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CONVOLUTION AND FOURIER-FEYNMAN TRANSFORMS

TIMOTHY HUFFMAN, CHULL PARK AND DAVID SKOUG

ABSTRACT. In this paper, for a class of functionals on Wiener space of the form $F(x) = \exp\{\int_0^T f(t, x(t)) dt\}$, we show that the Fourier-Feynman transform of the convolution product is a product of Fourier-Feynman transforms. This allows us to compute the transform of the convolution product without computing the convolution product.

1. Introduction and preliminaries. The concept of an L_1 analytic Fourier-Feynman transform was introduced by Brue in [1]. In [3] Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform. In [7] Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \leq p \leq 2$ which extended the results in [1, 3] and gave various relationships between the L_1 and L_2 theories. In [5], Huffman, Park and Skoug defined a convolution product for functionals on Wiener space and for a class of functionals of the type

$$F(x) = f\left(\int_0^T \alpha_1(t) dx(t), \dots, \int_0^T \alpha_n(t) dx(t)\right)$$

showed that the Fourier-Feynman transform of the convolution product was a product of Fourier-Feynman transforms.

In this paper we consider a class of functionals which play an important role in quantum mechanics, namely functionals of the form

$$F(x) = \exp\left\{\int_0^T f(t, x(t)) dt\right\}$$

for appropriate $f : [0, T] \times \mathfrak{R} \rightarrow \mathbf{C}$. For any two functionals in the class, we first establish the existence of their convolution product. We then show that the Fourier-Feynman transform of their convolution is

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a product of their transforms. For specific functionals, this allows us to compute the transform of their convolution product without actually computing their convolution product. Finally, in Section 5 we discuss the associativity of the convolution product.

In [3, 7] all of the functionals F on Wiener space and all of the complex-valued functions f on \mathfrak{R}^n were assumed to be Borel measurable. But, as was pointed out in [8, p. 170], the concept of scale-invariant measurability in Wiener space and Lebesgue measurability in \mathfrak{R}^n is precisely correct for the analytic Fourier-Feynman theory.

Let $C_0[0, T]$ denote Wiener space; that is, the space of real-valued continuous functions x on $[0, T]$ such that $x(0) = 0$. Let M denote the class of all Wiener measurable subsets of $C_0[0, T]$, and let m denote Wiener measure. $(C_0[0, T], M, m)$ is a complete measure space and we denote the Wiener integral of a functional F by

$$\int_{C_0[0, T]} F(x) m(dx).$$

A subset E of $C_0[0, T]$ is said to be scale-invariant measurable [4, 8] provided $\rho E \in M$ for each $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F \approx G$.

Let \mathbf{C} , \mathbf{C}_+ and \mathbf{C}_+^\sim denote respectively the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part. Let F be a \mathbf{C} -valued scale-invariant measurable functional on $C_0[0, T]$ such that

$$J(\lambda) = \int_{C_0[0, T]} F(\lambda^{-1/2} x) m(dx)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in \mathbf{C}_+ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0, T]$ with parameter λ , and for $\lambda \in \mathbf{C}_+$ we write

$$\int_{C_0[0, T]}^{anw\lambda} F(x) m(dx) = J^*(\lambda).$$

Let $q \neq 0$ be a real number, and let F be a functional such that $\int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx)$ exists for all $\lambda \in \mathbf{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q , and we write

$$\int_{C_0[0,T]}^{anf_q} F(x)m(dx) = \lim_{\lambda \rightarrow -iq} \int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx)$$

where $\lambda \rightarrow -iq$ through \mathbf{C}_+ .

Notation. i) For $\lambda \in \mathbf{C}_+$ and $y \in C_0[0, T]$, let

$$(1.1) \quad (T_\lambda(F))(y) = \int_{C_0[0,T]}^{anw_\lambda} F(x+y)m(dx).$$

ii) Given a number p with $1 \leq p \leq +\infty$, p and p' will always be related by $1/p + 1/p' = 1$.

iii) Let $1 < p \leq 2$, and let $\{H_n\}$ and H be scale-invariant measurable functionals such that for each $\rho > 0$,

$$(1.2) \quad \lim_{n \rightarrow \infty} \int_{C_0[0,T]} |H_n(\rho y) - H(\rho y)|^{p'} m(dy) = 0.$$

Then we write

$$(1.3) \quad \lim_{n \rightarrow \infty} (w_s^{p'}) (H_n) \approx H$$

and we call H the scale invariant limit in the mean of order p' . A similar definition is understood when n is replaced by the continuously varying parameter λ .

