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Physics, Chapter 33: Magnetic Properties of Matter

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Magnetic Properties of Matter

33-1 Introduction

Matter is composed of atoms consisting of positively charged nuclei and negative electrons. These electrons occur in shells, and the periodic nature of chemical properties of atoms as the atomic weight increases is a reflection of the fact that the chemical behavior of an atom depends largely upon the number of electrons in the outermost shell. In some parts of the periodic table (see Table 5 of Appendix A), electrons occupy places in an outer shell before an inner shell is completely filled; it is then observed that a number of different elements have very similar chemical properties. The same number of electrons lies in the same outermost shell of these different atoms, but the inner shells contain different numbers of electrons. The chemical similarity of the *rare-earth* elements, atomic numbers 57 to 71, may be explained on this basis. A similar state of affairs exists in the group of elements of atomic number 26 (iron), 27 (cobalt), and 28 (nickel), all of which contain 2 electrons in their outermost shell but have 6, 7, and 8 electrons respectively, in their next inner shell, where 10 electrons are required to fill that shell. As a consequence the electrons in the unfilled shell of one atom may exert an important influence on the electrons of the unfilled shell of a properly spaced adjacent atom in a crystal. The *ferromagnetism* of iron, nickel, and cobalt is explicable in terms of the electronic configuration of these partially filled shells.

The magnetic properties of matter arise from two sources. An electron in orbital motion about the nucleus constitutes a small circulating current, which generates a magnetic field. The electron moving in its orbit has an *angular momentum* about the nucleus. In addition to the magnetism due to its orbital motion, an electron has an intrinsic magnetic moment and an intrinsic angular momentum, owing to its *spin*.

In the absence of a magnetic field, the orbital magnetic moments and the intrinsic magnetic moments of different electrons are randomly oriented within matter. There may be relatively large local magnetic fields in small regions, but when these magnetic fields are averaged over a volume of even a cubic millimeter, the average field is zero, so that, macroscopically, no magnetism is displayed. Most materials are only very slightly magnetic in the presence of external magnetic fields. They are said to be either diamagnetic or paramagnetic. *A diamagnetic material is one for which* κ_m *is less than* 1. The magnetic effects induced in the material are opposed to the external field. We would expect materials to be diamagnetic if the orbital electronic effects predominated, for, in accordance with Lenz's law, the magnetic effects induced in a circuit must be in such a direction as to oppose the change in magnetic field in the substance. *A paramagnetic material is one for which* κ_m *is greater than* 1. When placed in a nonuniform field, a diamagnetic substance will experience a force directed from the stronger to the weaker part of the field. A paramagnetic substance will experience a force in the opposite direction. In general, in all materials except those called *ferromagnetic* (that is, those which behave like iron), the magnetic effects are quite small, and these materials may be treated as though their relative permeability κ_m is 1, to an accuracy of about 0.1 per cent. The magnetic behavior of most substances is not substantially different from vacuum. A *ferromagnetic* substance is attracted into a magnetic field with a *large* force. The relative permeability κ_m of a ferromagnetic substance may be as large as 10^4 or 10^5. Ferromagnetic substances are therefore special cases of the general class of paramagnetic substances. The subject of ferromagnetism is of great importance in electrical engineering, and is as complex as it is important. The properties of ferromagnetic substances form the basis of the practical design of motors, generators, transformers, magnetic amplifiers, tape recorders, loud-speakers, permanent magnets, and a host of other devices. We shall attempt to develop only some of the basic ideas of ferromagnetism, and to solve problems of the simplest type in which symmetry considerations enable us to see the important principles most clearly without the confusion of detail which cannot be neglected in practical engineering design.

33-2 Permeability

The basis of our study of the electrical properties of matter was the observed change in the capacitance of a parallel-plate capacitor when the space between the plates was filled with a dielectric. The dielectric constant was defined in this manner in Section 25-5 and was interpreted in subsequent sections in terms of the induced electric polarization. Lacking magnetic conductors, we must find some other way to define the magnetic properties of matter. One way is to compare the inductance of a long solenoid or toroid in vacuum with the inductance of that same solenoid or toroid when the space within the coil is filled with a medium. If L_0 is the inductance of a toroid in vacuum, and L is its inductance when filled with a particular material, then

$$\frac{L}{L_0} = \kappa_m, \tag{33-1}$$

where κ_m is the relative permeability of the medium. The inductance of a toroid in vacuum is given by the equation

$$L_0 = \frac{\mu_0 N^2 A}{s}. \tag{32-8}$$

Remembering that

$$\mu = \kappa_m \mu_0, \tag{29-16}$$

we may write

$$L = \frac{\mu N^2 A}{s} \tag{33-2}$$

for the inductance of a toroid filled with a magnetic medium.

Upon measurement of the inductance of a toroid filled with a ferromagnetic core, it is found that the inductance of the toroid is not constant but depends upon the magnitude of the current. That is, the permeability of the medium in the core is not constant but depends upon the magnetic field intensity within the toroid. Hence it is advantageous to study the variation of the permeability with H, the magnetic field intensity within the toroid.

