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INTEGRAL TRANSFORMS OF FUNCTIONALS IN $L_2(C_0[0, T])$

BYOUNG SOO KIM AND DAVID SKOUG

ABSTRACT. In this paper we give a necessary and sufficient condition that a functional $F(x)$ in $L_2(C_0[0, T])$ has an integral transform $\mathcal{F}_{\alpha, \beta}F(x)$ which also belongs to $L_2(C_0[0, T])$.

1. Introduction. Let $C_0[0, T]$ denote one-parameter Wiener space; that is, the space of all \mathbf{R} -valued continuous functions $x(t)$ on $[0, T]$ with $x(0) = 0$. Let \mathcal{M} denote the class of all Wiener measurable subsets of $C_0[0, T]$, and let m denote Wiener measure. $(C_0[0, T], \mathcal{M}, m)$ is a complete measure space, and we denote the Wiener integral of a Wiener integrable functional F by

$$\int_{C_0[0, T]} F(x)m(dx).$$

Let $L_2(C_0[0, T])$ be the space of all real or complex-valued functionals F satisfying

$$(1.1) \quad \int_{C_0[0, T]} |F(x)|^2 m(dx) < \infty.$$

Let $K = K[0, T]$ be the space of \mathbf{C} -valued continuous functions defined on $[0, T]$ which vanish at $t = 0$. Next we state the definition of the integral transform $\mathcal{F}_{\alpha, \beta}$ introduced in [6] and used in [5], for functionals F defined on K .

Definition 1. Let F be a functional defined on K . For each pair of nonzero complex numbers α and β , the integral transform $\mathcal{F}_{\alpha, \beta}F$ of F is defined by

$$(1.2) \quad \mathcal{F}_{\alpha, \beta}F(y) = \int_{C_0[0, T]} F(\alpha x + \beta y)m(dx), \quad y \in K,$$

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if it exists.

Remark 1. (i) When $\alpha = 1$ and $\beta = i$, $\mathcal{F}_{\alpha,\beta}F$ is the Fourier-Wiener transform introduced by Cameron in [1] and used by Cameron and Martin in [2].

(ii) When $\alpha = \sqrt{2}$ and $\beta = i$, $\mathcal{F}_{\alpha,\beta}F$ is the modified Fourier-Wiener transform introduced by Cameron and Martin in [3].

In this paper we give a necessary and sufficient condition that a functional $F(x)$ in $L_2(C_0[0, T])$ has an integral transform $\mathcal{F}_{\alpha,\beta}F$ also belonging to $L_2(C_0[0, T])$.

2. Integral transforms of the Fourier-Hermite functionals.

For $n = 0, 1, 2, \dots$, let $H_n(u)$ denote the Hermite polynomial

$$(2.1) \quad H_n(u) = (-1)^n (n!)^{-1/2} e^{u^2/2} \frac{d^n}{du^n} (e^{-u^2/2}).$$

Then, as is well known, the set

$$(2.2) \quad \{(2\pi)^{-1/4} H_n(u) e^{-u^2/4} : n = 0, 1, 2, \dots\}$$

is a CON set on \mathbf{R} .

Let $\{a_p(t) : p = 1, 2, \dots\}$ be a CON set of functions of bounded variation on $[0, T]$. Define

$$(2.3) \quad \Phi_{n,p}(x) = H_n \left(\int_0^T a_p(t) dx(t) \right), \quad n = 0, 1, 2, \dots, \quad p = 1, 2, \dots,$$

and

$$(2.4) \quad \Psi_{n_1, \dots, n_p}(x) = \Psi_{n_1, \dots, n_p, 0, \dots, 0}(x) = \Phi_{n_1, 1}(x) \cdots \Phi_{n_p, p}(x).$$

The functionals in (2.4) are called the Fourier-Hermite functionals.

