hooke's law. The forces exerted by the materials from which engineering structures are constructed are elastic in character. The forces exerted by adjacent atoms in a molecule may be approximated by Hooke's law. The study of simple harmonic motion is thus of considerable importance in engineering as well as in the physics of atoms and molecules.

12-3 Equations of Simple Harmonic Motion

Let us consider simple harmonic motion along the x axis. In the previous paragraph we have indicated that motion in which the restoring force was proportional to the displacement was simple harmonic; that is, the motion could be described in terms of sine or cosine functions. If a particle of mass m is subject to an elastic restoring force, we may write, from Newton's second law and from Equation (12-1),

$$ma = -kx,$$

and, writing for a its value using the notation of the calculus, $a = \dfrac{d^2x}{dt^2}$,

we have
$$m\frac{d^2x}{dt^2} = -kx.$$

Transposing, we find

$$m\frac{d^2x}{dt^2} + kx = 0.$$

Let us divide the equation by m and set

$$\frac{k}{m} = \omega^2, \tag{12-3}$$

so that the equation becomes

$$\frac{d^2x}{dt^2} + \omega^2 x = 0. \tag{12-4}$$

Equation (12-4) is a second-order differential equation, for it involves not only the variable x but also its second derivative with respect to the time, $\frac{d^2x}{dt^2}$. Such equations are commonly encountered in physics and engineering, and their solutions are often quite complex. For Equation (12-4) the solution is straightforward. First we observe that the equation contains a second derivative and will have to be integrated twice to find a solution for x as a function of time. The solution will therefore contain two constants of integration which will have to be evaluated in terms of the initial conditions of the problem. Knowing these facts in advance, we propose as a trial solution

$$x = A \cos(\omega t + \phi). \tag{12-5}$$

Remembering that $d(\cos x) = -\sin x\, dx$, and that $d(\sin x) = \cos x\, dx$, we find that

$$v = \frac{dx}{dt} = -A\omega \sin(\omega t + \phi) \tag{12-6}$$

and,

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = -A\omega^2 \cos(\omega t + \phi), \tag{12-7}$$

from which

$$a = -\omega^2 x. \tag{12-8}$$

The constants A and ϕ are the two constants of integration. The constant A is the *amplitude* of the simple harmonic motion; that is, it represents the largest value the displacement x can attain. The constant ϕ is called the *phase angle;* that is, the value of x when t is zero is given by $A \cos \phi$.

On substituting the values of x and $\frac{d^2x}{dt^2}$ from Equations (12-5) and (12-7) into the Equation (12-4), we find

$$-A\omega^2 \cos(\omega t + \phi) + \omega^2 A \cos(\omega t + \phi) = 0,$$

$$0 = 0,$$

so that Equation (12-5) is a correct solution to the differential Equation (12-4).

The period T for one complete oscillation is given by

$$T = \frac{2\pi}{\omega}, \tag{12-9}$$

for if the time is increased by T, the angle at which the cosine is to be evaluated is increased from $[\omega t + \phi]$ to $\left[\omega\left(t + \frac{2\pi}{\omega}\right) + \phi\right]$, an increase of 2π, which is the angular interval in which the sine or cosine repeats itself.

The number of oscillations per second is called the *frequency* f, which is given by

$$f = \frac{1}{T}, \tag{12-10}$$

and, substituting from Equation (12-9), we find

$$\omega = 2\pi f, \tag{12-11}$$

which is exactly the same as the relationship between angular velocity and the frequency for uniform circular motion.

From Equation (12-8) we note that the acceleration of a particle in simple harmonic motion is proportional to the displacement but is always in the opposite direction.

Illustrative Example. A particle attached to a spring has a frequency of 4 vibrations per second (abbreviated vib/sec) and an amplitude of 6 cm. Determine (a) the period of the vibration, (b) the maximum velocity of the particle, and (c) the maximum acceleration of the particle.

