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Physics, Chapter 32: Electromagnetic Induction

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Electromagnetic Induction

32-1 Motion of a Wire in a Magnetic Field

When a wire moves through a uniform magnetic field of induction B, in a direction at right angles to the field and to the wire itself, the electric charges within the conductor experience forces due to their motion through this magnetic field. The positive charges are held in place in the conductor by the action of interatomic forces, but the free electrons, usually one or two per atom, are caused to drift to one side of the conductor, thus setting up an electric field E within the conductor which opposes the further drift of electrons. The magnitude of this electric field E may be calculated by equating the force it exerts on a charge q, to the force on this charge due to its motion through the magnetic field of induction B; thus

$$Eq = Bqv,$$

from which

$$E = Bv.$$

If, as a result of the motion of the wire through the magnetic field, a charge q is moved a distance s along the wire against the internal electric field E, a quantity of work $\mathscr{W}$ is done by the agency moving the wire, given by the expression

$$\mathscr{W} = Eqs = Bvqs.$$

Thus an electromotive force is generated within the wire as a result of its motion through the magnetic field. The electromotive force across the ends of the wire is the work per unit charge done by the agency moving the wire. The emf $\mathscr{E}$ is thus

$$\mathscr{E} = \frac{\mathscr{W}}{q} = \frac{Bqvs}{q},$$

so that

$$\boxed{\mathscr{E} = Bsv.} \qquad (32\text{-}1)$$

As shown in Figure 32-1, the direction of the emf is the direction in which positive charges are made to move by the action of the magnetic field, and therefore is opposite to the direction of the induced electric field E within the wire.

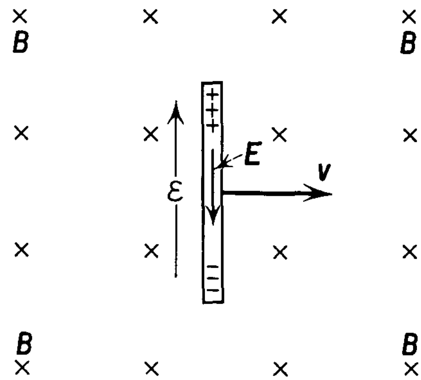


Fig. 32-1

To gain further insight into the effect of moving a wire through a magnetic field, let us suppose that the wire of length s slides over a fixed conductor a consisting of two parallel tracks which are electrically connected at one end, as shown in Figure 32-2. As the wire moves to the right with velocity v, the induced emf in the wire produces a current I in the closed circuit, in the direction of the emf $\mathcal{E}$ in the moving wire. We have seen in Section 31-4 that a wire carrying current in a direction perpendicular to the magnetic field experiences a force given by

$$F = BIs.$$

In the figure this force is directed to the left. In order to satisfy the principle of conservation of energy, the agency moving the wire to the right

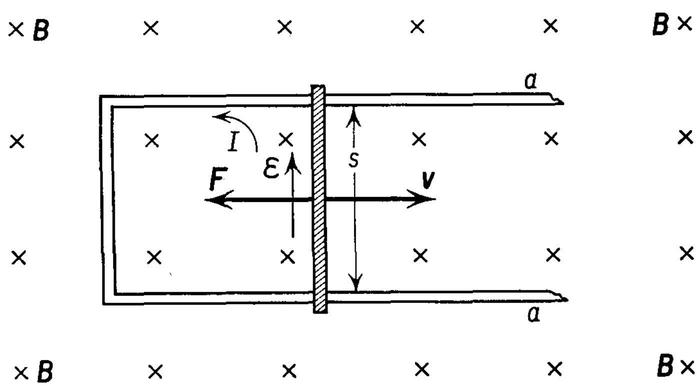


Fig. 32-2

must exert a force equal and opposite to the force F above, and expend mechanical power $\mathcal{P}$ such that

$$\mathcal{P} = Fv = BIsv.$$

At the same time the electrical power generated is

$$\mathcal{P} = \mathcal{E}I.$$

Equating the mechanical power expended and the electrical power generated, we find

$$\mathcal{E} = Bsv,$$

which is Equation (32-1).

In calculating the emf developed by a wire moving through a magnetic field, we have used two different points of view. The first calculation was made from essentially a microscopic point of view in which our attention was directed to the forces on isolated charges within the wire. The second calculation was made from a macroscopic viewpoint, in which our attention was directed to the force on the wire and to the emf. The same result was obtained in each case.

Many practical devices, such as electric generators and motors, are designed so that conductors move across magnetic fields. In using Equation (32-1) to discuss the operation of these devices, it must be remembered that B, s, and v were all considered to be perpendicular to one another. If they are not mutually perpendicular in a particular case, then only the components of the three quantities which are mutually perpendicular are to be considered. As the wire moves through the magnetic field, it is often described as "cutting" the lines of magnetic induction. Equation (32-1) then shows that the emf induced in a wire depends upon the number of lines of magnetic induction cut per unit time.

32-2 Magnetic Flux and Flux Density

It is convenient to represent the magnetic induction B by lines of magnetic induction, sometimes called lines of magnetic *flux*. The direction of the magnetic induction is tangent to the flux lines, and the magnitude of the magnetic induction is given in the usual way by the number of lines per unit area passing through a surface perpendicular to the flux lines. The total number of lines passing perpendicularly through an element of area is then called the *magnetic flux* Φ (capital phi) through that area. If the area of an element perpendicular to B is ΔA, then the flux $\Delta\Phi$ through that element is given by

$$\Delta\Phi = B\,\Delta A. \qquad \textbf{(32-2)}$$

In the mks system of units, the magnetic induction is stated in units of webers per square meter, and the area is stated in units of square meters. The flux is expressed in units of webers. For this reason the magnetic induction B is often referred to as the *flux density*.