In [4], Cameron and Martin showed that the Fourier-Hermite functionals form a CON set in $L_2(C_0[0, T])$. That is to say that every functional $F(x)$ in $L_2(C_0[0, T])$ has a Fourier-Hermite development which converges in the $L_2(C_0[0, T])$ sense to $F(x)$; namely, that

$$(2.5) \quad F(x) = \text{l.i.m.}_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N A_{n_1, \dots, n_N}^F \Psi_{n_1, \dots, n_N}(x),$$

where A_{n_1, \dots, n_N}^F is the Fourier-Hermite coefficient,

$$(2.6) \quad A_{n_1, \dots, n_N}^F = \int_{C_0[0, T]} F(x) \Psi_{n_1, \dots, n_N}(x) m(dx).$$

Throughout this paper, in order to ensure that various integrals exist, we will assume that $\beta = a + bi$ is a nonzero complex number satisfying the inequality

$$(2.7) \quad \operatorname{Re}(1 - \beta^2) = 1 + b^2 - a^2 > 0.$$

Note that $\operatorname{Re}(1 - \beta^2) = 1 + b^2 - a^2 > 0$ if and only if the point $(a, b) \in \mathbf{R}^2$ lies in the open region, determined by the hyperbola $a^2 - b^2 = 1$, containing the b -axis. Hence, for all $|\beta| \leq 1$, $\beta \neq \pm 1$, $\operatorname{Re}(1 - \beta^2) > 0$. Next we define

$$(2.8) \quad \alpha \equiv \sqrt{1 - \beta^2}, \quad -\pi/4 < \arg(\alpha) < \pi/4$$

and note that $\alpha^2 + \beta^2 = 1$ and $\operatorname{Re}(\alpha^2) = \operatorname{Re}(1 - \beta^2) > 0$.

Our first lemma plays a key role in finding the integral transform of the Fourier-Hermite functionals.

Lemma 1. *Let β be a nonzero complex number satisfying inequality (2.7) and let α be defined by equation (2.8). Let $r \in \mathbf{R}$. Then, for $n = 0, 1, 2, \dots$,*

$$(2.9) \quad \int_{\mathbf{R}} H_n(u) \exp \left\{ -\frac{1}{2\alpha^2}(u - r\beta)^2 \right\} du = \alpha\sqrt{2\pi} \beta^n H_n(r).$$

Proof. Since $H_n(u)$ is a polynomial of degree n and, since $\operatorname{Re}(\alpha^2) > 0$, the integral exists and

$$\begin{aligned} I_n &\equiv \int_{\mathbf{R}} H_n(u) \exp \left\{ -\frac{1}{2\alpha^2}(u - r\beta)^2 \right\} du \\ &= (-1)^n (n!)^{-1/2} \int_{\mathbf{R}} e^{u^2/2} \frac{d^n}{du^n} (e^{-u^2/2}) \exp \left\{ -\frac{1}{2\alpha^2}(u - r\beta)^2 \right\} du \\ &= (-1)^n (n!)^{-1/2} e^{r^2/2} \int_{\mathbf{R}} \exp \left\{ -\frac{\beta^2}{2\alpha^2} \left(u - \frac{r}{\beta} \right)^2 \right\} \frac{d^n}{du^n} (e^{-u^2/2}) du. \end{aligned}$$

Then, integrating by parts n times, we obtain

$$\begin{aligned}
 I_n &= (n!)^{-1/2} e^{r^2/2} \int_{\mathbf{R}} e^{-u^2/2} \frac{d^n}{du^n} \left(\exp \left\{ -\frac{\beta^2}{2\alpha^2} \left(u - \frac{r}{\beta} \right)^2 \right\} \right) du \\
 &= (n!)^{-1/2} (-\beta)^n e^{r^2/2} \int_{\mathbf{R}} e^{-u^2/2} \frac{d^n}{dr^n} \left(\exp \left\{ -\frac{\beta^2}{2\alpha^2} \left(u - \frac{r}{\beta} \right)^2 \right\} \right) du \\
 &= (n!)^{-1/2} (-\beta)^n e^{r^2/2} \frac{d^n}{dr^n} \left(\int_{\mathbf{R}} \exp \left\{ -\frac{1}{2\alpha^2} + \frac{r\beta}{\alpha^2} u - \frac{r^2}{2\alpha^2} \right\} du \right) \\
 &= (n!)^{-1/2} (-\beta)^n \alpha \sqrt{2\pi} e^{r^2/2} \frac{d^n}{dr^n} (e^{-r^2/2}) \\
 &= \alpha \sqrt{2\pi} \beta^n H_n(r),
 \end{aligned}$$

which completes the proof of Lemma 1. \square

Remark 2. Equation (2.9) holds for all $r \in \mathbf{C}$ since $H_n(r)$ is a polynomial of degree n and so both sides of equation (2.9) are analytic functions of r throughout \mathbf{C} .

Next, using Lemma 1, we obtain a formula for the integral transform of the Fourier-Hermite functionals given by equation (2.4).