We are finally ready to state the definition of the L_p analytic Fourier-Feynman transform [7] and our definition of the convolution product.

Definition. Let $q \neq 0$ be a real number. For $1 < p \leq 2$ we define the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F by the formula ($\lambda \in \mathbf{C}_+$),

$$(1.4) \quad (T_q^{(p)}(F))(y) = \lim_{\lambda \rightarrow -iq} (w_s^{p'}) (T_\lambda(F))(y)$$

whenever this limit exists. We define the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F by the formula

$$(1.5) \quad (T_q^{(1)}(F))(y) = \lim_{\lambda \rightarrow -iq} (T_\lambda(F))(y)$$

for s-a.e. y . We note that, for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e. We also note that if $T_q^{(p)}(F_1)$ exists and if $F_1 \approx F_2$, then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_2) \approx T_q^{(p)}(F_1)$.

Definition. Let F and G be functionals on $C_0[0, T]$. For $\lambda \in \mathbf{C}_+^\sim$ we define their convolution product (if it exists) by

$$(1.6) \quad (F * G)_\lambda(y) = \begin{cases} \int_{C_0[0, T]}^{anw_\lambda} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) m(dx), & \lambda \in \mathbf{C}_+ \\ \int_{C_0[0, T]}^{anf_q} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) m(dx), & \lambda = -iq, q \in \mathfrak{R} - \{0\}. \end{cases}$$

Remarks. i) When $\lambda = -iq$, we will denote $(F * G)_\lambda$ by $(F * G)_q$.

ii) Our definition of convolution is different than the definition given by Yeh in [10] and used by Yoo in [11]. For one thing, our convolution product is commutative, that is to say

$$(F * G)_\lambda(y) = (G * F)_\lambda(y).$$

In [10] and [11] Yeh and Yoo studied relationships between their convolution product and Fourier-Wiener transforms.

2. Transforms. First we describe the class of functionals $A = A_{pr}$ that we will be working with in this paper. For $1 \leq p \leq 2$ and $r \in (2p/(2p-1), \infty]$, let $L_{pr}([0, T] \times \mathfrak{R})$ be the space of all \mathbf{C} -valued Lebesgue measurable functions f on $[0, T] \times \mathfrak{R}$ such that $f(t, \cdot)$ is in $L_p(\mathfrak{R})$ for almost all $t \in [0, T]$ and as a function of t , $\|f(t, \cdot)\|_p$ is in $L_r([0, T])$. We define $A = A_{pr}$ to be the class of functionals F , such that for some $f \in L_{pr}([0, T] \times \mathfrak{R})$,

$$(2.1) \quad F(x) = \exp \left\{ \int_0^T f(t, x(t)) dx \right\}.$$

Then, see [8, pp. 170–171], $F(x)$ is defined s-a.e. and is scale-invariant measurable.

Theorem 2.1. *Let $F \in A$ be given by (2.1) with $f \in L_{pr}([0, T] \times \mathfrak{A})$. Then $T_\lambda(F)$ exists for all $\lambda \in \mathbf{C}_+$ and is given by*

$$\begin{aligned}
 (2.2) \quad (T_\lambda(F))(y) &= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n^*(T)} \int_{\mathfrak{A}^n} \prod_{j=1}^n \left[\left(\frac{\lambda}{2\pi(t_j - t_{j-1})} \right)^{1/2} \right. \\
 &\quad \left. \cdot f(t_j, u_j + y(t_j)) \exp \left\{ - \frac{\lambda(u_j - u_{j-1})^2}{2(t_j - t_{j-1})} \right\} \right] d\vec{u} d\vec{t} \\
 &= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{A}^n} \prod_{j=1}^n \left[\left(\frac{\lambda}{2\pi(t_j - t_{j-1})} \right)^{1/2} \right. \\
 &\quad \left. \cdot f(t_j, u_j) \exp \left\{ - \frac{\lambda[(u_j - u_{j-1}) - (y(t_j) - y(t_{j-1}))]^2}{2(t_j - t_{j-1})} \right\} \right] d\vec{u} d\vec{t}
 \end{aligned}$$