In the Gaussian system of units, the magnetic induction B is expressed in gausses, the area in square centimeters, and the flux in *maxwells*. We have already seen that

$$1 \text{ weber/m}^2 = 10^4 \text{ gausses},$$

so that

$$1 \text{ weber} = 10^4 \text{ gausses} \times 10^4 \text{ cm}^2,$$

or

$$1 \text{ weber} = 10^8 \text{ maxwells}.$$

Following the procedure we have used throughout the development of electricity and magnetism, unless otherwise indicated all equations are expressed in the mks system of units, and the principal ones will be restated in the Gaussian system of units in a table at the end of this chapter.

The magnetic flux Φ is a scalar quantity, but the magnetic induction $\mathbf{B}$ is a vector quantity; the area $\Delta\mathbf{A}$ may be considered as a vector quantity. In dealing with closed surfaces, as in Gauss's theorem in electrostatics, we considered the direction of an area as that of an outward drawn normal. Although in the present instance the area is not part of a closed surface, it may be thought of as a *film* or a *cap* bounded by a closed conducting boundary. In choosing the direction of the area vector, we must associate a positive direction of circulation around the boundary of the area in accordance with a *right-hand rule*. Thus if in Figure 32-2 the direction of the area vector is chosen as pointing toward the reader, the positive direction of the current in the wire may be found by directing the thumb of the right hand in the direction of the area vector. The curled fingers of the right hand indicate the positive direction of the current or the emf as in the counterclockwise direction. If the area vector is pointing into the paper, the positive direction of the current is clockwise. Following this convention, we may rewrite Equation (32-2) in vector form as the scalar product of $\mathbf{B}$ and $\Delta\mathbf{A}$ as

$$\Delta\Phi = \mathbf{B}\cdot\Delta\mathbf{A}. \tag{32-2a}$$

32-3 Faraday's Law of Electromagnetic Induction

The phenomenon of electromagnetic induction was discovered in 1831 by Michael Faraday (1791–1867) in England and independently by Joseph Henry (1797–1878) in the United States. One example of electromagnetic induction is the emf generated in a wire moving through a magnetic field, as discussed in Section 32-1. In Faraday's original experiment the apparatus consisted essentially of two neighboring circuits, shown in Figure 32-3; one circuit, which we shall call the primary circuit, contained a battery B, a coil P, and a key K, for opening and closing the circuit; the second circuit, or secondary circuit, consisted of a coil S and a galvanometer G. Faraday observed that when the key was closed, the galvanometer in the S circuit gave a momentary deflection and then returned to its zero position and remained there as long as the key was closed. When the key was opened, there was another momentary deflection of the galvanometer, opposite in direction to the previous deflection, and then the galvanometer needle returned to its zero position.

Analyzing this simple experiment, we find that when the key in the primary circuit was closed, a current started flowing through the primary coil P. This current produced a magnetic field in the neighborhood of P and also around the coil S; that is, a change was produced in the magnetic field around the coil S in the secondary circuit. The fact that the galva-

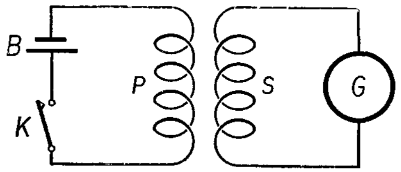


Fig. 32-3 Faraday's experiment on electromagnetic induction.

nometer showed only a momentary deflection can be interpreted by saying that a current was *induced* in the secondary circuit momentarily and that this induced current was due to the *change* in the magnetic field around the secondary circuit. As long as the current in the primary circuit remained constant, the magnetic field around both P and S remained constant, but the galvanometer read zero during this time. But when the magnetic field was again changed, say by opening the key, a current was again induced in the secondary circuit, this time in a direction opposite to that produced when the key was closed.

The results of the above experiment on electromagnetic induction can be explained qualitatively by stating that the change in the magnetic field around the secondary coil *induced an electromotive force* in the coil, and, since the coil is part of a closed circuit, this induced emf produced a current in the secondary circuit.

There are many ways in which the magnetic field around the coil S may be varied. Suppose, for example, that the key of the primary circuit is kept closed so that a steady current flows through the coil P. As long as the magnetic field around the secondary coil S remains constant, there will be no emf induced in it, and the galvanometer will read zero. But if we move S away from P so as to decrease the magnetic field around S, the galvanometer will show a deflection; similarly, if we move S toward P, the galvanometer will show a deflection but now in the opposite direction.

If we put a variable resistor in the primary circuit so that the current in it may be varied, and keep the distance between S and P constant, there will be an induced emf and hence an induced current in the secondary circuit whenever the current in the primary circuit is changed. When the current in the primary coil is increased, the induced current in the secondary coil will be in one direction; when the current in the primary coil is decreased, the induced current in the secondary coil will be in the opposite direction.