Theorem 2. *Let α and β be as in Lemma 1. Then, for each $y \in K$,*

$$(2.10) \quad \mathcal{F}_{\alpha,\beta} \Psi_{n_1, \dots, n_p}(y) = \beta^{n_1 + \dots + n_p} \Psi_{n_1, \dots, n_p}(y).$$

Proof. For $j = 1, 2, \dots$, let $r_j \equiv \int_0^T a_j(t) dy(t) = \langle a_j, y \rangle$, which we know exists for all $y \in K$ since a_j is of bounded variation on $[0, T]$. Then for every $y \in K$,

$$\begin{aligned}
 \mathcal{F}_{\alpha,\beta} \Psi_{n_1, \dots, n_p}(y) &= \int_{C_0[0,T]} \Psi_{n_1, \dots, n_p}(\alpha x + \beta y) m(dx) \\
 &= \int_{C_0[0,T]} H_{n_1}(\alpha \langle a_1, x \rangle + \beta \langle a_1, y \rangle) \cdots \\
 &\quad \cdots H_{n_p}(\alpha \langle a_p, x \rangle + \beta \langle a_p, y \rangle) m(dx) \\
 &= \prod_{j=1}^p \left[(2\pi)^{-1/2} \int_{\mathbf{R}} H_{n_j}(\alpha u_j + \beta r_j) e^{-u_j^2/2} du_j \right].
 \end{aligned}$$

Note that for all positive α and all $\beta \in \mathbf{C}$,

$$\int_{\mathbf{R}} H_n(\alpha u + \beta r) e^{-u^2/2} du = \frac{1}{\alpha} \int_{\mathbf{R}} H_n(u) e^{-(u-r\beta)^2/2\alpha^2} du.$$

But each side of the above expression is an analytic function of α throughout the region $\{\alpha \in \mathbf{C} : \text{Re}(\alpha^2) > 0\}$. Hence, by the uniqueness theorem for analytic functions, the above equality holds for all α with $\text{Re}(\alpha^2) > 0$ and all $\beta \in \mathbf{C}$ and so

$$\mathcal{F}_{\alpha,\beta} \Psi_{n_1,\dots,n_p}(y) = \prod_{j=1}^p \left[(2\pi\alpha^2)^{-1/2} \int_{\mathbf{R}} H_{n_j}(u_j) e^{-(u_j-r_j\beta)^2/2\alpha^2} du_j \right].$$

Then, using Lemma 1, we obtain equation (2.10), the desired result. \square

Our first corollary follows immediately from equation (2.10) and the fact that $\|\Psi_{n_1,\dots,n_p}\|_2 = 1$.

Corollary 3. *Let α and β be as in Lemma 1. Then*

$$(2.11) \quad \|\mathcal{F}_{\alpha,\beta} \Psi_{n_1,\dots,n_p}\|_2 = |\beta|^{n_1+\dots+n_p}.$$

Corollary 4. *Choosing $\alpha = \sqrt{2}$ and $\beta = i$ in equation (2.10) we obtain Lemma 5.1 [3, p. 104]; namely, that*

$$(2.12) \quad \mathcal{F}_{\sqrt{2},i} \Psi_{n_1,\dots,n_p}(y) = i^{n_1+\dots+n_p} \Psi_{n_1,\dots,n_p}(y)$$

for all $y \in K$.

3. Integral transforms of functionals belonging to $L_2(C_0[0, T])$.

For $F \in L_2(C_0[0, T])$ let (2.5) denote the Fourier-Hermite expression of $F(x)$ with the Fourier-Hermite coefficients A_{n_1,\dots,n_N}^F given by equation (2.6). For $N = 1, 2, \dots$, let

$$(3.1) \quad F_N(x) = \sum_{n_1,\dots,n_N=0}^N A_{n_1,\dots,n_N}^F \Psi_{n_1,\dots,n_N}(x).$$

Then, by Theorem 2, we know that for each $N = 1, 2, \dots$, $\mathcal{F}_{\alpha,\beta}F_N$ exists for all α and β as in Lemma 1, $\mathcal{F}_{\alpha,\beta}F_N$ is an element of $L_2(C_0[0, T])$ such that, for each $y \in K$,

$$(3.2) \quad \mathcal{F}_{\alpha,\beta}F_N(y) = \sum_{n_1, \dots, n_N=0}^N A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N} \Psi_{n_1, \dots, n_N}(y).$$