In order to understand properly the effect of the induced magnetization of the medium on the magnetic field within the toroid, let us first consider the behavior of a long rod of ferromagnetic material placed inside a long solenoid, as shown in Figure 33-1. When current is passed through

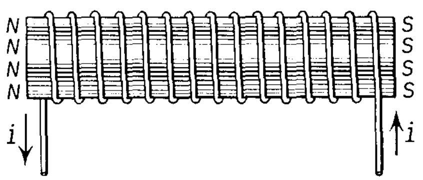


Fig. 33-1 A solenoid containing a ferromagnetic core.

the solenoid, there is a tendency for the intrinsic electronic magnetic moments of the electrons in the unfilled inner shells to align themselves with the direction of the field, like a collection of compass needles. Just as in the case of electric polarization, we may imagine the magnetized rod to be replaced by a layer of north poles at one end of the rod and a layer of south poles at the other end of the rod. The magnetic field intensity within the rod may then be calculated as being due to two causes. First there is the applied field intensity due to the current in the solenoid, and, second, there is the magnetic field intensity due to the induced poles in the material, at the ends of the rod. Let us suppose that the field is such as to induce a magnetic moment M per unit volume in the rod. The total magnetic moment in the rod is the product of the magnetization M by the volume of the rod. If the total strength of the induced pole at each end of the rod is represented by p, the magnitude of the induced pole strength may be found by representing the total magnetic moment in terms of the pole strength and in terms of the magnetization, and equating these two quantities. Thus we have

$$MAs = ps,$$

or

$$p = MA,$$

where A is the cross-sectional area of the rod and s is its length. We see that the induced pole strength does not depend upon the length of the rod but only on the magnetization and the area of the rod. If the rod is made very long, the induced poles contribute very little to the magnetic field intensity at the center of the solenoid, so that H at the center of the solenoid is the same, whether the solenoid is filled with magnetic material or is in vacuum. The same result is accomplished if the two ends of the solenoid are joined to form a toroid. In this case the rod has no free ends, so that there are no induced poles, and the value of H within the toroid is the same, regardless of whether the toroid is filled with matter or is in vacuum. For this reason experimental measurements are often made on a ringlike specimen, called a *Rowland ring*, after H. A. Rowland who first (1873) utilized toroids wound on iron rings to measure permeability.

While the magnetic field intensity H within the toroid is not changed when it is filled with a ferromagnetic material, the inductance, which depends upon the magnetic induction B, does change when the toroid is filled with matter. To study the changes in permeability with H we shall examine, through use of the Rowland ring, the way in which the magnetic induction within a substance changes with the imposed magnetic field intensity H.

33-3 Magnetic Measurements with a Rowland Ring

To measure the magnetic properties of a material by means of the Rowland-ring method, a specimen is machined in the form of a ring and is wound with a toroidal coil, called the primary coil, as shown in Figure 33-2. The primary coil is connected in series with a battery, an ammeter, and a rheostat; the latter is used to change the current in the primary coil. A small secondary winding is wound around the toroid and connected to a *ballistic galvanometer*. Such a galvanometer is constructed with a coil which has a relatively large moment of inertia, and a long period of vibration.

When a short burst of current passes through the galvanometer, there is very little rotation of the galvanometer coil until after the current has ceased. The angular impulse delivered to the galvanometer depends upon the torque, which is proportional to the current, and upon the time interval during which the torque is applied. The ballistic galvanometer thus re-

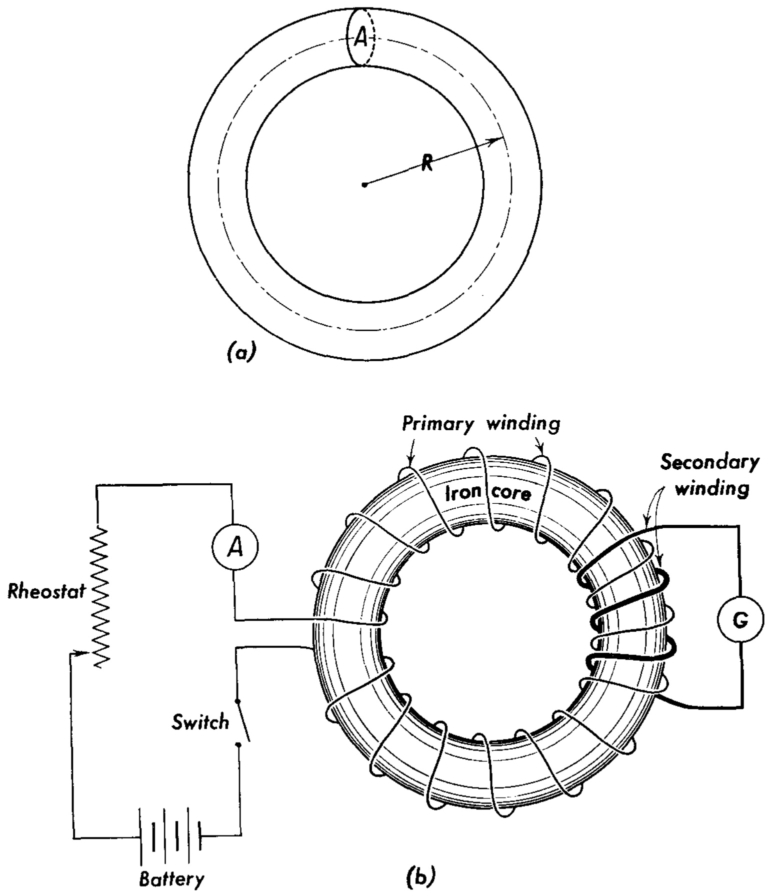


Fig. 33-2 Rowland ring.

ceives an angular impulse which depends upon the product of the current times the time, or the charge passing through it; its deflection is proportional to the total charge passing through in the time interval Δt, rather than the instantaneous value of the current.