where $\Delta_n(T) = \{\vec{t} = (t_1, \dots, t_n) \in [0, T]^n : 0 < t_1 < t_2 < \dots < t_n \leq T\}$, $\vec{u} = (u_1, \dots, u_n)$ and $t_0 \equiv 0 \equiv u_0$.

Proof. First note that

$$\begin{aligned}
 (2.3) \quad F(x) &= \exp \left\{ \int_0^T f(t, x(t)) dt \right\} \\
 &= 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\int_0^T f(t, x(t)) dt \right]^n \\
 &= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \prod_{j=1}^n [f(t_j, x(t_j))] d\vec{t}.
 \end{aligned}$$

Hence, for $\lambda > 0$, using a well-known Wiener integration formula, we obtain

$$\begin{aligned}
 (2.4) \quad (T_\lambda(F))(y) &= \int_{C_0[0, T]} F(\lambda^{-1/2}x + y) m(dx) \\
 &= 1 + \int_{C_0[0, T]} \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \prod_{j=1}^n \left[\left(\frac{\lambda}{2\pi(t_j - t_{j-1})} \right)^{1/2} \right. \\
 &\quad \left. \cdot f(t_j, u_j + y(t_j)) \exp \left\{ - \frac{\lambda(u_j - u_{j-1})^2}{2(t_j - t_{j-1})} \right\} \right] d\vec{u} d\vec{t}.
 \end{aligned}$$

Finally, by analytic continuation in λ , we obtain that equation (2.2) holds throughout \mathbf{C}_+ . \square

Theorem 2.2. *Let $F \in A$ be as in Theorem 2.1. Then for all $p \in [1, 2]$, the Fourier-Feynmann transform $T_q^{(p)}(F)$ exists for all real $q \neq 0$ and is given by the formula*

$$\begin{aligned}
 (2.5) \quad (T_q^{(p)}(F))(y) &= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^n} \prod_{j=1}^n \left[\left(\frac{-qi}{2\pi(t_j - t_{j-1})} \right)^{1/2} \right. \\
 &\quad \left. \cdot f(t_j, u_j + y(t_j)) \exp \left\{ \frac{qi(u_j - u_{j-1})^2}{2(t_j - t_{j-1})} \right\} \right] d\vec{u} d\vec{t} \\
 &= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^n} \prod_{j=1}^n \left[\left(\frac{-iq}{2\pi(t_j - t_{j-1})} \right)^{1/2} f(t_j, u_j) \right. \\
 &\quad \left. \cdot \exp \left\{ \frac{iq[(u_j - u_{j-1}) - (y(t_j) - y(t_{j-1}))]^2}{2(t_j - t_{j-1})} \right\} \right] d\vec{u} d\vec{t}
 \end{aligned}$$

Proof. By [7, Section 4] and [8, pp. 170–171], $T_q^{(p)}(F)$ exists for all real $q \neq 0$ and is scale-invariant measurable. Also note that because $\Phi(z) = e^z$ is an entire function of order 1 and because $f \in L_{pr}([0, T] \times \mathfrak{R})$, it was shown in [7] that, for any $q \in \mathfrak{R}$,

$$\sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^n} |q|^{n/2} \prod_{j=1}^n \left[\left(\frac{1}{2\pi(t_j - t_{j-1})} \right)^{1/2} |f(t_j, u_j)| \right] d\vec{u} d\vec{t} < \infty.$$

Thus, for all $y \in C_0[0, T]$, the series on the righthand side of equation (2.5) converges absolutely (and uniformly in q on compact subsets of $\mathfrak{R} - \{0\}$). Furthermore the series converges in the $L_p(C_0[0, T])$ mean. Thus, our representation (2.5) for $T_q^{(p)}(F)$ follows from equation (2.2). \square

3. Convolutions. In our first theorem, with F and G in $A = A_{pr}$, we obtain a series representation for the convolution product of F and G .