Furthermore,

$$(3.3) \quad \|\mathcal{F}_{\alpha,\beta}F_N\|_2^2 = \sum_{n_1, \dots, n_N=0}^N |A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N}|^2.$$

Definition 2. Let $F \in L_2(C_0[0, T])$ be given by (2.5). Then, for each nonzero complex numbers α and β , we define the integral transform $\mathcal{F}_{\alpha,\beta}F$ of F to be

$$(3.4) \quad \mathcal{F}_{\alpha,\beta}F(x) = \text{l.i.m.}_{N \rightarrow \infty} \mathcal{F}_{\alpha,\beta}F_N(x), \quad x \in C_0[0, T]$$

if it exists; that is to say, if

$$(3.5) \quad \lim_{N \rightarrow \infty} \int_{C_0[0, T]} |\mathcal{F}_{\alpha,\beta}F(x) - \mathcal{F}_{\alpha,\beta}F_N(x)|^2 m(dx) = 0.$$

Lemma 5. Let $F \in L_2(C_0[0, T])$ be given by equation (2.5) with Fourier-Hermite coefficients given by (2.6). Let α and β be as in Lemma 1 and assume that $\mathcal{F}_{\alpha,\beta}F$ exists and is in $L_2(C_0[0, T])$. Then

$$(3.6) \quad A_{n_1, \dots, n_N}^{\mathcal{F}_{\alpha,\beta}F} = A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N}$$

for each $N = 1, 2, \dots$.

Proof. Fix $N = 1, 2, \dots$. For any given $\varepsilon > 0$, take a natural number M satisfying $\|\mathcal{F}_{\alpha,\beta}F - \mathcal{F}_{\alpha,\beta}F_M\|_2 < \varepsilon$ and $M \geq N$. Then we have

$$\begin{aligned} & |A_{n_1, \dots, n_N}^{\mathcal{F}_{\alpha,\beta}F} - A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N}| \\ &= \left| \int_{C_0[0, T]} \mathcal{F}_{\alpha,\beta}F(x) \Psi_{n_1, \dots, n_N}(x) m(dx) - A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N} \right| \\ &\leq \left| \int_{C_0[0, T]} [\mathcal{F}_{\alpha,\beta}F(x) - \mathcal{F}_{\alpha,\beta}F_M(x)] \Psi_{n_1, \dots, n_N}(x) m(dx) \right| \\ &\quad + \left| \int_{C_0[0, T]} \mathcal{F}_{\alpha,\beta}F_M(x) \Psi_{n_1, \dots, n_N}(x) m(dx) - A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N} \right|. \end{aligned}$$

But, by the Hölder inequality,

$$\left| \int_{C_0[0,T]} [\mathcal{F}_{\alpha,\beta}F(x) - \mathcal{F}_{\alpha,\beta}F_M(x)] \Psi_{n_1,\dots,n_N}(x) m(dx) \right| \leq \|\mathcal{F}_{\alpha,\beta}F - \mathcal{F}_{\alpha,\beta}F_M\|_2 < \varepsilon$$

and from (3.2) we know that

$$\int_{C_0[0,T]} \mathcal{F}_{\alpha,\beta}F_M(x) \Psi_{n_1,\dots,n_N}(x) m(dx) = A_{n_1,\dots,n_N}^F \beta^{n_1+\dots+n_N}.$$

Hence

$$|\mathcal{A}_{n_1,\dots,n_N}^{\mathcal{F}_{\alpha,\beta}F} - A_{n_1,\dots,n_N}^F \beta^{n_1+\dots+n_N}| < \varepsilon$$

which establishes equation (3.6). \square

The following theorem is our main result. It gives a necessary and sufficient condition that a functional F in $L_2(C_0[0, T])$ has an integral transform $\mathcal{F}_{\alpha,\beta}F$ belonging to $L_2(C_0[0, T])$.