Suppose we start with an unmagnetized piece of iron and vary the current through the toroid in a series of small steps. At each step the magnetic field intensity H may be calculated from Equation (30-5), for, as we have seen, the magnetic intensity H within the toroid is unchanged by the presence of the iron core. Each time H is changed, there is a corresponding change in B and, consequently, a change in the flux through the secondary coil. From Faraday's law of induction, an emf is thus induced in the secondary coil.

Let us suppose that the change in B due to a change in H in a time interval Δt is represented by ΔB. If the cross-sectional area of the toroid is A, the change in flux $\Delta\Phi$ through the secondary coil is given by

$$\Delta\Phi = A\,\Delta B.$$

The induced emf in the secondary coil of N turns is

$$\mathcal{E} = -N\frac{\Delta\Phi}{\Delta t} = -NA\frac{\Delta B}{\Delta t}\cdot$$

If the total resistance of the secondary coil and galvanometer is R, the magnitude of the current induced in the secondary circuit is given by Ohm's law as

$$i = \frac{\mathcal{E}}{R} = \frac{NA}{R}\frac{\Delta B}{\Delta t}\cdot$$

Multiplying through by the time interval Δt, we find that the charge Δq flowing through the galvanometer when the current in the primary winding of the toroid is changed is given by

$$\Delta q = -\frac{NA}{R}\Delta B. \tag{33-3}$$

Thus each time the current through the primary coil is changed, a measurement of the charge flowing through the ballistic galvanometer enables us to determine the change in B, ΔB, from the constants of the measuring circuit.

If we start with $H = 0$ and with unmagnetized iron in which the magnetization M is zero, then since

$$B = \mu_0 H + M, \tag{29-14a}$$

the magnetic induction within the coil is zero initially. By measuring the successive changes in B as H is varied, we get a series of values of B, as shown in Figure 33-3. The magnetic induction B increases slowly at first, with increasing H, then increases more rapidly until the flat portion of the curve is reached at A. Along the flat portion of the curve the iron is said to be *saturated*. Very little further magnetization in the iron is observed when H is increased, and the increase in B is due to the increase in H itself. The curve appears flat because of the difference in the scales used for B and

H. The relative permeability κ_m may be obtained from such a curve and the relationship

$$B = \kappa_m \mu_0 H,$$

when the scales of the axes are represented in mks units. The shape of the curve and the permeability itself depend upon the types of iron used. In some laboratory samples of iron, a relative permeability as high as 10^6 has been achieved.

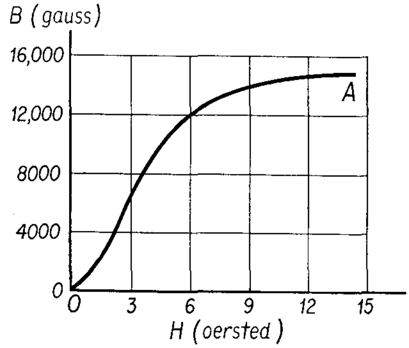
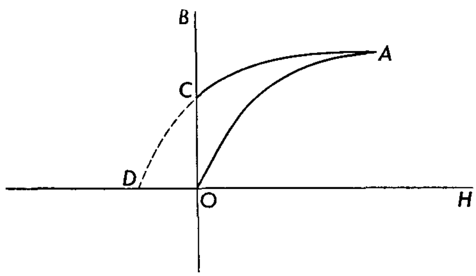


Fig. 33-3 Magnetization curve of soft iron.

33-4 Hysteresis

Let us take a piece of unmagnetized iron in the form of a ring of circular cross section, wind a toroid around it, then wind a secondary coil around it, and connect it to a battery and meters, as shown in Figure 33-2. If we

Fig. 33-4 Hysteresis.



send current through the coil and magnetize the iron until the magnetic induction has reached its saturation value, as shown in Figure 33-4, the variation of B with H follows the magnetization curve OA. If we now decrease the current in small steps, it is found that the demagnetization curve does not follow the original curve OA but, instead, follows the curve AC. The value of the magnetic induction remaining when the current in the toroid is reduced to zero is given by OC. The iron is now permanently magnetized, and the value of the induction OC is called the *retentivity*. In permanent magnets a high retentivity is required, while in other applications a low value of the retentivity is required.

To reduce the magnetic induction to zero, that is, to *demagnetize* the iron, it is necessary to reverse the current in the magnetizing coil in order to reverse the direction of H. When H has reached the negative value

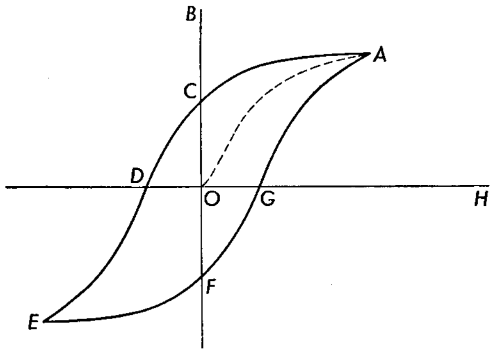


Fig. 33-5 Hysteresis loop in cycle of magnetization of iron.

given by OD, the induction within the iron is reduced to zero. The value of H needed to reduce B to zero is called the *coercive force*. The fact that B lags behind its previous value while H returns to its former value is called *hysteresis*.

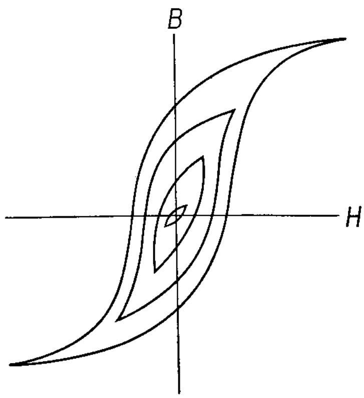


Fig. 33-6 Hysteresis curves obtained by using alternating current of successively smaller values in the primary circuit.

