2019

MATH 314: Linear Algebra-A Peer Review of Teaching Project Benchmark Portfolio

Xavier Pérez-Giménez
University of Nebraska-Lincoln, xperez@unl.edu

Follow this and additional works at: https://digitalcommons.unl.edu/prtunl
Part of the Higher Education Commons, and the Higher Education and Teaching Commons

Pérez-Giménez, Xavier, "MATH 314: Linear Algebra-A Peer Review of Teaching Project Benchmark Portfolio" (2019). UNL Faculty Course Portfolios. 145.
https://digitalcommons.unl.edu/prtunl/145

This Portfolio is brought to you for free and open access by the Peer Review of Teaching Project at DigitalCommons@University of Nebraska - Lincoln. It has been accepted for inclusion in UNL Faculty Course Portfolios by an authorized administrator of DigitalCommons@University of Nebraska - Lincoln.
MATH 314: Linear Algebra—A Peer Review of Teaching Project Benchmark Portfolio

Xavier Pérez-Giménez

June 4, 2019

Contents

1 Introduction and Objectives of Course Portfolio 3

2 Benchmark Memo 1: Reflections on the Course Syllabus 3
   2.1 Description of the course ......................................................... 3
      2.1.1 Purpose ................................................................. 4
      2.1.2 Prerequisites .......................................................... 4
      2.1.3 Students .................................................................. 4
      2.1.4 The course and the broader curriculum ......................... 5
   2.2 Objectives of the course ......................................................... 6
      2.2.1 Knowledge ............................................................... 6
      2.2.2 Abilities ................................................................. 6
      2.2.3 Conceptual Understanding ............................................. 7
      2.2.4 Perspectives and Attitudes ............................................ 7
      2.2.5 Impact on the larger society ........................................... 7

3 Benchmark Memo 2: Description of Course Activities 8
   3.1 Teaching Methods and Classroom Time .................................. 8
      3.1.1 How Are The Methods Used in Class? ............................... 8
      3.1.2 Course Activities Outside of Class ................................. 8
   3.2 How Do The Methods Facilitate Course Goals? ....................... 9
   3.3 Course Materials ................................................................. 9

4 Benchmark Memo 3: Documentation and Analysis of Student Learning 10
   4.1 Evidence of students meeting learning goals .......................... 10
      4.1.1 Knowledge and Abilities .............................................. 10
      4.1.2 Conceptual Understanding .......................................... 10
   4.2 Quantitative analysis of student performance ........................ 11
      4.2.1 Student time dedication .............................................. 12
      4.2.2 Impact of Quizzes and Homework on Student Learning .... 15

5 Planned Changes and Assessment of the Portfolio Process 23

1
6 Appendix

6.1 Course Syllabus .................................................... 23
6.2 Sample Exams ...................................................... 23
6.3 Sample Quizzes ..................................................... 23
6.4 Additional Handouts .............................................. 23
6.5 Sample Work from Students ................................. 23
1 Introduction and Objectives of Course Portfolio

I enrolled in the Peer-Review-of-Teaching-Project as an opportunity to reflect about my pedagogical goals and methods and learn from the experience of other faculty members. Writing this portfolio will allow me to examine how my teaching goals align with my course activities and document the impact that those have on student learning. I hope this helps me make more informed instructional choices in the future, and enhance the learning outcomes of my students.

My target course is Math 314 (Linear Algebra). This course is a core requirement in our mathematics major and also a prerequisite for many courses in multiple STEM disciplines. As a result, its typical enrollment includes students with different backgrounds and learning goals. For some of these students, this course is the first and only time that they will be exposed to mathematical proofs, while for some others this course serves as a preparation for more advanced mathematical courses.

2 Benchmark Memo 1: Reflections on the Course Syllabus

2.1 Description of the course

<table>
<thead>
<tr>
<th>UNL Course Bulletin Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fundamental concepts of linear algebra, including properties of matrix arithmetic, systems of linear equations, vector spaces, inner products, determinants, eigenvalues and eigenvectors, and diagonalization.</td>
</tr>
</tbody>
</table>

One can say that Linear Algebra is a mathematical branch studying system of linear equations, for example

\[
\begin{align*}
4x + 5y + 6z &= 7 \\
2x + 6y + z &= 2 \\
x + y + z &= 4
\end{align*}
\]

Linear Algebra has many applications to diverse scientific areas, and even has powerful uses in computer graphics, linguistics, audio engineering, artificial intelligence, humanitarian aide, and legal studies etc.

This course is a transition course from computational courses, like calculus, to more theoretical ones. You will need to understand definitions and theorems, be able to apply them, and sometimes, prove theorems. The material in the course will tend to be more mathematically subtle than that encountered in your previous math courses, and will consequently require a significant effort on your part to master.

The course covers sections from Chapters 1 to 7 of the text (see Section 3.3): systems of linear equations, matrix algebra, determinants, vector spaces, eigenvalues and eigenvectors, orthogonality, and inner product spaces, and quadratic forms.
2.1.1 Purpose

This course serves three major purposes. First: to provide students with a solid theoretical framework and set of tools that are fundamental in mathematics and other STEM fields. Second: to expose students to some abstract mathematical objects and simple proofs in preparation for more advanced courses. Third: to make explicit connections between the course material and real-world practical applications.

2.1.2 Prerequisites

In order to enroll in Math 314, students need to have a grade of P, C, or better in Math 106 (Calculus I) and Math 107 (Calculus II) or their corresponding honors versions. Strictly speaking, the material from Math 106–107 is not relevant to Math 314, but having these calculus courses as a prerequisite ensures that students taking this course have a reasonable level of mathematical maturity.

By contrast, neither Math 208 (Calculus III) nor Math 221 (Differential Equations) are formal prerequisites of Math 314, but some of their content is relevant to this course. For instance, the 3-dimensional graphical representation skills acquired in Math 208 are helpful for visualizing multidimensional linear mappings and quadratic forms. Furthermore, Math 221 introduces several central concepts in linear algebra (such as linear combinations, linear independence and eigenvalues), and applies them to the solution of differential equations. Therefore, I believe that it is useful for students to take Math 208 and 221 before or concurrently with Math 314, and in fact many students choose to do so. (In my opinion, it would also make sense to require Math 314 as a prerequisite for Math 221, since this would allow to cover some topics in Math 221 in a more systematic way, but this is out of the scope of this portfolio.)

2.1.3 Students

The typical students of this course are mostly engineering, mathematics and computer science majors in their junior year (but sophomore and senior level students are also common). For instance, in the Spring 2019 term, there was an enrollment of 31 students, of which 5 were sophomore, 17 were junior and 9 were senior. Furthermore, 14 of these students were engineering majors (with notably 6 mechanical engineers), 8 were majoring in mathematics, 6 in computer science, 3 in physics, 1 in psychology and 1 in graphic design (some of these students were enrolled in dual majors). There were 6 students enrolled in the Honors Program, but none of them took this course for honors credit.

An initial survey showed that 87.1% of the students had completed (or were concurrently taking) Math 208 (Calculus III), and similarly 74.2% of the students had taken or were taking Math 221 (Differential Equations). None of these courses are formal prerequisites of Math 314, but some of their content is relevant to this course (see the earlier discussion in the Prerequisites Section).
2.1.4 The course and the broader curriculum

This course is one of the core requirements of our mathematics major. Moreover, it is a prerequisite for 14 400-level courses in the mathematics program:

- MATH 405 (Discrete and Finite Mathematics for High School Teaching),
- MATH 409 (Math for High School Teachers II, Using Math to Understand Our World),
- MATH 415 (Theory of Linear Transformations),
- MATH 428 (Principles of Operations Research),
- MATH 432 (Linear Optimization),
- MATH 433 (Nonlinear Optimization),
- MATH 435 (Math in the City),
- MATH 441 (Approximation of Functions),
- MATH 442 (Methods of Applied Mathematics I),
- MATH 447 (Numerical Linear Algebra),
- MATH 456 (Differential Geometry I),
- MATH 471 (Introduction to Topology),
- MATH 487 (Probability Theory),
- MATH 489 (Stochastic Processes).

It is also a prerequisite for 6 400-level courses and one 900-level graduate course in the computer science program:

- CSCE 412 (Data Visualization),
- CSCE 432 (High-Performance Processor Architectures),
- CSCE 441 (Approximation of Functions),
- CSCE 447 (Numerical Linear Algebra),
- CSCE 470 (Computer Graphics),
- CSCE 477 (Cryptography and Computer Security),
- CSCE 970 (Pattern Recognition).

Finally, Math 314 is a prerequisite for 3 courses in mechanical engineering, one course in electrical & computer engineering and one course in statistics:
• MECH 350 (Introduction to Dynamics and Control of Engineering Systems),
• MECH 431 (Computational Heat Transfer and Fluid Flow),
• MECH 888 (Nonlinear Optimization in Engineering),
• ECEN 935 (Computational Intelligence),
• STAT 970 (Linear Models).

ACE Student Learning Outcomes: This course is part of UNL’s Achievement-Centered Education (ACE) general education program. In particular, it satisfies ACE Outcome 3: “Use mathematical, computational, statistical, or formal reasoning (including reasoning based on principles of logic) to solve problems, draw inferences, and determine reasonableness.”

Graduate credit: The course can also be taken at the 800-level as Math 814. For this, an extra project or assignment should be designed. No student in the Spring 2019 term took the course for graduate credit.

2.2 Objectives of the course

2.2.1 Knowledge

Students should come away from the course with a working knowledge of the basic concepts and tools in linear algebra. More specifically, they should be familiar with some of the following ideas:

• Systems of equations and the Gaussian elimination algorithm.
• Matrices, vectors and operations.
• General vector spaces, subspaces, linear transformations, bases and coordinates.
• Eigenvalues, eigenvectors and diagonalization of square matrices.
• Orthogonality, projections and applications.
• Symmetric matrices and quadratic forms.

2.2.2 Abilities

Students should be able to solve problems in linear algebra using the tools discussed in class. They should also be able to prove simple mathematical statements. Finally, they should be able to apply linear algebra to model and solve real-world problems.
2.2.3 Conceptual Understanding

One of the main difficulties of this course is that there often exist many equivalent formulations of one same statement in terms of seemingly unrelated algebraic objects. For instance, the fact that a set of vectors is linearly independent can be restated in terms of properties of matrices and systems of equations. Students should be able to establish connections between algebraic objects and identify these different but equivalent formulations.

The course also introduces abstract objects (such as general vector spaces), which are obtained as a generalization of more concrete objects that students are already familiar with. This poses an extra challenge to many students, who are not used to this level of abstraction. By the end of the course, students should grasp the definitions and main properties of these abstract objects.

Finally, while this is not a proof-based course, students are exposed (possibly for the first time) to some simple proofs. Students are expected to understand the proofs covered during the course and become comfortable with the idea of formal arguments.

2.2.4 Perspectives and Attitudes

It is common among students, even mathematics majors, to have a negative attitude towards proofs and mathematical abstraction. In my opinion, this is often due to lack of exposure. This course offers a gentle introduction to some abstract mathematical structures and gives students a glimpse at elementary proofs. This should raise students awareness of the beauty of mathematical formalism, motivate them to look further beyond routine mathematical calculations, and prepare them for more advanced mathematical courses. This course is by no means a rigorous introduction to proofs. This role is played by Math 309, Math 310 or Math 325, which our mathematics undergraduates typically take after Math 314. On the other hand, for some non-mathematics majors, this is a terminal course in mathematics, so their experience in Math 314 may play an important role in their future attitude towards mathematics.

Another common attitude among students (specially those in earlier years) is that learning is a passive process; i.e. it is the instructor’s job to teach them the material during the lectures. This course attempts to change this misconception by moving the focus of the learning process back to the student. Students are required to read sections of textbook independently, and are strongly encouraged to look for alternative resources (on-line or in paper). Moreover, they are expected to solve additional homework problems (that are not necessarily collected or graded), and make sure they are up-to-date with all the ideas and skills relevant to each topic of the course.

2.2.5 Impact on the larger society

Linear algebra is a fundamental subject in mathematics with endless applications in each and every field of science and engineering, ranging from theoretical foundations to real-world technological applications. Therefore, it is of vital importance to educate future professionals in many disciplines with a solid background in linear algebra and the associated problem-solving skills.
3 Benchmark Memo 2: Description of Course Activities

3.1 Teaching Methods and Classroom Time

After some serious consideration, I decided to follow a traditional lecture-based instructional approach for most of the course. Designing an efficient structure for the lectures is crucial, since time is limited (2.5 hours a week in three 50-minute lectures). A variety of methods is employed including lecture, question and answer, weekly quizzes and problem sessions.

3.1.1 How Are The Methods Used in Class?

I start every lecture with a summary of the lecture plan. I often combine this with a brief review of the previous material, and show how the new content is connected to or builds upon it. Moreover, at the beginning of each new topic, I motivate it by discussing potential real-world applications, and encourage students to do some further independent research on the connections between that topic and their own fields of interest.

I make the lecture as participative as possible by asking frequent questions to students, but without pressing them into getting the right answer right away. Sometimes wrong answers stimulate fruitful discussions which may lead to a better conceptual understanding of the material. I try to create an atmosphere in which students feel comfortable to discuss and discover the subject all together. I also noticed that in all these discussions students appreciate being addressed by their names, so I make an effort to learn as many as I can.

Since most students are not very comfortable with abstract definitions and formal proofs, I try to go from concrete to abstract whenever it is possible. For instance, I may start by presenting a numerical example which encapsulates most of the key ideas of a general proof, and then ask students to help me generalize that to the more abstract setting.

As mentioned in Section 2.2.3, frequently in linear algebra one same statement can be formulated in several equivalent ways involving different mathematical objects. It is sometimes hard by students to grasp the connections between these equivalent formulations, so I make a conscious effort to stress that out during the lectures. For instance, whenever I make a new claim, I try to state in as many different ways as possible and ask students to help me with this endeavor.

Once a week there is a 10–15 minute quiz at the end of the lecture. This encourages students to study on a weekly basis and stay up-to-date with the material. Students can check their own notes as they work on the problems.

Occasionally, lectures are replaced by problem or review sessions. During these sessions, I prepare a worksheet with problems that we solve together. I also encourage students ahead of time to prepare and bring their own problems or discussion topics.

3.1.2 Course Activities Outside of Class

Daily homework is a core component of this course. Since grading resources are limited, this term I am experimenting with partial collection of assignments. Students are split into two groups, A and B. During the first half of the term (which, roughly speaking, goes until the
first midterm), only assignments from group A are collected and graded. Students from group B are also supposed to work on the homework problems, but their work is not collected. On the second half of the term (until the second midterm), the roles are reversed and only group B submits their work. Besides reducing the grading work, this allows me to measure whether the obligation of submitting assignments affects students’ attitude towards homework and ultimately has an impact on their learning. Note that all students from both groups have access to the solutions, which I usually post every day after collecting the assignments, and they are encouraged to meet me during office hours to obtain feedback on their work.

In order to promote independent studying skills, students are once in a while required to read some designated sections from the textbook on their own. They are provided with a list of concepts and skills that they are expected to learn from those readings. Then, at the lecture, some time is devoted to clarify questions and make sure that all students have assimilated those ideas.

### 3.2 How Do The Methods Facilitate Course Goals?

All the student abilities described in Section 2.2.2 are discussed and practiced during the lectures. Particular care is taken with proofs or new ideas that require some degree of abstract formalism. In that case, often we start from a more concrete example that captures the essential ideas of the general problem and work our way from there.

Homework assignments and quizzes provide another opportunity for students to practice their skills and test their conceptual understanding of the subject. They contain a variety of types of questions: some problems are purely computational and follow from a routine application of the methods learned in class; some others require a deeper level of conceptual understanding of the subject (True/False questions are particularly suitable for that); some other problems involve proving a simple claim, similar to other statements proved during the lectures; finally, some questions concern real-world applications and require some modeling of the problem by using linear algebraic ingredients.