Theorem 6. *Let $F \in L_2(C_0[0, T])$ be given by equation (2.5) with Fourier-Hermite coefficients given by (2.6). Let α and β be as in Lemma 1. Then $\mathcal{F}_{\alpha,\beta}F$ exists and is an element of $L_2(C_0[0, T])$ if and only if*

$$(3.7) \quad \lim_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N |A_{n_1, \dots, n_N}^F \beta^{n_1+\dots+n_N}|^2 < \infty.$$

Furthermore, if (3.7) holds, then the Fourier-Hermite expression of $\mathcal{F}_{\alpha,\beta}F$ is given by

$$(3.8) \quad \mathcal{F}_{\alpha,\beta}F(y) = \text{l.i.m.}_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N A_{n_1, \dots, n_N}^F \beta^{n_1+\dots+n_N} \Psi_{n_1, \dots, n_N}(y).$$

Proof. Assume that $\mathcal{F}_{\alpha,\beta}F$ exists and is an element of $L_2(C_0[0, T])$. By (3.5) we have that, for any given $\varepsilon > 0$,

$$\int_{C_0[0,T]} |\mathcal{F}_{\alpha,\beta}F(x) - \mathcal{F}_{\alpha,\beta}F_N(x)|^2 m(dx) < \varepsilon$$

for sufficiently large N , and so

$$\begin{aligned} & \left(\sum_{n_1, \dots, n_N=0}^N |A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N}|^2 \right)^{1/2} \\ &= \|\mathcal{F}_{\alpha, \beta} F_N\|_2 \\ &\leq \|\mathcal{F}_{\alpha, \beta} F\|_2 + \|\mathcal{F}_{\alpha, \beta} F - \mathcal{F}_{\alpha, \beta} F_N\|_2 \\ &\leq \|\mathcal{F}_{\alpha, \beta} F\|_2 + \varepsilon. \end{aligned}$$

Hence we have

$$\lim_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N |A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N}|^2 \leq \|\mathcal{F}_{\alpha, \beta} F\|_2^2 < \infty.$$

To prove the converse, suppose that (3.7) holds. Let $M > N$, let

$$I_M = \{(n_1, \dots, n_M) : n_1, \dots, n_M = 0, 1, \dots, M\},$$

and let

$$\begin{aligned} I_N &= \{(n_1, \dots, n_M) : n_1, \dots, n_N = 0, 1, \dots, N \\ &\text{and } n_{N+1} = \dots = n_M = 0\}. \end{aligned}$$

Then

$$\begin{aligned} & \int_{C_0[0, T]} |\mathcal{F}_{\alpha, \beta} F_M(x) - \mathcal{F}_{\alpha, \beta} F_N(x)|^2 m(dx) \\ &= \int_{C_0[0, T]} \left| \sum_{I_M - I_N} A_{n_1, \dots, n_M}^F \beta^{n_1 + \dots + n_M} \Psi_{n_1, \dots, n_M}(x) \right|^2 m(dx) \\ &= \sum_{I_M - I_N} |A_{n_1, \dots, n_M}^F \beta^{n_1 + \dots + n_M}|^2 \\ &= \sum_{n_1, \dots, n_M=0}^M |A_{n_1, \dots, n_M}^F \beta^{n_1 + \dots + n_M}|^2 - \sum_{n_1, \dots, n_N=0}^N |A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N}|^2 \end{aligned}$$

which goes to 0 as $M, N \rightarrow \infty$. Hence $\{\mathcal{F}_{\alpha, \beta} F_N\}$ is a Cauchy sequence in $L_2(C_0[0, T])$ and, since $L_2(C_0[0, T])$ is complete,

$$\mathcal{F}_{\alpha, \beta} F(x) = \lim_{N \rightarrow \infty} \mathcal{F}_{\alpha, \beta} F_N(x), \quad x \in C_0[0, T]$$

exists and is an element of $L_2(C_0[0, T])$. \square

Corollary 7. *Let F, α and β be as in Theorem 6. Furthermore, assume that $|\beta| \leq 1$. Then $\mathcal{F}_{\alpha, \beta} F$ exists, belongs to $L_2(C_0[0, T])$, and*

$$\begin{aligned}
 \|\mathcal{F}_{\alpha, \beta} F\|_2^2 &= \lim_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N |A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N}|^2 \\
 &\leq \lim_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N |A_{n_1, \dots, n_N}^F|^2 = \|F\|_2^2.
 \end{aligned}
 \tag{3.9}$$

In addition,

$$\|\mathcal{F}_{\alpha, \beta} F\|_2 = \|F\|_2
 \tag{3.10}$$

if and only if $|\beta| = 1$.