All ferromagnetic materials exhibit the phenomenon of hysteresis. One of the effects of hysteresis is that the value of B for any given value of H is not always the same but rather depends upon the magnetic history of the material. If an alternating current is sent through the toroid wound on a ferromagnetic core, the B–H curve will be similar to Figure 33-5 for each complete cycle of current. From Equation (32-14) the energy per unit volume in the magnetic field depends upon the product BH. While the iron is being magnetized, energy is being stored in the magnetic field; when the iron is demagnetized, some of that energy is recovered as electrical energy. In traversing one cycle of the hysteresis loop, the energy per unit volume of iron dissipated as heat is the area inside the hysteresis loop. In the design of a-c machinery, it is important that a type of iron be used in which the area enclosed within the hysteresis loop is small, so that a minimum of energy is lost through this mechanism.

A simple way of demagnetizing a substance is to place it inside a coil and pass alternating current through the coil. The amplitude of the alternating current is decreased slowly, so that the hysteresis loop gets smaller and smaller in successive cycles, as shown in Figure 33-6, until finally, when the current in the coil is zero, the values of B and H are zero.

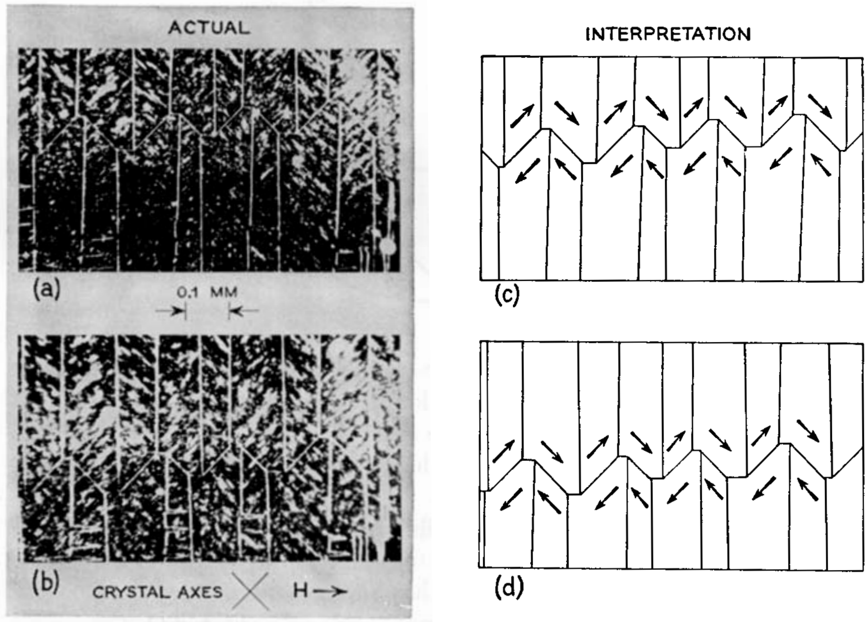


Fig. 33-7 (a) Photograph of magnetic domains of a magnet; (c) diagram of (a) showing the directions of magnetization at the boundaries of the domains. (b) Photograph showing the movement of the domain walls when an external magnetic field H is applied; (d) diagram of (b) showing that the domains that are magnetized favorably with respect to H grow at the expense of the other domains. (Photograph by H. J. Williams and R. M. Bozorth; reproduced by permission of *Electrical Engineering*, **68**, 1949, 471.)

The hysteresis loop is explained by means of the *domain theory of magnetization*, first stated by P. Weiss in 1907. According to this theory a ferromagnetic material is composed of many small regions, or domains, each magnetized to saturation, and about 5×10^{-3} cm in width. When a suspension of colloidal magnetite is applied to a highly polished piece of iron while it is being magnetized, the colloidal particles collect near the domain boundaries and may be observed with a microscope, as shown in Figure 33-7.

In the unmagnetized state the directions in which the domains are magnetized are distributed at random or in some other way such that the resultant magnetization of the specimen is zero. Changes in the total magnetization of the specimen are produced by changes in the direction of magnetization of the domains or by motion of the boundaries of the walls of the domains, as shown in Figure 33-8. At weak fields the magnetization proceeds by boundary displacement. When the material is very

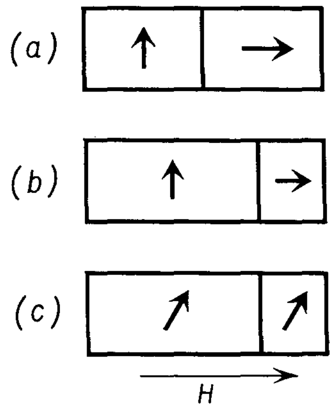


Fig. 33-8 (a) Original magnetization of two adjacent domains. (b) Change in magnetization as a result of the growth of one domain at the expense of the other. (c) Change in magnetization resulting from a change in the direction of magnetization within the domains.

pure and homogeneous, the boundaries of the domains are easily changed, and so the coercive force is small and the permeability is high. When impurities are present, or the material is in a state of internal stress, the material is inhomogeneous, so that the boundaries of the domains are less easily displaced. The initial permeability is lowered, and the coercive force is increased. By sintering a magnetic material of very fine powders, or by precipitating an impurity or another metallurgical phase, the displacement of domain boundaries is hindered, and good permanent magnets are made which require a large demagnetizing field to alter their magnetization. At high magnetic fields the magnetization is accomplished by rotation of the direction of magnetization within the domains. When the direction of magnetization in all the domains is parallel to the applied magnetic field, the material is saturated.