The independent reading assignments and the homework submission system in two groups A and B reinforce the idea that students should take responsibility for their learning process and that the instructor’s role is mainly to provide guidance and facilitate this process.

### 3.3 Course Materials


This textbook is the standard choice of the department for Math 314. It is well-organized and the material is presented concisely and with clarity. It includes a proof for most of the theorems (even though they are not always easy to grasp by many students), plenty of examples and a good selection of problems. The book comes with additional on-line resources, which expand the topics introduced at the beginning of each chapter, adding real-world applications and ideas for projects.
Study Guide: For each lecture (which roughly corresponds to one section in the textbook), I provide students with a list of key ideas that they need to understand and a set of skills that they need to master.

Additional handouts: During the course I complement the material from the textbook with handouts on specific topics.

Solutions and Sample Exams: At the end of each lecture I provide students with the solutions to the daily assignment and/or quiz. In addition, I share sample exams from previous years (with and without solutions).

The Internet: I encourage students to search for their own on-line resources and take advantage of 21st century learning opportunities.

4 Benchmark Memo 3: Documentation and Analysis of Student Learning

4.1 Evidence of students meeting learning goals

4.1.1 Knowledge and Abilities

Most students developed a satisfactory working knowledge of basic concepts and tools in linear algebra, and were able to use them to solve problems, especially those involving routine computational skills. This is demonstrated by the reasonably high scores obtained in the final exam on questions of this nature. For instance, the median scores for questions 1, 2, 5, 6, 7, 8, 9 and 10 of the final exam are respectively 60%, 75%, 87.5%, 83.3%, 100%, 75%, 90% and 62.5%. (The statements of all these exam questions along with their solutions can be found in Section 6.2.)

Similarly, most students showed an adequate level of competence in real-world applications of linear algebra and modeling problems. For example, the median scores for assignment 25, problem 4 in the first midterm and problem 11 in the final exam are respectively 91.7%, 100% and 75%, which provides evidence for such claim. Moreover, note that the topics covered in the 3 aforementioned questions were not discussed in class in full detail, but assigned to students for independent reading, which demonstrates that students took these readings seriously.

4.1.2 Conceptual Understanding

Unfortunately, when it comes to conceptual understanding of the theoretical underpinnings of the subject, many students did not meet the expected learning goals. A clear example of that is illustrated by the poor median scores obtained in the True/False questions of both the second midterm and the final exam, which were respectively 12.5% and 60%. Note that by answering True or False at random one should expect to get around 50% of the answers right, which is much better than 12.5% (!).
Likewise, many students struggled with proving simple statements and obtained really poor scores on these type of questions consistently throughout the exams. In an attempt to mitigate this, I provided the students with a list of 5 proof-type questions, one of which would appear in the final exam. Even then, the median score for question 3 in the final (which was the one from that list) was only 37.5%.

4.2 Quantitative analysis of student performance

Final course grades were assigned following the rules described in the syllabus with minor adjustments: that is, 35% of the course grade comes from the final exam, $35\% = 2 \times 17.5\%$ from the two midterms and 30% from the homework and quizzes together (where the lowest few scores in the assignments or quizzes are dropped). We refer the reader to Figures 1, 2 and 14–18 for more details on the grades for each of the course components.

Regarding course grades, 11 out of the 31 students were in the A range (with notably 5 A+’s), 11 students were in the B range and 8 students in the C range. One single student obtained a D grade, but that student was warned multiple times that this would be a likely outcome after missing most of the lectures and quizzes. This is a reasonable grade distribution for Math 314, although it would have been of course better to have fewer students in the C range.

![Course Grades](image)

Figure 1: Course grade distribution (Median = 83.26, Q1 = 76.18, Q3 = 90.42).
4.2.1 Student time dedication

Students were asked to fill a brief survey about the time they devoted to each component of the course. The goal was to obtain some data to analyze the correlations between student dedication and performance in the course. The survey was voluntary but not anonymous, and it was answered by 19 out of 31 students. We include the results of this survey in Figures 3–7.
Student Survey:

1. How many hours a week on average do you devote to the course (outside of class)?

2. How often do you complete the homework problems when your group is NOT required to turn them in?
   0 1 2 3 4 5 (0=never, 5=every week)

3. How often do you study for the weekly quizzes?
   0 1 2 3 4 5 (0=never, 5=every week)

4. How often do you read Canvas materials (handouts, assignment/quiz solutions, sample exams)?
   0 1 2 3 4 5 (0=never, 5=every week)

5. How often do you study from the textbook?
   0 1 2 3 4 5 (0=never, 5=every week)

Figure 3: Question 1. How many hours a week on average do you devote to the course (outside of class)?
It appears at first sight that none of the answers to these survey questions determine the global course performance of students. For instance, Figure 8 shows a negative but not very significant correlation between the average number of hours of weekly study (Question 1) and the course grade. This lack of correlation may be due the fact that students have diverse backgrounds and levels of mathematical ability, and thus the time they need to devote to the course varies significantly for each individual.

Replacing Question 1 by any other question yields similar inconclusive results. In fact, whether or not a student participated in the survey seems to be a much better predictor of their final grade. Indeed, the median grade 88.60 among those 19 students who answered the survey is significantly higher than the median grade 76.17 among those who did not.

However, a closer analysis reveals more interesting patterns. For instance, Figure 9 shows that students who consistently prepare for their weekly quizzes tend to do better on those quizzes. (Note that here we do not measure how much time a student devotes every week, but rather their regularity throughout the term.) Furthermore, Figure 10 reveals another interesting fact about homework. Recall that students are split into two groups (A and B) such that, for the first half of the term, only students from group A have their assignments collected and, for the second half, only those in group B turn in their assignments. It turns out that students who consistently worked on homework problems when these were not due for their group also ended up doing better when their assignments were collected and graded.
4.2.2 Impact of Quizzes and Homework on Student Learning

According to Figures 11 and 12, students who obtained higher grades on quizzes and homework also did better on their final exam. Combining this fact with the discussion in the previous section supports the idea that preparing for (and taking) quizzes and working on homework problems have both a positive impact on student learning. Moreover, this appears to be more significant in the case of quizzes.

In Figure 13, we compare the grades obtained in Midterms 1 and 2 for each student of group A and B, in an attempt to analyze the relevance of collecting homework. Note that among those students who scored higher in the first midterm than in the second one, there are slightly more students in group A than in B. Since these were precisely the students who submitted assignments before the first Midterm, this may indicate that collecting assignments from students is better than just having them work on the problems without submitting. However, the differences between the two groups are minimal and thus not very significant.
Figure 6: Question 4. How often do you read Canvas materials (handouts, assignment/quiz solutions, sample exams)? (0=never, 5=every week)

Figure 7: Question 5. How often do you study from the textbook? (0=never, 5=every week)
Figure 8: Course grades and weekly study

Figure 9: Quiz preparation
Figure 10: Homework consistency

Figure 11: Impact of quizzes
Figure 12: Impact of homework

Figure 13: Homework collection — comparison between groups A and B
Figure 14: Homework grades

Figure 15: Quiz grades
Figure 16: Midterm 1 grades

Figure 17: Midterm 2 grades
Figure 18: Final exam grades
5 Planned Changes and Assessment of the Portfolio Process

In view of the struggles that many students had to meet some of the learning goals regarding conceptual understanding of abstract ideas and proofs, it would be advisable to either redesign the learning methods specific to achieving those goals or set different expectations. It would also be interesting to give students an open-ended project in which they would apply some linear algebra tools to their favorite field (or at least they try to understand how these methods are typically applied). This would certainly be a requirement for students taking the course for either honors or graduate credit.

6 Appendix

Below we include the following course materials:

6.1 Course Syllabus
6.2 Sample Exams
6.3 Sample Quizzes
6.4 Additional Handouts
6.5 Sample Work from Students
1 Instructor

Dr. Xavier Pérez-Giménez
Assistant Professor
Department of Mathematics
University of Nebraska – Lincoln
Lincoln, NE 68588-0130, USA

Email: xperez@unl.edu
Website: http://www.math.unl.edu/~xperezgimenez2/
Office: Avery Hall 333
Office Hours: TR 2:30 pm – 4:00 pm (or by email appointment)
Grader: TBA

2 Textbook


3 ACE Outcome 3

This course satisfies ACE Outcome 3: “Use mathematical, computational, statistical, or formal reasoning (including reasoning based on principles of logic) to solve problems, draw inferences, and determine reasonableness.” Your instructor will provide examples, you will discuss them in class, and you will practice with numerous homework problems. The exams will test how well you’ve mastered the material. The final exam will be the primary means of assessing your achievement of ACE Outcome 3.

4 Contacting me

The best way to contact with me is by email, xperez@unl.edu. Please put “[MATH 314]” at the beginning of the title and make sure to include your whole name in your email. Using your official UNL email to contact me is strongly recommended. My office is Avery Hall 333. My office hours are TR 2:30 pm – 4:00 pm. If you want to meet at another time, please email me in advance, and we will try to schedule a time to meet.

5 Course Description

One can say that Linear Algebra is a mathematical branch studying system of linear equations, for example

\[
\begin{align*}
4x + 5y + 6z &= 7 \\
2x + 6y + z &= 2 \\
x + y + z &= 4
\end{align*}
\]

Linear Algebra has many applications to diverse scientific areas, and even has powerful uses in computer graphics, linguistics, audio engineering, artificial intelligence, humanitarian aide, and legal studies etc.

This course is a transition course from computational courses, like calculus, to more theoretical ones. You will need to understand definitions and theorems, be able to apply them, and sometimes, prove theorems. The material in the course will tend to be more mathematically subtle than that encountered in your previous math courses, and will consequently require a significant effort on your part to master.

The course covers sections from Chapters 1 to 7 of the text: systems of linear equations, matrix algebra, determinants, vector spaces, eigenvalues and eigenvectors, orthogonality, and inner product spaces, and quadratic forms.
6 Homework and quizzes

Homework is designed to help students understand the materials and to prepare them for exams. We would have homework almost every week. Pop quizzes will be given if needed. There are no make-up quizzes.

Collaboration is encouraged in this course. However, copying someone else’s work and submitting it as your own is unacceptable. This act of academic dishonesty will be prosecuted in accordance with university policy.

Besides homework and quizzes, you are expected to read the appropriate sections of the text before coming to the class. You are also expected to work through the indicated exercises after the corresponding material is presented in class, and before the next class meeting.

7 Calculators and Electronics

You are not allowed to have on your person during exams or quizzes any devices that can access the internet or communicate in any way. Cell phones, Apple watches, etc. should be put away in backpacks/purses. Calculators, laptops, tablets, cell phones, and other non-medical electronic devices are not permitted during exams unless otherwise stated. During class, cell phones should be set on vibrate or off. If you need to take a call, send a text message, etc., please quietly leave the classroom to do so, so that you do not distract other students. You are welcome to return to class quietly when you are finished. If you wish to take notes using an electronic device, you must first demonstrate to me that you can type or write fast enough to do so properly, and that you can do it without distracting others, before the privilege to use such devices may be granted. If you are found to be abusing this privilege, you risk forfeiting it.

8 Grading

Your minimal course grade will be computed as follows.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Homework+Quizzes:</td>
<td>30%</td>
<td></td>
</tr>
<tr>
<td>Midterms:</td>
<td>2 x 17.5% = 35%</td>
<td></td>
</tr>
<tr>
<td>Final Exam:</td>
<td>35%</td>
<td></td>
</tr>
<tr>
<td>Total:</td>
<td>100%</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Grade</th>
<th>Percentage Range</th>
<th>Grade</th>
<th>Percentage Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>A+</td>
<td>95.00% - 100%</td>
<td>B-</td>
<td>76.66% - 79.99%</td>
</tr>
<tr>
<td>A</td>
<td>90.00% - 94.99%</td>
<td>C+</td>
<td>73.33% - 76.65%</td>
</tr>
<tr>
<td>A-</td>
<td>86.66% - 89.99%</td>
<td>C</td>
<td>70.00% - 73.32%</td>
</tr>
<tr>
<td>B+</td>
<td>83.33% - 86.65%</td>
<td>C-</td>
<td>66.66% - 69.99%</td>
</tr>
<tr>
<td>B</td>
<td>80.00% - 83.32%</td>
<td>D+</td>
<td>63.33% - 66.65%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>D</td>
<td>60.00% - 63.32%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>56.66% - 59.99%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>56.66% - 59.99%</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>56.66% - 59.99%</td>
</tr>
</tbody>
</table>

9 Attendance

Daily attendance for class lectures is expected and is extremely important. While attendance is not recorded, missing even one class will put you behind. Note that there is a strong correlation between class absences and poor grades. You are responsible for all material and announcements in class regardless of whether or not you attended. You are also responsible for making arrangements with another classmate to find out what you missed. You should not ask me to go over material you missed (due to tardiness or absences) during office hours or over email.

10 Exams

There are three exams: two midterms and a final. Students are expected to arrange their personal and work schedule to allow them to take the final exam at the scheduled time. No student will be permitted to take the final exam early.

Make-up exams will only be given with written evidence of an official university excused absence.
11 Incompletes

A grade of “Incomplete” may be considered if all but a small portion of the class has been successfully completed, but the student in question is prevented from completing the course by a severe, unexpected, and documented event. Students who are simply behind in their work should consider dropping the course.

12 ADA Statement

Students with disabilities are encouraged to contact the instructor for a confidential discussion of their individual needs for academic accommodation. It is the policy of the University of Nebraska-Lincoln to provide flexible and individualized accommodation to students with documented disabilities that may affect their ability to fully participate in course activities or to meet course requirements. To receive accommodation services, students must be registered with the Services for Students with Disabilities (SSD) office, 132 Canfield Administration, 472-3787 voice or TTY.

13 Grade Questions

Any questions regarding grading/scoring of homework, exams, or projects must be made within two class days from when they were handed back, or no change in grade will be made.

Because of privacy rights, I cannot discuss grades over email or telephone. Please do not email or call me asking about your grade. I will not be able to give you any information. Of course, I am happy to discuss grades in my office.

14 Special Dates

Aug. 31, 2018 (Friday): Last day to withdraw from this course and not have it appear on your transcript.
Oct. 12, 2018 (Friday): Last day to change your grade option to or from Pass/No Pass.
Nov. 09, 2018 (Friday): Last day to drop this course and receive a grade of W. (No permission required.) After this date, you cannot drop.

15 Departmental Grading Appeals Policy

Students who believe their academic evaluation has been prejudiced or capricious have recourse for appeals to (in order) the instructor, the departmental chair, the departmental appeals committee, and the college appeals committee.

16 Course Evaluation

The Department of Mathematics Course Evaluation Form will be available during the last two weeks of class. You will get an email when the form becomes available. Evaluations are anonymous and instructors do not see any of the responses until after final grades have been submitted. Evaluations are important—the department uses evaluations to improve instruction. Please complete the evaluation and take the time to do so thoughtfully.
### Tentative schedule:

The following table shows the material expected to be covered for each week of the semester. The exercises listed here are only recommended problems, they are not your official assignments. Note that what is shown here is approximate; your instructor may change the dates for each assignment and/or exam. It is your responsibility to keep track of the course details and schedule for your section.