Theorem 3.1. *Let $F \in A$ be given by (2.1), and let $G \in A$ be given by*

$$(3.1) \quad G(x) = \exp \left\{ \int_0^T g(t, x(t)) dt \right\}$$

*with $g \in L_{pr}([0, T] \times \mathfrak{R})$. Then their convolution product $(F * G)_\lambda$ exists for all $\lambda \in \mathbf{C}_+$ and is given by*

$$(3.2) \quad (F * G)_\lambda(y) = 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^n} \prod_{j=1}^n \left[\left(\frac{\lambda}{2\pi(t_j - t_{j-1})} \right)^{1/2} \cdot \exp \left\{ - \frac{\lambda(u_j - u_{j-1})^2}{2(t_j - t_{j-1})} \right\} \left\{ f \left(t_j, \frac{y(t_j) + u_j}{\sqrt{2}} \right) + g \left(t_j, \frac{y(t_j) - u_j}{\sqrt{2}} \right) \right\} \right] d\vec{u} d\vec{t}.$$

Proof. For $\lambda > 0$, using a well-known Wiener integration formula, we obtain

$$\begin{aligned} (F * G)_\lambda(y) &= \int_{C_0[0,T]} F \left(\frac{y + \lambda^{-1/2}x}{\sqrt{2}} \right) G \left(\frac{y - \lambda^{-1/2}x}{\sqrt{2}} \right) m(dx) \\ &= \int_{C_0[0,T]} \exp \left\{ \int_0^T f \left(t, \frac{y(t) + \lambda^{-1/2}x(t)}{\sqrt{2}} \right) dt \right\} \\ &\quad \cdot \exp \left\{ \int_0^T g \left(t, \frac{y(t) - \lambda^{-1/2}x(t)}{\sqrt{2}} \right) dt \right\} m(dx) \\ &= \int_{C_0[0,T]} \exp \left\{ \int_0^T \left[f \left(t, \frac{y(t) + \lambda^{-1/2}x(t)}{\sqrt{2}} \right) + g \left(t, \frac{y(t) - \lambda^{-1/2}x(t)}{\sqrt{2}} \right) \right] dt \right\} m(dx) \\ &= \int_{C_0[0,T]} \left\{ 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \left[\int_0^T \left[f \left(t, \frac{y(t) + \lambda^{-1/2}x(t)}{\sqrt{2}} \right) + g \left(t, \frac{y(t) - \lambda^{-1/2}x(t)}{\sqrt{2}} \right) \right] dt \right]^n \right\} m(dx) \end{aligned}$$

$$\begin{aligned}
&= 1 + \sum_{n=1}^{\infty} \int_{C_0[0,T]} \int_{\Delta_n(T)} \prod_{j=1}^n \left[f\left(t_j, \frac{y(t_j) + \lambda^{-1/2}x(t_j)}{\sqrt{2}}\right) \right. \\
&\quad \left. + g\left(t_j, \frac{y(t_j) - \lambda^{-1/2}x(t_j)}{\sqrt{2}}\right) \right] d\vec{t} m(dx) \\
&= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^n} \prod_{j=1}^n \left[\left(\frac{1}{2\pi(t_j - t_{j-1})}\right)^{1/2} \right. \\
&\quad \left. \cdot \exp\left\{-\frac{(u_j - u_{j-1})^2}{2(t_j - t_{j-1})}\right\} \right. \\
&\quad \left. \cdot \left\{ f\left(t_j, \frac{y(t_j) + \lambda^{-1/2}u_j}{\sqrt{2}}\right) + g\left(t_j, \frac{y(t_j) - \lambda^{-1/2}u_j}{\sqrt{2}}\right) \right\} \right] d\vec{u} d\vec{t} \\
&= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^n} \prod_{j=1}^n \left[\left(\frac{\lambda}{2\pi(t_j - t_{j-1})}\right)^{1/2} \right. \\
&\quad \left. \cdot \exp\left\{-\frac{\lambda(u_j - u_{j-1})^2}{2(t_j - t_{j-1})}\right\} \right. \\
&\quad \left. \cdot \left\{ f\left(t_j, \frac{y(t_j) + u_j}{\sqrt{2}}\right) + g\left(t_j, \frac{y(t_j) - u_j}{\sqrt{2}}\right) \right\} \right] d\vec{u} d\vec{t}.
\end{aligned}$$