The motion of the boundary walls of the magnetic domains does not occur smoothly, particularly along the steep portion of the magnetization curve. This irregular motion produces sudden changes in the magnetization of the specimen as the magnetizing force H is changed. If the galvanometer in the secondary circuit of Figure 33-2 is replaced by an amplifier and loud-speaker, a succession of clicks will be heard when the magnetization is changing. This effect, known as the *Barkhausen effect,* was discovered by H. Barkhausen in 1919.

From Figure 33-6 we see that the magnetization which remains in a specimen when the applied magnetic field intensity is reduced to zero depends upon the magnitude of the applied field. Through the residual magnetization the specimen remembers the amplitude of the applied field.

The magnetic memory is the basis of magnetic tape recorders and of many present-day electronic computers.

When a specimen is magnetized, it behaves as though there were magnetic poles at the boundaries of the specimen. If there is a crack in the specimen, poles of opposite polarity appear on the adjacent faces of the crack. This is the basis of magnetic inspection of ferrous machine parts, widely used in aircraft maintenance and production. A part is magnetized and is then flushed with light oil which carries a suspension of magnetic powder. The powder tends to cling to the crack, enabling an inspector to identify a defective part. This is also the basis of the techniques used in making photographs of domains such as Figure 33-7.

33-5 Other Magnetic Effects

The intrinsic magnetic moment of the electron, associated with electron spin, rather than the orbital motion of the electron, is responsible for ferromagnetism. Electrons in completed shells are arranged so that he total magnetic moment of the shell is zero. In iron, cobalt, and nickel the electrons in the unfilled inner shell are responsible for ferromagnetism.

In ferromagnetic materials the inner electrons of neighboring atoms are bound by forces called *exchange forces* which depend upon the orientation of the spins of these inner electrons. In ferromagnetic materials the electrons of adjacent atoms are held parallel, whereas in other materials these forces generally tend to align the spins so that they are antiparallel, that is, in opposite directions, so that adjacent atoms tend to neutralize each other's magnetic effects. When iron is heated to such a temperature that the thermal energy of the electrons exceeds the energy associated with the exchange force, the electrons are no longer able to maintain their parallel orientation, so that the iron is no longer ferromagnetic. The temperature at which this magnetic transition occurs is called the *Curie temperature*, which, in iron, is 760°C.

One would expect that ferromagnetism would also depend upon the separation and arrangement of iron atoms in the crystal lattice. If the atoms were sufficiently far apart, or were not laid out according to a proper pattern, it would be impossible for adjacent atoms to influence each other; indeed, this is the case. In single crystals the magnetization curve is different in different crystal directions. Furthermore, while iron is ferromagnetic, a type of stainless steel containing 18 per cent chromium and 8 per cent nickel is nonmagnetic. Similarly for the two common oxides of iron, one, called magnetite, Fe_3O_4, is magnetic, while the other, called hematite, Fe_2O_3, is nonmagnetic. It is also possible to make alloys which display ferromagnetic properties out of elements, such as copper, manganese, and aluminum, which themselves are not ferromagnetic. These are called *Heusler's alloys*. The way in which the atoms are arranged in the alloy or in a crystal is of fundamental importance in determining its ferromagnetic properties.

Two extremely interesting effects have been observed, using macroscopic specimens, which confirm the theory that the spin of the electron and its inherent magnetic moment are responsible for ferromagnetism. Let us suppose that a rod of ferromagnetic material is suspended by a fiber, inside a solenoid, as shown in Figure 33-9. When current is passed through the solenoid, the orientation of the magnetic moment of a large number of electrons is changed. This implies that the angular-momentum vector has been changed for each of these electrons. According to the principle of conservation of angular momentum, a system which has experienced no external torques must retain a constant value of its angular momentum, and so the rod itself must rotate in such a direction that the total angular momentum of the system, made up of the crystal lattice and the electrons, remains equal to zero. This is called the *Einstein-de Haas effect,* in which a specimen is observed to rotate when it is being magnetized.

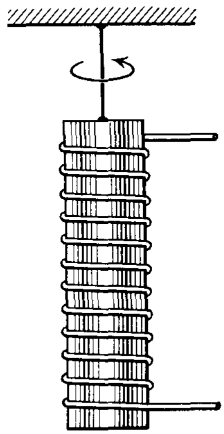


Fig. 33-9 Demonstration of the Einstein-de Haas effect. Magnetic specimen will rotate when it is being magnetized.

The inverse of this effect is called the *Barnett effect.* In the Barnett effect a specimen of ferromagnetic material is mechanically rotated, and the specimen may be observed to become magnetized.

Both of the above phenomena are classed as *gyromagnetic effects.*

33-6 Magnetic Circuits

Lines of magnetic induction always form closed loops. If we think of a tube of magnetic induction as a tube bounded by lines of induction, the lines of induction may never cross each other, and the total number of lines of induction contained within such a tube must be constant. *Thus the magnetic flux within a tube of induction is constant.* This is the first principle used in the calculation of a *magnetic circuit,* which may be considered as a *closed path of magnetic material.*

Let us consider the work done in carrying a magnetic pole around a wire carrying current. The magnetic field intensity at a distance a from a long straight wire is given by

$$H = \frac{I}{2\pi a} \cdot \qquad \textbf{(30-1a)}$$

Suppose that a pole of strength p is carried around the circle of radius a, concentric with the wire carrying current into the plane of the paper in Figure 33-10, opposite to the direction of the magnetic field. The pole