<table>
<thead>
<tr>
<th>Week of</th>
<th>Section</th>
<th>Recommended (but not necessarily mandatory) Exercises</th>
</tr>
</thead>
<tbody>
<tr>
<td>January 7</td>
<td>1.1 Systems of Linear Equations</td>
<td>1, 3, 5, 9, 10, 11, 15, 18, 19, 20, 23, 24, 25, 31</td>
</tr>
<tr>
<td>January 7</td>
<td>1.2 Row Reduction and Echelon Forms</td>
<td>1, 3, 7, 11, 13, 15, 17, 19, 21, 22, 23, 24, 25, 26</td>
</tr>
<tr>
<td>January 7</td>
<td>1.3 Vector Equations</td>
<td>1, 3, 5, 7, 9, 11, 13, 14, 15, 17, 18, 19, 23, 24, 25, 28</td>
</tr>
<tr>
<td>January 14</td>
<td>1.4 The Matrix Equation $Ax = b$</td>
<td>1, 3, 7, 9, 11, 13, 14, 15, 17–24</td>
</tr>
<tr>
<td>January 14</td>
<td>1.5 Solution Sets of Linear Systems</td>
<td>2, 5, 6, 7, 8, 9, 11, 12, 13, 15, 16, 20, 23, 24, 25, 40</td>
</tr>
<tr>
<td>January 14</td>
<td>1.6 Applications</td>
<td>3(a,b), 7, 14</td>
</tr>
<tr>
<td>January 21</td>
<td>1.7 Linear Independence</td>
<td>1, 3, 5, 7, 8, 9, 13, 14, 15, 17, 19, 21, 22, 23, 24, 28, 30</td>
</tr>
<tr>
<td>January 21</td>
<td>1.8 Introduction to Linear Transformations</td>
<td>1, 2, 3, 5, 7, 9, 11, 13–16, 19, 21, 22, 32, 33, 34</td>
</tr>
<tr>
<td>January 28</td>
<td>1.9 The Matrix of a Linear Transformation</td>
<td>1, 5, 7, 8, 13, 15, 17, 22–25, 38</td>
</tr>
<tr>
<td>January 28</td>
<td>2.1 Matrix Operations</td>
<td>1, 3, 5, 7–11, 15, 16, 19, 22, 24</td>
</tr>
<tr>
<td>January 28</td>
<td>2.2 The Inverse of a Matrix</td>
<td>1, 3, 5, 7, 8, 9, 10, 13, 20, 21, 23, 24, 29, 31, 32, 33</td>
</tr>
<tr>
<td>February 4</td>
<td>2.3 Characterization of Invertible Matrices</td>
<td>1–7(odd), 11, 12, 13, 16, 17, 19, 22, 33, 37</td>
</tr>
<tr>
<td>February 4</td>
<td>2.5 Matrix Factorizations</td>
<td>3, 5, 9, 11, 19</td>
</tr>
<tr>
<td>February 11</td>
<td>Midterm Exam I</td>
<td></td>
</tr>
<tr>
<td>February 11</td>
<td>3.1 Introduction to Determinants</td>
<td>1–13 (odd), 39, 40</td>
</tr>
<tr>
<td>February 11</td>
<td>3.2 Properties of Determinants</td>
<td>1–8, 11, 15, 18, 19, 25, 27, 28, 31</td>
</tr>
<tr>
<td>February 18</td>
<td>4.1 Vector Spaces and Subspaces</td>
<td>1–15, 17, 19, 20, 21, 23, 24, 25, 27</td>
</tr>
<tr>
<td>February 18</td>
<td>4.2 Null Spaces, Column spaces</td>
<td>1, 2, 3, 5, 7, 11, 12, 15, 17, 19, 20, 21, 25–28, 30, 35, 37</td>
</tr>
<tr>
<td>February 18</td>
<td>4.3 Linearly Independent Sets; Bases</td>
<td>1–19 (odd), 21–25, 31, 32</td>
</tr>
<tr>
<td>February 25</td>
<td>4.4 Coordinate Systems</td>
<td>1, 3, 5, 7, 8, 11, 13, 15, 16, 27, 28, 29</td>
</tr>
<tr>
<td>February 25</td>
<td>4.5 The Dimension of a Vector Space</td>
<td>1–5, 7–17 (odd), 19, 20, 21, 29, 30, 31</td>
</tr>
<tr>
<td>February 25</td>
<td>4.6 Rank</td>
<td>1, 3, 4, 5–15 (odd), 17, 18, 19, 21, 25, 27–29</td>
</tr>
<tr>
<td>March 4</td>
<td>4.7 Change of Basis</td>
<td>1–9 (odd), 11, 12, 13, 15</td>
</tr>
<tr>
<td>March 4</td>
<td>4.9 Applications/Catch Up</td>
<td>1, 3, 5, 9, 11</td>
</tr>
<tr>
<td>March 4</td>
<td>5.1 Eigenvectors and Eigenvalues</td>
<td>1–15 (odd), 19, 21, 22, 23, 24, 25, 27, 31, 33</td>
</tr>
<tr>
<td>March 11</td>
<td>5.2 The Characteristic Equation</td>
<td>1, 3, 7, 9, 11, 13, 17, 21, 22, 23, 24</td>
</tr>
<tr>
<td>March 11</td>
<td>5.3 Diagonalization</td>
<td>1, 3, 5, 7, 11, 15, 16, 19, 21, 22, 23, 24, 25, 27, 29</td>
</tr>
<tr>
<td>March 11</td>
<td>5.4 Eigenvectors and Linear Transformations</td>
<td>1, 3, 5, 8, 9, 11, 13, 19, 23, 27</td>
</tr>
<tr>
<td>March 18</td>
<td>Spring vacation</td>
<td></td>
</tr>
<tr>
<td>March 25</td>
<td>5.5 Complex Eigenvalues</td>
<td>1, 5, 9, 13, 16</td>
</tr>
<tr>
<td>March 25</td>
<td>Catch Up and Review</td>
<td></td>
</tr>
<tr>
<td>April 1</td>
<td>Midterm Exam II</td>
<td></td>
</tr>
<tr>
<td>April 1</td>
<td>6.1 Inner Product, Length and Orthogonality</td>
<td>1–19 (odd), 20, 25–31</td>
</tr>
<tr>
<td>April 1</td>
<td>6.2 Orthogonal Sets</td>
<td>1, 5, 9, 11, 13, 15, 17, 23, 24, 27–29</td>
</tr>
<tr>
<td>April 1</td>
<td>6.3 Orthogonal Projections</td>
<td>1, 5, 7, 9, 11, 13, 15, 17, 21, 22, 23, 24</td>
</tr>
<tr>
<td>April 8</td>
<td>6.4 The Gram-Schmidt Process</td>
<td>1, 5, 9, 11, 15, 17, 18, 19, 22</td>
</tr>
<tr>
<td>April 8</td>
<td>6.5 Least-Squares Problems</td>
<td>1, 3, 5, 7, 11, 15, 17, 18, 19, 21</td>
</tr>
<tr>
<td>April 8</td>
<td>6.6 Applications</td>
<td>1, 3, 7a, 9</td>
</tr>
<tr>
<td>April 15</td>
<td>7.1 Diagonalization of Symmetric Matrices</td>
<td>1–19 (odd), 23, 25, 26, 28, 29, 36</td>
</tr>
<tr>
<td>April 15</td>
<td>7.2 Quadratic Forms</td>
<td>1–13 (odd), 21, 22, 23, 24</td>
</tr>
<tr>
<td>April 15</td>
<td>7.4 Singular Value Decomposition</td>
<td>1, 3, 5, 7, 9, 11, 12, 13, 17, 18, 23</td>
</tr>
<tr>
<td>April 22</td>
<td>Catch Up and Review for Final Exam</td>
<td></td>
</tr>
<tr>
<td>April 29</td>
<td>Final Exam Week</td>
<td></td>
</tr>
</tbody>
</table>

The Final Exam is on Friday, May 3, 7.30 am – 9.30 am
1. (6 points) Suppose that the matrix below is the augmented matrix of a linear system. Determine the value(s) of $h$ for which the linear system is: a) inconsistent, b) consistent with one single solution or c) consistent with many solutions.

\[
\begin{bmatrix}
1 & -7 & 5 & -2 \\
0 & h - 2 & 4 & 12 \\
0 & 0 & h & 6 \\
\end{bmatrix}
\]

$h \neq 0, 2$: **consistent with one solution**, since there is a pivot position at each row of the coefficient matrix.

$h = 0$: **inconsistent**, since there is a pivot position at the last column of the augmented matrix.

$h = 2$: \[
\begin{bmatrix}
1 & -7 & 5 & -2 \\
0 & 0 & 4 & 12 \\
0 & 0 & 2 & 6 \\
\end{bmatrix}
\sim
\begin{bmatrix}
1 & -7 & 5 & -2 \\
0 & 0 & 4 & 12 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
consistent with many solutions ($x_2$ is a free variable).

2. (4 points) Show whether or not the following transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is linear.

$T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 - 3x_2 \\ 7|x_2| \end{bmatrix}$

$T\left(-\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} -2x_1 + 3x_2 \\ 7|x_2| \end{bmatrix} + \begin{bmatrix} 2x_1 - 3x_2 \\ 7|x_2| \end{bmatrix} = -T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right)$

so $T$ is **not linear**
3. (8 points)

(a) Determine whether vectors \([1, 2, 3]^T\), \([0, 6, 7]^T\) and \([-1, -1]^T\) form a linearly independent set.

Let \(A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & -1 \end{bmatrix}\). It has one pivot at each row, so \(A \vec{x} = \vec{0}\) only has the trivial solution \(\vec{x} = \vec{0}\).

So columns of \(A\) are linearly independent.

(b) Use part a) to answer each of the following questions:

- Is \(A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 6 & 7 \\ 0 & 0 & -1 \end{bmatrix}\) an invertible matrix?

Yes, since its columns are linearly independent.

- Is \(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\) in the subset of \(\mathbb{R}^3\) spanned by the columns of \(A\)?

Yes. In fact, the columns of \(A\) span all vectors in \(\mathbb{R}^3\), since \(A\) is invertible.
4. (5 points) Write down a matrix equation that describes the traffic flow in this network. Note: you are not asked to solve the system.

\[
\begin{align*}
\begin{cases}
\mathbf{x}_1 - \mathbf{x}_3 - \mathbf{x}_4 &= 40 \quad (A) \\
\mathbf{x}_1 + \mathbf{x}_2 &= 200 \quad (B) \\
\mathbf{x}_2 + \mathbf{x}_3 - \mathbf{x}_5 &= 100 \quad (C) \\
\mathbf{x}_4 + \mathbf{x}_5 &= 60 \quad (D)
\end{cases}
\end{align*}
\]

Matrix equation:

\[
\begin{pmatrix}
1 & 0 & -1 & -1 & 0 \\
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & -1 \\
0 & 0 & 0 & 1 & 1
\end{pmatrix}
\begin{pmatrix}
\mathbf{x}_1 \\
\mathbf{x}_2 \\
\mathbf{x}_3 \\
\mathbf{x}_4 \\
\mathbf{x}_5
\end{pmatrix}
= 
\begin{pmatrix}
40 \\
200 \\
100 \\
60
\end{pmatrix}
\]

5. (5 points) Find the standard matrix A of the linear transformation \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) that reflects vectors through the line \( \mathbf{x}_2 = -\mathbf{x}_1 \).

\[
\begin{align*}
T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} \\
T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) &= \begin{bmatrix} -1 \\ 0 \end{bmatrix}
\end{align*}
\]

So the standard matrix of \( T \) is

\[
\begin{bmatrix}
0 & -1 \\
-1 & 0
\end{bmatrix}
\]
6. (6 points) Let \( T : \mathbb{R}^5 \to \mathbb{R}^3 \) be the linear transformation whose standard matrix is
\[
A = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 \\
0 & 6 & 7 & 8 & 9 \\
0 & 0 & -1 & -4 & 2
\end{bmatrix}.
\]
(a) Is \( T \) one-to-one? Justify your answer.

\[
A \vec{x} = \vec{0} \text{ has many solutions, since variables } x_4, x_5 \text{ are free.}
\]

Therefore, \( T \) is not 1-to-1.

(b) Is \( T \) onto \( \mathbb{R}^3 \)? Justify your answer.

\[
A \vec{x} = \vec{b} \text{ is consistent for every } \vec{b} \text{ in } \mathbb{R}^5,
\]

since \( A \) has one pivot at each row.

Then, \( T \) is onto \( \mathbb{R}^3 \).

7. (10 points) Mark the following statements as true (T) or false (F).

(a) \( T \) or F. The reduced row-echelon form of a matrix is unique.
(b) T or F. The set \( \text{Span}\{\vec{u}, \vec{v}\} \) is always visualized as a plane through the origin.
(c) T or F. A homogeneous equation is always consistent.
(d) T or F. If \( \vec{u} \) and \( \vec{v} \) are linearly independent, and if \( \vec{w} \) is in \( \text{Span}\{\vec{u}, \vec{v}\} \), then \( \{\vec{u}, \vec{v}, \vec{w}\} \) is linearly dependent.
(e) T or F. If \( A \) is a \( 5 \times 3 \) matrix, and \( T \) is a transformation defined by \( T(\vec{x}) = A\vec{x} \), then the domain of \( T \) is \( \mathbb{R}^5 \).
(f) T or F. The vector \( \vec{b} \) is a linear combination of the columns of a matrix \( A \) if and only if the equation \( A\vec{x} = \vec{b} \) has at least one solution.
(g) T or F. \( T(x_1, x_2) = (x_1 + 7, x_2 + 7) \) is not a linear transformation.
(h) T or F. A mapping \( T : \mathbb{R}^n \to \mathbb{R}^m \) is one-to-one if each vector in \( \mathbb{R}^n \) maps to a unique vector in \( \mathbb{R}^m \).
(i) T or F. If \( A \) and \( B \) are \( m \times n \) matrices, then both \( AB^T \) and \( A^TB \) are defined.
(j) T or F. For \( n \times n \) matrices \( A, B, \) and \( C \), if \( CA = CB \), then \( A = B \).
8. (6 points) Solve only one of the following two problems:

(a) Find the LU-factorization of the following matrix:

\[
A = \begin{bmatrix}
1 & -1 & 0 \\
-2 & 5 & 3 \\
3 & -6 & 2
\end{bmatrix}
\]

(b) Find the inverse of the following matrix:

\[
A = \begin{bmatrix}
0 & 2 & 2 \\
5 & 5 & 5 \\
0 & 0 & -3
\end{bmatrix}
\]

\[
A = LU \quad \text{with} \quad L = \begin{bmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
3 & -1 & 1
\end{bmatrix} \quad \& \quad U = \begin{bmatrix}
1 & -1 & 0 \\
0 & 3 & 3 \\
0 & 0 & 5
\end{bmatrix}
\]

\[
\begin{align*}
A_1 & = \begin{bmatrix}
1 & -1 & 0 \\
-2 & 5 & 3 \\
3 & -6 & 2
\end{bmatrix} \\
& \sim \begin{bmatrix}
1 & -1 & 0 \\
0 & 3 & 3 \\
0 & -1 & 1
\end{bmatrix} \\
& \sim \begin{bmatrix}
1 & -1 & 0 \\
0 & 3 & 3 \\
0 & 0 & 5
\end{bmatrix}
\end{align*}
\]

\[
A = \begin{bmatrix}
0 & 2 & 2 \\
5 & 5 & 5 \\
0 & 0 & -3
\end{bmatrix} \\
\sim \begin{bmatrix}
0 & 2 & 2 \\
5 & 5 & 5 \\
0 & 0 & -3
\end{bmatrix} \\
\sim \begin{bmatrix}
0 & 2 & 2 \\
0 & 5 & 5 \\
0 & 0 & -3
\end{bmatrix} \\
\sim \begin{bmatrix}
0 & 1 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \\
\sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \\
\sim \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \\
\sim \begin{bmatrix}
1 & 0 & 0 \\
-1/2 & 1/2 & 0 \\
0 & 0 & -1/3
\end{bmatrix}
\]

\[
A^{-1} = \begin{bmatrix}
-1/2 & \sqrt{5} & 0 \\
\sqrt{2} & 0 & \sqrt{3} \\
0 & 0 & -1/3
\end{bmatrix}
\]
Page for scratch work. Please indicate in the problem if you have work here.
1. (300 points & US$1,000,000)\(^1\)

Let \(\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}\) be the Riemann zeta function.