Now, by analytic continuation in λ , we obtain that equation (3.2) is valid throughout \mathbf{C}_+ . \square

In our next theorem we obtain a series representation for $T_\lambda(F * G)_\lambda$.

Theorem 3.2. *Let f, g, F and G be as in Theorem 3.1. Then, for all $\lambda \in \mathbf{C}_+$, $T_\lambda(F * G)_\lambda$ exists and is given by the formula*

$$\begin{aligned}
(3.3) \quad (T_\lambda(F * G)_\lambda)(z) &= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^{2n}} \prod_{j=1}^n \left[\left\{ f(t_j, v_j + z(t_j)/\sqrt{2}) \right. \right. \\
&\quad \left. \left. + g(t_j, r_j + z(t_j)/\sqrt{2}) \right\} \left(\frac{\lambda}{2\pi(t_j - t_{j-1})}\right) \right. \\
&\quad \left. \cdot \exp\left\{-\frac{\lambda(v_j - v_{j-1})^2}{2(t_j - t_{j-1})} - \frac{\lambda(r_j - r_{j-1})^2}{2(t_j - t_{j-1})}\right\} \right] d\vec{r} d\vec{v} d\vec{t}.
\end{aligned}$$

Proof. For $\lambda > 0$, using (3.2) we see that

$$\begin{aligned}
 (T_\lambda(F * G)_\lambda)(z) &= \int_{C_0[0,T]} (F * G)_\lambda(\lambda^{-1/2}x + z)m(dx) \\
 &= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^n} \int_{C_0[0,T]} \prod_{j=1}^n \left[\left(\frac{\lambda}{2\pi(t_j - t_{j-1})} \right)^{1/2} \right. \\
 &\quad \cdot \exp \left\{ - \frac{\lambda(u_j - u_{j-1})^2}{2(t_j - t_{j-1})} \right\} \\
 &\quad \cdot \left\{ f \left(t_j, \frac{\lambda^{-1/2}x(t_j) + z(t_j) + u_j}{\sqrt{2}} \right) \right. \\
 &\quad \left. \left. + g \left(t_j, \frac{\lambda^{-1/2}x(t_j) + z(t_j) - u_j}{\sqrt{2}} \right) \right\} m(dx) \right] d\vec{u} d\vec{t} \\
 &= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^{2n}} \prod_{j=1}^n \left[\left(\frac{\lambda}{2\pi(t_j - t_{j-1})} \right) \right. \\
 &\quad \cdot \exp \left\{ - \frac{\lambda(u_j - u_{j-1})^2}{2(t_j - t_{j-1})} - \frac{\lambda(w_j - w_{j-1})^2}{2(t_j - t_{j-1})} \right\} \\
 &\quad \cdot \left\{ f \left(t_j, \frac{z(t_j) + w_j + u_j}{\sqrt{2}} \right) \right. \\
 &\quad \left. \left. + g \left(t_j, \frac{z(t_j) + w_j - u_j}{\sqrt{2}} \right) \right\} \right] d\vec{w} d\vec{u} d\vec{t}.
 \end{aligned}$$

Next, in the above expression we make the substitutions

$$v_j = \frac{w_j + u_j}{\sqrt{2}} \quad \text{and} \quad r_j = \frac{w_j - u_j}{\sqrt{2}}$$

for $j = 1, 2, \dots, n$. The Jacobian of this transformation is one and for $j = 1, \dots, n$, we have that

$$(u_j - u_{j-1})^2 + (w_j - w_{j-1})^2 = (v_j - v_{j-1})^2 + (r_j - r_{j-1})^2.$$