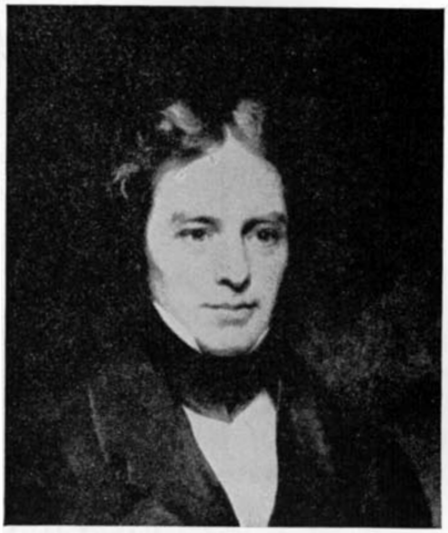


Fig. 32-4 Michael Faraday (1791–1867). Chemist and physicist. Discovered the laws of electrolysis and electromagnetic induction. Introduced the concept of lines of force to help understand the phenomena associated with electric and magnetic fields. (Courtesy of *Scripta Mathematica.*)

The results of many experiments on electromagnetic induction can be stated in the form of a law known as *Faraday's law of electromagnetic induction* as follows:

The electromotive force induced in each turn of wire in any circuit is equal to the time rate of decrease of the magnetic flux through it, or, in the form of an equation:

$$\boxed{\mathcal{E} = -\frac{d\Phi}{dt}.} \tag{32-3}$$

Faraday's law may be considered as one of the fundamental empirical laws of electromagnetism, or it may be derived by applying Ampère's law and the principle of conservation of energy to typical cases. To show this, let us reconsider the case of a wire moving perpendicularly with constant velocity through a magnetic field of flux density **B**, as shown in Figure 32-2. If the wire moves a distance Δx to the right, the change in area is

$$\Delta A = s\,\Delta x.$$

In accordance with the sign convention, the positive direction of the normal to ΔA is upward toward the reader. The flux density **B** is directed downward, so that $\mathbf{B}\cdot\Delta\mathbf{A}$, the change in flux $\Delta\Phi$ through this area, is

$$\Delta\Phi = -Bs\,\Delta x.$$

If this change in flux takes place in a short time interval Δt, then

$$\frac{\Delta\Phi}{\Delta t} = -Bs\frac{\Delta x}{\Delta t},$$

and, in the limit of short time intervals, we find

$$\frac{d\Phi}{dt} = -Bs\frac{dx}{dt} = -Bsv.$$

Substituting for the quantity on the right-hand side of the above equation from Equation (32-1), we find

$$\mathcal{E} = -\frac{d\Phi}{dt}\cdot \tag{32-3}$$

Although we have derived the result for a moving wire, Equation (32-3) has been found to be true for the emf induced in any closed circuit when the flux through that circuit changes with time.

Whenever the magnetic flux passing through a single turn of wire is changing, the instantaneous emf developed in the loop is given by Equation (32-3), regardless of the reason for the change in the flux. If the coil consists of N turns of wire, an emf will be induced in each turn by the changing magnetic flux. The turns of a coil may be considered as connected in series, so that the emf in the coil will be the sum of the emf's induced in the individual turns. If the rate of change of magnetic flux is the same through each turn, then the total emf induced in the coil will be given by

$$\boxed{\mathcal{E} = -N\frac{d\Phi}{dt}\cdot} \tag{32-4}$$

Illustrative Example. A coil containing 750 turns of wire is wound on a rectangular frame 20 cm by 30 cm. The magnetic induction normal to the area of the coil is 0.3 weber/m^2. The magnetic field is reduced to zero at a uniform rate in 0.25 sec. Determine the magnitude of the emf induced in the coil.

The total flux through the coil is

$$\begin{aligned}\Phi &= BA\\ &= 0.3\,\frac{\text{weber}}{\text{m}^2}\times 0.06\ \text{m}^2\\ &= 0.018\ \text{weber}\end{aligned}$$

The rate of change of flux is given by

$$\frac{d\Phi}{dt} = \frac{0.018 \text{ weber}}{0.25 \text{ sec}}$$

$$= 0.072 \text{ weber/sec.}$$

Since $N = 750$ turns, we have, from Equation (32-4), neglecting the sign of the emf,

$$\mathcal{E} = 750 \times 0.072 \text{ volts,}$$

$$\mathcal{E} = 54 \text{ volts.}$$

32-4 Lenz's Law

The method for determining the direction of the current in a coil produced by an induced emf was first clearly stated by H. F. E. Lenz (1804-1864) in 1834; it is based upon the application of the principle of conservation of energy to the process of electromagnetic induction. Lenz's law states that *the induced current is in such a direction as to oppose, by its magnetic action, whatever change produces the current.*

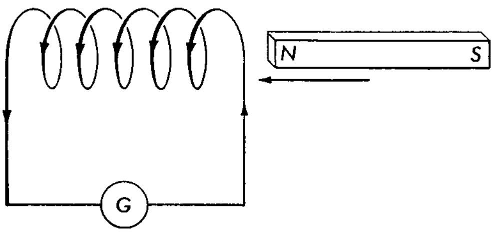


Fig. 32-5 An emf is induced in the coil by the motion of a bar magnet. Direction of the induced current is shown by the arrows on the wire.