Prove that every non-trivial zero of \(\zeta\) is of the form \(z = 1/2 + ix\) with \(x \in \mathbb{R}\).

I ran out of time...

2. (a) (4 points) Compute the determinant of the following matrix

\[
A = \begin{bmatrix}
4 & -5 & 3 \\
2 & 0 & -1 \\
2 & -3 & 2 \\
\end{bmatrix}.
\]

\[
\det A = -2 \begin{vmatrix}
-5 & 3 \\
-3 & 2 \\
\end{vmatrix} + 0 + \begin{vmatrix}
4 & -5 \\
2 & -3 \\
\end{vmatrix} = \\
= -2(-1) + (-2) = 0
\]

(b) (2 points) Is the transpose \(A^T\) invertible? Why or why not?

No, \(A^T\) is not invertible, since \(\det A = 0\)

(c) (2 points) Is rank \(A = 3\)? Why or why not?

No, rank \(A < 3\), since \(A\) is \(3 \times 3\) and \(\det A = 0\).

\(^1\)Happy April fool's day! This is the famous Riemann Hypothesis, one of the Millennium Prize Problems. The first person to solve it will be awarded a US$1 million prize by the Clay Mathematics Institute.
3. (a) (4 points) Find a basis of \( H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \), where 
\[
\vec{v}_1 = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix}
\]
\[
\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.
\]
Justify your answer.

\[
\vec{v}_3 = -\frac{3}{2} \vec{v}_1, \text{ so it can be removed from the set.}
\]

\[
H = \text{Span} \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \} = \text{Span} \{ \vec{v}_1, \vec{v}_2 \}.
\]

\( \vec{v}_1 \) and \( \vec{v}_2 \) are linearly independent, since they are not multiple of each other.

So \( \mathbf{B} = \{ \vec{v}_1, \vec{v}_2 \} \) is a basis of \( H \).

(b) (4 points) Let \( \vec{w}_1 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \) and \( \vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} \).

Extend \( \{ \vec{w}_1, \vec{w}_2 \} \) to a basis of \( \mathbb{R}^3 \).

Let \( \vec{e}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \), \( \vec{w}_1, \vec{w}_2, \vec{e}_3 \) are linearly independent.

Since \( \det \begin{bmatrix} -3 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -4 & 0 \end{bmatrix} = -10 \neq 0 \). So \( \{ \vec{w}_1, \vec{w}_2, \vec{e}_3 \} \) is a basis of \( \mathbb{R}^3 \).

4. (6 points) Let \( B = \{ \vec{b}_1, \vec{b}_2 \} \) and \( C = \{ \vec{c}_1, \vec{c}_2 \} \) be bases for \( \mathbb{R}^2 \), where

\[
\vec{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \text{and} \quad \vec{c}_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}, \quad \vec{c}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.
\]

Find \( P_{c \leftarrow B} \), that is, the change-of-coordinates matrix from \( B \) to \( C \).

\[
\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 4 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 3 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 0 & -1 & 1 \end{bmatrix}
\]

\[
\sim \begin{bmatrix} 10 & 3 & 2 \\ 0 & 1 & -1 \end{bmatrix}, \quad \text{so} \quad P_{c \leftarrow B} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}.
\]
5. (a) (4 points) Find a basis of Nul(A), where \( \mathbf{A} = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \).

Solve \( \mathbf{A} \mathbf{x} = \mathbf{0} \).

\( \mathbf{A} = \begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{bmatrix} \).

\[ \begin{align*}
\mathbf{x}_1 &= 7 \mathbf{x}_3 - 6 \mathbf{x}_4 \\
\mathbf{x}_2 &= -4 \mathbf{x}_3 + 2 \mathbf{x}_4 \\
\mathbf{x}_3, \mathbf{x}_4 & \text{ free}
\end{align*} \]

\( \mathbf{x} = \mathbf{x}_3 \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} + \mathbf{x}_4 \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \).

So \( \left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\} \) is basis of Nul(\( \mathbf{A} \)).

(b) (2 points) Determine rank \( \mathbf{A} \).

\( \text{rank } \mathbf{A} = \# \text{ columns} - \dim \text{Nul}(\mathbf{A}) = 4 - 2 = 2 \).

6. (a) (4 points) Find the characteristic polynomial and the eigenvalues of \( \mathbf{A} = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix} \).

\[ \mathbf{P}_A(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}) = \begin{vmatrix} -3-\lambda & -2 \\ 4 & 1-\lambda \end{vmatrix} = \lambda^2 + 2 \lambda + 5 \]

\[ \mathbf{P}_A(\lambda) = 0 \rightarrow \lambda = -1 \pm 2i \]

Eigenvalues: \(-1 \pm 2i\).

(b) (2 points) Is the matrix above diagonalizable? Justify your answer.

\begin{itemize}
  \item Yes. Since the eigenvalues are different, we can find 2
  \item linearly independent eigenvectors, and thus \( \mathbf{A} \) is diagonalizable.
\end{itemize}

(c) (4 points) Find a basis for the eigenspace of \( \mathbf{B} = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} \) corresponding to eigenvalue \( \lambda = 5 \).

\[ (\mathbf{B} - 5 \mathbf{I}) \mathbf{x} = \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0} . \]

By visual inspection, \( \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) works.

So \( (5, [1]) \) eigenpair, and \( \left\{ [1], [2] \right\} \) is basis of \( E_5 = \text{Nul}(\mathbf{B} - 5 \mathbf{I}) \).
7. (8 points) Let $A$ be an $m \times n$ matrix. Prove that Null($A$) is a vector subspace of $\mathbb{R}^n$.

   (1) Suppose $\vec{x}, \vec{y}$ are in Null($A$).
   Then $\vec{x}, \vec{y}$ are in $\mathbb{R}^n$ and $A\vec{x} = \vec{0}$, $A\vec{y} = \vec{0}$.
   So $A(\vec{x} + \vec{y}) = A\vec{x} + A\vec{y} = \vec{0}$. So $\vec{x} + \vec{y}$ also in Null($A$),

   (2) Suppose $\vec{x}$ in Null($A$). Then $A\vec{x} = \vec{0}$
   So $A(c\vec{x}) = c(A\vec{x}) = c\vec{0} = \vec{0}$, so $c\vec{x}$ also in Null($A$).

Hence, Null($A$) is a subspace of $\mathbb{R}^n$.

8. (10 points) Mark the following statements as true (T) or false (F). You do NOT need to explain your answers.

   (a) T or F. The rank of a matrix equals the number of nonzero rows.

   (b) T or F. For any square matrix $A$ we have: $\det(-A) = -\det(A)$.

   (c) T or F. The change-of-coordinates matrix between two bases is always invertible.

   (d) T or F. If the matrix $A$ is diagonalizable, then $A^3$ is diagonalizable as well.

   (e) T or F. If $\lambda$ is an eigenvalue of $A$, then $\lambda$ is an eigenvalue of $A^2$

   (f) T or F. If $\vec{x}$ is an eigenvector of $A$, then $\vec{x}$ is an eigenvector of $A^2$

   (g) T or F. If $A$ is $m \times n$ and rank($A$) = $n$, then the columns of $A$ form a basis for $\mathbb{R}^m$.

   (h) T or F. The set of all polynomials forms a vector space (with usual addition and scalar multiplication).

   (i) T or F. The set of all continuous functions $f$ satisfying $f(0) = 1$ forms a vector space (with usual addition and scalar multiplication).

   (j) T or F. If $A$ is row-equivalent to the identity matrix, then $A$ is diagonalizable.
1. (10 points) Let \( A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -2 & -1 \\ 2 & 2 & 1 \end{bmatrix} \).

(a) Show that the eigenvalues of \( A \) are \(-1, 0 \) and \( 3 \).

\[
P_A(\lambda) = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 1 & -2-\lambda & -1 \\ 2 & 2 & 1-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2+\lambda) = 0 \implies \lambda = -1, 0, 3
\]

(b) Find a basis for the eigenspace of \( A \) corresponding to eigenvalue \( 3 \).

\[
(A - 3I) \vec{x} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -5 & -1 \\ 2 & 2 & -2 \end{bmatrix} \vec{x} = \vec{0}
\]

\[
\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \text{ works!}
\]

\[\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \} \text{ basis for } E_3 = \text{Nul}(A - 3I)
\]

(c) Is \( A \) diagonalizable? Justify your answer. \text{Yes, since all eigenvalues are different}

(d) Is \( A \) invertible? Justify your answer. \text{No, since } 0 \text{ is an eigenvalue}
2. (8 points) Let \( A = \begin{bmatrix} -3 & -6 & 3 & 15 & -9 \\ 2 & 4 & -2 & -10 & 6 \\ 0 & 2 & 7 & 0 & 3 \end{bmatrix} \) 
\( \sim \begin{bmatrix} 1 & 2 & -1 & -5 & 3 \\ 0 & 2 & 7 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \)

(a) Find a basis of Row(\( A \)).

basis of Row(\( A \)) : \[ \begin{bmatrix} 1 \\ 2 \\ 3 \\ -1 \\ -5 \\ 3 \end{bmatrix} , \begin{bmatrix} 0 \\ 2 \\ 7 \\ 0 \\ 3 \end{bmatrix} \]

(b) Determine the dimension of Null(\( A \)). (You do not need to find a basis.)

\[ \dim \text{ Null}(A) = \# \text{ columns} - \text{Rank} \ A \]
\[ = 5 - 2 = 3 \]

3. (8 points) Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation with standard matrix \( A \). Prove that if \( A\bar{x} = \bar{0} \) has only the trivial solution \( \bar{x} = \bar{0} \), then \( T \) must be one-to-one.

Hint: To show that \( T \) is one-to-one, first suppose that we have some \( \vec{v} \) and \( \vec{w} \) in \( \mathbb{R}^n \) with \( T(\vec{v}) = T(\vec{w}) \), and then conclude that \( \vec{v} \) and \( \vec{w} \) must be equal.

Let \( \vec{v}, \vec{w} \) in \( \mathbb{R}^n \) with \( T(\vec{v}) = T(\vec{w}) \)

Then \( \vec{0} = T(\vec{v}) - T(\vec{w}) = T(\vec{v} - \vec{w}) \).

\( T \) linear mapping

So \( A(\vec{v} - \vec{w}) = \vec{0} \). We must have \( \vec{v} - \vec{w} = \vec{0} \), so \( \vec{v} = \vec{w} \).

Then, \( T \) is 1-to-1.
4. (10 points) Let \( P_n \) be the vector space of all polynomials of degree at most \( n \), under the usual rules of polynomial addition and scalar multiplication.

(a) Let \( W \) be the set of all polynomials \( p(x) \) in \( P_n \) such that \( p(2) = 0 \). Show that \( W \) is a vector subspace of \( P_n \).

\[
\text{Way 1: } \quad \text{Let } p(t), q(t) \in W \text{ and } c \in \mathbb{R}.
\]

i) \((p + q)(2) = p(2) + q(2) = 0 \implies p + q \in W\)

ii) \((cp)(2) = c \cdot p(2) = 0 \implies cp \in W\)

So \( W \) is a subspace.

\[
\text{Way 2: } \quad p(t) = a_n t^n + \cdots + a_1 t + a_0 \text{ in } W
\]

means \( a_n 2^n + \cdots + a_1 2 + a_0 = 0 \), so \( a_0 = -2 a_n - \cdots - 2 a_1 \)

So every \( p(t) \) in \( W \) is of the form

\[
p = a_n t^n + \cdots + a_1 t - 2 a_n - \cdots - 2 a_1 = a_n (t^n - 2^n) + \cdots + a_1 (t - 2)
\]

so \( W = \text{Span} \{ t^n - 2^n, \ldots, t - 2 \} \) and thus is a subspace.

(b) If in the definition of \( W \) above we had replaced \( "p(2) = 0" \) with \( "p(2) = 1" \), would it still be a subspace? Why or why not?

No, since the 0 polynomial \( 0(t) = 0 \)

does not satisfy \( 0(2) = 1 \).
5. (8 points) Consider the vectors \( \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \), \( \vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \) and \( \vec{u}_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \).

(a) Show that \( \{\vec{u}_1, \vec{u}_2, \vec{u}_3\} \) is an orthogonal basis for \( \mathbb{R}^3 \).

\[
\langle \vec{u}_1, \vec{u}_2 \rangle = 1 - 1 = 0 \\
\langle \vec{u}_1, \vec{u}_3 \rangle = 1 - 1 = 0 \\
\langle \vec{u}_2, \vec{u}_3 \rangle = 1 - 2 + 1 = 0
\]

So \( \vec{u}_1, \vec{u}_2, \vec{u}_3 \) are orthogonal and nonzero, so they are also l.i. Then \( \{\vec{u}_1, \vec{u}_2, \vec{u}_3\} \) is an orthogonal basis of \( \mathbb{R}^3 \).

(b) Express \( \vec{x} = \begin{bmatrix} 2 \\ -7 \\ 0 \end{bmatrix} \) as a linear combination of \( \vec{u}_1, \vec{u}_2, \vec{u}_3 \). Hint: there is a short way to do this without solving any system of equations.

\[
\vec{x} = \frac{\langle \vec{x}, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1 + \frac{\langle \vec{x}, \vec{u}_2 \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle} \vec{u}_2 + \frac{\langle \vec{x}, \vec{u}_3 \rangle}{\langle \vec{u}_3, \vec{u}_3 \rangle} \vec{u}_3 =
\]

\[
= \frac{2}{2} \vec{u}_1 + \frac{-12}{6} \vec{u}_2 + \frac{9}{3} \vec{u}_3
\]

\[
= \vec{u}_1 - 2 \vec{u}_2 + 3 \vec{u}_3
\]

6. (6 points) Consider the following vectors in \( \mathbb{R}^3 \).

\[
\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u}_2 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \vec{y} = \begin{bmatrix} 5 \\ 0 \\ -1 \end{bmatrix}.
\]

Note that \( \vec{u}_1 \) and \( \vec{u}_2 \) are orthogonal. Find the closest point (i.e. best approximation) to \( \vec{y} \) in the subspace \( W = \text{Span}\{\vec{u}_1, \vec{u}_2\} \).

The closest point is given by

\[
\hat{y} = \text{proj}_W \vec{y} = \frac{\langle \vec{y}, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1 + \frac{\langle \vec{y}, \vec{u}_2 \rangle}{\langle \vec{u}_2, \vec{u}_2 \rangle} \vec{u}_2
\]

\[
= \frac{4}{2} \vec{u}_1 + \frac{-6}{6} \vec{u}_2 = 2\vec{u}_1 - \vec{u}_2 = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}
\]
7. (6 points) Consider the following two vectors in $\mathbb{R}^3$.

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}.$$ 

Construct an **orthogonal basis** for $\text{Span}\{\mathbf{x}_1, \mathbf{x}_2\}$. (Hint: You learned a process for doing this.)

$$\mathbf{u}_1 = \mathbf{x}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$$

$$\mathbf{u}_2 = \mathbf{x}_2 - \text{proj}_{\text{Span} \mathbf{u}_1} \mathbf{x}_2 = \mathbf{x}_2 - \frac{10}{5} \mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\left\{ \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\} \text{ orthogonal basis for } \text{Span} \mathbf{x}_1, \mathbf{x}_2 \right\}$$

8. (8 points) Find the least-squares solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ for

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \\ 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 0 \end{bmatrix}.$$ 

**Way 1:**

$$A^TA = \begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^T\mathbf{b} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$$

normal equations:

$$\begin{bmatrix} 6 & 0 \\ 0 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$$

So $$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

**Way 2:** (Note columns of $A$ are orthogonal)

$$\hat{\mathbf{b}} = \text{proj}_{\text{col}(A)} \mathbf{b} =$$

$$= \frac{6}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \frac{5}{5} \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} = A \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$A \mathbf{x} = \hat{\mathbf{b}} \quad \text{(we want)}$$

So $$\mathbf{x} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$
9. (10 points) Let \( Q(\vec{x}) = 2x_1x_2 - 2x_1^2 - 2x_2^2 \) be a quadratic form.