Thus, for $\lambda > 0$,

$$\begin{aligned}
 (T_\lambda(F * G)_\lambda)(z) &= 1 + \sum_{n=1}^{\infty} \int_{\Delta_n(T)} \int_{\mathfrak{R}^{2n}} \prod_{j=1}^n \left[\left(\frac{\lambda}{2\pi(t_j - t_{j-1})} \right) \right. \\
 &\quad \cdot \exp \left\{ - \frac{\lambda(v_j - v_{j-1})^2}{2(t_j - t_{j-1})} - \frac{\lambda(r_j - r_{j-1})^2}{2(t_j - t_{j-1})} \right\} \\
 &\quad \cdot \left\{ f \left(t_j, v_j + \frac{z(t_j)}{\sqrt{2}} \right) + g \left(t_j, r_j + \frac{z(t_j)}{\sqrt{2}} \right) \right\} \right] d\vec{r} d\vec{v} d\vec{t}.
 \end{aligned}$$

Finally, by analytic continuation in λ , we obtain that equation (3.3) is valid through \mathbf{C}_+ . \square

4. Transforms of convolutions.

Lemma 4.1. *If $T_\lambda(F)$, $T_\lambda(G)$ and $T_\lambda(F * G)_\lambda$ exist for $\lambda > 0$, then*

$$(4.1) \quad (T_\lambda(F * G)_\lambda)(z) = (T_\lambda(F))(z/\sqrt{2})(T_\lambda(G))(z/\sqrt{2}).$$

Proof. For $\lambda > 0$, using (1.1) and (1.6) we see that

$$\begin{aligned} (T_\lambda(F * G)_\lambda)(z) &= \int_{C_0[0,T]} (F * G)_\lambda(z + \lambda^{-1/2}y)m(dy) \\ &= \int_{C_0^2[0,T]} F(z/\sqrt{2} + \lambda^{-1/2}(y + x)/\sqrt{2}) \\ &\quad \cdot G(z/\sqrt{2} + \lambda^{-1/2}(y - x)/\sqrt{2})m(dy)m(dx). \end{aligned}$$

But $w_1 = (y + x)/\sqrt{2}$ and $w_2 = (y - x)/\sqrt{2}$ are independent standard Wiener processes and $m \times m$ is rotation invariant in $C_0^2[0, T]$, and hence,

$$\begin{aligned} (T_\lambda(F * G)_\lambda)(z) &= \int_{C_0^2[0,T]} F(z/\sqrt{2} + \lambda^{-1/2}w_1) \\ &\quad \cdot G(z/\sqrt{2} + \lambda^{-1/2}w_2)m(dw_1)m(dw_2) \\ &= \int_{C_0[0,T]} F(z/\sqrt{2} + \lambda^{-1/2}w_1)m(dw_1) \\ &\quad \cdot \int_{C_0[0,T]} G(z/\sqrt{2} + \lambda^{-1/2}w_2)m(dw_2) \\ &= (T_\lambda(F))(z/\sqrt{2})(T_\lambda(G))(z/\sqrt{2}). \end{aligned}$$

As a corollary to Lemma 4.1, we obtain an interesting relationship involving convolutions and analytic Wiener integrals. \square

Theorem 4.1. *Let F and G be as in Theorem 3.1. Then for all $\lambda \in \mathbf{C}_+$,*

$$(4.2) \quad (T_\lambda(F * G)_\lambda)(z) = (T_\lambda(F))(z/\sqrt{2})(T_\lambda(G))(z/\sqrt{2}).$$

Proof. By Lemma 4.1, equation (4.2) holds for all $\lambda > 0$. The result now follows because $T_\lambda(F)$, $T_\lambda(G)$ and $T_\lambda(F * G)_\lambda$ all have analytic extensions throughout \mathbf{C}_+ . \square

Our next theorem shows that the Fourier-Feynman transform of the convolution product is the product of their transforms.

Theorem 4.2. *Let f, g, F and G be as in Theorem 3.1. Then, for all real $q \neq 0$,*

$$(4.3) \quad (T_q^{(p)}(F * G)_q)(z) = (T_q^{(p)}(F))(z/\sqrt{2})(T_q^{(p)}(G))(z/\sqrt{2})$$

for $1 \leq p \leq 2$.