(a) From Equation (12-10) we have

$$T = \frac{1}{f} = \frac{1}{4\text{ vib/sec}}$$
$$= \frac{1}{4}\,\frac{\text{sec}}{\text{vib}},$$

and since vibration is a dimensionless quantity, a pure number,

$$T = \tfrac{1}{4}\text{ sec.}$$

(b) The maximum velocity may be obtained from Equation (12-6) by observing that the largest numerical value of v occurs when $\cos(\omega t + \phi) = 1$. Thus we have

$$v_{\max} = A\omega.$$

From Equation (12-11)

$$\omega = 2\pi f = 8\pi\,\frac{\text{radians}}{\text{sec}}.$$

Thus, substituting for A its value of 6 cm, we find

$$v_{\max} = 6\text{ cm} \times 8\pi\text{ sec}^{-1}$$
$$= 48\pi\,\frac{\text{cm}}{\text{sec}}.$$

(c) The greatest numerical value of the acceleration is

$$a_{\max} = A\omega^2$$
$$= 6\text{ cm} \times 64\pi^2\text{ sec}^{-2}$$
$$= 384\pi^2\,\frac{\text{cm}}{\text{sec}^2}.$$

If, instead of the cosine function a sine function had been used as a trial solution, we would have found that the sine function would have proved equally satisfactory. Formally there is no essential difference between a sine-function solution and a cosine-function solution; one can be changed into the other by simply adding $+90°$ or $-90°$ to the angle, for

$$\sin(\theta + 90°) = \cos\theta,$$

and

$$\cos(\theta - 90°) = \sin\theta.$$

While a description of simple harmonic motion has been achieved in the present section through the use of the calculus, another way to gain insight into simple harmonic motion is to compare it to uniform motion in a circle, called the *reference circle*.

12-4 The Reference Circle

When an object moves in uniform circular motion, its projection onto the x or y axis moves in simple harmonic motion. The projected position corresponds to the position of a particle moving in simple harmonic motion; the projected acceleration vector corresponds to the acceleration of a particle moving in simple harmonic motion.

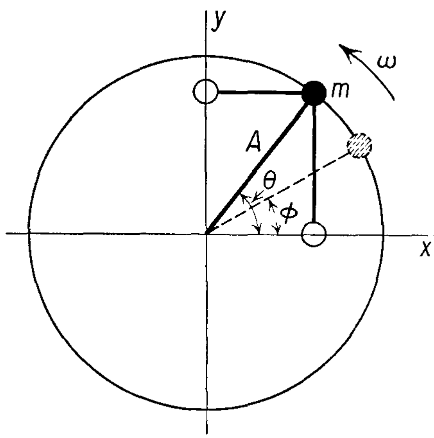

Fig. 12-3 The reference circle. A is the radius of the circle.

Referring to Figure 12-3, let us suppose that at time $t = 0$, an object in uniform circular motion (solid circle) is located at a position on the circle of radius A given by the angle ϕ, and that it rotates in the counterclockwise direction with uniform angular speed ω. The angular position of the object at a subsequent time t is given by the angle θ such that $\theta = \omega t + \phi$. The x coordinate of the object is

$$x = A\cos(\omega t + \phi), \tag{12-12}$$

while the y coordinate is given by

$$y = A\sin(\omega t + \phi), \tag{12-13}$$

but these are also the coordinates of the projection of the circular motion onto the x and y axes, respectively.

Comparing Equations (12-12) and (12-5), we see that the projection of the motion in the reference circle onto the x axis is precisely the same as the simple harmonic motion, provided that the radius of the reference circle is made equal to the amplitude of the simple harmonic motion, and the rotational speed of the object in the reference circle is made equal to the *angular frequency* ω of the simple harmonic motion. The reference-circle analogue clarifies the meaning of the phase angle ϕ in the simple harmonic motion.

We see also that either a sine function or a cosine function is suitable for the description of simple harmonic motion. The projected motion along the y axis is of the same frequency and amplitude as that along the x axis,

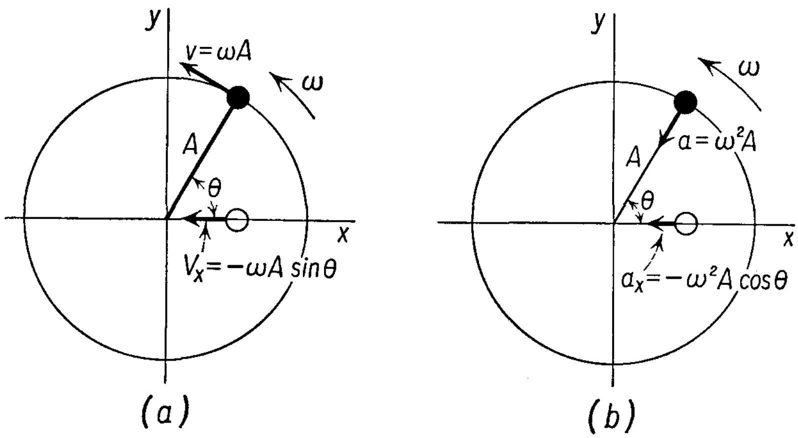

Fig. 12-4 Use of the reference circle to determine (a) the velocity of the particle (white) in simple harmonic motion, and (b) the acceleration of this particle.

but the two motions are 90° out of phase with each other. When the x projection is at the origin, the y projection has its maximum value A.