(a) Find a symmetric \( 2 \times 2 \) matrix \( A \) such that \( Q(\vec{x}) = \vec{x}^T A \vec{x} \) for all \( \vec{x} \in \mathbb{R}^2 \).

\[
A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}
\]

(b) Is \( Q \) positive-definite, negative-definite, or indefinite? Explain why.

\[
P_A(\lambda) = \lambda^2 + 4\lambda + 3 = 0 \quad \rightarrow \quad \lambda = -1, -3
\]

Eigenvalues \(-1, -3 < 0 \rightarrow Q \) is negative-definite

(c) Is there any vector \( \vec{x} \) such that \( Q(\vec{x}) > 0 \)?

No, since \( Q \) is negative-definite, \( Q(\vec{x}) \) \( \begin{cases} < 0 & \text{if } \vec{x} \neq \vec{0} \\ = 0 & \text{if } \vec{x} = \vec{0} \end{cases} \)

10. (8 points)

(a) Find an orthogonal diagonalization of the following matrix \( A \), which has eigenvectors \( \vec{x}_1, \vec{x}_2 \) and \( \vec{x}_3 \).

\[
A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & 1 \\ 3 & 1 & 1 \end{bmatrix}, \quad \vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \vec{x}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \vec{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}
\]

\[
A \vec{x}_1 = -2 \vec{x}_1, \quad A \vec{x}_2 = 2 \vec{x}_2, \quad A \vec{x}_3 = 5 \vec{x}_3
\]

\[
\begin{cases} \text{eigenvectors } -2, 2, 5 \text{ are different} & \Rightarrow \text{eigenvectors } \vec{x}_1, \vec{x}_2, \vec{x}_3 \text{ are orthogonal.} \\
A \text{ symmetric} \end{cases}
\]

So \( A = PD P^T \) with \( D = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix} \), \( P = \begin{bmatrix} \sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ 0 & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix} \)

(b) Indicate how you would compute \( A^{100} \) efficiently.

\[
A^{100} = (PD P^T)^{100} = PD^{100} P^T = P \begin{bmatrix} (-2)^{100} & 0 \\ 0 & 2^{100} \end{bmatrix} P^T
\]
11. **(8 points)** The weather in Lincoln is either good or bad on any given day. If the weather is good today, there is a \(4/5\) chance the weather will also be good tomorrow and a \(1/5\) chance the weather will be bad. If the weather is bad today, it will be good tomorrow with probability \(1/2\) and bad with probability \(1/2\).

(a) Find the **stochastic matrix** that describes this situation.

\[
P = \begin{bmatrix}
\frac{4}{5} & \frac{1}{2} \\
\frac{1}{5} & \frac{1}{2}
\end{bmatrix}
\]

(b) Find the **steady-state vector**.

\[
P \vec{x} = \vec{x}, \quad \begin{bmatrix}
-\frac{4}{5} & \frac{1}{2} \\
\frac{1}{5} & -\frac{1}{2}
\end{bmatrix} \vec{x} = \vec{0}, \quad \text{so} \quad \vec{x} = \begin{bmatrix} \frac{5}{7} \\ \frac{2}{7} \end{bmatrix}
\]

(c) In the long run, how likely is it for the weather in Lincoln to be good on a given day?

Since all entries in \(P\) are positive, \(\lim_{k \to \infty} P^k \vec{x}_0 = \begin{bmatrix} \frac{5}{7} \\ \frac{2}{7} \end{bmatrix}\) for any initial state \(\vec{x}_0\). So, there is a \(\frac{5}{7}\) chance of good weather.

12. **(10 points)** Mark the following statements as true (T) or false (F).

(a) T or F. If \(\lambda\) is an eigenvalue of \(A\), then \(\lambda^3\) is an eigenvalue of \(A^3\).

(b) T or F. Every square matrix with real entries has at least one real eigenvalue.

(c) T or F. An \(n \times n\) matrix is orthogonally diagonalizable if and only if it is symmetric.

(d) T or F. Any set of 6 vectors in \(\mathbb{R}^5\) is linearly dependent.

(e) T or F. The derivative operation \(\frac{d}{dx}\) is a linear operation on polynomial functions.

(f) T or F. If \(\det(A) \neq 0\), then \(A\) is diagonalizable.

(g) T or F. If \(A\) is \(m \times n\), then \(\text{rank}(A) + \text{dim}(\text{Null}(A)) = n\).

(h) T or F. An \(2 \times 2\) matrix can sometimes have only a single one-dimensional eigenspace.

(i) T or F. If \(A = A^T\), and if vectors \(\vec{u}\) and \(\vec{v}\) satisfy \(A\vec{u} = 3\vec{u}\) and \(A\vec{v} = 4\vec{v}\), then \(\langle \vec{u}, \vec{v} \rangle = 0\).

(j) T or F. A positive definite matrix has strictly positive eigenvalues.
Page for scratch work. Please indicate in the problem if you have work here.
1. Find the general solution of the linear system whose augmented matrix is

\[
\begin{bmatrix}
1 & 3 & 4 & 7 \\
3 & 9 & 7 & 6
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 3 & 4 & 7 \\
3 & 9 & 7 & 6
\end{bmatrix} \sim
\begin{bmatrix}
1 & 3 & 4 & 7 \\
0 & 0 & -5 & -15
\end{bmatrix} \sim
\begin{bmatrix}
1 & 3 & 4 & 7 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

\[
\begin{bmatrix}
1 & 3 & 0 & -5 \\
0 & 0 & 1 & 3
\end{bmatrix}
\]

solution:
\[
\begin{cases}
\alpha_1 = -5 - 3\alpha_2, \\
\alpha_2 \text{ free} \\
\alpha_3 = 3
\end{cases}
\]

2. Suppose the coefficient matrix of a system of linear equations has a pivot position in every row. Explain why the system is consistent.

Since the coefficient matrix has a pivot in every row, there cannot be a pivot in the last column of the augmented matrix.

Therefore, the reduced echelon form does not have any row like this

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 1
\end{bmatrix}
\]

and the system is consistent.
1. Obtain a system of linear equations satisfied by $x_1, \ldots, x_5$ when the chemical reaction below is balanced. Use the following order of the elements: Na, H, C, O.

Do **not** solve the system.

$$x_1 \text{NaHCO}_3 + x_2 \text{H}_3\text{C}_6\text{H}_5\text{O}_7 \rightarrow x_3 \text{Na}_3\text{C}_6\text{H}_5\text{O}_7 + x_4 \text{H}_2\text{O} + x_5 \text{CO}_2$$

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 8 \\ 6 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

System $\mathbf{Ax} = \mathbf{0}$ with $\mathbf{A} = \begin{bmatrix} 1 & 0 & -3 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 \\ 1 & 6 & -6 & 0 & -1 \\ 3 & 7 & -7 & -1 & -2 \end{bmatrix}$

2. Find the value(s) of $h$ for which the vectors are linearly dependent. Justify your answer.

$$\begin{bmatrix} 1 \\ 5 \\ -3 \\ -9 \\ h \end{bmatrix}, \begin{bmatrix} -2 \\ -9 \\ 6 \\ -9 \end{bmatrix}, \begin{bmatrix} 3 \\ h \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -9 & h & 0 \\ -3 & 6 & -9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & 0 \\ 5 & -9 & h & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The homogeneous system $x_1 \begin{bmatrix} 1 \\ 5 \\ -3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ -9 \\ 6 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ h \\ -9 \end{bmatrix}$ has many solutions besides the trivial one $x_1 = x_2 = x_3 = 0$.

Therefore, vectors are linearly dependent for every $h$.
1. Compute the determinant of $A$ by cofactor expansions. At each step, choose a row or column that involves the least amount of computations.

$$A = \begin{bmatrix} 4 & 0 & -7 & 3 & -5 \\ 0 & 0 & 2 & 0 & 0 \\ 7 & 3 & -6 & 4 & -8 \\ 5 & 0 & 5 & 2 & -3 \\ 0 & 0 & 9 & -1 & 2 \end{bmatrix}$$

$$\det A = -2 \begin{vmatrix} 4 \ 7 \ 5 \ 0 \ 0 \ \end{vmatrix} - \begin{vmatrix} 0 \ 4 \ 2 \ 5 \ 1 \ \end{vmatrix} = -2 \cdot 3 \begin{vmatrix} 5 \ -2 \ 0 \ -1 \ 2 \ \end{vmatrix} =$$

$$= -6 \begin{vmatrix} 4 \ 2 \ -3 \ -5 \ \end{vmatrix} - \begin{vmatrix} 3 \ -5 \ \end{vmatrix} = -6(4 \cdot 1 - 5 \cdot 1) = 6$$
2. Compute the determinant of $A$ by row reduction to echelon form.

$$
A = \begin{bmatrix}
-2 & 5 & -7 & 2 & -6 \\
2 & -4 & 8 & 5 & 9 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & -3 \\
0 & 0 & -3 & -2 & 5 \\
\end{bmatrix} \sim \begin{bmatrix}
-2 & 5 & -7 & 2 & -6 \\
0 & 1 & 1 & 7 & 3 \\
0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & -3 \\
0 & 0 & -3 & -2 & 5 \\
\end{bmatrix} \\
\sim \begin{bmatrix}
-2 & 5 & -7 & 2 & -6 \\
0 & 1 & 1 & 7 & 3 \\
0 & 0 & -3 & -2 & 5 \\
0 & 0 & 0 & 1 & -3 \\
0 & 0 & 0 & 0 & -1 \\
\end{bmatrix} \text{ \text{ row exchange } n_3 \leftrightarrow n_5}
$$

$$
\det A = (-1) \begin{vmatrix}
-2 & 1 & -3 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
\end{vmatrix} = 6 \\
\text{ row exchange }
$$

3. Use your previous answer to determine if the following vectors are a linearly dependent set.

$$
\begin{bmatrix}
-2 \\
2 \\
0 \\
0 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
5 \\
-4 \\
0 \\
0 \\
0 \\
\end{bmatrix}, \begin{bmatrix}
-7 \\
8 \\
0 \\
0 \\
-3 \\
\end{bmatrix}, \begin{bmatrix}
2 \\
5 \\
0 \\
0 \\
1 \\
\end{bmatrix}, \begin{bmatrix}
-6 \\
9 \\
1 \\
0 \\
-3 \\
\end{bmatrix}
$$

since $\det A = 6 \neq 0$ \\
its columns are \underline{linearly independent}
1. The set \( B = \{1 + 2t, 2 - t\} \) is a basis for \( P_1 \).

Find the coordinate vector of \( p(t) = -1 + 8t \) relative to \( B \).

Find \( \alpha_1, \alpha_2 \) such that

\[
-1 + 8t = \alpha_1 (1 + 2t) + \alpha_2 (2 - t)
\]

\[
= (\alpha_1 + 2\alpha_2) + (2\alpha_1 - \alpha_2) t
\]

So

\[
\begin{align*}
\alpha_1 + 2\alpha_2 &= -1 \\
2\alpha_1 - \alpha_2 &= 8
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
\alpha_1 = 3 \\
\alpha_2 = -2
\end{cases}
\]

So \( [p]_B = \begin{bmatrix} 3 \\ -2 \end{bmatrix} \)

2. Find the dimension of the subspace spanned by the following vectors.

\[
A = \begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 2 & 1 & -2 & 1 \\ 0 & -5 & -20 & 15 \end{bmatrix}
\]

\[
A \sim \begin{bmatrix} 1 & 3 & 9 & -7 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & -20 & 15 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 & 3 \\ 0 & 1 & 4 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

\( A \) has 2 pivot columns, so \( \text{Col}(A) \) has dimension 2.
1. Find a basis for the eigenspace of $A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$ corresponding to eigenvalue 5.

$$\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \mathbf{v} = \mathbf{0} \quad \Rightarrow \quad \mathbf{v} = t \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ for } t \in \mathbb{R}$$

So $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $E_5 = \text{Nul}(A - 5I)$

2. Find the characteristic polynomial and the eigenvalues of $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$

$$P_A(\lambda) = \det(A - \lambda I) = \lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda = \begin{cases} 3 \\ -1 \end{cases}$$

Eigenvalues $3, -1$

3. The matrix $A$ below is factored in the form $PDP^{-1}$. Find the eigenvalues of $A$ and a basis for each eigenspace.

$$\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 1/2 & -3/4 \end{bmatrix} \begin{bmatrix} 1/4 & 1/2 & 1/4 \\ 0 & 1/2 & -1/2 \\ 0 & 0 & 1/4 \end{bmatrix}$$

Eigenvalues of $A$: $5, 1, 1$

Basis for $E_5 = \text{Nul}(A - 5I)$: $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \right\}$

Basis for $E_1 = \text{Nul}(A - I)$: $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$
1. Consider an orthogonal set of vectors \( \{\overrightarrow{u}_1, \overrightarrow{u}_2\} \), where \( \overrightarrow{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \) and \( \overrightarrow{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \).

Find the distance from \( \overrightarrow{y} = \begin{bmatrix} -3 \\ -3 \\ 9 \end{bmatrix} \) to the plane \( W \) in \( \mathbb{R}^3 \) spanned by \( \overrightarrow{u}_1 \) and \( \overrightarrow{u}_2 \).

\[
\overrightarrow{y} = \frac{\langle \overrightarrow{y}, \overrightarrow{u}_1 \rangle}{\langle \overrightarrow{u}_1, \overrightarrow{u}_1 \rangle} \overrightarrow{u}_1 + \frac{\langle \overrightarrow{y}, \overrightarrow{u}_2 \rangle}{\langle \overrightarrow{u}_2, \overrightarrow{u}_2 \rangle} \overrightarrow{u}_2 = \frac{12}{6} \overrightarrow{u}_1 + \frac{15}{5} \overrightarrow{u}_2
\]

\[
= 2 \overrightarrow{u}_1 + 3 \overrightarrow{u}_2 = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}
\]

so \( \overrightarrow{y} - \hat{\overrightarrow{y}} = \begin{bmatrix} -5 \\ -2 \\ 1 \end{bmatrix} \) and the distance from \( \overrightarrow{y} \) to \( W \) is

\[
\| \overrightarrow{y} - \hat{\overrightarrow{y}} \| = \sqrt{20}
\]
2. Let \( \vec{b}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} \) and \( \vec{b}_2 = \begin{bmatrix} 8 \\ 5 \\ -6 \end{bmatrix} \). Let \( W \) be a subspace of \( \mathbb{R}^3 \) with basis \( \{ \vec{b}_1, \vec{b}_2 \} \).

(a) Use the Gram-Schmidt process to produce an orthogonal basis for \( W \).

\[
\vec{u}_1 = \vec{b}_1 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}
\]

\[
\vec{u}_2 = \vec{b}_2 - \frac{\langle \vec{b}_2, \vec{u}_1 \rangle}{\langle \vec{u}_1, \vec{u}_1 \rangle} \vec{u}_1 = \vec{b}_2 - \frac{30}{10} \vec{u}_1
\]

\[
= \vec{b}_2 - 3 \vec{u}_1 = \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix}
\]

Orthogonal basis for \( W \):
\( \{ \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} \} \)

(b) Find an orthonormal basis for \( W \).

\[
\left\{ \frac{1}{\sqrt{10}} \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \quad \frac{1}{\sqrt{35}} \begin{bmatrix} -1 \\ 5 \\ -3 \end{bmatrix} \right\}
\]
Vector Spaces

Definition: A (real) vector space $V$ is a nonempty set of vectors with two operations (addition of vectors and multiplication by a scalar) such that for any $\vec{u}, \vec{v}, \vec{w}$ in $V$ and any scalars $c, d$ in $\mathbb{R}$:

i) $\vec{u} + \vec{v}$ is in $V$

ii) $\vec{u} + \vec{v} = \vec{v} + \vec{u}$

iii) $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

iv) There is $\vec{0}$ in $V$ such that $\vec{u} + \vec{0} = \vec{0} + \vec{u} = \vec{u}$

v) For each $\vec{u}$ in $V$ there is $-\vec{u}$ in $V$ such that $\vec{u} + (-\vec{u}) = (-\vec{u}) + \vec{u} = \vec{0}$

(we write $\vec{v} + (-\vec{u}) = \vec{v} - \vec{u}$)

vi) $c\vec{u}$ is in $V$

vii) $c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v}$

viii) $(c + d)\vec{u} = c\vec{u} + d\vec{u}$

ix) $c(d\vec{u}) = (cd)\vec{u}$

x) $1\vec{u} = \vec{u}$

Additional properties:

1. $0\vec{u} = \vec{0}$

2. $c\vec{0} = \vec{0}$

3. $(-1)\vec{u} = -\vec{u}$

Definition: A subspace $H$ of a vector space $V$ is a subset of $V$ that is also a vector space.