Another way of stating Lenz's law is that the direction of the induced current is such as to oppose the change in the magnetic flux in the circuit. If for any reason whatever there is an increase in the magnetic flux through the circuit, the induced current will be in such a direction as to set up a magnetic field to oppose the increase in the magnetic flux through it. Similarly, if there is a decrease in the magnetic flux through the circuit, the induced current will be in such a direction that it will set up a magnetic field which will oppose the decrease in the flux through it. In the case of the moving wire of Figure 32-2, the motion of the wire to the right tended to increase the magnetic flux enclosed within the closed circuit. The increase in magnetic flux was directed into the plane of the paper. According to Lenz's law the induced current in this circuit had to be in the counterclockwise direction, as shown in the figure. The magnetic field generated by the induced current was directed out of the plane of the paper, so as to oppose the change in the flux through the circuit.

Let us suppose that a bar magnet is brought near a coil whose terminals are connected to a galvanometer, as shown in Figure 32-5. When the north pole of the magnet approaches the coil, the galvanometer registers a current in one direction. When the magnet is removed from the coil, the galvanometer registers a current in the opposite direction. We may find the direction of the induced currents by the application of Lenz's law. When the north pole of the bar magnet is brought near one end of the coil, the induced current in the coil will be in such a direction as to set up a magnetic field which will oppose the motion of the north pole toward it; that is, the magnetic field caused by the induced current will repel the north pole of the bar magnet. If the north pole of the magnet is moved away from one end of the coil, the induced current will set up a magnetic field so as to attract the north pole of the bar magnet and oppose its motion away from the coil. Thus work must be done in moving the bar magnet with respect to the coil because of the force which is generated when the magnet is moved. This work is transformed into electric energy, as evidenced by the existence of an induced current in the coil, and the conversion of this energy into heat.

32-5 Electric Dynamo or Generator

The moving wire of Figure 32-2 is a simple form of electric generator. In this illustrative dynamo, mechanical work is done on the wire, and this is converted into electrical energy. More practical forms of generators are usually constructed so that the motion of the conductor is rotational rather than translational. In the simplest case a coil of N turns is rotated about an axis in the plane of the coil, in a magnetic field which is perpendicular to the axis of rotation, as shown in Figure 32-6. The two ends of the coil are connected to two insulated conducting rings called *slip rings*, mounted on the axis and rotating with the coil. Two blocks of carbon, called *brushes*, press against these rings as they rotate and provide electrical contact with the external circuit.

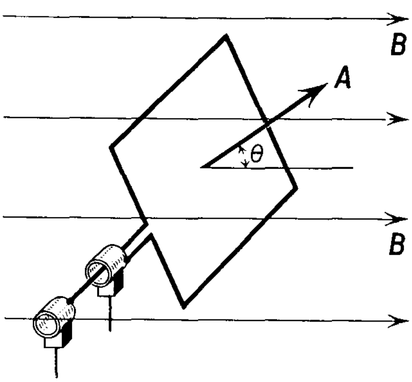


Fig. 32-6

Let us suppose that the coil of area A is rotating in a field of uniform flux density B. If the coil is oriented so that the normal to the plane of the coil makes an angle θ with the lines of flux, the component of the magnetic field which is normal to the plane of the coil is $B \cos \theta$, so that the flux passing perpendicularly through the coil is

$$\Phi = BA \cos \theta.$$

The emf $\mathcal{E}$ generated in a coil of N turns, as a result of the changing flux through the coil as it rotates, may be obtained from Equation (32-4) as

$$\begin{aligned} \mathcal{E} &= -N \frac{d\Phi}{dt} \\ &= -NBA \frac{d}{dt}(\cos \theta) \\ &= NBA \sin \theta \frac{d\theta}{dt} \cdot \end{aligned}$$

Let us assume that the coil rotates with uniform angular velocity ω, and that at time $t = 0$, the angle $\theta = 0°$. We have

$$\theta = \omega t,$$

and

$$\frac{d\theta}{dt} = \omega,$$

so that

$$\mathcal{E} = NBA\omega \sin \omega t. \tag{32-5}$$

The emf is zero at time $t = 0$ and varies as a sine function of the time. The emf changes direction after each half revolution of the coil. The emf is said to be an *alternating* emf. The maximum value of the emf occurs when $\theta = \omega t = 90°$; that is, when the plane of the coil is parallel to the

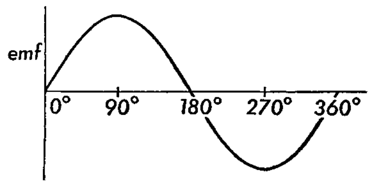


Fig. 32-7 Graph of the alternating emf induced in the coil of the generator during one revolution.

magnetic field. The emf is zero when the plane of the coil is perpendicular to the magnetic field ($\theta = 0°$). The variation in the emf induced in the coil is one revolution, as shown in Figure 32-7. The rate of change of the magnetic flux through the coil is greatest when it is passing through the position where it is parallel to the field, and is zero when the coil is perpendicular to the field.

In some small generators, such as those operated by hand and used to supply current to ring bells in rural telephones, the magnetic field may be supplied by a permanent magnet; these generators are called *magnetos*.

In most generators the magnetic field is produced by current in *field coils*. This current may be supplied by a battery or it may be supplied by the generator itself. Instead of a single rotating coil of wire, there are usually several coils, each consisting of many turns of wire, wound on an iron core, rotating in the magnetic field. The whole assembly is called an *armature*. Most large generators have a complicated field structure, with two or more pairs of poles. The induced emf in the coils of an armature is always an alternating emf, and the current in these coils is always an alternating current.

