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TRANSLATION THEOREMS FOR FOURIER-FEYNMAN TRANSFORMS AND CONDITIONAL FOURIER-FEYNMAN TRANSFORMS

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TRANSLATION THEOREMS FOR FOURIER-FEYNMAN TRANSFORMS AND CONDITIONAL FOURIER-FEYNMAN TRANSFORMS

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1. Introduction. Translation theorems for Wiener integrals were given by Cameron and Martin in [**3**] and by Cameron and Graves in [**2**]. Translation theorems for analytic Feynman integrals were given by Cameron and Storvick in [**4**], [**7**] and translation theorems for Feynman integrals on abstract Wiener and Hilbert spaces were given by Chung and Kang in [**12**].

The concept of an L_1 analytic Fourier-Feynman transform (FFT) was introduced by Brue in [**1**]. In [**5**], Cameron and Storvick introduced an L_2 FFT. In [**20**], Johnson and Skoug developed an L_p FFT for $1 \le p \le 2$ which extended the results in [**1**], [**5**] and gave various relationships between the L_1 and L_2 theories. In [**15**]–[**17**], Huffman, Park and Skoug obtained various results involving the FFT and the convolution product, and in [**18**] used the concept of the (generalized) Feynman integral [**13**], [**24**] to define a (generalized) FFT (GFFT) and a generalized convolution product. Very recently [**26**], Park and Skoug studied (generalized) conditional FFT's (GCFFT's) and conditional convolution products.

In this paper we establish translation theorems for GFFT's and GCFFT's. In Section 3 we establish a translation theorem for the GFFT of very general functionals F defined on Wiener space $C_0[0,T]$, and in Section 4 we obtain a general translation theorem for GCFFT's. We then proceed to show that these general translation theorems apply to two well-known classes of functionals; namely, the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [**6**], and the space $\mathcal{B}_n^{(p)}$ consisting of functionals of the form

$$F(x) = f(\langle \alpha_1, x\rangle, \dots, \langle \alpha_n, x\rangle)$$

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where $\langle \alpha_j, x\rangle$ denotes the Paley-Wiener-Zygmund stochastic integral $\int_0^T \alpha_j(s)\,dx(s)$.

In defining the FFT [**5**], [**15**], [**19**] of F, one starts with, for $\lambda > 0$, the Wiener integral

$$E_x[F(y+\lambda^{-1/2}x)] = \int_{C_0[0,T]} F(y+\lambda^{-1/2}x)m(dx) \tag{1.1}$$

and then extends analytically in λ to the right-half complex plane. In [**18**], [**26**] and in this paper, in defining the GFFT we start with the Wiener integral

$$E_x[F(y+\lambda^{-1/2}z(x,\cdot))] = \int_{C_0[0,T]} F(y+\lambda^{-\frac{1}{2}}z(x,\cdot))m(dx) \tag{1.2}$$

where z is the Gaussian process

$$z(x,t) = \int_0^t h(s)\,dx(s) \tag{1.3}$$

with $h \in L_2[0,T]$ and $\int_0^t h(s)\,dx(s)$ is the Paley-Wiener-Zygmund stochastic integral. Of course if $h(t) \equiv 1$ on $[0,T]$, then $z(x,t) = x(t)$ and so the (generalized) Wiener integral in (1.2) reduces to the ordinary Wiener integral given by (1.1).

2. Definitions and preliminaries. Let $C_0[0,T]$ denote Wiener space; that is, the space of all $\mathbf{R}$-valued continuous functions $x(t)$ on [0,T] with $x(0) = 0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$ and let m denote Wiener measure. A subset B of $C_0[0,T]$ is said to be scale-invariant measurable [**9**], [**21**] provided $\rho B \in \mathcal{M}$ for all $\rho > 0$ and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for all $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e). If two functionals F and G are equal s-a.e., we write $F \approx G$.

For a detailed discussion of scale-invariant measurability and its relation with other topics, see [**21**]. In [**27**], Segal gives an interesting discussion of the relation between scale change in Wiener space and certain questions in quantum field theory.

Throughout this paper, we assume that every functional F we consider is s-a.e. defined, is scale-invariant measurable and, for each $\lambda > 0$, $F(\lambda^{-1/2}z(x,\cdot))$ is Wiener integrable in x on $C_0[0,T]$.

Let h be an element of $L_2[0,T]$ with $\|h\| > 0$, let $z(x,t)$ be given by (1.3) and let

$$a(t) = \int_0^t h^2(s)\,ds.$$

Then z is a Gaussian process with mean zero and covariance function

$$E_x[z(x,s)z(x,t)] = a(\min\{s,t\}).$$

Next we state the definition of the (generalized) analytic Feynman integral [**13**], [**18**]. Let $\mathbf{C}_+ = \{\lambda \in \mathbf{C} : \operatorname{Re}\lambda > 0\}$ and let $\tilde{\mathbf{C}}_+ = \{\lambda \in \mathbf{C} : \lambda \neq 0 \text{ and } \operatorname{Re}\lambda \geq 0\}$. Let $J(\lambda) = E[F(\lambda^{-1/2}z(x,\cdot))]$. If a function $J^*(\lambda)$ exists analytic in λ on $\mathbf{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is called the (generalized) analytic Wiener integral of F with parameter λ, and for $\lambda \in \mathbf{C}_+$, we write

$$E_x^{\mathrm{anw}_\lambda}[F(z(x,\cdot))] = J^*(\lambda). \tag{2.1}$$

Let real $q \neq 0$ be given. Then we define the (generalized) analytic Feynman integral of F with parameter q by $(\lambda \in \mathbf{C}_+)$

$$E_x^{\mathrm{anf}_q}[F(z(x,\cdot))] = \lim_{\lambda \to -iq} E_x^{\mathrm{anw}_\lambda}[F(z(x,\cdot))] \tag{2.2}$$

if the limit exists.