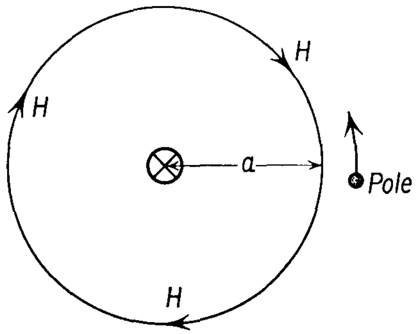


Fig. 33-10 Work done in carrying a magnetic pole of strength p around a wire carrying current. The direction of the current is into the paper.

is carried a distance $2\pi a$ against the force exerted by the magnetic field, so that the work done $\mathscr{W}$ is given by

$$\mathscr{W} = \int pH\,ds = Ip;$$

therefore

$$\frac{\mathscr{W}}{p} = \int H\,ds = I. \tag{33-4}$$

The work per unit pole in carrying a pole around a wire carrying current does not depend upon the radius of the circle but only upon the current in the wire. Any arbitrary path followed in carrying the pole around the wire may be approximated by a combination of radial paths, in which no work is done, and circular paths, in which the work done does not depend upon the radius but only upon the fraction of the circle traversed, as shown in Figure 33-11. By analogy with electromotive force, the work per unit pole done in carrying a north pole around a closed path is called the *magnetomotive force*, abbreviated mmf, and represented by $\mathscr{F}$.

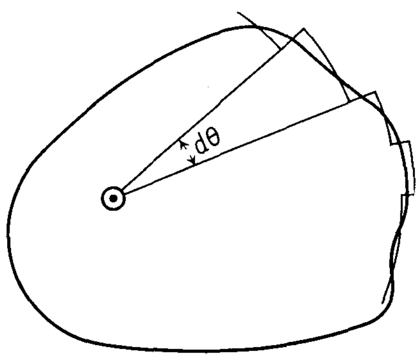


Fig. 33-11 Approximation of any path around a current by radial and circular displacements.

If the closed path encircles a number of wires carrying current, the work done in carrying a unit pole about each wire is given by Equation

(33-4) and, in, mks units, the total work per unit pole done is equal to the sum of the currents. This is the second fundamental principle for the design of magnetic circuits. The first principle relates to the magnetic induction B, while the second is concerned with the magnetic intensity H. In order to relate these, we must make use of the concept of permeability, imposing a third condition upon the magnetic circuit. These three conditions may be expressed in the following equations.

$$B = \mu H = \kappa_m \mu_0 H. \tag{33-5a}$$

$$\mathcal{J} = \int H\, ds = \sum I. \tag{33-5b}$$

$$\Phi = BA = \text{const.} \tag{33-5c}$$

Let us consider the case of a magnetic circuit made up of a ring-shaped core around which a uniform toroid has been wound. We shall imagine that the core has been cut so that a small section of the core may be removed. We shall first compute the induction in the core when the core is intact with a current I in the winding. Next we shall calculate the induction in the core and gap when the cut section is removed. This is the usual case of an electromagnet with an air gap.

Case 1. A uniform toroid of N turns is wound on a core of length s and cross-sectional area A. Find the induction within the core when there is a steady current I in the toroid.

We may assume that all of the magnetic flux is confined to the volume of the toroid, so that the core itself may be considered as a tube of induction. The magnetic intensity is constant around the core. Applying Equation (33-5b) and integrating around the mean circumference s of the core, we find

$$\mathcal{J} = Hs = NI,$$

or

$$H = \frac{NI}{s},$$

a result we have previously stated as Equation (30-5). The magnetic induction within the core may be obtained from Equation (33-5a) as

$$B = \mu H = \frac{\mu NI}{s}.$$

The permeability of iron varies greatly from specimen to specimen, depending upon heat treatment, purity, internal stress, and magnetic intensity. In Figure 33-12 two graphs of the relative permeability of Armco iron are shown displaying the great variation in relative permeability with H and with heat treatment. In order to determine B, it is necessary to have such a curve for the particular iron being used. Alternatively, if the iron was initially unmagnetized and the magnetization curve is available, as in Figure 33-3, H may be computed from the formula above and B read from the curve. Appropriate conversion factors must be applied to Figure 33-3 if mks units are used, as in the formulas developed in the text of this chapter. We recall that

$$B: \quad 1 \text{ weber/m}^2 = 10^4 \text{ gausses,}$$

and

$$H: \quad 1 \text{ amp/m} = 4\pi \times 10^{-3} \text{ oersted.}$$

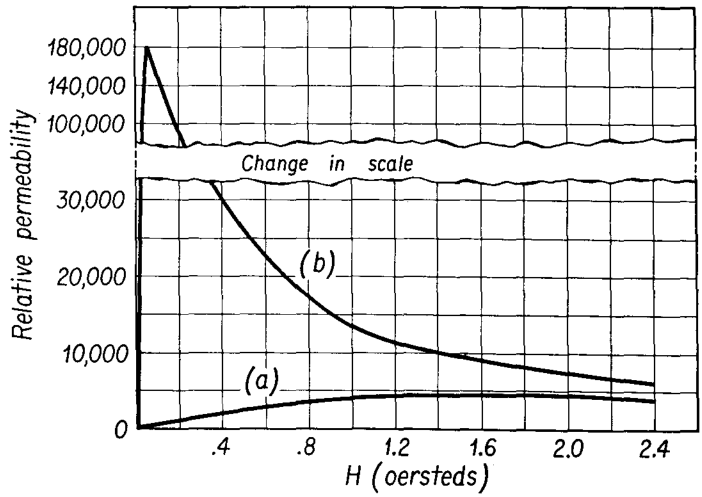


Fig. 33-12 Relative permeability of soft magnet iron (a) with standard annealing and (b) specially annealed. (Courtesy of Armco Steel Corporation.)