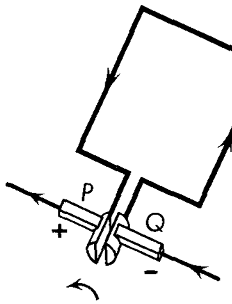


Fig. 32-8 Split-ring commutator. P and Q are brushes in contact with the segments of the commutator.

For some purposes it is desired to have a current which does not reverse its direction, as in electroplating baths. For such cases it is necessary to change the alternating current developed in the armature to current which is always in the same direction in the outside circuit. In a simple form of d-c generator, this is accomplished by a *split-ring commutator*, shown in Figure 32-8. The two ends of the armature are connected to the two insulated halves of the split ring. As the coil rotates, a given brush is always connected to that part of the coil moving in a particular direction through the field, so that one of the brushes is always the positive terminal of the generator and the other brush is always the negative terminal. The current from an armature with a single turn is not constant but is pulsating, as shown in Figure 32-9. The wave form is essentially a sine wave with the negative half cycles reversed.

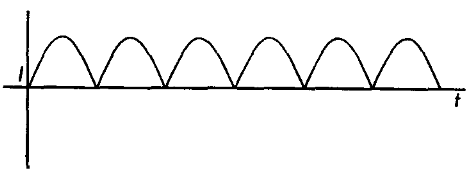


Fig. 32-9 Direct current from a single coil.

In modern d-c generators the armature consists of many coils connected in series, and the commutator contains many segments. Figure 32-10 shows the current from a generator containing two coils. The small variations in the current are referred to as commutator ripple.

A simple generator coil rotated in an unknown magnetic field may be used to measure the magnetic induction, using Equation (32-5), if the dimensions of the coil and the speed of rotation are known, and the emf generated by the rotation of the coil is measured.

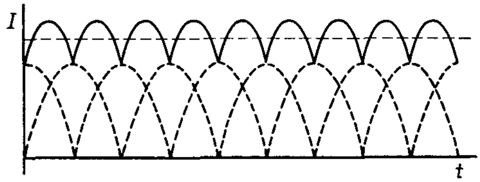


Fig. 32-10 Direct current from a d-c generator having a commutator with many segments. The dotted lines show the current from the two pairs of segments, while the solid line shows the current from such a generator. Note that the current is not constant but fluctuates about some average value shown as a dashed horizontal line.

32-6 Self-Inductance

The law of electromagnetic induction states that an emf is induced in any circuit in which the magnetic flux is changing. The manner in which the change in the magnetic flux is produced does not matter; the changes may be induced by external currents or magnets, or by changes in the circuit itself. The coil of a motor is caused to rotate by current passing through the armature, and as a result of the rotation of the armature, the flux through the coil is changed and a *back emf* is developed. When current is sent through a coil, a magnetic field is established through it, and any changes in the current generate changes in the magnetic flux through the coil. These changes in flux induce an emf in the coil, which, according to Lenz's law, must be in such a direction as to oppose the change in current. The emf induced in the coil is proportional to the rate of change of current in it, or, in the form of an equation,

$$\boxed{\mathcal{E} = -L\frac{di}{dt}.} \qquad (32\text{-}6)$$

The constant of proportionality L represents a property of the coil which depends upon its dimensions and its geometrical shape; L is called the *self-inductance* of the coil.

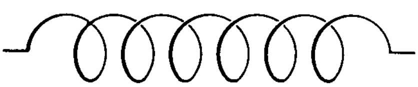


Fig. 32-11 Schematic representation of an inductor.

The minus sign is used to express the fact that the emf induced in a coil by a *change* in current is opposite to the direction of the *change*. In the mks system the unit of inductance is the *henry*, after Joseph Henry. Thus a coil has an inductance of 1 henry if an emf of 1 volt is induced in the coil when the current through it is changing at the rate of 1 amp/sec. A device having inductance is called an *inductor* and is represented schematically in Figure 32-11.

Another interpretation of the self-inductance L of a circuit may be obtained by comparing Equations (32-4) and (32-6), yielding

$$L\frac{di}{dt} = N\frac{d\Phi}{dt},$$

from which

$$Li = N\Phi,$$

so that

$$L = \frac{N\Phi}{i}. \tag{32-7}$$

The quantity $N\Phi$ is called the *flux linkage* of the circuit, hence the self-inductance L is the flux linkage per unit current of a circuit.

Let us determine the self-inductance of a uniformly wound toroid of N turns, mean length s, and cross-sectional area A. The magnetic field intensity within the toroid is uniform and given by

$$H = \frac{Ni}{s}. \tag{30-5}$$

When a toroid is in vacuum, the magnetic induction within the toroid is given by the equation

$$B = \frac{\mu_0 Ni}{s},$$

and the flux of magnetic induction within the toroid is

$$\Phi = \frac{\mu_0 NiA}{s}.$$

Since

$$L = \frac{N\Phi}{i}, \tag{32-7}$$

we get

$$L = \frac{\mu_0 N^2 A}{s}. \tag{32-8}$$

Thus the self-inductance of a toroid in air is a property of the geometry of the toroid, just as the capacitance of a capacitor is a property of its geometry. Any conducting element in an electrical circuit has the property of inductance. The conductors which connect the various parts of an electric circuit also generate a magnetic field when current passes through them. The inductance associated with the leads is often called *stray* inductance or distributed inductance.