Next we state the definition of the GFFT given in [**18**], [**26**] using (2.1) and (2.2) above. For $\lambda > 0$ and $y \in C_0[0,T]$, let

$$T_\lambda(F)(y) = E_x^{\mathrm{anw}_\lambda}[F(y + z(x,\cdot))]. \tag{2.3}$$

In the standard Fourier theory the integrals involved are often interpreted in the mean; a similar concept is useful in the FFT theory [**20**, p. 104]. Let $p \in (1,2]$ and let p and p' be related by $1/p + 1/p' = 1$. Let $\{H_n\}$ and H be scale-invariant measurable functionals such that, for each $\rho > 0$,

$$\lim_{n\to\infty} E[|H_n(\rho y) - H(\rho y)|^{p'}] = 0.$$

Then we write

$$H \approx \text{l.i.m.}_{n\to\infty} H_n$$

and we call H the scale-invariant limit in the mean of order p'. A similar definition is understood when n is replaced by the continuously varying parameter λ. Let real $q \neq 0$ be given. For $1 < p \le 2$ we define the L_p analytic GFFT, $T_q^{(p)}(F)$ of F, by the formula, $\lambda \in \mathbf{C}_+$,

$$T_q^{(p)}(F)(y) = \text{l.i.m.}_{\lambda\to -iq} T_\lambda(F)(y) \tag{2.4}$$

if it exists. We define the L_1 analytic GFFT, $T_q^{(1)}(F)$ of F, by the formula, $\lambda \in \mathbf{C}_+$,

$$T_q^{(1)}(F)(y) = \lim_{\lambda\to -iq} T_\lambda(F)(y) \tag{2.5}$$

if it exists. We note that for $1 \le p \le 2$, $T_q^{(p)}(F)$ is defined only s-a.e. We also note that if $T_q^{(p)}(F)$ exists and if $F \approx G$, then $T_q^{(p)}(G)$ exists and $T_q^{(p)}(G) \approx T_q^{(p)}(F)$.

The following Wiener integration formula is used throughout this paper

$$E\bigg[\exp\bigg\{\frac{i}{\sqrt{\lambda}}\langle u, x\rangle\bigg\}\bigg] = \exp\bigg\{-\frac{\|u\|^2}{2\lambda}\bigg\} \tag{2.6}$$

for $\lambda > 0$ and $u \in L_2[0,T]$.

3. A general translation theorem. Throughout this paper we will always translate by

$$x_0(t) = \int_0^t \beta(s)\,ds, \quad \beta \in L_2[0,T]. \tag{3.1}$$

In our first result we obtain a translation theorem for the GFFT of very general functionals F.

Theorem 3.1. *Let $p \in [1,2]$ be given, and let $F : C_0[0,T] \to \mathbf{C}$ be such that the* GFFT, *$T_q^{(p)}(F)$ of F exists for all real $q \neq 0$. Let*

x_0 be given (3.1) and let $z(x,t)$ be given by (1.3) with $h \in L_\infty[0,T]$, $(\beta/h) \in L_2[0,T]$ and $(\beta/h^2) \in L_2[0,T]$. Then

$$T_q^{(p)}(F)(y+x_0) \approx \exp\left\{iq\left\langle \frac{\beta}{h^2}, y\right\rangle + \frac{iq}{2}\left\|\frac{\beta}{h}\right\|^2\right\} T_q^{(p)}(F^*)(y) \tag{3.2}$$

where

$$\begin{aligned} F^*(z(x,\cdot)) &= \exp\left\{-iq\int_0^T \frac{\beta(s)}{h^2(s)}\,dz(x,s)\right\}F(z(x,\cdot)) \\ &= \exp\left\{-iq\left\langle\frac{\beta}{h}, x\right\rangle\right\}F(z(x,\cdot)). \end{aligned} \tag{3.3}$$

Proof. We will give the proof for the case $p \in (1,2]$. The case $p = 1$ is similar, but somewhat easier. For $\lambda > 0$, using (3.3) we see that

$$\begin{aligned} I &\equiv T_\lambda(F^*)(y) \\ &= E_x[F^*(y + \lambda^{-1/2}z(x,\cdot))] \\ &= \exp\left\{-iq\left\langle\frac{\beta}{h^2}, y\right\rangle\right\} \\ &\quad \cdot E_x\left[\exp\left\{-iq\lambda^{-1/2}\left\langle\frac{\beta}{h^2}, z(x,\cdot)\right\rangle\right\}F(y+\lambda^{-1/2}z(x,\cdot))\right]. \end{aligned}$$

Using the translation theorem in the form

$$E[F(x)] = \exp\left\{-\frac{\|u_0'\|^2}{2}\right\}E[F(x+u_0)\exp\{-\langle u_0', x\rangle\}]$$

with $u_0(t) = \lambda^{1/2}\int_0^t \beta(s)/h(s)\,ds$ and $x_0(t) = \int_0^t \beta(s)\,ds = \lambda^{-1/2}\int_0^t h(s)\,du_0(s)$, we get that

$$\begin{aligned} I = \exp&\left\{-iq\left\langle\frac{\beta}{h^2}, y\right\rangle - \frac{\lambda}{2}\left\|\frac{\beta}{h}\right\|^2 - iq\lambda^{-1/2}\left\langle\frac{\beta}{h^2}, z(u_0,\cdot)\right\rangle\right\} \\ &\cdot E_x\left[\exp\left\{-iq\lambda^{-1/2}\left\langle\frac{\beta}{h^2}, z(x,\cdot)\right\rangle - \lambda^{1/2}\left\langle\frac{\beta}{h}, x\right\rangle\right\}\right. \\ &\qquad\qquad \left. \cdot F(y+\lambda^{-1/2}z(x,\cdot) + \lambda^{-1/2}z(u_0,\cdot))\right]. \end{aligned}$$