To check that a subset $H$ is a subspace of $V$, we only need to verify:

i) $\vec{0}$ in $H$

ii) for any $\vec{u}, \vec{v}$ in $H$, $\vec{u} + \vec{v}$ is also in $H$

iii) for any $\vec{u}$ in $H$ and any scalar $c$, $c\vec{u}$ is also in $H$

(Note that for a nonempty $H$, property (iii) implies (i).)
Matrix operations (properties)

Sum and scalar multiplication

For any $m \times n$ matrices $A, B, C$ and any scalars $c, d$ in $\mathbb{R}$:

i) $A + B = B + A$

ii) $(A + B) + C = A + (B + C)$

iii) $A + O = A$, where $O = \begin{bmatrix} 0 & 0 & \cdots \\ 0 & 0 & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}$ is the zero $m \times n$ matrix.

iv) $c(A + B) = cA + cB$

v) $(c + d)A = cA + dA$

vi) $c(dA) = (cd)A$

Matrix multiplication

For any matrices $A, B, C$ of the appropriate dimensions and any scalar $c$ in $\mathbb{R}$:

i) $(AB)C = A(BC)$

ii) $A(B + C) = AB + AC$

iii) $(A + B)C = AC + BC$

iv) $c(AB) = (cA)B = A(cB)$

v) $I_mA = A = AI_n$,

where $A$ is an $m \times n$ matrix, $I_m$ is the $m \times m$ identity and $I_n$ is the $n \times n$ identity.

Warning!

1) $AB \neq BA$ (in general)

2) $AB = AC \Rightarrow B = C$ (no cancellation rule)

3) $AB = O \Rightarrow A = O$ or $B = O$
Transpose

For any matrices $A, B$ of the appropriate dimensions and any scalar $c$ in $\mathbb{R}$:

i) $(A^T)^T = A$

ii) $(A + B)^T = A^T + B^T$

iii) $(cA)^T = cA^T$

iv) $(AB)^T = B^T A^T$

Inverse and transpose

For any square $n \times n$ matrices $A, B$:

i) $(A^{-1})^{-1} = A$

ii) $(AB)^{-1} = B^{-1} A^{-1}$

iii) $(A^T)^{-1} = (A^{-1})^T$
Characterization of invertible matrices

Theorem
Let $A$ be a square $n \times n$ matrix.
The following statements are all equivalent:

a) $A$ is invertible.

b) $A$ is row equivalent to the identity $I_n$.

c) $A$ has $n$ pivot positions.

d) $A\vec{x} = \vec{0}$ has only the trivial solution.

e) The columns of $A$ are linearly independent.

f) The linear transformation $\vec{x} \rightarrow A\vec{x}$ is one-to-one.

g) $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$ in $\mathbb{R}^n$.

h) The columns of $A$ span $\mathbb{R}^n$.

i) The linear transformation $\vec{x} \rightarrow A\vec{x}$ is onto $\mathbb{R}^n$.

j) There is an $n \times n$ matrix $C$ with $CA = I_n$.

k) There is an $n \times n$ matrix $B$ with $AB = I_n$.

l) $A^T$ is invertible.

Note. Not valid if $A$ is not square.

Update Statements (a,b,..,l) are also equivalent to:

m) $\det A \neq 0$. 
Update 2  Statements (a,b,\ldots,m) are also equivalent to:

n) \text{rank } A = \text{dim } \text{Col}(A) = n.

o) \text{Col}(A) = \mathbb{R}^n.

p) \text{nullity } A = \text{dim } \text{Nul}(A) = 0.

q) \text{Nul}(A) = \{\vec{0}\}.

r) \text{Row}(A) = \mathbb{R}^n.
Classification of quadratic forms

Let $A$ be a symmetric matrix, and $Q(\vec{x}) = \vec{x}^T A \vec{x}$ be a quadratic form.

We can classify a quadratic form $Q$ (or equivalently a symmetric matrix $A$) as follows.

a) $Q$ (or $A$) is **positive definite**:

$Q(\vec{x}) > 0$ for all $\vec{x} \neq \vec{0}$ $\iff$ all eigenvalues of $A$ are positive

e.g. $z = Q(x, y) = 3x^2 + 2y^2$

b) $Q$ (or $A$) is **negative definite**:

$Q(\vec{x}) < 0$ for all $\vec{x} \neq \vec{0}$ $\iff$ all eigenvalues of $A$ are negative

e.g. $z = Q(x, y) = -3x^2 - 2y^2$

c) $Q$ (or $A$) is **indefinite**:

$Q(\vec{x})$ takes both positive and negative values $\iff$ $A$ has positive and negative eigenvalues

e.g. $z = Q(x, y) = 3x^2 - 2y^2$

**Note:** positive/negative semidefinite are defined similarly, replacing $>$ by $\geq$ and $<$ by $\leq$. 
1. \((300\text{ points} \& \text{US}\$1,000,000)\)\(^1\)

Let \(\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}\) be the Riemann zeta function.

Prove that every non-trivial zero of \(\zeta\) is of the form \(z = 1/2 + ix\) with \(x \in \mathbb{R}\).

2. (a) (4 points) Compute the determinant of the following matrix

\[
A = \begin{bmatrix}
4 & -5 & 3 \\
2 & 0 & -1 \\
2 & -3 & 2
\end{bmatrix}
\]

\[
\begin{vmatrix}
4 & 0 & -1 \\
-3 & 2 & -5 \\
2 & 2 & 4 + 3
\end{vmatrix}
= 12 + 30 - 18 = 24
\]

\[
\det A = 24
\]

(b) (2 points) Is the transpose \(A^T\) invertible? Why or why not?

Yes, since \(\det A \neq 0\) \(A\) is invertible, which means \(A^T\) is invertible.

(c) (2 points) Is rank \(A = 3\)? Why or why not?

\(\text{Rank} = \# \text{ of pivot columns}. \) Yes, rank \(A = 3\), because \(A\) is invertible and invertible matrices have "\(n\)" pivot columns.

\(^1\)Happy April fool's day! This is the famous Riemann Hypothesis, one of the Millennium Prize Problems. The first person to solve it will be awarded a US\$1 million prize by the Clay Mathematics Institute.
3. (a) (4 points) Find a basis of \( H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \), where \( \vec{v}_1 = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix} \), \( \vec{v}_2 = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} \) and \( \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \). Justify your answer.

\[ \begin{bmatrix} -2 & 3 & 1 \\ 4 & -6 & 2 \\ 2 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} -2 & 3 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -2 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \]

2 pivots, so \( \dim H = 2 \) and basis of \( H = \{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \} \)

(b) (4 points) Let \( \vec{w}_1 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \) and \( \vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} \). Extend \( \{\vec{w}_1, \vec{w}_2\} \) to a basis of \( \mathbb{R}^3 \).

Justify your answer.

basis is extended with \( I \), since there are no entries in row 2, \( \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \) must be a vector in the basis.

\[ \text{basis of } \mathbb{R}^3 = \{ \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \} \]

4. (6 points) Let \( B = \{\vec{b}_1, \vec{b}_2\} \) and \( C = \{\vec{c}_1, \vec{c}_2\} \) be bases for \( \mathbb{R}^2 \), where

\( \vec{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \) and \( \vec{c}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \).

Find \( P \), that is, the change-of-coordinates matrix from \( B \) to \( C \).

\[ \begin{bmatrix} 1 & 2 \\ 1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

\[ P = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix} \]
5. (4 points) Find a basis of Nul(A), where \( A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \).

\[
\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -7 & 6 \\ 0 & 1 & 4 & -2 \end{bmatrix} \quad \begin{aligned} x_1 &= 7x_3 - 6x_4 \\ x_2 &= 4x_3 + 2x_4 \end{aligned}
\]

\[
x_3 \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix} \quad \text{basis } \text{Nul}(A) = \left\{ \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \\ 1 \end{bmatrix} \right\}
\]

\[
x_3 = \text{free} \\
x_4 = \text{free}
\]

(b) (2 points) Determine rank A.

\[
\text{Rank} = n - \dim \text{Nul}(A) = 4 - 2 = 2
\]

6. (a) (4 points) Find the characteristic polynomial and the eigenvalues of \( A = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix} \).

\[
\det(A - \lambda I) = \det \begin{bmatrix} -3-\lambda & -2 \\ 4 & 1-\lambda \end{bmatrix} = 0
\]

\[
(3-\lambda)(1-\lambda) - 4(-2) = -3 + 3\lambda - \lambda + \lambda^2 + 8 = \lambda^2 + 2\lambda + 5
\]

\[
\lambda = -2 \pm \frac{(-4)(1)(2)}{2} = -1 \pm i
\]

(b) (2 points) Is the matrix above diagonalizable? Justify your answer.

\[
\text{No, because the eigenvalues aren't real.}
\]

(c) (4 points) Find a basis for the eigenspace of \( B = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} \) corresponding to eigenvalue \( \lambda = 5 \).

\[
B v_1 = \lambda v_1
\]

\[
v_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\]

basis of eigenspace = \{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \}
7. (8 points) Let $A$ be an $m \times n$ matrix. Prove that $\text{Nul}(A)$ is a vector subspace of $\mathbb{R}^n$.

Since $A$ is $m \times n$, then there are $n$ variables in $A$. $\text{Nul}(A)$ will always be in $\mathbb{R}^n$ because $\text{Nul}(A)$ will always have as many entries as there are variables. $\text{Nul}(A)$ is the subspace made of the solutions to $A\vec{x} = \vec{0}$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$A\vec{x} = \vec{0}$$

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$x_1 = -2x_4$

$x_2 = -3x_4$

$x_3 = 2x_4$

$x_4$ is free

This is in $\mathbb{R}^4$, $n = 4$

$\text{Nul}(A) = \begin{bmatrix} -2 \\ -3 \\ 2 \\ 1 \end{bmatrix}$

8. (10 points) Mark the following statements as true (T) or false (F). You do NOT need to explain your answers.

(a) T or F. The rank of a matrix equals the number of nonzero rows.

(b) T or F. For any square matrix $A$ we have: $\det(-A) = -\det(A)$.

(c) T or F. The change-of-coordinates matrix between two bases is always invertible.

(d) T or F. If the matrix $A$ is diagonalizable, then $A^3$ is diagonalizable as well.

(e) T or F. If $\lambda$ is an eigenvalue of $A$, then $\lambda$ is an eigenvalue of $A^2$.

(f) T or F. If $\vec{x}$ is an eigenvector of $A$, then $\vec{x}$ is an eigenvector of $A^2$.

(g) T or F. If $A$ is $m \times n$ and $\text{rank}(A) = n$, then the columns of $A$ form a basis for $\mathbb{R}^m$.

(h) T or F. The set of all polynomials forms a vector space (with usual addition and scalar multiplication).

(i) T or F. The set of all continuous functions $f$ satisfying $f(0) = 1$ forms a vector space (with usual addition and scalar multiplication).

(j) T or F. If $A$ is row-equivalent to the identity matrix, then $A$ is diagonalizable.
1. (300 points & US$1,000,000)\(^1\)

Let \( \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \) be the Riemann zeta function.

Prove that every non-trivial zero of \( \zeta \) is of the form \( z = 1/2 + ix \) with \( x \in \mathbb{R} \).

2. (a) (4 points) Compute the determinant of the following matrix

\[
A = \begin{bmatrix}
4 & -5 & 3 \\
2 & 0 & -1 \\
2 & -3 & 2
\end{bmatrix}
\]

\[
\det(A) = 5 \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 3 \\ 2 & -1 \end{vmatrix} = 5(4+2) + 3(-4-6) = 30 + (-30) = 0
\]

(b) (2 points) Is the transpose \( A^T \) invertible? Why or why not?

No because \( A \) is linearly dependent

(c) (2 points) Is rank \( A = 3 \)? Why or why not?

No, because rank \( A = 3 \) implies 3 pivot columns which implies linear independence, but \( \det(A) = 0 \).

---

\(^1\)Happy April fool's day! This is the famous Riemann Hypothesis, one of the Millennium Prize Problems. The first person to solve it will be awarded a US$1 million prize by the Clay Mathematics Institute.
3. (a) (4 points) Find a basis of \( H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \), where \( \vec{v}_1 = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix}\) and \( \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\). Justify your answer.

\[
\begin{bmatrix} -2 & 3 & 1 \\ 4 & -6 & 2 \\ 2 & -3 & 3 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} -2 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\( r_3 = r_2 + r_1 \)  
\( r_2 = r_1 \)  

\( -2x_1 + 3x_2 + x_3 = 0 \)  
\( 4x_3 = 0 \)

\( \text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ -6 \\ -3 \end{bmatrix} \right\} \)

(b) (4 points) Let \( \vec{w}_1 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \) and \( \vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} \). Extend \( \left\{ \vec{w}_1, \vec{w}_2 \right\} \) to a basis of \( \mathbb{R}^3 \).

Justify your answer.

\[
\begin{bmatrix} -3 & 0 & 1 \\ 6 & 1 & 0 \\ 2 & -4 & 2 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}
\]

\( \text{The basis is} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \)

4. (6 points) Let \( B = \left\{ \vec{b}_1, \vec{b}_2 \right\} \) and \( C = \left\{ \vec{c}_1, \vec{c}_2 \right\} \) be bases for \( \mathbb{R}^2 \), where

\[
\vec{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \quad \vec{c}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}
\]

Find \( P_{\text{C \to \text{B}}} \), that is, the change-of-coordinates matrix from \( B \) to \( C \).

\[
P_{\text{C \to \text{B}}} = \begin{bmatrix} b_1 \end{bmatrix}_C \begin{bmatrix} b_2 \end{bmatrix}_C \xrightarrow{\text{OK}} \begin{bmatrix} 2 & 3 \\ -1 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}
\]

\[
P_{\text{C \to \text{B}}} = \begin{bmatrix} \frac{6}{15} & \frac{3}{15} \\ \frac{1}{15} & \frac{2}{15} \end{bmatrix}
\]

\[
P_{\text{C \to \text{B}}} = \begin{bmatrix} \frac{6}{15} & \frac{3}{15} \\ \frac{1}{15} & \frac{2}{15} \end{bmatrix}
\]
5. (a) (4 points) Find a basis of $\text{Nul}(A)$, where $A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \\ \end{bmatrix}$.

\[
\begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \\ \end{pmatrix} \xrightarrow{r_1 - 3r_2} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \\ \end{pmatrix} \xrightarrow{r_1 - 5r_3} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \\ \end{pmatrix} \xrightarrow{r_2 + 4r_3} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \\ \end{pmatrix} \xrightarrow{r_3 + 2r_4} \begin{pmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \\ \end{pmatrix}
\]

(b) (2 points) Determine $\text{rank } A$.

\[
\text{Rank } A = 2
\]

6. (a) (4 points) Find the characteristic polynomial and the eigenvalues of $A = \begin{bmatrix} -3 & -2 \\ 4 & 1 \\ \end{bmatrix}$.

\[
\begin{pmatrix} -3 - \lambda & 1 \\ 4 & -1 - \lambda \\ \end{pmatrix} + 8 = 0
\]

\[
-3 + 3\lambda - \lambda + x^2 + 8 = \lambda^2 + 2\lambda + 5
\]

\[
\lambda = \frac{-2 \pm \sqrt{16 - 4(5)}}{2} = -1 \pm 2i
\]

(b) (2 points) Is the matrix above diagonalizable? Justify your answer.

\[
A = PDP^{-1}
\]

\[
\text{yes, because there are 2 eigenvalues and therefore } D = \begin{bmatrix} -1 + 2i & 0 \\ 0 & -1 - 2i \\ \end{bmatrix}
\]

(c) (4 points) Find a basis for the eigenspace of $B = \begin{bmatrix} 1 & 2 \\ 0 & 5 \\ \end{bmatrix}$ corresponding to eigenvalue 5.

\[
\text{Nul}(B - 5I) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \end{pmatrix}
\]

\[
\text{Span} \left\{ \begin{pmatrix} 1 \\ 2 \\ \end{pmatrix} \right\}
\]
7. (8 points) Let $A$ be an $m \times n$ matrix. Prove that $\text{Nul}(A)$ is a vector subspace of $\mathbb{R}^n$.