From Equation (32-8) we see that μ_0 may be expressed in terms of the unit of inductance as

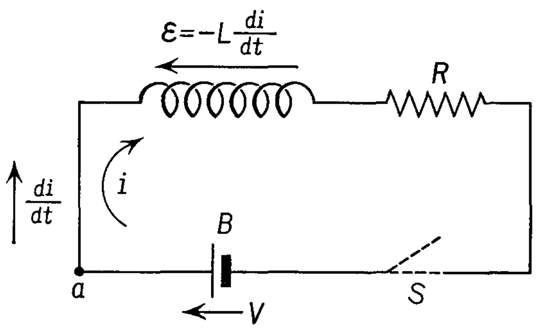
$$\mu_0 = 4\pi \times 10^{-7}\,\frac{\text{henry}}{\text{meter}}, \tag{32-9}$$

and, indeed, these are the units in which μ_0 is most commonly expressed.

32-7 Inductive Transients

Let us consider the simple circuit of Figure 32-12 in which an inductor L and a resistor R are connected in series to a battery B. Initially the switch S is open. We may analyze the behavior of the circuit by applying Kirchhoff's laws (Section 27-6) to the circuit at any instant after the switch is closed. Let us suppose that at a particular time t the current in the circuit

Fig. 32-12



is in the clockwise direction, as shown in the figure, and that the current is increasing so that the direction of the rate of change of current is in the same direction as the current itself. Starting at the point a and applying Kirchhoff's laws by moving a probe charge around the circuit in the direction of the current, we find

$$-L\frac{di}{dt} - iR + V = 0, \tag{32-10}$$

where V is the emf of the battery, and $-L\ (di/dt)$ is the back emf in the inductor.

Equation (32-10) is a differential equation whose solution is given by

$$i = \frac{V}{R}(1 - e^{-\frac{R}{L}t}). \tag{32-11}$$

We may verify this solution by differentiation of Equation (32-11) and substitution into Equation (32-10).

$$\frac{di}{dt} = \frac{V}{R}\frac{R}{L}e^{-\frac{R}{L}t}. \tag{32-12}$$

Substituting for i and for di/dt from Equations (32-11) and (32-12) into Equation (32-10), we find

$$-L\frac{V}{L}e^{(-R/L)t} - V + Ve^{(-R/L)t} + V = 0,$$

establishing the correctness of the solution.

The current in the circuit is zero at the instant the switch is closed, and gradually increases to a maximum value which is determined only by the magnitude of the resistance and the emf of the cell, as shown in Figure 32-13. Initially, the rate of change of current is very large, and the emf induced in the inductor limits the flow of current. At the instant the switch is closed, the emf of the inductor is equal and opposite to the emf of the

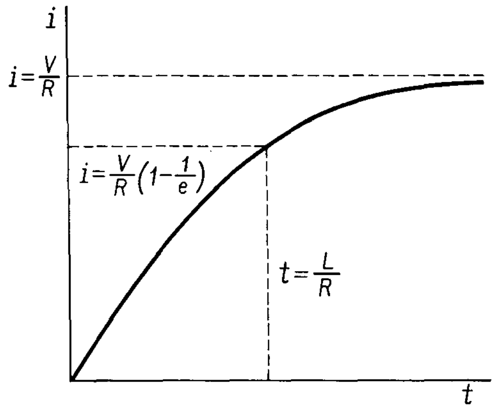


Fig. 32-13 Growth of current in a circuit containing inductance and resistance.

battery, so that the current is zero. When the current reaches a steady value and is no longer changing, there is no induced emf, and the current is determined by Ohm's law. At a time $t = L/R$, called the *time constant* of the circuit, the current has reached to within $1/e$ of its maximum value.

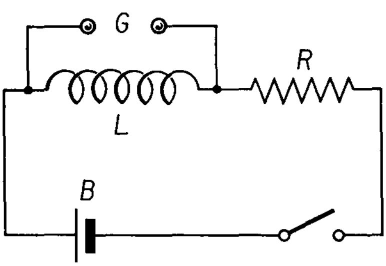


Fig. 32-14 A spark-gap G placed across the terminals of an inductor.

The curve describing the current as a function of time is called a *transient*, for it describes the current shortly after the switch is closed rather than the steady-state current that is established after a long interval of time.

In the process of establishing a current in the circuit, a magnetic field is established in the inductor. If the switch is suddenly opened after the current has reached a steady value, an emf will be induced in the inductor whose value will depend upon the rate of change of current. The more quickly the current is interrupted, the greater will be the induced emf. If a spark gap is placed across the terminals of the inductor, as shown in Figure 32-14, a spark may pass between the terminals of the spark gap when the switch is opened, because of the large induced emf in the inductor. The energy of the magnetic field is then dissipated into the heat, sound, and radiant energy generated at the spark gap. The emf generated upon opening a circuit containing inductance is often called a switching transient, and is responsible for the large arcs which are often observed when electrical switches are opened.