Noting that $\langle \beta/h^2, z(x,\cdot)\rangle = \langle \beta/h, x\rangle$, $\langle \beta/h^2, z(u_0,\cdot)\rangle = \lambda^{1/2}\|\beta/h\|^2$, and that $z(u_0,t) = \lambda^{1/2}x_0(t)$, we obtain that

$$\begin{aligned} I &= \exp\left\{- iq\left\langle \frac{\beta}{h^2}, y\right\rangle - \frac{1}{2}(\lambda+2iq)\left\|\frac{\beta}{h}\right\|^2\right\} \\ &\quad\cdot E_x\left[\exp\left\{- \lambda^{-1/2}(iq+\lambda)\left\langle\frac{\beta}{h}, x\right\rangle\right\}\right. \\ &\qquad\left.\cdot F(y+x_0+\lambda^{-1/2}z(x,\cdot))\right]. \end{aligned} \tag{3.4}$$

Using Hölder's inequality, we get that

$$\begin{aligned} E_x\bigg[\bigg|\bigg(\exp\bigg\{- \lambda^{-1/2}(iq+\lambda)\bigg\langle\frac{\beta}{h}, x\bigg\rangle\bigg\} - 1\bigg)F(y+x_0+\lambda^{-1/2}z(x,\cdot))\bigg|\bigg] \\ \le \left(E_x\left[\left|\exp\left\{- \lambda^{-1/2}(iq+\lambda)\left\langle\frac{\beta}{h}, x\right\rangle\right\} - 1\right|^{p'}\right]\right)^{1/p'} \\ \cdot (E_x[|F(y+x_0+\lambda^{-1/2}z(x,\cdot))|^p])^{1/p}. \end{aligned}$$

Note that each factor in the last expression has a limit as $\lambda \to -iq$ in $\mathbf{C}_+$, and that

$$\left(E_x\left[\left|\exp\left\{- \lambda^{-1/2}(iq+\lambda)\left\langle\frac{\beta}{h}, x\right\rangle\right\} - 1\right|^{p'}\right]\right)^{1/p'} \longrightarrow 0$$

as $\lambda \to -iq$ in $\mathbf{C}_+$. Hence

$$\begin{aligned} \text{l.i.m.}_{\lambda\to -iq}E_x\left[\exp\left\{- \lambda^{-1/2}(iq+\lambda)\left\langle\frac{\beta}{h}, x\right\rangle\right\}F(y+x_0+\lambda^{-1/2}z(x,\cdot))\right] \\ = \text{l.i.m.}_{\lambda\to -iq}E_x[F(y+x_0+\lambda^{-1/2}z(x,\cdot))] \\ = \text{l.i.m.}_{\lambda\to -iq}T_\lambda(F)(y+x_0). \end{aligned}$$

Hence, letting $\lambda \to -iq$ in (3.4) yields (3.2) as desired. □

In our first corollary below we will see that the translation formula (3.2) holds for the GFFT of functionals in the Banach algebra $\mathcal{S}$

introduced by Cameron and Storvick in [**6**]. The Banach algebra $\mathcal{S}$ consists of functionals expressible in the form

$$F(x) = \int_{L_2[0,T]} \exp\{i\langle u, x\rangle\}\, df(u) \tag{3.5}$$

for s-a.e. $x \in C_0[0,T]$ where f is an element of $M(L_2[0,T])$, the space of all $\mathbf{C}$-valued countably additive finite Borel measures on $L_2[0,T]$. Further work on $\mathcal{S}$ shows that it contains many functionals of interest in Feynman integration theory [**8**], [**10**], [**22**], [**25**], [**28**].

Corollary 3.1. *Let $F \in \mathcal{S}$ be given by* (3.5)*, and let x_0 be given by* (3.1)*. Let z, h and β be as in Theorem* 3.1*. Then for all $p \in [1,2]$ and all real $q \neq 0$,*

$$T_q^{(p)}(F)(y + x_0) \approx \exp\left\{ iq\left\langle \frac{\beta}{h^2}, y\right\rangle + \frac{iq}{2}\left\|\frac{\beta}{h}\right\|^2 \right\} T_q^{(p)}(F^*)(y) \tag{3.6}$$

where F^* is given by (3.3)*.*

Proof. This corollary follows from Theorem 3.1 above since, by [**18**, Theorem 3.1], $T_q^{(p)}(F)$ exists for all $p \in [1,2]$ and all real $q \neq 0$. □

In our next theorem we observe that the two sides of (3.6) are identically equal for every $y \in C_0[0,T]$ of the form

$$y(t) = \int_0^t \phi(s)\, ds, \quad 0 \le t \le T \tag{3.7}$$

for some $\phi \in L_2[0,T]$.

Theorem 3.2. *Let F, F^*, z and x_0 be as in Corollary* 3.1*, and let y be given by* (3.7)*. Then for all $p \in [1,2]$ and all real $q \neq 0$,*

$$T_q^{(p)}(F)(y + x_0) \,=\, \exp\left\{ iq\left\langle \frac{\beta}{h^2}, y\right\rangle + \frac{iq}{2}\left\|\frac{\beta}{h}\right\|^2 \right\} T_q^{(p)}(F^*)(y). \tag{3.8}$$

Proof. We first note that y and $y+x_0$ are both absolutely continuous on $[0,T]$ and their derivatives are elements of $L_2[0,T]$. Then, direct calculations show that $T_q^{(p)}(F)(y+x_0)$ and $T_q^{(p)}(F^*)(y)$ both exist for every y of the form (3.7) and satisfy equation (3.8). □

By choosing $y(t)\equiv 0$ and $h(t)\equiv 1$ on $[0,T]$ in Theorem 3.2 above, we obtain Theorem 4 of [**7**] as a corollary since $h(t)\equiv 1$ implies that $z(x,t)=x(t)$.