To make the case for the reference circle complete, let us find the velocity and acceleration of the projected particle, shown as a white circle in Figure 12-4. The black particle moving in uniform circular motion with angular speed ω has a linear speed $v = \omega A$ directed tangentially. The projection of this vector onto the x axis is the velocity of the white projected particle moving in simple harmonic motion in the x direction. From the figure

$$v_x = -\omega A \sin\theta = -A\omega \sin(\omega t + \phi),$$

which is identical with Equation (12-6). An object moving in uniform circular motion must experience a centripetal acceleration $a = \omega^2 A$. From Figure 12-4(b) the acceleration of the projected (white) particle is the projection of the acceleration of the real (black) particle onto the x axis.

Thus we have

$$a_x = -\omega^2 A \cos\theta = -A\omega^2 \cos(\omega t + \phi),$$

which is identical with Equation (12-7).

Remembering that $A \cos\theta = A \cos(\omega t + \phi) = x$, and substituting in the above equation, we find that the acceleration of the white particle on the x axis is

$$a = -\omega^2 x, \tag{12-14}$$

which is identical with Equation (12-8).

The basic condition for simple harmonic motion was that the restoring force was proportional to the displacement, or that $F = -kx$. Substituting for F from Newton's second law, we found that for a particle of mass m subject to an elastic restoring force,

$$ma = -kx,$$

or

$$a = -\frac{k}{m}x, \tag{12-15}$$

and again we see that the acceleration of the projected particle on the x axis, given by Equation (12-14), is identical with that experienced by a real particle which is subject to an elastic restoring force as given in Equation (12-15), provided that,

$$\omega^2 = \frac{k}{m},$$

which is identical with Equation (12-3). The device of the reference circle is a very useful method for solving problems in simple harmonic motion at the level of this text. In using the reference circle, it must be remembered that the radius of the reference circle is equal to the amplitude of the simple harmonic motion it is chosen to represent; the frequency of the rotational motion in the reference circle must be equal to the frequency of the simple harmonic motion; and the phase angle ϕ must be chosen so that the projected particle is at the proper location at time $t = 0$.

Illustrative Example. A particle attached to a spring has a frequency of 4 vib/sec and an amplitude of 6 cm. Determine (a) the period of the vibration, (b) the maximum velocity of the particle, (c) the velocity of the particle when its displacement is 2 cm, (d) the acceleration of the particle when its displacement is 2 cm, (e) the maximum acceleration of the particle, and (f) the time required by the particle to move from a displacement of +2 cm to a displacement of +4 cm.

(a) The period of the vibration is the reciprocal of the frequency of vibration. Hence

$$T = \frac{1}{f} = \frac{1}{4}\text{ sec.}$$

To solve the remainder of the problem, we make use of the reference circle. Let us suppose that the simple harmonic motion takes place along the x axis, and let us construct a reference circle of radius equal to the amplitude of the simple harmonic motion; thus the radius of the reference circle is 6 cm. The real particle in simple harmonic motion will be referred to as the white particle, while

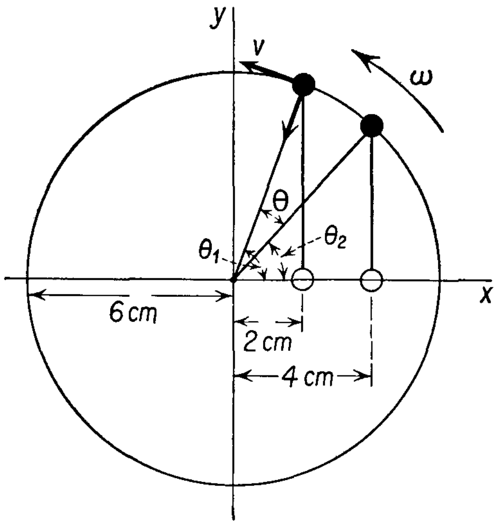

Fig. 12-5

the imagined particle moving in the reference circle will be referred to as the black particle, corresponding to the manner in which they have been drawn in Figure 12-5. The frequency with which both particles move is given by the frequency of the simple harmonic motion which is 4 vib/sec. This implies that the angular speed of the black particle is $\omega = 2\pi f = 8\pi$ radians/sec, which, for convenience, we imagine to be in the counterclockwise direction.