32-8 Energy Stored in an Inductor

Let us calculate the energy of an inductor when there is a steady-state current I in it. During the transient interval when the current is changing from zero to the maximum value I, the emf of the inductor is given by

$$\mathcal{E} = -L\frac{di}{dt}. \tag{32-6}$$

An amount of power $\mathcal{P}$ is expended by an external source of electrical energy to establish this current. The applied potential difference is opposite to the direction of the induced emf so that

$$\mathcal{P} = -\mathcal{E}i = Li\frac{di}{dt}.$$

The work $d\mathcal{W}$ done by an outside agency in driving current through the inductor against the induced emf in a time dt is

$$d\mathcal{W} = \mathcal{P}\,dt = Li\frac{di}{dt}\,dt,$$

or

$$d\mathcal{W} = Li\,di.$$

Integrating between the limits of $i = 0$ and $i = I$, the final current through the inductor, we have,

$$\mathcal{W} = \int_0^I Li\,di,$$

yielding

$$\mathcal{W} = \tfrac{1}{2}LI^2 \tag{32-13}$$

for the energy of an inductor. When L is in henrys and I is in amperes, $\mathcal{W}$ is in joules.

When the current in a circuit builds up from zero to a value I, energy is supplied to the magnetic field, its value being $\frac{1}{2}LI^2$. As long as the current remains constant, no additional energy is supplied to the magnetic field. In the circuit of Figure 32-12, all of the energy supplied during the steady state is transformed into heat. When the current decreases from I to zero, the magnetic field also decreases to zero; the energy that was stored in the magnetic field is returned to the circuit.

We can use the equation for the energy of an inductor to determine the energy per unit volume $\mathscr{W}_V$ in a magnetic field by considering an inductor in the form of a toroid. Its magnetic field is confined entirely to the volume within the toroid. The inductance of a toroid is given by

$$L = \frac{\mu_0 N^2 A}{s}, \tag{32-8}$$

hence the energy stored in the magnetic field of the toroid is, from Equation (32-13),

$$\mathscr{W} = \frac{1}{2}\frac{\mu_0 N^2 A}{s} I^2.$$

The magnetic field intensity within the toroid is given by

$$H = \frac{NI}{s}. \tag{30-5}$$

Substituting for N from Equation (30-5) into the above equation for the energy of the toroid, we have

$$\mathscr{W} = \tfrac{1}{2}\mu_0 H^2 A s.$$

The volume within the toroid is given by the product of its cross-sectional area A by its mean circumference s. Thus the energy per unit volume $\mathscr{W}_V$ of the magnetic field is given by

$$\mathscr{W}_V = \tfrac{1}{2}\mu_0 H^2 = \tfrac{1}{2}BH. \tag{32-14}$$

Notice that our procedure for finding the energy stored in the magnetic field has been very similar to the calculation by which we found the energy per unit volume stored in the electric field. In the case of the magnetic field, we utilized the energy in the field of a toroid, while in the electric field we utilized the energy in the field of a capacitor. Recalling that result from Equation (25-5), we may write the energy per unit volume $\mathscr{W}_V$ in the electromagnetic field as

$$\mathscr{W}_V = \tfrac{1}{2}(\epsilon_0 E^2 + \mu_0 H^2). \tag{32-15a}$$

This formula may be rewritten as

$$\mathscr{W}_V = \tfrac{1}{2}(DE + BH). \tag{32-15b}$$

In the mks system of units, the energy density $\mathscr{W}_V$ in the electromagnetic field is in joules per cubic meter when the electric field is expressed in volts per meter and the magnetic field intensity is expressed in amperes per meter.

TABLE 32-1 PRINCIPAL EQUATIONS IN MKS AND GAUSSIAN UNITS

Equation	MKS	Gaussian	
(32-2)	$\Delta\Phi = B\,\Delta A$	Same form as mks	Flux change
(32-3)	$\mathcal{E} = -\frac{d\Phi}{dt}$	$\mathcal{E} = -\frac{1}{c}\frac{d\Phi}{dt}$	Electromagnetic induction
(32-4)	$\mathcal{E} = -N\frac{d\Phi}{dt}$	$\mathcal{E} = -\frac{N}{c}\frac{d\Phi}{dt}$	N-turn coil
(32-6)	$\mathcal{E} = -L\frac{di}{dt}$	Same form as mks	Inductance
(32-8)	$L = \frac{\mu_0 N^2 A}{s}$	$L = \frac{4\pi N^2 A}{sc^2}$	Toroid or long solenoid
(32-13)	$\mathscr{W} = \frac{1}{2}LI^2$	Same form as mks	Energy
(32-14)	$\mathscr{W}_V = \frac{1}{2}\mu_0 H^2 = \frac{1}{2}BH$	$\mathscr{W}_V = \frac{H^2}{8\pi} = \frac{BH}{8\pi}$	Energy density in vacuum
(32-15b)	$\mathscr{W}_V = \frac{1}{2}(DE + BH)$	$\mathscr{W}_V = \frac{1}{8\pi}(DE + BH)$	Energy density