Corollary 3.2. *Let F, F^* and x_0 be as in Theorem* 3.2. *Then for all real $q\neq 0$,*

$$\begin{aligned} E_x^{\mathrm{anf}_q}[F(x+x_0)] &= \exp\left\{\frac{iq\|\beta\|^2}{2}\right\}E_x^{\mathrm{anf}_q}[F^*(x)] \\ &= \exp\left\{\frac{iq\|\beta\|^2}{2}\right\}E_x^{\mathrm{anf}_q}[\exp\{-iq\langle\beta,x\rangle\}F(x)]. \end{aligned}$$

Next we want to briefly discuss another class of functionals to which our general translation theorem applies. Let $h\in L_2[0,T]$ and let $z(x,t)$ be given by (1.3). Then choose $\{\alpha_1,\dots,\alpha_n\}$ from $L_2[0,T]$ such that

(a) $\{\alpha_1,\dots,\alpha_n\}$ are orthogonal on $[0,T]$, and

(b) $\{\alpha_1 h,\dots,\alpha_n h\}$ are orthonormal on $[0,T]$.

Remark 3.1. One way to do this would be to choose $0=t_0<t_1<\dots<t_n=T$ with

$$\text{Lebesgue measure }\{\{\text{ support of } h\}\cap[t_{j-1},t_j]\}>0$$

for $j=1,\dots,n$, and then letting

$$\alpha_j(s)=\left(\int_{t_{j-1}}^{t_j}h^2(s)\,ds\right)^{-1/2}\chi_{[t_{j-1},t_j]}(s).$$

Now let $\mathcal{B}_n^{(p)}$ be the space of all functionals F on $C_0[0,T]$ of the form

$$F(x)=f(\langle\alpha_1,x\rangle,\dots,\langle\alpha_n,x\rangle) \tag{3.9}$$

s-a.e. where $f \in L_p(\mathbf{R}^n)$ and the α_j's satisfy (a) and (b) above.

Corollary 3.3. *Let $p \in [1,2]$, let x_0 be given by (3.1), and let $z(x,t)$ be given by (1.3) with $h \in L_\infty[0,T]$, $(\beta/h) \in L_2[0,T]$ and $(\beta/h^2) \in L_2[0,T]$. Let $F \in \mathcal{B}_n^{(p)}$ be given by (3.9), and let F^* be given by (3.3). Then, for all real $q \neq 0$,*

$$(3.10) \quad T_q^{(p)}(F)(y+x_0) \approx \exp\left\{iq\left\langle \frac{\beta}{h^2}, y\right\rangle + \frac{iq}{2}\left\|\frac{\beta}{h}\right\|^2\right\} T_q^{(p)}(F^*)(y).$$

Remark 3.2. In our proof below we use Lemmas 1.1 and 1.2 of [**19**, pp. 98–102]. These two lemmas are true without the dimension restriction $\nu < (2p/(2-p))$ (in our notation $\nu = n$); in fact for each $p \in [1,2]$, these two lemmas are valid for all integers $\nu > 0$.

Proof of Corollary 3.3. In view of Theorem 3.1, it will suffice to show that $T_q^{(p)}(F)$ exists for all $p \in [1,2]$ and all real $q \neq 0$.

For $\lambda > 0$ we obtain that

$$\begin{aligned}
&T_\lambda(F)(y+x_0) \\
&= E_x[F(y+x_0+\lambda^{-1/2}z(x,\cdot))] \\
&= E_x[f(\langle\alpha_1, y+x_0\rangle + \lambda^{-1/2}\langle\alpha_1 h, x\rangle, \dots, \langle\alpha_n, y+x_0\rangle + \lambda^{-1/2}\langle\alpha_n h, x\rangle)] \\
&= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbf{R}^n} f(\vec{u}) \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n (u_j - \langle\alpha_j, y+x_0\rangle)^2\right\} d\vec{u} \\
&= g(\lambda; \langle\vec{\alpha}, y+x_0\rangle)
\end{aligned}$$

for s-a.e. $y \in C_0[0,T]$, where $\vec{u} = (u_1, \dots, u_n)$, $\langle\vec{\alpha}, y+x_0\rangle = (\langle\alpha_1, y+x_0\rangle, \dots, \langle\alpha_n, y+x_0\rangle)$, and where

$$(3.11) \qquad g(\lambda; \vec{w}) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbf{R}^n} f(\vec{u}) \exp\left\{-\frac{\lambda}{2}\|\vec{u}-\vec{w}\|^2\right\} d\vec{u}.$$

Clearly $g(\lambda; \langle\vec{\alpha}, y+x_0\rangle)$ is an analytic function of λ throughout $\mathbf{C}_+$.

For the case $p = 1$, an application of the dominated convergence theorem shows that $T_q^{(1)}(F)$ exists for all real $q \neq 0$ and that

$$\begin{aligned} T_q^{(1)}(F)(y+x_0) &\approx g(-iq;\langle\vec{\alpha}, y+x_0\rangle) \\ &\approx \left(\frac{q}{2\pi i}\right)^{n/2}\int_{\mathbf{R}^n} f(\vec{u})\exp\left\{\frac{iq}{2}\|\vec{u}-\langle\vec{\alpha}, y+x_0\rangle\|^2\right\} d\vec{u}. \end{aligned}$$