Proof. As was noted in the proof of Theorem 2.2, the series expansions for $(T_q^{(p)}(F))(z/\sqrt{2})$ and $(T_q^{(p)}(G))(z/\sqrt{2})$ both converge absolutely for all $z \in C_0[0, T]$. Hence the righthand side of (4.3) is a bounded continuous function of λ on \mathbf{C}_+ for all $z \in C_0[0, T]$. Hence, using equation (4.1), $T_q^{(p)}(F * G)_q$ exists and is given by equation (4.3) for all desired values of p and q . \square

Corollary. *Let f and F be as in Theorem 2.1. Then for all real $q \neq 0$,*

$$(T_q^{(p)}(F * F)_q)(z) = [(T_q^{(p)}(F))(z/\sqrt{2})]^2$$

for $1 \leq p \leq 2$.

Remark. Formula (4.3) is useful in that it permits one to calculate $T_q^{(p)}(F * G)_q$ without actually calculating $(F * G)_q$. In practice, $T_\lambda(F)$ and $T_\lambda(G)$ are usually easier to calculate than are $(F * G)_\lambda$ and $T_\lambda(F * G)_\lambda$.

Examples. We finish this section by finding $T_q^{(p)}(F_j * F_k)_q$ for various functionals on Wiener space $C_0[0, T]$. In view of equation (4.3), we need only compute the transforms of the various functionals F_j . The results are summarized below where the expressions for $T_\lambda(F_j)$ are valid for all $\lambda \in \mathbf{C}_+$ unless otherwise indicated.

Let α be any real function in $L_2[0, T]$, and let $c \in \mathbf{C}$. Then

TABLE 1.

$F_j(x)$	$T_\lambda(F_j)(z)$
$F_1(x) = 1$	1
$F_2(x) = \int_0^T \alpha(t) dx(t)$	$\int_0^T \alpha(t) dz(t)$
$F_3(x) = \int_0^T x^2(t) dt$	$\int_0^T z^2(t) dt + \frac{T^2}{(2\lambda)}$
$F_4(x) = [\int_0^T \alpha(t) dx(t)]^2$	$\int_0^T \alpha(t) dz(t)]^2 + \frac{\ \alpha\ ^2}{\lambda}$
$F_5(x) = \exp\{c \int_0^T \alpha(t) dx(t)\}$	$\exp\{c \int_0^T \alpha(t) dz(t) + \frac{c^2 \ \alpha\ ^2}{(2\lambda)}\}$
$F_6(x) = \exp\{c [\int_0^T \alpha(t) dx(T)]^2\}$	$\sqrt{\frac{\lambda}{(\lambda - 2c\ \alpha\ ^2)}} \exp\{(\frac{c\lambda}{(\lambda - 2c\ \alpha\ ^2)}) (\int_0^r \alpha(t) dz(t))^2\}$ provided $\text{Re}(c/\lambda) < (2\ \alpha\ ^2)^{-1}$
$F_7(x) = \exp\{c \int_0^r x(t) dt\}$	$\exp\{c \int_0^r z(t) dt + \frac{c^2 T^3}{(6\lambda)}\}$

Now, using Table 1, together with equation (4.3), one can immediately compute $T_q^{(p)}(F_j * F_k)_q$ for $j, k \in \{1, 2, \dots, 7\}$. For example,

$$(T_q^{(p)}(F_3 * F_7)_q)(z) = \left[\int_0^T \frac{z^2(t)dt}{2} + \frac{iT^2}{2q} \right] \exp \left\{ c \int_0^T \frac{z(t)dt}{\sqrt{2}} + \frac{ic^2 T^3}{6q} \right\}.$$

5. Associativity of the convolution product. We finish this paper with a brief discussion of the associativity of the convolution product. Recall that the convolution product is commutative; that is to say, $(F * G)_\lambda = (G * F)_\lambda$ for all F, G and λ .