(b) The black particle moves with constant angular speed and therefore with constant linear speed, but its velocity vector is constantly changing in direction. Since the speed of the white particle is the projection onto the x axis of the velocity vector of the black particle, the white particle will move with greatest speed when the black particle is moving parallel to the x axis. This occurs at the two positions where the circle intersects the y axis, that is, when the white particle passes through its equilibrium position. At this position the speeds of the two particles are identical, so that the maximum speed of the white particle is equal to the constant linear speed of the black particle. Thus

$$v = \omega A = 8\pi \times 6 \frac{\text{cm}}{\text{sec}} = 48\pi \frac{\text{cm}}{\text{sec}},$$

$$v = 151 \frac{\text{cm}}{\text{sec}}.$$

(c) When the white particle is displaced 2 cm from the equilibrium position, the position angle of the black particle is θ_1 such that

$$\theta_1 = \text{arc cos}\left(\tfrac{2}{6}\right) = 1.23 \text{ radians.}$$

To find the velocity of the white particle, we must find the x component of the velocity of the black particle, which is given by

$$\begin{aligned} v &= v \sin \theta_1 \\ &= 1.51 \times 0.942 \frac{\text{cm}}{\text{sec}} \\ &= 142 \frac{\text{cm}}{\text{sec}} \cdot \end{aligned}$$

(d) The acceleration of the white particle is the x component of the acceleration of the black particle and is therefore given by

$$a_x = -a \cos \theta_1,$$

where the minus sign indicates the direction of the acceleration vector. For the black particle in circular motion,

$$\begin{aligned} a &= \omega^2 A = (8\pi)^2 \times 6 \frac{\text{cm}}{\text{sec}^2} \\ &= 3{,}789 \frac{\text{cm}}{\text{sec}^2}, \end{aligned}$$

so that

$$\begin{aligned} a_x &= -3{,}789 \times \frac{2}{6} \frac{\text{cm}}{\text{sec}^2} \\ &= -1{,}263 \frac{\text{cm}}{\text{sec}^2} \cdot \end{aligned}$$

(e) The maximum acceleration of the white particle will occur when the centripetal acceleration of the black particle is in the direction of the x axis; that is, where the circle intersects the x axis. At these points the acceleration of the two particles is identical. Thus the maximum acceleration of the white particle in simple harmonic motion is

$$a = 3{,}789 \frac{\text{cm}}{\text{sec}^2},$$

occurring at the position of maximum displacement.

(f) When the white particle is at a displacement of 2 cm, we have already seen that the black particle is located at angle θ_1. When the displacement of the white particle is 4 cm, the black particle is located at angle θ_2. The angular displacement of the black particle in rotating from θ_2 to θ_1 is Θ (capital theta).

$$\Theta = \theta_1 - \theta_2.$$

We have already seen that $\theta_1 = 1.23$ radians, while

$$\theta_2 = \text{arc cos}\left(\tfrac{4}{6}\right) = 0.84 \text{ radian.}$$

Thus

$$\begin{aligned} \Theta &= 1.23 \text{ radians} - 0.84 \text{ radian} \\ &= 0.39 \text{ radian.} \end{aligned}$$

Knowing that ω, the angular speed of the black particle, is 8π radians/sec, we find the time for the black particle to travel from the first to the second position as

$$t = \frac{\Theta}{\omega}$$

$$= \frac{0.39 \text{ radian}}{8\pi \text{ radians/sec}},$$

so that

$$t = 0.016 \text{ sec}.$$

This problem may also be solved by the substitution of appropriate numbers for A, ω, and ϕ in Equations (12-5), (12-6), and (12-7), as in a previous example. From the statement of the problem, $A = 6$ cm, $\omega = 2\pi f = 8\pi \text{ sec}^{-1}$, and if we wish to have the initial position of the particle at the undisplaced or equilibrium position when $t = 0$, we would set $\phi = \pi/2$ radians. The maximum values of the velocity and acceleration are given by $v = A\omega$ and $a = A\omega^2$. The value of the velocity and acceleration when the displacement was 2 cm could be found by substituting $x = 2$ cm in Equation (12-5), solving for t, and substituting that value of the t in Equations (12-6) and (12-7). To find the time at which the displacement is 4 cm, we set $x = 4$ cm in Equation (12-5), and to find the elapsed time in traveling between a displacement of 2 cm and a displacement of 4 cm, we would subtract the two times. The geometric procedure using the reference circle and the procedure involving the use of the formulas derived by mathematical analysis yield equivalent results. Note that in the analytic treatment and in the reference circle the particle in simple harmonic motion could be thought to move along either the x or the y axis, at our convenience.