For the case $p \in (1,2]$, Lemma 1.1 of [**19**] tells us that for all $\lambda \in \tilde{\mathbf{C}}_+$, $g(\lambda;\vec{w})$ is an element of $L_{p'}(\mathbf{R}^n)$ with $\|g(\lambda;\cdot)\|_{p'} \leq \|f\|_p(|\lambda|/2\pi)^{n(1-p)/2p}$. In addition, by Lemma 1.2 of [**19**], we have that $\|g(\lambda;\cdot)-g(-iq;\cdot)\|_{p'} \to 0$ as $\lambda \to -iq$ through values in $\mathbf{C}_+$. Hence for all $\rho > 0$,

$$\begin{aligned} E_y[|g(\lambda;\langle\vec{\alpha},\rho y+x_0\rangle) - g(-iq;\langle\vec{\alpha},\rho y+x_0\rangle)|^{p'}] \\ \leq \rho^{-n}\|g(\lambda;\cdot)-g(-iq;\cdot)\|_{p'}^{p'} \end{aligned}$$

which goes to zero as $\lambda \to -iq$ through $\mathbf{C}_+$. Hence for all $p \in [1,2]$, $T_q^{(p)}(F)$ exists and we have that

$$\begin{aligned} T_q^{(p)}(F)(y+x_0) &\approx g(-iq;\langle\vec{\alpha}, y+x_0\rangle) \\ &\approx \left(\frac{q}{2\pi i}\right)^{n/2}\int_{\mathbf{R}^n} f(\vec{u})\exp\left\{\frac{iq}{2}\|\vec{u}-\langle\vec{\alpha}, y+x_0\rangle\|^2\right\} d\vec{u} \\ &\approx \left(\frac{q}{2\pi i}\right)^{n/2}\int_{\mathbf{R}^n} f(\vec{u}+\langle\vec{\alpha}, y+x_0\rangle)\exp\left\{\frac{iq}{2}\|\vec{u}\|^2\right\} d\vec{u}. \qquad (3.12) \end{aligned}$$

$\square$

Remark 3.3. For $F \in \mathcal{B}_n^{(p)}$ given by (3.9) and F^* given by (3.3), using the Gram-Schmidt orthogonalization procedure, one can show that

$$\begin{aligned} T_q^{(p)}(F^*)(y) \approx \left(\frac{q}{2\pi i}\right)^{n/2} \exp\left\{-iq\left\langle\frac{\beta}{h^2}, y\right\rangle - \frac{iq}{2}\left\|\frac{\beta}{h}\right\|^2 + \frac{iq}{2}\sum_{j=1}^n\langle\alpha_j, x_0\rangle^2\right\} \\ \cdot\int_{\mathbf{R}^n} f(\langle\vec{\alpha}, y\rangle+\vec{u})\exp\left\{-iq\sum_{j=1}^n\langle\alpha_j,x_0\rangle u_j + \frac{iq}{2}\sum_{j=1}^n u_j^2\right\} d\vec{u}. \end{aligned}$$

Again, as in Theorem 3.2 above, it turns out that the two sides of (3.10) are identically equal for every y of the form (3.7).

Theorem 3.3. *Let F, F^*, z and x_0 be as in Corollary* 3.3, *and let y be given by* (3.7). *Then for all $p \in [1,2]$ and all real $q \neq 0$,*

$$T_q^{(p)}(F)(y+x_0) = \exp\left\{iq\left\langle \frac{\beta}{h^2}, y\right\rangle + \frac{iq}{2}\left\|\frac{\beta}{h}\right\|^2\right\}T_q^{(p)}(F^*)(y). \tag{3.13}$$

4. Translation theorems for conditional transforms. In this section we will first establish a translation theorem for the GCFFT of very general functionals F. Then, as corollaries we will show that this translation formula also holds for the GCFFT of functionals in the classes $\mathcal{S}$ and $\mathcal{B}_n^{(p)}$ discussed in Section 3. For some related work involving conditional integrals and transforms, see [**11**], [**13**], [**14**], [**23**], [**24**], [**26**], [**29**]. Throughout this section we will always condition by

$$X(x) = z(x,T). \tag{4.1}$$

First we will state the appropriate definitions of conditional integrals and transforms [**13**], [**14**], [**26**]. For $\lambda > 0$ and $\eta \in \mathbf{R}$ let

$$J_\lambda(\eta) = E[F(\lambda^{-1/2}z(x,\cdot))|\lambda^{-1/2}z(x,T) = \eta] \tag{4.2}$$

denote the (generalized) conditional Wiener integral of $F(\lambda^{-1/2}z(x,\cdot))$ given $\lambda^{-1/2}z(x,T)$. If for almost all $\eta \in \mathbf{R}$, there exists a function $J_\lambda^*(\eta)$, analytic in λ on $\mathbf{C}_+$ such that $J_\lambda^*(\eta) = J_\lambda(\eta)$ for $\lambda > 0$, then $J_\lambda^*(\eta)$ is defined to be the conditional analytic Wiener integral of $F(z(x,\cdot))$ given $z(x,T)$ with parameter λ and for $\lambda \in \mathbf{C}_+$ we write

$$J_\lambda^*(\eta) = E_x^{\text{anw}_\lambda}[F(z(x,\cdot))|z(x,T) = \eta]. \tag{4.3}$$

If, for fixed real $q \neq 0$, $\lim_{\lambda\to -iq} J_\lambda^*(\eta)$ exists for almost all $\eta \in \mathbf{R}$, we denote the value of this limit by

$$E_x^{\text{anf}_q}[F(z(x,\cdot))|z(x,T) = \eta] \tag{4.4}$$

and call it the (generalized) conditional analytic Feynman integral of F given X with parameter q.