Case 2. A uniform toroid of N turns is wound on a core of mean circumference l_c having an air gap of length l_a. The core and gap have cross-sectional area A. Find the induction within the core and the gap when there is a current I in the toroid.

Once again we shall neglect any fringing field (although here this is a more serious approximation which, in practical problems, may give rise to considerable error) and shall consider that the boundary of the toroid is a tube of induction. We denote the magnetic intensity in the core by H_c and the magnetic intensity in the air gap by H_a. Similarly, the magnetic induction in the core is B_c, and the induction in the air gap is B_a. Applying Equation (33-5b), we find

$$\mathcal{J} = H_c l_c + H_a l_a = NI.$$

From Equation (33-5c) we find

$$B_c A_c = B_a A_a = \Phi.$$

If the permeability of the core is μ_c and that of the gap is μ_a, we may combine these two equations through Equation (33-5c) to find

$$\Phi = \frac{NI}{\dfrac{l_a}{\mu_a A_a} + \dfrac{l_c}{\mu_c A_c}}. \tag{33-6}$$

Equation (33-6) is often compared to Ohm's law for electric circuits. The flux Φ is thought to be analogous to the current, the quantity NI is the *magnetomotive force* $\mathcal{F}$, in analogy with the electromotive force, and the quantity $l/\mu A$ is called the *reluctance* $\mathcal{R}$ in analogy with the resistance. This is generally written as

$$\Phi = \frac{\mathcal{F}}{\mathcal{R}}, \tag{33-7}$$

where these quantities are given as

$$\mathcal{F} = NI \qquad \text{(ampere turns)},$$

$$\mathcal{R} = \sum \frac{l}{\mu A} \qquad \text{(ampere turns/weber)},$$

$$\Phi = BA \qquad \text{(webers)}.$$

In general, the permeability is not known as a function of the current in the coil; in such a case, if the desired induction in the gap is known, the induction in the core may first be computed from the formulas. The magnetic intensity in the core appropriate to this value of induction may then be obtained from the magnetization curve, giving the permeability, and the current required may then be obtained from the formulas.

Since the current literature of magnetism is largely written in terms of gausses and oersteds, we will calculate an example in Gaussian units, for which appropriate equations are listed in Table 33-1 and units are listed in Table 33-2.

Illustrative Example. The average circumference of a Rowland ring of soft iron is 50.1 cm. The ring is cut by an air gap 0.1 cm wide. The cross-sectional area of the ring (and gap) is 5 cm^2. The ring is wound with 2,000 turns of wire. Find the current required to produce an induction of 8,000 gausses in the gap.

Assuming that all of the flux in the core passes through the gap, and noting that the area of the gap is the same as the area of the core, the induction in the core is also 8,000 gausses. From Figure 33-3 a magnetic intensity of $H = 4$ oersteds produces this induction in soft iron. Thus, at an induction of 8,000 gausses, the relative permeability κ_m is given by Gaussian units

$$\kappa_m = \frac{B}{H}$$

$$= \frac{8{,}000 \text{ gausses}}{4 \text{ oersteds}} = 2{,}000.$$

The relative permeability of the air gap is 1. Thus the reluctance $\mathcal{R}$ of the magnetic circuit is

$$\mathcal{R} = \sum \frac{l}{\kappa_m a},$$

$$\mathcal{R} = \frac{50 \text{ cm}}{2{,}000 \times 5 \text{ cm}^2} + \frac{0.1 \text{ cm}}{1 \times 5 \text{ cm}^2},$$

$$\mathcal{R} = 0.025 \text{ gilbert}.$$

The magnetomotive force $\mathcal{F}$ is given by

$$\mathcal{F} = \frac{4\pi NI}{c},$$

$$\mathcal{F} = \frac{4\pi \times 2{,}000 \times I}{c}.$$

The required flux Φ is given by

$$\Phi = BA = 8{,}000 \text{ gausses} \times 5 \text{ cm}^2,$$

$$\Phi = 40{,}000 \text{ maxwells}.$$

We relate these quantities by the equation

$$\Phi = \frac{\mathcal{F}}{\mathcal{R}},$$

$$40{,}000 = \frac{4\pi \times 2{,}000 \times I}{c \times 0.025}.$$

Thus

$$I = \frac{c}{8\pi} \text{ statamperes}.$$

Remembering that 1 amp $= 3 \times 10^9$ statamperes, the current through the coil needed to produce the required induction in the gap is

$$I = \frac{10}{8\pi} \text{ amp}$$

$$= 0.399 \text{ amp}.$$

Note that the reluctance of the air gap is much higher than the reluctance of the iron path. In practical magnet problems it is necessary to take the fringing field into account. In general, something less than one half of the flux which passes through the iron also passes through the air gap.