12-5 Angular Harmonic Motion

A system capable of rotating about a fixed axis will move with angular harmonic motion when the torque G which acts on it is proportional to its angular displacement θ and opposite in direction to it, as given by

$$G = -\kappa\theta. \tag{12-2}$$

If I is its moment of inertia about this axis, we can write

$$G = I\alpha,$$

where α is its angular acceleration. Solving the above equations for α, we get

$$\alpha = -\frac{\kappa}{I}\theta, \tag{12-16}$$

which shows that the angular acceleration α is proportional to the angular displacement and opposite in direction to it. Equation (12-16) can be compared with Equation (12-8) for linear simple harmonic motion; Equation (12-16) is its analogue and is the equation for *angular harmonic motion.*

The period for angular harmonic motion is then given by

$$T = 2\pi\sqrt{\frac{I}{\kappa}} \tag{12-17}$$

for the period of a system of moment of inertia I and restoring constant κ moving with angular harmonic motion about a fixed axis.

The equations for the angular displacement and the angular velocity as functions of the time can be obtained in exactly the same manner as those for the linear displacement and the linear velocity, and will be their respective analogues.

12-6 The Pendulum

The motion of a pendulum is another example of periodic motion. When the amplitude of oscillation of a pendulum is small, the pendulum motion may be approximated as simple harmonic. This property was discovered by Galileo, and was first applied to the construction of a clock by the Dutch physicist Christian Huygens (1629–1695) in 1657.

The pendulum appears in two forms—the *simple pendulum* consisting of a string of negligible weight, one end of which is attached to some fixed support while the other end is attached to a small ball called a *pendulum bob*, and the *physical pendulum*, a rigid body which is supported at some point above its center of gravity. When at rest, the bob is vertically beneath the point of support, as shown in Figure 12-6, and is in equilibrium under the action of two forces, its weight mg and the tension S in the string. When pulled aside to some position A and released, it travels in a circular arc through its equilibrium position C to a point B on the other side. When the pendulum was moved to A, it was actually lifted through a height h. From the principle of conservation of energy, the points A and B must be at equal heights h above the point C. In the absence of frictional forces, the motion would continue indefinitely, but of course, no device can be built which completely eliminates frictional forces, hence the amplitude gradually diminishes as mechanical energy is converted to other forms of energy.

To derive an expression for the period of the simple pendulum, we observe that the torque on the simple pendulum which tends to rotate it about the point of support is due to the weight of the pendulum bob and is given by

$$G = -mgL\sin\theta. \tag{12-18}$$

Comparing the torque on a simple pendulum with the condition for an elastic restoring torque in Equation (12-2), we see that the torque acting on a simple pendulum is proportional not to the angular displacement but to the sine of the angular displacement, so that the motion is not simple harmonic. But if the displacement is restricted to small angles, the sine of θ is practically equal to θ, and we may write

$$G = -mgL\theta. \tag{12-19}$$

For small displacements from the equilibrium position, the pendulum moves under the influence of restoring torque proportional to the angular dis-

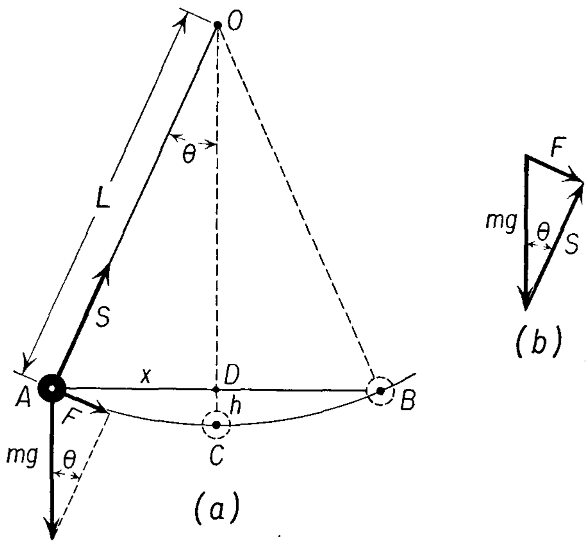

Fig. 12-6 The simple pendulum.

placement, and its motion is simple harmonic. Considering the pendulum as a rigid body, of moment of inertia $I = mL^2$, we may write

$$G = I\alpha,$$

and, eliminating G from the above equations, we find

$$\alpha = -\frac{mgL}{I}\theta. \tag{12-20}$$

By combining Equation (12-20) with Equation (12-16), we can see that the period is

$$T = 2\pi\sqrt{\frac{I}{mgL}}, \tag{12-21}$$

and, substituting for I its value mL^2, we find that

$$T = 2\pi\sqrt{\frac{L}{g}}. \tag{12-22}$$

Thus the period of a simple pendulum does not depend upon the mass of the pendulum bob but only upon its length and the value of g. The simple pendulum therefore provides one method of determining the value of g.