Remark 4.1. In [**24**], Park and Skoug give a formula for expressing conditional Wiener integrals in terms of ordinary Wiener integrals; namely, that for $\lambda > 0$,

$$\begin{aligned} &E[F(\lambda^{-1/2}z(x,\cdot))|\lambda^{-1/2}z(x,T)=\eta] \\ &\quad = E_x\left[F\left(\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}\frac{a(\cdot)}{a(T)}z(x,T)+\frac{a(\cdot)}{a(T)}\eta\right)\right]. \end{aligned} \tag{4.5}$$

Thus we have that

$$\begin{aligned} &E_x^{\mathrm{anw}_\lambda}[F(z(x,\cdot))|z(x,T)=\eta] \\ &\quad = E_x^{\mathrm{anw}_\lambda}\left[F\left(z(x,\cdot)-\frac{a(\cdot)}{a(T)}z(x,T)+\frac{a(\cdot)}{a(T)}\eta\right)\right] \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &E_x^{\mathrm{anf}_q}[F(z(x,\cdot))|z(x,T)=\eta] \\ &\quad = E_x^{\mathrm{anf}_q}\left[F\left(z(x,\cdot)-\frac{a(\cdot)}{a(T)}z(x,T)+\frac{a(\cdot)\eta}{a(T)}\right)\right] \end{aligned} \tag{4.7}$$

where in (4.6) and (4.7) the existence of either side implies the existence of the other side and its equality.

Next we define the GCFFT. For $\lambda \in \mathbf{C}_+$ and $y \in C_0[0,T]$, let $T_\lambda(F|X)(y,\eta)$ denote the conditional analytic Wiener integral of $F(y+z(x,\cdot))$ given $X(x) = z(x,T)$, that is to say,

$$\begin{aligned} T_\lambda(F|X)(y,\eta) &= E_x^{\mathrm{anw}_\lambda}[F(y+z(x,\cdot))|z(x,T)=\eta] \\ &= E_x^{\mathrm{anw}_\lambda}\left[F\left(y+z(x,\cdot)-\frac{a(\cdot)}{a(T)}z(x,T)+\frac{a(\cdot)}{a(T)}\eta\right)\right]. \end{aligned} \tag{4.8}$$

For $1 < p \le 2$ we define the L_p analytic GCFFT, $T_q^{(p)}(F|X)(y,\eta)$ by the formula

$$T_q^{(p)}(F|X)(y,\eta) = \mathrm{l.i.m.}_{\lambda\to -iq} T_\lambda(F|X)(y,\eta) \tag{4.9}$$

if it exists, and we define the L_1 analytic GCFFT of F by the formula

$$T_q^{(1)}(F|X)(y,\eta) = \lim_{\lambda\to -iq} T_\lambda(F|X)(y,\eta) \tag{4.10}$$

if it exists.

Remark 4.2. Using Remark 4.1 above, it follows that for all functionals F in the classes $\mathcal{S}$ and $\mathcal{B}_n^{(p)}$, the GCFFT $T_q^{(p)}(F|X)$ exists and is given by the formula

$$T_q^{(p)}(F|X)(y,\eta) = E_x^{\mathrm{anf}_q}\left[F\left(y+z(x,\cdot)-\frac{a(\cdot)}{a(T)}z(x,T)+\frac{a(\cdot)}{a(T)}\eta\right)\right] \tag{4.11}$$

for all $p \in [1,2]$ and all real $q \neq 0$.

In our first theorem we obtain a very general translation theorem that gives an interesting relationship between the conditional transforms $T_q^{(p)}(F|X)$ and $T_q^{(p)}(F^*|X)$.

Theorem 4.1. *Let $p \in [1,2]$ be given, and let $F : C_0[0,T] \to \mathbf{C}$ be such that the* GCFFT, *$T_q^{(p)}(F|X)$ of F exists for all real $q \neq 0$. Let $X(x)$ be given by* (4.1). *Let x_0 be given by* (3.1) *and $z(x,t)$ by* (1.3) *with $h \in L_\infty[0,T]$, $\beta/h \in L_2[0,T]$ and $\beta/h^2 \in L_2[0,T]$. Then for all real $q \neq 0$,*

$$\begin{aligned} T_q^{(p)}(F|X)(y+x_0,\eta) &\approx \exp\left\{iq\left\langle\frac{\beta}{h^2},y\right\rangle+\frac{iq}{2}\left\|\frac{\beta}{h}\right\|^2+\frac{iqx_0(T)}{a(T)}\left(\eta+\frac{x_0(T)}{2}\right)\right\} \\ &\quad\cdot T_q^{(p)}(F^*|X)(y,\eta+x_0(T)) \end{aligned} \tag{4.12}$$

where F^* is given by equation (3.3).

Proof. Again we will give the proof for the case $p \in (1,2]$; the case $p=1$ is similar, but somewhat easier. For $\lambda > 0$ and $\eta_1 \in \mathbf{R}$, using (3.3) and (4.5) we see that

$$\begin{aligned} I &\equiv T_\lambda(F^*|X)(y,\eta_1) \\ &= E_x[F^*(y+\lambda^{-1/2}z(x,\cdot))|\lambda^{-1/2}z(x,T)=\eta_1] \end{aligned}$$

$$
\begin{aligned}
&= E_x\bigg[F^*\bigg(y+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}+\eta_1\frac{a(\cdot)}{a(T)}\bigg)\bigg]\\
&= \exp\bigg\{-iq\bigg\langle\frac{\beta}{h^2},y\bigg\rangle-iq\eta_1\frac{x_0(T)}{a(T)}\bigg\}\\
&\quad\cdot E_x\bigg[\exp\bigg\{-iq\lambda^{-1/2}\bigg\langle\frac{\beta}{h^2},z(x,\cdot)\bigg\rangle+iq\lambda^{-1/2}x_0(T)\frac{z(x,T)}{a(T)}\bigg\}\\
&\qquad\cdot F\bigg(y+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}+\eta_1\frac{a(\cdot)}{a(T)}\bigg)\bigg].
\end{aligned}
$$