In the following we will restrict our attention to the case $\lambda > 0$; extensions to \mathbf{C}_+ follow by analytic continuation in λ . Furthermore, we will assume that the scale-invariant measurable functionals F, G, H and K are nice enough so that their various transforms and convolution

products all exist throughout \mathbf{C}_+^\sim . Then $T_\lambda(F) \approx T_\lambda(G)$ implies that $F \approx G$.

i) In general the convolution product defined by equation (1.6) is not associative; that is to say,

$$((F * G)_\lambda * H)_\lambda \neq (F * (G * H)_\lambda)_\lambda$$

since, by Lemma 4.1,

$$\begin{aligned} (T_\lambda((F * (G * H)_\lambda)_\lambda))(z) &= (T_\lambda(F))(z/\sqrt{2}) \\ &\quad \cdot (T_\lambda(G))(z/2)(T_\lambda(H))(z/2) \\ &\neq (T_\lambda(F))(z/2) \\ &\quad \cdot (T_\lambda(G))(z/2)(T_\lambda(H))(z/\sqrt{2}) \\ &= (T_\lambda(((F * G)_\lambda * H)_\lambda))(z). \end{aligned}$$

In particular, for $G(z) \equiv H(z) \equiv 1$ on $C_0[0, T]$,

$$\begin{aligned} (T_\lambda((F * (1 * 1)_\lambda)_\lambda))(z) &= (T_\lambda(F))(z/\sqrt{2}) \\ &\neq (T_\lambda(F))(z/2) \\ &= (T_\lambda(((F * 1)_\lambda * 1)_\lambda))(z). \end{aligned}$$

ii) However, if for each functional F on $C_0[0, T]$, we define the functional F_λ^* on $C_0[0, T]$ by

$$(5.1) \quad F_\lambda^*(z) = (1 * F)_\lambda(z),$$

then we have the associativity result

$$(5.2) \quad (F_\lambda^* * (G * H)_\lambda)_\lambda = ((F * G)_\lambda * H_\lambda^*)_\lambda$$

since, by equation (4.1),

$$\begin{aligned} (T_\lambda((F_\lambda^* * (G * H)_\lambda)_\lambda))(z) &= (T_\lambda(F))(z/2) \\ &\quad \cdot (T_\lambda(G))(z/2)(T_\lambda(H))(z/2) \\ &= (T_\lambda(((F * G)_\lambda * H_\lambda^*)_\lambda))(z). \end{aligned}$$

iii) Next, if we define F_λ^{**} on $C_0[0, T]$ by

$$(5.3) \quad F_\lambda^{**}(z) = (1 * F_\lambda^*)_\lambda(z),$$

then we have the associativity result

$$(5.4) \quad (((F * G)_\lambda * H_\lambda^*)_\lambda * K_\lambda^{**})_\lambda = (F_\lambda^{**} * ((G * H)_\lambda * K_\lambda^*)_\lambda)_\lambda$$

since, by equation (4.1),

$$\begin{aligned} & (T_\lambda(((F * G)_\lambda * H_\lambda^*)_\lambda * K_\lambda^{**}))_\lambda(z) \\ &= (T_\lambda(F))(z/(2\sqrt{2}))(T_\lambda(G))(z/(2\sqrt{2})) \\ & \quad \cdot (T_\lambda(H))(z/(2\sqrt{2}))(T_\lambda(K))(z/(2\sqrt{2})) \\ &= (T_\lambda((F_\lambda^{**} * ((G * H)_\lambda * K_\lambda^*)_\lambda))_\lambda(z). \end{aligned}$$

Remark. iv) Both sides of (5.2) are, of course, also equal to $(G_\lambda^* * (F * H)_\lambda)_\lambda$, while both sides of (5.4) are also equal to

$$(G_\lambda^{**} * ((F * K)_\lambda * H_\lambda^*)_\lambda)_\lambda$$

as well as nine other similar expressions.

v) The procedures used in ii) and iii) above can easily be extended to more factors by defining $F_\lambda^{***}(z) = (1 * F_\lambda^{**})_\lambda(z)$, $F_\lambda^{****}(z) = (1 * F_\lambda^{***})_\lambda(z)$, etc.

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