In Figure 12-7 a physical pendulum has been drawn in which the point O is the point of support and the point C is the position of the center of gravity. The distance between C and O is represented by h. If the mass of the physical pendulum is m and its moment of inertia about an axis through O is I, the equations for the torque and the angular acceleration are identical with those for the simple pendulum except that h replaces L. The period of a physical pendulum is then given by

$$T = 2\pi \sqrt{\frac{I}{mgh}}. \qquad (12\text{-}23)$$

Comparing Equations (12-22) and (12-23), we find that a physical pendulum will have the same period as a simple pendulum of length L when

$$L = \frac{I}{mh}. \qquad (12\text{-}24)$$

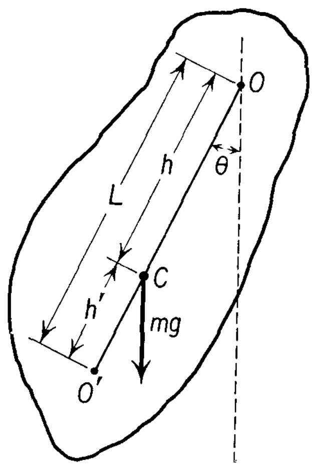

Fig. 12-7 The physical pendulum. The distances h and h' are measured from C to O and O' respectively. $L = h + h'$ is the equivalent simple pendulum length.

L is called the *equivalent simple pendulum length* of the physical pendulum and is shown in Figure 12-7 as the distance between O and a point O' below the center of gravity. The point O' is called the *center of oscillation* and has the interesting property that, if the pendulum is suspended about O' as an axis, its period will be exactly the same as it is when suspended about O as an axis. An accurate determination of the locations of the two points O and O' makes possible an accurate determination of the equivalent simple pendulum length L and is one of the best methods for determining g.

The physical pendulum is often used for pendulum clocks. In order for such a pendulum clock to keep accurate time, it is necessary that the pendulum be so constructed that the moment of inertia and the length L are constant and remain the same in spite of temperature variations. Such pendulums are said to be *compensated*.

It is appropriate to note here that a clock or watch consists of two separate mechanisms. One of these beats off equal time intervals, and the other counts them and moves the hands. In a pendulum clock a physical pendulum is the means of measuring equal time intervals, while in a watch or chronometer the torsional vibrations of the balance wheel perform this necessary function. The pendulum clock is adjusted to keep correct time by altering both h and I, while in the watch the coarse adjustment is made by varying the moment of inertia of the balance wheel, and the fine adjustment is made by altering the torsion constant κ of the watch spring. In both cases the primary characteristic of simple harmonic motion that is essential to the measurement of time is the fact that the period does not depend on the amplitude of vibration. Such motion is called *isochronous*, meaning that the oscillation requires the same time for all possible amplitudes.

12-7 Resonance

The frequency with which a pendulum or an elastic object vibrates when it is displaced from its equilibrium position and released is called the *natural frequency* of the system. Thus a piano string or a diving board will vibrate with its natural frequency after being displaced from an

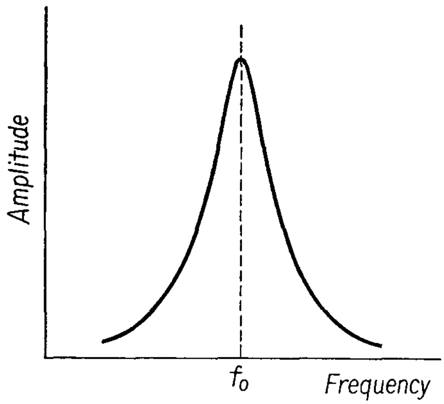

Fig. 12-8 The amplitude of vibration of an oscillator of natural frequency f_0 depends upon the frequency f with which it is driven. At resonance, $f = f_0$, and the oscillator vibrates with maximum amplitude.

equilibrium position. If such a mechanical oscillator is not simply struck a blow, or displaced from an equilibrium position and then released, but is driven by a force which varies periodically, it is required to move with the frequency of the driving force. The amplitude with which the oscillator will vibrate under the influence of such a force will depend markedly upon the driving frequency. When the frequency with which the oscillator is driven is far from its natural frequency, the amplitude of vibration will be quite small, but when the driving frequency is the same as the natural frequency of the oscillator, it will be excited to large amplitudes, as shown in Figure 12-8. This condition is known as *resonance*. At resonance the amplitude of the oscillations will depend upon the degree of *damping* in the oscillator itself, that is, on the amount of internal friction in the oscillator.