TABLE 32-2 CONVERSION FACTORS RELATING MKS AND GAUSSIAN UNITS

Quantity	Symbol	MKS Unit	Gaussian Unit	
Flux	Φ	1 weber	$= 10^8$ maxwells	(emu)
Inductance	L	1 henry	$= \frac{1}{9 \times 10^{11}}$ stathenry	(esu)
Pole	p	1 weber	$= \frac{10^8}{4\pi}$ unit pole	(emu)
Magnetic intensity	H	$1\,\frac{\text{nt}}{\text{weber}} = 1\,\frac{\text{amp}}{\text{m}}$	$= 4\pi \times 10^{-3}$ oersted	(emu)
Magnetic induction	B	1 weber/m^2	$= 10^4$ gausses	(emu)
Electric intensity	E	$1\,\frac{\text{nt}}{\text{coul}} = 1\,\frac{\text{volt}}{\text{m}}$	$= \frac{10^{-4}}{3}\frac{\text{statvolt}}{\text{cm}} = \frac{10^{-4}}{3}\frac{\text{dyne}}{\text{stcoul}}$	(esu)
Electric displacement	D	1 coul/m^2	$= 3 \times 10^5$ statcoul/cm^2	(esu)

Problems

32-1. The magnetic flux in a coil having 40 turns changes steadily from zero to 20,000 maxwells in 2 sec. Find the induced emf in the coil.

32-2. The magnetic flux through a coil of 125 turns is changed at a constant rate from zero to 0.40 weber in 2.5 sec. Determine the emf induced in this coil.

32-3. A circular coil of 50 turns and radius 15 cm lies in the x-y plane. The magnetic induction is changed in 0.001 sec at a constant rate from zero to a value whose x component is 0.3 weber/m^2, whose y component is 0.4 weber/m^2, and whose z component is 0.5 weber/m^2. Find the emf induced in the coil.

32-4. A wire 50 cm long is at rest along the x axis. A large magnet generating a uniform field directed along the $+y$ direction of 0.2 weber/m^2 is moved in the $+z$ direction with a speed of 30 m/sec. Find the magnitude and the direction of the emf induced in the wire.

32-5. A wire 75 cm long is moved in the y direction at a speed of 25 m/sec, so that the wire is always parallel to the x axis. The magnetic field has components $B_x = 0.2$ weber/m^2, $B_y = -0.3$ weber/m^2, and $B_z = 0.4$ weber/m^2. Find the emf induced in the wire.

32-6. A short solenoid, connected to a galvanometer, stands on one end upon a table. The north pole of a long bar magnet is brought down from above into the solenoid. Apply Lenz's law to find if the direction of the current induced in the solenoid is clockwise or counterclockwise, as viewed from above.

32-7. A rectangular coil of wire having 10 turns with dimensions of 20 cm by 30 cm is rotating at constant speed of 600 rpm in a magnetic field in which the magnetic induction is 600 gausses. The axis of rotation is perpendicular to the field. Find the maximum value of the emf produced.

32-8. Part of a closed circuit consists of a straight wire 1.5 m long moving at a speed of 2 m/sec perpendicular to a magnetic field of 10,000 gausses. (a) What is the emf induced in the circuit? (b) What is the force on the wire when the induced current is 5 amp?

32-9. A coil of 300 concentrated turns and an area of 800 cm^2 is lying flat on a horizontal table. When the coil is turned over through 180° in 0.10 sec, the average induced emf is 0.024 volt. What is the vertical component of the magnetic intensity of the earth's magnetic field?

32-10. A rectangular coil 12 cm by 25 cm and containing 15 turns is rotating at a constant speed of 1,800 rpm in a magnetic field in which the magnetic induction is 0.15 weber/m^2. The axis of rotation is perpendicular to the field. (a) Determine the maximum emf induced in this coil. (b) If the zero of time is taken at the point where the coil is parallel to the magnetic induction, find the emf in the coil when it has rotated by 53°.

32-11. When the current in a coil is changed at a constant rate of 5 amp/sec, the emf induced in the coil is 0.25 volt. Determine the self-inductance of the coil.

32-12. Derive a formula for the self-inductance of a long solenoid. Assume that the field is uniform everywhere within the solenoid.

32-13. What is the self-inductance of a solenoid 50 cm long and 5 cm in diameter, wound with 400 turns of wire?

32-14. What is the self-inductance of a toroid of mean circumference 25 cm, wound with 500 turns of wire, if the cross-sectional area of each turn is 2 cm^2?

32-15. An inductor has an inductance of 0.01 henry and an internal resistance of 5 ohms. The inductor is connected to the terminals of a battery having an emf of 12 volts. What will be the current in the circuit (a) in 0.001 sec? (b) In 0.01 sec? (c) In 0.1 sec? (d) Determine the time constant of this circuit.

32-16. An inductor of inductance 0.1 henry is connected in series with a 50-ohm resistor. This series combination is connected across the terminals of a 100-volt battery. What will be the energy stored in the magnetic field of the inductor when the current reaches a steady value?

32-17. Solve Equation (32-10) by separating the variables so that it becomes

$$\frac{di}{i - \frac{V}{R}} = -\frac{R}{L}\,dt,$$

and integrating to obtain Equation (32-11). Evaluate the constant of integration, letting $i = 0$ when $t = 0$.

32-18. Suppose that a switching arrangement is used in the circuit of Figure 32-12 so that the battery is removed from the circuit and a connecting wire is substituted in its place. (a) Show that Equation (32-10) becomes

$$L\frac{di}{dt} + iR = 0.$$

(b) Solve this equation for the current as a function of the time. Evaluate the constant of integration letting $i = I$ when $t = 0$. (c) Plot a graph of this equation and compare it with the graph of Figure 32-13.

32-19. Referring to the toroid of Problem 32-14, calculate (a) the energy in the magnetic field when the current is 10 amp and (b) the energy per unit volume of this field.