Using the translation theorem in the form

$$
E[F(x)]=\exp\bigg\{-\frac{\|u_0'\|^2}{2}\bigg\}E[F(x+u_0)\exp\{-\langle u_0',x\rangle\}]
$$

with $u_0(t)=\lambda^{1/2}\int_0^t\beta(s)/h(s)\,ds$ and $x_0(t)=\int_0^t\beta(s)\,ds=\lambda^{-1/2}\int_0^t h(s)\,du_0(s)$, we obtain that

$$
\begin{aligned}
I &= \exp\bigg\{-iq\bigg\langle\frac{\beta}{h^2},y\bigg\rangle-iq\eta_1\frac{x_0(T)}{a(T)}-\frac{\lambda}{2}\bigg\|\frac{\beta}{h}\bigg\|^2\\
&\qquad -iq\lambda^{-1/2}\bigg\langle\frac{\beta}{h^2},z(u_0,\cdot)\bigg\rangle+iq\lambda^{-1/2}x_0(T)\frac{z(u_0,T)}{a(T)}\bigg\}\\
&\quad\cdot E_x\bigg[\exp\bigg\{-iq\lambda^{-1/2}\bigg\langle\frac{\beta}{h^2},z(x,\cdot)\bigg\rangle\\
&\qquad\qquad +iq\lambda^{-1/2}x_0(T)\frac{z(x,T)}{a(T)}-\lambda^{1/2}\bigg\langle\frac{\beta}{h},x\bigg\rangle\bigg\}\\
&\qquad\qquad \cdot F\bigg(y+x_0+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}\\
&\qquad\qquad\qquad -\lambda^{-1/2}z(u_0,T)\frac{a(\cdot)}{a(T)}+\eta_1\frac{a(\cdot)}{a(T)}\bigg)\bigg].
\end{aligned}
$$

Next observing that $\langle\beta/h^2,z(x,\cdot)\rangle=\langle\beta/h,x\rangle$, $\langle\beta/h^2,z(u_0,\cdot)\rangle=\lambda^{1/2}\|\beta/h\|^2$, $z(u_0,T)=\lambda^{1/2}x_0(T)$, and then setting $\eta_1=\eta+x_0(T)$, we obtain that

$$
\begin{aligned}
I &= T_\lambda(F^*|X)(y,\eta+x_0(T)) \\
&= \exp\bigg\{-iq\Big\langle\frac{\beta}{h^2},y\Big\rangle - iq\eta\frac{x_0(T)}{a(T)} - \Big(\frac{\lambda}{2}+iq\Big)\Big\|\frac{\beta}{h}\Big\|^2\bigg\} \\
&\quad\cdot E_x\bigg[\exp\bigg\{-\lambda^{-1/2}(iq+\lambda)\Big\langle\frac{\beta}{h},x\Big\rangle + iq\lambda^{-1/2}x_0(T)\frac{z(x,T)}{a(T)}\bigg\} \\
&\qquad\cdot F\Big(y+x_0+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}+\eta\frac{a(\cdot)}{a(T)}\Big)\bigg].
\end{aligned}
\tag{4.13}
$$

Since $T_q^{(p)}(F|X)$ exists for each $q \in \mathbf{R}$ with $q \neq 0$, we know that $T_\lambda^{(p)}(F|X)$ exists for each $\lambda \in \mathbf{C}_+$. Thus

$$
E_x\bigg[\Big|F\Big(y+x_0+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}+\eta\frac{a(\cdot)}{a(T)}\Big)\Big|^p\bigg]
$$

exists. Using Hölder's inequality, we see that

$$
\begin{aligned}
&E_x\bigg[\bigg|\bigg(\exp\bigg\{-\lambda^{-1/2}(iq+\lambda)\Big\langle\frac{\beta}{h},x\Big\rangle\bigg\}-1\bigg)\exp\bigg\{iq\lambda^{-1/2}x_0(T)\frac{z(x,T)}{a(T)}\bigg\} \\
&\qquad\qquad\cdot F\Big(y+x_0+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}+\eta\frac{a(\cdot)}{a(T)}\Big)\bigg|\bigg] \\
&\le \bigg(E_x\bigg[\bigg|\exp\bigg\{-\lambda^{-1/2}(iq+\lambda)\Big\langle\frac{\beta}{h},x\Big\rangle\bigg\}-1\bigg|^{p'}\bigg]\bigg)^{1/p'} \\
&\quad\cdot\bigg(E_x\bigg[\bigg|\exp\bigg\{iq\lambda^{-1/2}x_0(T)\frac{z(x,T)}{a(T)}\bigg\} \\
&\qquad\cdot F\Big(y+x_0+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}+\eta\frac{a(\cdot)}{a(T)}\Big)\bigg|^p\bigg]\bigg)^{1/p}.
\end{aligned}
$$

Note that $z(x,T)$ and $z(x,\cdot)-z(x,T)a(\cdot)/a(T)$ are independent processes. Hence

$$
\begin{aligned}
&E_x\bigg[\bigg|\exp\bigg\{iq\lambda^{-1/2}x_0(T)\frac{z(x,T)}{a(T)}\bigg\} \\
&\qquad\cdot F\Big(y+x_0+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}+\eta\frac{a(\cdot)}{a(T)}\Big)\bigg|^p\bigg]
\end{aligned}
$$

$$= E_x\left[\left|\exp\left\{iq\lambda^{-1/2}x_0(T)\frac{z(x,T)}{a(T)}\right\}\right|^p\right]$$
$$\cdot E_x\left[\left|F\left(y+x_0+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}+\eta\frac{a(\cdot)}{a(T)}\right)\right|^p\right]$$
$$= E_x\left[\left|\exp\left\{iq\lambda^{-1/2}x_0(T)\frac{z(x,T)}{a(T)}\right\}\right|^p\right]$$
$$\cdot E_x[\,|F(y+x_0+\lambda^{-1/2}z(x,\cdot))|^p\,|\,\lambda^{-1/2}z(x,T)=\eta].$$