Although we shall not undertake the analysis of resonant motion, we shall mention some examples of resonance of importance in engineering. Any elastic structure which is deflected to large amplitudes by a periodic driving force is thereby exposed to large alternating stresses and is likely to fail at far smaller loads than the same structure could safely withstand under static loading, through a process called *fatigue failure.*

As an illustration, consider an airplane engine that is designed to operate at N rev/sec. Because it is impossible to build a perfectly balanced engine, the rotation of the engine acts as a driving force which drives the

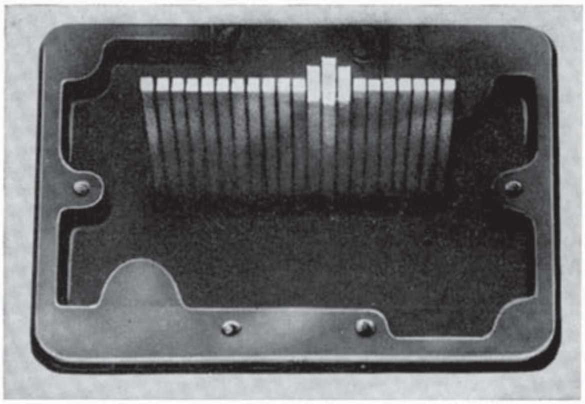

Fig. 12-9 Reed comb of a Frahm tachometer. The reeds are tuned in intervals of 50 vib/min from 3,000 at the left to 4,000 at the right. The assemblage was in contact with the case of a motor rotating at 3,600 rpm. (Courtesy of James G. Biddle Co.)

entire airplane at a frequency of N vib/sec. Any part of the airplane whose natural frequency is close to N will be excited to large amplitudes. If, by inadvertence, the natural frequency of the wing, or the propeller, were the same as that of the engine, failure of these parts would occur in a much shorter time than might otherwise be the case. In such circumstances a part can often be made stronger, that is, can be made to last longer, by removing material from it in such a way as to alter its natural frequency.

The phenomenon of resonance can be used in the measurement of the rotational frequency of a motor. Figure 12-9 shows a reed tachometer consisting of a comb of "harmonica" reeds which have been adjusted so that their natural frequencies are close to each other. If this *tachometer* is placed on the floor or table adjacent to a motor, some reeds will be driven by the vibration of the motor; the natural frequency of the reed driven to largest amplitude will measure the frequency of rotation of the motor. Such a tachometer may be mounted almost anywhere on board a single-engine airplane to indicate the engine speed in revolutions per minute.

Problems

12-1. A body whose mass is 500 gm hangs from a vertical spring whose constant is 200,000 dynes/cm. The body is pulled down a distance of 6 cm and released. Determine (a) the period of the motion, (b) the resultant force on the body when at the 6-cm point, and (c) the acceleration at this position.

12-2. In Problem 12-1 determine (a) the velocity of the body, when its displacement is 3 cm, and (b) its maximum velocity. Obtain your answers by considerations of simple harmonic motion and also by applying the principle of conservation of energy.

12-3. A body which has a mass of 60 gm is attached to a helical spring 25 cm long and, when lowered gently, stretches the spring 5 cm. The body is then pulled down an additional 8 cm and released, thus setting the spring in vibration. (a) What is the constant of the spring? (b) What is the period of oscillation?

12-4. A body which has a mass of 40 gm is attached to a spring, and the system is then set into vibration. The measured value of the period of vibration is 0.50 sec. (a) Determine the constant of the spring. (b) Determine the velocity of the body at the equilibrium position if the amplitude is 6 cm. (c) Determine its maximum acceleration.

12-5. When a cylinder whose mass is 4.0 kg is hung from a spring and set into motion, the frequency is 2.4 vib/sec. When another cylinder is substituted for the first one, the frequency of vibration is 3.2 vib/sec. Determine the mass of the second cylinder.