The $\text{Nul}(A) = \text{Span} \{ \vec{x} : A\vec{x} = \vec{0} \}$

Because $A$ has $n$ columns the solution set of $A\vec{x} = \vec{0} \in \mathbb{R}^n$.

Spanned sets are always subspaces.

$\text{Nul}(A)$ is a subspace of $\mathbb{R}^n$.

It is true that $\text{Nul}(A) = \text{Span} \{ \vec{x} : A\vec{x} = \vec{0} \}$ without Span.

show that $\vec{x}, \vec{y}$ in $\text{Nul}(A) \Rightarrow \vec{x} + \vec{y}$ in $\text{Nul}(A)$.

8. (10 points) Mark the following statements as true (T) or false (F). You do NOT need to explain your answers.

(a) T or F. The rank of a matrix equals the number of nonzero rows.

(b) T or F. For any square matrix $A$ we have: $\det(-A) = -\det(A)$.

(c) T or F. The change-of-coordinates matrix between two bases is always invertible.

(d) T or F. If the matrix $A$ is diagonalizable, then $A^3$ is diagonalizable as well.

(e) T or F. If $\lambda$ is an eigenvalue of $A$, then $\lambda$ is an eigenvalue of $A^2$

(f) T or F. If $\vec{x}$ is an eigenvector of $A$, then $\vec{x}$ is an eigenvector of $A^2$

(g) T or F. If $A$ is $m \times n$ and rank$(A) = n$, then the columns of $A$ form a basis for $\mathbb{R}^n$.

(h) T or F. The set of all polynomials forms a vector space (with usual addition and scalar multiplication).

(i) T or F. The set of all continuous functions $f$ satisfying $f(0) = 1$ forms a vector space (with usual addition and scalar multiplication).

(j) T or F. If $A$ is row-equivalent to the identity matrix, then $A$ is diagonalizable.
1. (300 points & US$1,000,000)$^{1}$

Let \( \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \) be the Riemann zeta function.

Prove that every non-trivial zero of \( \zeta \) is of the form \( z = 1/2 + ix \) with \( x \in \mathbb{R} \).

2. (a) (4 points) Compute the determinant of the following matrix

\[
A = \begin{bmatrix}
4 & -5 & 3 \\
2 & 0 & -1 \\
2 & -3 & 2 \\
\end{bmatrix}
\]

\[
= -2 \begin{vmatrix} -5 & 3 \\ 2 & -3 \end{vmatrix} + 1 \begin{vmatrix} 4 & -5 \\ 2 & -3 \end{vmatrix}
\]

\[
= -2(-10 + 9) + 1(-12 + 10)
\]

\[
= -2(-1) + 1(-2)
\]

\[
= 2 + 2 = 0
\]

(b) (2 points) Is the transpose \( A^T \) invertible? Why or why not?

No \( \text{bc} \) the \( \det A = 0 \)

(c) (2 points) Is \( \text{rank} A = 3 \)? Why or why not?

\[
\begin{bmatrix}
4 & -5 & 3 \\
2 & -3 & 2 \\
2 & 0 & -1 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
4 & -5 & 3 \\
0 & -\frac{1}{2} & \frac{1}{2} \\
0 & 0 & -3 \\
\end{bmatrix}
\rightarrow
\begin{bmatrix}
4 & -5 & 3 \\
0 & -\frac{1}{2} & \frac{1}{2} \\
0 & 0 & 0 \\
\end{bmatrix}
\]

\[
\text{No \( \text{bc} \) not 3 pivot columns}
\]

\[
\frac{1}{2}k_2 - k_3 \\
\frac{1}{2}k_2 + k_3 \\
-k_3
\]

\[1\text{Happy April fool's day! This is the famous Riemann Hypothesis, one of the Millennium Prize Problems. The first person to solve it will be awarded a US$1 million prize by the Clay Mathematics Institute.}\]
3. (4 points) Find a basis of $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where $\vec{v}_1 = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix}$ and

$$\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$ Justify your answer.

$$\begin{bmatrix} -2 & 3 & 1 \\ 4 & -6 & 2 \\ 2 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} -2 & 3 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$L_2 \Rightarrow L_1 + 2L_1$

$$\text{Solve: } -2x_1 + 3x_2 = 0, \quad 2x_1 - 3x_2 = 0, \quad x_3 = 0$$

$$\Rightarrow \text{inconsistent}$$

(b) (4 points) Let $\vec{w}_1 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$. Extend $\{\vec{w}_1, \vec{w}_2\}$ to a basis of $\mathbb{R}^3$.

Justify your answer.

$$\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} \Rightarrow \text{2 vectors do not span } \mathbb{R}^3.$$

4. (6 points) Let $B = \{\vec{b}_1, \vec{b}_2\}$ and $C = \{\vec{c}_1, \vec{c}_2\}$ be bases for $\mathbb{R}^2$, where

$$\vec{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \text{and} \quad \vec{c}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \vec{c}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$ Find $P_{c_1 \rightarrow B}$, that is, the change-of-coordinates matrix from $B$ to $C$.

$$\begin{bmatrix} 2 \\ -1 \end{bmatrix}_B \Rightarrow \begin{bmatrix} 1 \\ 1 \end{bmatrix}_C = \begin{bmatrix} \frac{1}{2} \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 6 \end{bmatrix}_B \Rightarrow \begin{bmatrix} 1 \\ 4 \end{bmatrix}_C = \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$$
5. (a) (4 points) Find a basis of \( \text{Nul}(A) \), where \( A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \). 

\[
\begin{align*}
\begin{bmatrix} 7 \\ -4 \\ 1 \\ 0 \end{bmatrix} & \text{ and } \\
\begin{bmatrix} -3 \\ 2 \\ 0 \end{bmatrix} & \text{ is a basis of } \text{Nul}(A) 
\end{align*}
\]

(b) (2 points) Determine rank \( A \).

\[ \text{rank } A = 2 \] 

6. (a) (4 points) Find the characteristic polynomial and the eigenvalues of \( A = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix} \).

\[
\begin{align*}
(\lambda + 3)(\lambda + 1) + 8 & = 0 \\
-3\lambda - 4 + A^2 & = 0 \\
\text{CP: } \lambda^2 + 2\lambda + 5 & = 0
\end{align*}
\]

(b) (2 points) Is the matrix above diagonalizable? Justify your answer.

No, \( \lambda \) has complex eigenvalues.

(c) (4 points) Find a basis for the eigenspace of \( B = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} \) corresponding to eigenvalue \( \lambda = 5 \).

\[
\begin{align*}
(A - 5I) & = \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \\
\text{and } & \\
-4x_1 + 2x_2 = 0 \\
x_1 & = \frac{1}{2} x_2 \\
x_2 = 1 & \Rightarrow \text{span } \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
x_2 & = 2 \Rightarrow \text{span } \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\end{align*}
\]

basis for \( \lambda = 5 \):

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

\[
\text{dim } \text{span } \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2
\]
7. (8 points) Let $A$ be an $m \times n$ matrix. Prove that $\text{Nul}(A)$ is a vector subspace of $\mathbb{R}^n$.

$$\text{rank } A = \# \text{ piv. pos. } = \dim \text{col } A = \dim \text{row } A$$

$$\dim \text{null } A = n - \text{rank } A$$

$\text{Nul}(A)$ is a vector subspace of $\mathbb{R}^n$ b/c $\text{Nul } A$ spans $\mathbb{R}^n$.

8. (10 points) Mark the following statements as true (T) or false (F). You do NOT need to explain your answers.

(a) T or F. The rank of a matrix equals the number of nonzero rows.

(b) T or F. For any square matrix $A$ we have: $\det(-A) = -\det(A)$.

(c) T or F. The change-of-coordinates matrix between two bases is always invertible.

(d) T or F. If the matrix $A$ is diagonalizable, then $A^3$ is diagonalizable as well.

(e) T or F. If $\lambda$ is an eigenvalue of $A$, then $\lambda$ is an eigenvalue of $A^2$

(f) T or F. If $\vec{x}$ is an eigenvector of $A$, then $\vec{x}$ is an eigenvector of $A^2$

(g) T or F. If $A$ is $m \times n$ and $\text{rank}(A) = n$, then the columns of $A$ form a basis for $\mathbb{R}^m$.

(h) T or F. The set of all polynomials forms a vector space (with usual addition and scalar multiplication).

(i) T or F. The set of all continuous functions $f$ satisfying $f(0) = 1$ forms a vector space (with usual addition and scalar multiplication).

(j) T or F. If $A$ is row-equivalent to the identity matrix, then $A$ is diagonalizable.
1. (300 points & US$1,000,000)$^{1}$

Let $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ be the Riemann zeta function.

Prove that every non-trivial zero of $\zeta$ is of the form $z = 1/2 + ix$ with $x \in \mathbb{R}$.

2. (a) (4 points) Compute the determinant of the following matrix

$$A = \begin{bmatrix} 4 & -5 & 3 \\ 2 & 0 & -1 \\ 2 & -3 & 2 \end{bmatrix}$$

$$\det A = \left((4 \cdot -3) + (2 \cdot -1) + (2 \cdot 3)\right) - \left((-5 \cdot 2) + (4 \cdot -1) + (2 \cdot -3)\right)$$

$$\begin{align*}
&= \left(-8 - 18\right) - \left(-20 + 12\right) \\
&= -26 + 8 \\
&= -18
\end{align*}$$

(b) (2 points) Is the transpose $A^T$ invertible? Why or why not?

No b/c $\det A \neq 0$

(c) (2 points) Is rank $A = 3$? Why or why not?

No b/c $\det A \neq 0$

$^{1}$Happy April fool's day! This is the famous Riemann Hypothesis, one of the Millennium Prize Problems. The first person to solve it will be awarded a US$1 million prize by the Clay Mathematics Institute.
3. (a) (4 points) Find a basis of $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where $\vec{v}_1 = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Justify your answer. Since $\vec{v}_2 = 1.5 \vec{v}_1$, the basis is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ or $\{\vec{v}_1, 1.5 \vec{v}_1, \vec{v}_3\}$.

(b) (4 points) Let $\vec{w}_1 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$. Extend $\{\vec{w}_1, \vec{w}_2\}$ to a basis of $\mathbb{R}^3$. Extend $\{\vec{w}_1, \vec{w}_2\}$ to a basis of $\mathbb{R}^3$. B/c they are linearly independent, $\begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$ both are needed to form a basis.

You need to add a 3rd vector.

4. (6 points) Let $B = \{\vec{b}_1, \vec{b}_2\}$ and $C = \{\vec{c}_1, \vec{c}_2\}$ be bases for $\mathbb{R}^2$, where

$\vec{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\vec{c}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Find $P_{c\rightarrow B}$, that is, the change-of-coordinates matrix from $B$ to $C$. $P_{c\rightarrow B}$ =

$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$
5. (a) (4 points) Find a basis of \( \text{Nul}(A) \), where \( A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \).

\[
\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = 7x_3 - 6x_4
\]
\[
\begin{bmatrix} 1 \\ 0 \\ 7 \\ 6 \end{bmatrix}
\]

(b) (2 points) Determine \( \text{rank}(A) \).

\[
\text{rank}(A) = 2 \quad \text{why?}
\]

6. (a) (4 points) Find the characteristic polynomial and the eigenvalues of \( A = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix} \).

\[
\det(\lambda I - A) = \begin{vmatrix} -3 - \lambda & -2 \\ 4 & 1 - \lambda \end{vmatrix} = \begin{vmatrix} 1 - \lambda \end{vmatrix} - \begin{vmatrix} -2 \end{vmatrix} = \begin{vmatrix} 0 \end{vmatrix} - \begin{vmatrix} 6 \end{vmatrix} = \begin{vmatrix} 0 \end{vmatrix} - \begin{vmatrix} 6 \end{vmatrix} = \begin{vmatrix} -6 \end{vmatrix} = -6
\]

\[
\chi_A(\lambda) = -6 = (\lambda + 2)(\lambda - 3)
\]

(b) (2 points) Is the matrix above diagonalizable? Justify your answer.

\[
\text{Yes, } \text{but } \text{det}(A - \lambda I) = 0 \quad \text{wrong reason}
\]

(c) (4 points) Find a basis for the eigenspace of \( B = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} \) corresponding to eigenvalue 5.

\[
\text{Nul}(B - 5I) = \text{Nul}\left[\begin{bmatrix} -4 & 2 \\ 0 & -5 \end{bmatrix}\right]
\]

\[
-4x_1 = -2x_2
\]
\[
x_1 = \frac{1}{2} x_2
\]

basis = \[\begin{bmatrix} 1 \\ 1 \end{bmatrix}\]
7. (8 points) Let \( A \) be an \( m \times n \) matrix. Prove that \( \text{Nul}(A) \) is a vector subspace of \( \mathbb{R}^n \).

\[ \text{Nul}(A) \text{ means } A\vec{r} = \vec{0} \text{ so } \vec{0} \text{ is in Nul(}A) \]

\[ cA\vec{r} = \vec{0} \text{ will work & modify } A \]

\[ p(0) + q(0) = 0 \]

8. (10 points) Mark the following statements as true (T) or false (F). You do NOT need to explain your answers.

(a) T or F. The rank of a matrix equals the number of nonzero rows.

(b) T or F. For any square matrix \( A \) we have: \( \det(-A) = -\det(A) \).

(c) T or F. The change-of-coordinates matrix between two bases is always invertible.

(d) T or F. If the matrix \( A \) is diagonalizable, then \( A^3 \) is diagonalizable as well.

(e) T or F. If \( \lambda \) is an eigenvalue of \( A \), then \( \lambda \) is an eigenvalue of \( A^2 \)

(f) T or F. If \( \vec{x} \) is an eigenvector of \( A \), then \( \vec{x} \) is an eigenvector of \( A^2 \)

(g) T or F. If \( A \) is \( m \times n \) and \( \text{rank}(A) = n \), then the columns of \( A \) form a basis for \( \mathbb{R}^m \).

(h) T or F. The set of all polynomials forms a vector space (with usual addition and scalar multiplication).

(i) T or F. The set of all continuous functions \( f \) satisfying \( f(0) = 1 \) forms a vector space (with usual addition and scalar multiplication).

(j) T or F. If \( A \) is row-equivalent to the identity matrix, then \( A \) is diagonalizable.
\begin{align*}
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \\
2e^2 &= 2 \\
1^2 &= 1 \\
2\ln 2 &= 2\ln 1 + 1 \\
2\ln 2 &= 0 \\
\ln 2 &= 0 \\
e^{2\ln 2} &= e^0 \\
2 &= 1 \\
\text{Call the institute, I've done it}
\end{align*}
1. (300 points & US$1,000,000)$^1$

Let $\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$ be the Riemann zeta function.