Furthermore each factor in the last expression above has a limit as $\lambda \to -iq$ in $\mathbf{C}_+$. Therefore, the last expression is bounded in a deleted neighborhood of $-iq$ intersected with $\mathbf{C}_+$. Since $E_x[\,|\exp\{-\lambda^{1/2}(iq+\lambda)\langle\beta/h,x\rangle\}-1|^{p'}]\to 0$ as $\lambda\to -iq$ in $\mathbf{C}_+$, we conclude that

$$E_x\left[\exp\left\{-\lambda^{-1/2}(iq+\lambda)\left\langle\frac{\beta}{h},x\right\rangle+iq\lambda^{-1/2}x_0(T)\frac{z(x,T)}{a(T)}\right\}\right.$$
$$\left.\cdot F\left(y+x_0+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}+\eta\frac{a(\cdot)}{a(T)}\right)\right]$$

and

$$E_x\left[\exp\left\{iq\lambda^{-1/2}x_0(T)\frac{z(x,T)}{a(T)}\right\}\right.$$
$$\left.\cdot F\left(y+x_0+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}+\eta\frac{a(\cdot)}{a(T)}\right)\right]$$

have the same transform as $\lambda\to -iq$ in $\mathbf{C}_+$. Using the independence between $z(x,T)$ and $z(x,\cdot)-z(x,T)a(\cdot)/a(T)$ again, we see that

$$E_x\left[\exp\left\{iq\lambda^{-1/2}x_0(T)\frac{z(x,T)}{a(T)}\right\}\right.$$
$$\left.\cdot F\left(y+x_0+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}+\eta\frac{a(\cdot)}{a(T)}\right)\right]$$

(4.14)

$$= E_x\left[\exp\left\{iq\lambda^{-1/2}x_0(T)\frac{z(x,T)}{a(T)}\right\}\right]$$
$$\cdot E_x\left[F\left(y+x_0+\lambda^{-1/2}z(x,\cdot)-\lambda^{-1/2}z(x,T)\frac{a(\cdot)}{a(T)}+\eta\frac{a(\cdot)}{a(T)}\right)\right]$$

$$= \exp\left\{ -\frac{q^2 x_0^2(T)}{2\lambda a(T)}\right\} T_\lambda(F|X)(y+x_0,\eta).$$

Thus, using (4.14) and letting $\lambda \to -iq$ in (4.13), we obtain (4.12) which concludes the proof of Theorem 4.1. □

Next we observe that formula (4.12) holds for all functionals in the classes $\mathcal{S}$ and $\mathcal{B}_n^{(p)}$.

Corollary 4.1. *Let $F \in \mathcal{S}$ be given by* (3.5) *and $X(x)$ by* (4.1). *Let x_0, z, h and β be as in Theorem* 4.1. *Then for all $p \in [1,2]$ and all real $q \neq 0$,*

$$\begin{aligned} T_q^{(p)}&(F|X)(y+x_0,\eta) \\ &\approx \exp\left\{ iq\left\langle \frac{\beta}{h^2}, y\right\rangle + \frac{iq}{2}\left\|\frac{\beta}{h}\right\|^2 + \frac{iqx_0(T)}{a(T)}\left(\eta + \frac{x_0(T)}{2}\right)\right\} \\ &\quad \cdot T_q^{(p)}(F^*|X)(y,\eta+x_0(T)) \end{aligned}$$

where F^* is given by equation (3.3).

Proof. This corollary follows from Theorem 4.1 since, by Remark 4.2 above, $T_q^{(p)}(F|X)$ exists for all $p \in [1,2]$ and all real $q \neq 0$. □

Remark 4.3. For $F \in \mathcal{S}$ given by (3.5), direct calculations show that

$$\begin{aligned} T_q^{(p)}&(F|X)(y+x_0,\eta) \\ &\approx \int_{L_2[0,T]} \exp\left\{ i\langle u, y+x_0\rangle + ib\eta - \frac{i}{2q}\int_0^T [u(s)-b]^2 h^2(s)\,ds\right\} df(u), \end{aligned}$$

and that

$$\begin{aligned} T_q^{(p)}&(F^*|X)(y,\eta) \\ &\approx \exp\left\{ -iq\left\langle \frac{\beta}{h^2}, y\right\rangle - \frac{iqx_0(T)\eta}{a(T)} - \frac{iq}{2}\left\|\frac{\beta}{h}\right\|^2 + \frac{iqx_0^2(T)}{2a(T)}\right\} \\ &\quad \cdot \int_{L_2[0,T]} \exp\bigg\{ -ibx_0(T) + i\langle u, y+x_0\rangle \\ &\qquad\qquad + ib\eta - \frac{i}{2q}\int_0^T [u(s)-b]^2 h^2(s)\,ds\bigg\}\, df(u) \end{aligned}$$

where

$$b = \frac{1}{a(T)} \int_0^T u(s)h^2(s)\,ds = \frac{(u,h^2)}{a(T)}.$$

Corollary 4.2. *Let $X(x)$, x_0, z, h and β be as in Theorem* 4.1. *Let $p \in [1,2]$, let $F \in \mathcal{B}_n^{(p)}$ be given by* (3.9), *let F^* be given by* (3.3). *Then for all real $q \neq 0$, $T_q^{(p)}(F|X)$ and $T_q^{(p)}(F^*|X)$ exist and are related by formula* (4.15).

Proof. This corollary also follows immediately from Theorem 4.1 since by Remark 4.2 above, $T_q^{(p)}(F|X)$ exists for all $p \in [1,2]$ and all real $q \neq 0$. □

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