12-6. Determine the period of a simple pendulum, oscillating with small amplitude, when the length of the pendulum is 75 cm.

12-7. A simple pendulum 1.0 m long, having a mass of 250 gm, is displaced through an angle of 10° and released. Determine (a) the resultant force acting on the pendulum bob at this position of maximum displacement, (b) its maximum angular acceleration, (c) its maximum angular velocity, (d) the tension in the pendulum string when the displacement is 5°, and (e) the velocity and acceleration of the pendulum bob when the displacement is 5°.

12-8. (a) Determine the period of vibration of a pendulum 80 cm long at a place where $g = 980$ cm/sec^2. (b) What length of pendulum at the same place will have half this period?

12-9. A "seconds" pendulum has a period of 2 sec. A seconds pendulum which kept accurate time at a place where g was 980 cm/sec^2 is found to lose 2 min/day at a new location. Find g at this new location.

12-10. An object moves in simple harmonic motion with period of 4 sec and amplitude 1 m. (a) What is the frequency of the motion? (b) What is the velocity when the displacement is 30 cm? (c) What is the acceleration when the displacement is -60 cm? (d) How long a time is required for the object to move from the point where the displacement is 30 cm to the point where the displacement is -60 cm?

12-11. An object moves in rotational simple harmonic motion with period 4 sec and amplitude 1 radian. (a) What is the frequency of the motion? (b) What is the angular velocity when the angular displacement is 0.30 radian? (c) What is the angular acceleration when the angular displacement is -0.60 radian? (d) How long a time is required for the object to move from an angular displacement of 0.30 radian to a point where the angular displacement is -0.60 radian?

12-12. Two simple harmonic oscillators have the same frequency of 4 cycles/sec and the same amplitude of 10 cm. At a particular time it is observed that the first oscillator is moving to the right and the second oscillator is moving to the left along the same line when both oscillators have the same displacement of 6 cm. (a) How far were they apart 0.25 sec later? (b) How far were they apart 0.10 sec after the initial observation?

12-13. Two simple harmonic oscillators, having the same amplitude of 10 cm in the y direction, are displaced so the first has a positive initial displacement of 10 cm, while the second has a negative initial displacement of 10 cm. The first oscillator has a frequency of 8 cycles/sec, while the second has a frequency of 4 cycles/sec. The oscillators are released at the same instant. Find the time at which both oscillators are at the same position.

12-14. A spring of spring constant 50 lb/ft has one end fastened to the wall and the other end fastened to a rectangular block of wood weighing 32 lb. The spring is horizontal, and the block rests on a horizontal table. The coefficient of friction between the block and the table is 0.1. The block is initially displaced so that the spring is stretched 1 ft from its equilibrium position and is released. What will be the speed of the block when it passes through the equilibrium position? [NOTE: The motion is not simple harmonic, for there is loss of mechanical energy through friction. Solve from energy considerations.]

12-15. A meter stick of mass 400 gm is pivoted 30 cm from one end and is allowed to oscillate as a physical pendulum. The width of the meter stick is 2 cm, and its thickness is $\frac{1}{2}$ cm. Find the period of the oscillation.

12-16. A circular disk 1 cm thick and of radius 20 cm, having a mass of 4 kg, is suspended from a pivot 2 cm from the edge of the disk and is allowed to oscillate while it hangs downward. Find the period of the disk for small oscillations.

12-17. A particle moves in simple harmonic motion in the x direction with amplitude 10 cm and frequency 5 cycles/sec. The particle is moving in the positive x direction at a displacement of $x = +5$ cm when $t = 0$. Find the position of the particle when $t = 0.02$ sec.

12-18. Two identical springs are laid side by side on a horizontal frictionless table, each having a spring constant of 1 lb/in. A sphere of mass 0.1 slug is connected to the free end of one spring, while a second sphere of mass 0.2 slug is connected to the free end of the second spring. Both masses are drawn aside a distance of 6 in. and released. What is the displacement of the sphere of mass 0.2 slug when the displacement of the other sphere is -3 in.?

12-19. Find the period of vibration of a cylinder of radius r, height h, and density ρ which is floating upright, partially immersed in a fluid of density ρ_0.

12-20. A wire is bent in the form of the arc of a circle of radius R, and is mounted so that it is in the vertical plane. A bead is placed upon the wire and released. In the absence of friction between the bead and the wire, show that the bead will oscillate with period

$$T = 2\pi(R/g)^{1/2}$$

provided that its initial displacement is sufficiently small.