Prove that every non-trivial zero of $\zeta$ is of the form $z = 1/2 + ix$ with $x \in \mathbb{R}$.

The margins of this paper are too narrow for my proof...

2. (a) (4 points) Compute the determinant of the following matrix

\[
A = \begin{bmatrix}
4 & -5 & 3 \\
2 & 0 & -1 \\
2 & -3 & 2
\end{bmatrix}
\]

\[
\begin{align*}
\det A &= 4 \begin{vmatrix}
2 & -1 \\
2 & -3
\end{vmatrix} - 5 \begin{vmatrix}
2 & -1 \\
2 & 2
\end{vmatrix} + 3 \begin{vmatrix}
2 & -1 \\
2 & 2
\end{vmatrix} \\
&= 4(0 - 2) - 5(4 + 2) + 3(-6) \\
&= -12 + 30 - 18 \\
&= 0
\end{align*}
\]

So: \[\text{Let } A = 0.\]

(b) (2 points) Is the transpose $A^T$ invertible? Why or why not?

Since $A$ is not invertible ($\det A = 0 \Rightarrow$ not invertible), $A^T$ (the transpose) is not invertible, either. \[\text{NO}\]

(c) (2 points) Is rank $A = 3$? Why or why not?

Since $\det (A) = 0$, not all columns are linearly independent. Since the rank is the # of independent columns, and there are only 2 independent columns, rank $A \neq 3$. \[\text{NO}\]

$^1$Happy April fool's day! This is the famous Riemann Hypothesis, one of the Millennium Prize Problems. The first person to solve it will be awarded a US$1$ million prize by the Clay Mathematics Institute.
3. (a) (4 points) Find a basis of $H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$, where $\vec{v}_1 = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix}$ and $\vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Justify your answer.

Notice that $\vec{v}_3 = 1.5 \vec{v}_1 - \vec{v}_2$. Since the basis is the smallest collection of linearly independent vectors that span $H$, and $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent (by observation — easy to verify), and since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is the smallest spanning set (since only 2 linearly independent components of the definition of $H$)

The basis $B$ of $H$: $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\} = \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{3} \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

(b) (4 points) Let $\vec{w}_1 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix}$ and $\vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix}$. Extend $\{\vec{w}_1, \vec{w}_2\}$ to a basis of $\mathbb{R}^3$.

Justify your answer. First, notice that $\vec{w}_1$ and $\vec{w}_2$ are linearly independent (by inspection — obvious).

Then, recall that for $\mathbb{R}^n$, a set of $n$ linearly independent vectors in $\mathbb{R}^n$ forms a basis. Then, observe that $\vec{w}_1$ and $\vec{w}_2$ do not have an $x_2$ component. Then, let $\vec{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Since this is linearly independent from $\vec{w}_1$ and $\vec{w}_2$, $\vec{w}_1, \vec{w}_2, \vec{w}_3$ form a basis by the above.

4. (6 points) Let $B = \{\vec{b}_1, \vec{b}_2\}$ and $C = \{\vec{c}_1, \vec{c}_2\}$ be bases for $\mathbb{R}^2$, where

$\vec{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\vec{b}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ and $\vec{c}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{c}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$.

Find $P_{C\rightarrow B}$ that is, the change-of-coordinates matrix from $B$ to $C$.

$$
\begin{bmatrix}
\vec{b}_1^c \\
\vec{b}_2^c
\end{bmatrix} = P_{C\rightarrow B}
$$

Using work from right, we know $P_{C\rightarrow B} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}$ by composing $[\vec{b}_1^c]$ and $[\vec{b}_2^c]$ together.

$$
\Rightarrow P_{C\rightarrow B} = \begin{bmatrix} 3 & 2 \\ -1 & 1 \end{bmatrix}
$$
5. (a) (4 points) Find a basis of \( \text{Nul}(A) \), where \( A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \).

Remember: \( \text{null}(A) \) is spanned by linearly dependent columns (i.e., non-pivot columns) of \( A \). So, \( \text{Nul}(A) = \text{Span} \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \end{bmatrix} \right\} \).

Since there are linearly independent columns by observation, a basis is a set of spanning independent vectors.

\text{Basis of Nul}(A) \quad B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \end{bmatrix} \right\}.

(b) (2 points) Determine rank \( A \).

\text{Rank} + \text{nullity} = \text{columns} \quad \text{nullity} = \text{dim}(\text{Nul}(A))

= 2. \quad \text{Since columns} = 4, \quad \text{rank} + 2 = 4. \quad \text{So:} \quad \text{rank} = 2.

6. (a) (4 points) Find the characteristic polynomial and the eigenvalues of \( A = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix} \).

\text{det} \left[ \begin{bmatrix} -3-\lambda & -2 \\ 4 & 1-\lambda \end{bmatrix} \right] = (-3-\lambda)(1-\lambda) - (-2)(4)

= 3 - 2\lambda + \lambda^2 + 8 = \lambda^2 + 2\lambda + 5 = 0.

\text{Characteristic polynomial:} \quad \lambda^2 + 2\lambda + 5 = 0.

\lambda = \frac{-2 \pm \sqrt{(-2)^2 - 4(1)(5)}}{2} = -1 \pm \sqrt{-4} \quad \text{eigenvalues}.

\lambda = -1 + 2i, \quad -1 - 2i.

(b) (2 points) Is the matrix above diagonalizable? Justify your answer.

Since eigenvalues exist, these exist eigenvectors for both. So, we can make \( PDP^{-1} \) from eigenvectors. So, since \( D \) is diagonal, \( \text{Y \in \mathbb{S} \quad \text{and are different!} \) \)

(c) (4 points) Find a basis for the eigenspace of \( B = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} \) corresponding to eigenvalue \( \lambda = \frac{1}{2} \).

\( (B - \lambda I)x = 0 \) \quad \Rightarrow \quad \begin{bmatrix} -\frac{1}{2} & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0

\Rightarrow -4x_1 + 2x_2 = 0 \Rightarrow -4x_1 = -2x_2 \Rightarrow x_1 = \frac{1}{2} x_2, \quad x_2 \text{ free}

\Rightarrow x = \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}

\Rightarrow \text{Basis} = \left\{ \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} \right\}.
7. (8 points) Let \( A \) be an \( m \times n \) matrix. Prove that \( \text{Null}(A) \) is a vector subspace of \( \mathbb{R}^n \).

To prove subspace: if \( p, q \in \text{Null}(A) \) then \( p + q \in \text{Null}(A) \).

If \( c \in \mathbb{R} \), \( c \cdot p \in \text{Null}(A) \).

Remember: \( \text{Null}(A) \) is a span of the non-independent columns of \( A \).

\[ \text{Null}(A) = \text{Span} \{ n_1, \ldots, n_i, \ldots, n_z \} \]

For some value \( i \). If no null space of \( A \), then \( \text{Null}(A) = \text{Span} \{ \} = \{0\} \).

By definition, if \( p \) and \( q \in \text{Span} \{ n_1, \ldots, n_i, \ldots, n_z \} \), then \( p + q \in \text{Span} \{ n_1, \ldots, n_i, \ldots, n_z \} \).

So, we meet property (1).

Similarly, by definition of "span", if \( p \in \text{Span} \{ n_1, \ldots, n_i, \ldots, n_z \} \), then \( c \cdot p \in \text{Span} \{ n_1, \ldots, n_i, \ldots, n_z \} \), \( c \in \mathbb{R} \). So, we meet property (2).

Since \( \{ \text{for } p, q \in \text{Null}(A), c \in \mathbb{R} \} (p + q \in \text{Null}(A) \text{ and } c \cdot p \in \text{Null}(A)) \),

\[ \text{Null}(A) \text{ is a vector subspace of } \mathbb{R}^n \]

So, we have proven what we need to.

8. (10 points) Mark the following statements as true (T) or false (F). You do NOT need to explain your answers.

(a) \( \text{T or F} \): The rank of a matrix equals the number of nonzero rows.

(b) \( \text{T or F} \): For any square matrix \( A \) we have: \( \det(-A) = -\det(A) \). Suppose \( 2 \times 2 \) diagonal.

(c) \( \text{T or F} \): The change-of-coordinates matrix between two bases is always invertible.

(d) \( \text{T or F} \): If the matrix \( A \) is diagonalizable, then \( A^3 \) is diagonalizable as well.

(e) \( \text{T or F} \): If \( \lambda \) is an eigenvalue of \( A \), then \( \lambda \) is an eigenvalue of \( A^2 \)

(f) \( \text{T or F} \): If \( \vec{x} \) is an eigenvector of \( A \), then \( \vec{x} \) is an eigenvector of \( A^2 \)

(g) \( \text{T or F} \): If \( A \) is \( m \times n \) and \( \text{rank}(A) = n \), then the columns of \( A \) form a basis for \( \mathbb{R}^n \).

(h) \( \text{T or F} \): The set of all polynomials forms a vector space (with usual addition and scalar multiplication).

(i) \( \text{T or F} \): The set of all continuous functions \( f \) satisfying \( f(0) = 1 \) forms a vector space (with usual addition and scalar multiplication).

(j) \( \text{T or F} \): If \( A \) is row-equivalent to the identity matrix, then \( A \) is diagonalizable.
Page for scratch work. Please indicate in the problem if you have work here.

\[
\begin{align*}
\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 - \lambda & -2 \\ 0 & -2 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} - \lambda & 0 \\ 0 & \frac{1}{2} \end{bmatrix} & = 0 \\
\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 - \lambda & 1 - 2 \\ 0 & 1 - 2 \end{bmatrix} & = 1, 2 \\
\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} & = \begin{bmatrix} 4 - \lambda & 0 \\ 0 & 1 - 2 \end{bmatrix} = 1, 2 \\
\text{for } \lambda \\
\text{and } 2 = 1, 4
\end{align*}
\]
1. (300 points & US$1,000,000)\(^1\)

Let \(\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}\) be the Riemann zeta function.

Prove that every non-trivial zero of \(\zeta\) is of the form \(z = 1/2 + ix\) with \(x \in \mathbb{R}\).

2. (a) (4 points) Compute the determinant of the following matrix

\[
A = \begin{bmatrix}
4 & -5 & 3 \\
2 & 0 & -1 \\
2 & 3 & 2 \\
\end{bmatrix}
\]

Expand the 1st row

\[\det A = 4 \cdot (0 - 3) - (-5) \cdot (4 + 2) + 3 \cdot (-6) = 0\]

(b) (2 points) Is the transpose \(A^T\) invertible? Why or why not?

No. Since \(A\) is not invertible.

(c) (2 points) Is rank \(A = 3\)? Why or why not?

No. Since \(\det A = 0\), \(A\) is not invertible, so \(\text{rank } A < 3\).

---

\(^1\)Happy April fool's day! This is the famous Riemann Hypothesis, one of the Millennium Prize Problems. The first person to solve it will be awarded a US$1 million prize by the Clay Mathematics Institute.
3. (a) (4 points) Find a basis of \( H = \text{Span}\{\vec{v}_1, \vec{v}_2, \vec{v}_3\} \), where \( \vec{v}_1 = \begin{bmatrix} -2 \\ 4 \\ 2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 3 \\ -6 \\ -3 \end{bmatrix} \) and \( \vec{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \). Justify your answer.

\[
\begin{bmatrix} -2 & 3 & 1 \\ 4 & -6 & 2 \\ 2 & -3 & 3 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} -2 & 3 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 4 \end{bmatrix} \xrightarrow{\text{RREF}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}
\]

\[
\begin{bmatrix} -2 & 3 \\ 0 & 4 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ has the pivot position. So the bases of } H \text{ would be } \left\{ \vec{v}_1, \vec{v}_2 \right\}, \text{ since } \vec{v}_3 \text{ is a linear combination of } \vec{v}_1 \text{ and } \vec{v}_2.
\]

(b) (4 points) Let \( \vec{w}_1 = \begin{bmatrix} -3 \\ 0 \\ 2 \end{bmatrix} \) and \( \vec{w}_2 = \begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} \). Extend \( \{\vec{w}_1, \vec{w}_2\} \) to a basis of \( \mathbb{R}^3 \).

Justify your answer.

We can find a vector that is linearly independent to \( \vec{w}_1 \) and \( \vec{w}_2 \).

4. (6 points) Let \( B = \left\{ \vec{b}_1, \vec{b}_2 \right\} \) and \( C = \left\{ \vec{c}_1, \vec{c}_2 \right\} \) be bases for \( \mathbb{R}^2 \), where

\[
\vec{b}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \quad \text{and} \quad \vec{c}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{c}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}.
\]

Find \( P_{C \rightarrow B} \), that is, the change-of-coordinates matrix from \( B \) to \( C \).
5. (a) (4 points) Find a basis of \( \text{Nul}(A) \), where \( A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \).

\[
\begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 0 & 7/3 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 0 & 1 & -6/7 \end{bmatrix}
\]

\[
\begin{aligned}
&x_1 + 3x_2 + 5x_3 = 0 \\
x_2 - 6/7x_4 = 0
\end{aligned}
\]

\(x_2\) and \(x_4\) are free, so \(x_1 = -3x_2 - \frac{30}{7}x_4\) and \(x_3 = \frac{6}{7}x_4\).

(b) (2 points) Determine \(\text{rank}(A)\).

6. (a) (4 points) Find the characteristic polynomial and the eigenvalues of \( A = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix} \).

\[
A = \begin{bmatrix} -3 & -2 \\ 4 & 1 \end{bmatrix}, \quad A - \lambda I = \begin{bmatrix} -3 - \lambda & -2 \\ 4 & 1 - \lambda \end{bmatrix}
\]

\[
\det(A - \lambda I) = (-3 - \lambda)(1 - \lambda) + 8 = 0
\]

\[
\lambda^2 + 2\lambda + 5 = 0
\]

(b) (2 points) Is the matrix above diagonalizable? Justify your answer.

No.

(c) (4 points) Find a basis for the eigenspace of \( B = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} \) corresponding to eigenvalue 5.

\[
\lambda = 5
\]

\[
A - 5I = \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}
\]

\[
\begin{aligned}
&-4x_1 + 2x_2 = 0 \\
x_2 \text{ is free}
\end{aligned}
\]

\[
\begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}
\]

So \(x_2\) is free.
7. (8 points) Let $A$ be an $m \times n$ matrix. Prove that $\text{Nul}(A)$ is a vector subspace of $\mathbb{R}^n$.

If $A$ is a $m \times n$ matrix, $\dim \text{Nul}(A)$ would be $n$.

And the zero vector would be included in $\text{Nul}(A)$.

So $\text{Nul}(A)$ is a vector subspace of $\mathbb{R}^n$.

---

8. (10 points) Mark the following statements as true (T) or false (F). You do NOT need to explain your answers.

(a) T or F. The rank of a matrix equals the number of nonzero rows.

(b) T or F. For any square matrix $A$ we have: $\det(-A) = -\det(A)$.

(c) T or F. The change-of-coordinates matrix between two bases is always invertible.

(d) T or F. If the matrix $A$ is diagonalizable, then $A^3$ is diagonalizable as well.

(e) T or F. If $\lambda$ is an eigenvalue of $A$, then $\lambda$ is an eigenvalue of $A^2$

(f) T or F. If $\vec{x}$ is an eigenvector of $A$, then $\vec{x}$ is an eigenvector of $A^2$

(g) T or F. If $A$ is $m \times n$ and $\text{rank}(A) = n$, then the columns of $A$ form a basis for $\mathbb{R}^m$.

(h) T or F. The set of all polynomials forms a vector space (with usual addition and scalar multiplication).

(i) T or F. The set of all continuous functions $f$ satisfying $f(0) = 1$ forms a vector space (with usual addition and scalar multiplication).

(j) T or F. If $A$ is row-equivalent to the identity matrix, then $A$ is diagonalizable.