Corollary 8. *Let p be a fixed positive integer, and let $F(x) = f(\langle a_1, x \rangle, \dots, \langle a_p, x \rangle)$ where f is such that*

$$f(u_1, \dots, u_p) \exp \left\{ -\frac{1}{4} \sum_{j=1}^p u_j^2 \right\} \in L_2(\mathbf{R}^p).
 \tag{3.11}$$

Let α and β be as in Lemma 1. Then the integral transform $\mathcal{F}_{\alpha, \beta} F$ exists and is an element of $L_2(C_0[0, T])$. (Note that in this case we don't need the restriction $|\beta| \leq 1$).

Proof. Since zero is an admissible value for each n_j in the Fourier-Hermite coefficient A_{n_1, \dots, n_N}^F of F , we need only consider the two cases $N = p$ and $N > p$. Then, using (2.2), (2.3), (2.4) and (2.6), a direct calculation shows that, for all nonnegative indices n_1, \dots, n_N ,

$$A_{n_1, \dots, n_N}^F = \begin{cases} 0 & N > p \text{ and } n_N > 0, \\ \left(\frac{1}{2\pi}\right)^{p/2} \int_{\mathbf{R}^p} f(u_1, \dots, u_p) H_{n_1}(u_1) \dots & \\ H_{n_p}(u_p) \exp \left\{ -\frac{1}{2} \sum_{j=1}^p u_j^2 \right\} d\vec{u} & N = p. \end{cases}$$

Moreover, by (3.11), we know that $|A_{n_1, \dots, n_p}^F| < \infty$ for all nonnegative indices n_1, \dots, n_p . Hence,

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N |A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N}|^2 \\ \leq |\beta|^{2p^2} \sum_{n_1, \dots, n_p=0}^p |A_{n_1, \dots, n_p}^F|^2 < \infty, \end{aligned}$$

and so, by Theorem 6, $\mathcal{F}_{\alpha, \beta} F$ exists and is an element of $L_2(C_0[0, T])$. \square

Next, choosing $\alpha = \sqrt{2}$ and $\beta = i$, we obtain the main theorem of [3].

Corollary 9. *Every functional $F(x) \in L_2(C_0[0, T])$ has a Fourier-Wiener transform $G(y) \in L_2(C_0[0, T])$. The functional $G(y)$ has $F(-x)$ as its transform and F and G satisfy Plancherel's relation*

$$(3.12) \quad \int_{C_0[0, T]} |F(x)|^2 m(dx) = \int_{C_0[0, T]} |G(y)|^2 m(dy).$$

Proof. Using Corollary 7 and Theorem 6, we obtain that $G(y) \in L_2(C_0[0, T])$ is given by

$$G(y) = \text{l.i.m.}_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N A_{n_1, \dots, n_N}^F i^{n_1 + \dots + n_N} \Psi_{n_1, \dots, n_N}(y),$$

and that

$$\mathcal{F}_{\sqrt{2}, i} G(y) = \text{l.i.m.}_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N A_{n_1, \dots, n_N}^F (-1)^{n_1 + \dots + n_N} \Psi_{n_1, \dots, n_N}(y).$$

But it is easy to see that

$$(-1)^{n_1 + \dots + n_N} \Psi_{n_1, \dots, n_N}(y) = \Psi_{n_1, \dots, n_N}(-y)$$

and so $\mathcal{F}_{\sqrt{2},i}G(y) = F(-y)$. Equation (3.12) then follows immediately from the Fourier-Hermite expressions for $F(x)$ and $G(y)$ and the fact that $\{\Psi_{n_1, \dots, n_N}\}$ is an orthonormal set. \square

Remark 3. In [7], Lee, for the abstract Wiener space (H, B, p_1) (also see [5]) and the class $\mathcal{E}_a(B)$ of the restrictions to B of exponential type analytic functionals on $[B]$, the complexification of B , established the following theorem [7, Theorem 2.6].

Theorem 2.6 [7]. For $F \in \mathcal{E}_a(B)$,

$$\int_B |\mathcal{F}_{\alpha,\beta}F(y)|^2 p_1(dy) = \int_B |F(y)|^2 p_1(dy)$$

if and only if $\alpha^2 + \beta^2 = 1$ and $|\beta| = 1$.

He then pointed out that Theorem 2.6 ensures that $\mathcal{F}_{\alpha,\beta}$ can be extended from $\mathcal{E}_a(B)$ to $L_2(p_1)$ as a unitary operator.