Problems

33-1. Find the self-inductance of a toroid of 500 turns wound over an iron ring whose relative permeability is given by Figure 33-12, (a) when the current through the coil is 0.1 amp, (b) when the current through the coil is 0.2 amp, and

TABLE 33-1 PRINCIPAL EQUATIONS IN MKS AND GAUSSIAN UNITS

Equation	MKS	Gaussian	
(33-1)	$\frac{L}{L_0} = \kappa_m$	Same as mks	Relative permeability
(33-2)	$L = \frac{\mu N^2 A}{s}$	$L = \frac{\kappa_m 4\pi N^2 A}{sc^2}$	Long solenoid or toroid
(29-14)	$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}$	$\mathbf{B} = \mathbf{H} + 4\pi\mathbf{M}$	
(33-5a)	$B = \mu H = \kappa_m \mu_0 H$	$B = \kappa_m H$	
(33-5b)	$\mathcal{F} = \int H\, ds = \sum I$	$\mathcal{F} = \int H\, ds = \sum \frac{4\pi I}{c}$	Mmf
(33-5c)	$\Phi = BA$	Same as mks	Flux
(33-7)	$\Phi = \frac{\mathcal{F}}{\mathcal{R}}$	Same as mks	
	$\mathcal{F} = NI$	$\mathcal{F} = \frac{4\pi NI}{c}$	Mmf
	$\mathcal{R} = \sum \frac{l}{\mu a}$	$\mathcal{R} = \sum \frac{l}{\kappa_m a}$	Reluctance
(32-14)	$\mathcal{W}_V = \frac{1}{2} BH$	$\mathcal{W}_V = \frac{1}{8\pi} BH$	Magnetic energy density

TABLE 33-2 CONVERSION FACTORS RELATING MKS AND GAUSSIAN UNITS

Quantity	Symbol	MKS Unit	Gaussian Unit	
Magnetic intensity	H	$1 \frac{\text{amp}}{\text{m}}$	$= 4\pi \times 10^{-3}$ oersted	(emu)
Flux density	B	$1 \frac{\text{weber}}{\text{m}^2}$	$= 10^4$ gausses	(emu)
Flux	Φ	1 weber	$= 10^8$ maxwells	(emu)
Magnetomotive force	$\mathcal{F}$	1 amp turn	$= \frac{4\pi}{10}$ gilbert	(emu)
Reluctance	$\mathcal{R}$	$1 \frac{\text{amp turn}}{\text{weber}}$	$= 4\pi \times 10^{-9}$ emu	(emu)
Magnetization	M	$1 \frac{\text{weber}}{\text{m}^2}$	$= \frac{10^4}{4\pi} \frac{\text{unit poles}}{\text{cm}^2}$	(emu)

$$\mu_0 = 4\pi \times 10^{-7} \frac{\text{henry}}{\text{m}} = 4\pi \times 10^{-7} \frac{\text{weber}}{\text{nt m}^2} = 4\pi \times 10^{-7} \frac{\text{nt}}{\text{amp}^2} = 4\pi \times 10^{-7} \frac{\text{weber}}{\text{amp m}}$$

(c) when the current through the coil is 0.4 amp. The cross-sectional area of the ring is 1 cm^2, and its mean circumference is 125 cm.

33-2. A coil of wire contains 8 turns and has a resistance of 24 ohms. The coil is connected to a ballistic galvanometer which has a resistance of 60 ohms. If the magnetic flux through the coil is changed from 0 to 0.003 weber, (a) determine the charge which flows through the circuit. (b) If the sensitivity of the galvanometer is 25 μcoul/cm, determine the galvanometer deflection in centimeters.

33-3. A small coil of 200 turns of wire, having a circular area of 5 cm^2 and a resistance of 12 ohms, is used as an exploring coil to measure the magnetic field between the poles of a magnet. The coil is connected to a ballistic galvanometer whose resistance is 36 ohms and whose sensitivity is 0.18 μcoul/cm. (a) The coil is thrust into the magnetic field with the plane of the coil perpendicular to the lines of induction. The observed galvanometer deflection is 6.30 cm. Determine the magnetic induction. (b) While the coil is in this field, it is rotated through 180° about a diameter as axis. Determine the deflection of the galvanometer.

33-4. A solenoid 80 cm long has 500 turns and a cross-sectional area of 3.0 cm^2. A short secondary coil of 20 turns is wound around the middle of the first solenoid. The secondary coil, of resistance 1.3 ohms, is connected to a ballistic galvanometer of resistance 26.2 ohms. Determine the charge which flows through the secondary coil (a) when the switch is closed and the current through the primary rises to 2 amp and (b) when the current in the primary is increased from 2 amp to 3 amp.

33-5. A Rowland ring, wound with 1,000 turns of wire and having a mean circumference of 50 cm, carries a current of 4 amp. The relative permeability of the core is 800. (a) What is the magnetic intensity in the core? (b) What is the induction in the core? (c) What is the magnetization of the core?

33-6. Repeat Problem 33-5 in the case that the ring has been cut so that it has a gap 1 mm wide.

33-7. A ring of magnet iron, whose magnetization curve is shown in Figure 33-3, is wound with a toroidal coil of 250 turns. The mean circumference of the ring is 15 cm, and its cross-sectional area is 5 cm^2. Find the current in the coil required to produce an induction of 2,000 gausses.

33-8. A piece of magnet iron is used as the core of a solenoid. The magnetic field intensity inside the solenoid is 5 oersteds, and the induction within the core is 2,000 gausses. (a) What is the magnetic energy per unit volume within the iron? (b) What is the relative permeability of the iron at this induction? (c) What is the magnetization of the iron?

33-9. A long, straight, hollow tubular conductor of radius r carries a current I uniformly distributed around the conductor. Find the magnetic field intensity (a) at a point P_1 inside the tube at a distance a less than r, and (b) at a point P_2 outside at a distance b greater than r. [HINT: Carry a unit pole around a circular path concentric with the tube and apply a symmetry argument.]

33-10. Repeat the calculation in the illustrative example of Section 33-6 in mks units.