4. Further results. Recall that, throughout this paper, we have assumed that $\beta = a + bi$ was a nonzero complex number satisfying inequality (2.7); namely, that $\text{Re}(1 - \beta^2) > 0$. Furthermore, in Corollary 7, we showed that if β also satisfies the inequality $|\beta| \leq 1$, then $\mathcal{F}_{\alpha,\beta}$ exists as an element of $L_2(C_0[0, T])$ for all $F \in L_2(C_0[0, T])$ with α given by (2.8). In the example below we show that for any complex number β with $|\beta| > 1$ and $\text{Re}(1 - \beta^2) > 0$, there exists a functional $F \in L_2(C_0[0, T])$ (of course F depends on β) such that $\mathcal{F}_{\alpha,\beta}F$ doesn't exist as an element of $L_2(C_0[0, T])$.

Example 10. Let $\beta = a + bi$ be such that $|\beta| = k > 1$ and $\text{Re}(1 - \beta^2) > 0$. Let α be given by (2.8). Let $\Psi_{(0)}(x) \equiv \Psi_0(x)$, $\Psi_{(1)}(x) \equiv \Psi_1(x)$, $\Psi_{(2)}(x) \equiv \Psi_{1,1}(x)$, $\Psi_{(3)}(x) \equiv \Psi_{1,1,1}(x)$, etc. For $N = 1, 2, \dots$, let

$$(4.1) \quad F_N(x) = \sum_{n=0}^N k^{-n} \Psi_{(n)}(x).$$

Then, since $\{\Psi_{(n)} : n = 0, 1, 2, \dots\}$ is an orthonormal set of functionals in $L_2(C_0[0, T])$,

$$\int_{C_0[0, T]} |F_N(x)|^2 m(dx) = \sum_{n=0}^N k^{-2n} = \frac{k^2}{k^2 - 1} \left[1 - \left(\frac{1}{k^2} \right)^{N+1} \right],$$

and so

$$\lim_{N \rightarrow \infty} \|F_N\|_2^2 = \frac{k^2}{k^2 - 1}.$$

But, for $N > M$,

$$\|F_N - F_M\|_2^2 = \sum_{n=M+1}^N \frac{1}{k^{2n}} \rightarrow 0$$

as $M, N \rightarrow \infty$. Hence $\{F_N\}_{N=1}^\infty$ is a Cauchy sequence in $L_2(C_0[0, T])$ and, since $L_2(C_0[0, T])$ is complete,

$$F(x) \equiv \text{l.i.m.}_{N \rightarrow \infty} F_N(x)$$

is an element of $L_2(C_0[0, T])$. In fact,

$$F(x) = \sum_{n=0}^{\infty} k^{-n} \Psi_{(n)}(x)$$

is the Fourier-Hermite series for F ; i.e., the Fourier-Hermite coefficients for F are $A_0^F = 1$, and for $n_p \neq 0$,

$$(4.2) \quad A_{n_1, \dots, n_p}^F = \begin{cases} 0 & n_1 n_2 \dots n_p \neq 1 \\ k^{-p} & n_1 n_2 \dots n_p = 1. \end{cases}$$

Next, using (4.2) and the fact that $|\beta| = k$, we observe that

$$\begin{aligned} \sum_{n_1, \dots, n_N=0}^N |A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N}|^2 &= |A_0^F \beta^0|^2 + |A_1^F \beta^1|^2 + |A_{1,1}^F \beta^2|^2 \\ &\quad + \dots + |A_{1, \dots, 1}^F \beta^N|^2 \\ &= 1 + 1 + 1 + \dots + 1 = N + 1. \end{aligned}$$

Hence, by Theorem 6, $\mathcal{F}_{\alpha,\beta}F$ doesn't exist as an element of $L_2(C_0[0, T])$. However, note that $\mathcal{F}_{\alpha,\beta}F$ does exist as an element of $L_2(C_0[0, T])$ if $|\beta| < k$ with $\text{Re}(1 - \beta^2) > 0$ and $\alpha = \sqrt{1 - \beta^2}$. \square

Our final results in this paper involve the inverse transform of $\mathcal{F}_{\alpha,\beta}$. In order to ensure the existence of the inverse transform of $\mathcal{F}_{\alpha,\beta}$, we need to put an additional assumption on $\beta = a + bi$; namely, that

$$(4.3) \quad \text{Re} \left(1 - \frac{1}{\beta^2} \right) > 0.$$

But $\text{Re}[1 - (1/\beta^2)] > 0 \Leftrightarrow (a^2 + b^2)^2 - (a^2 - b^2) > 0$. But the graph of $(a^2 + b^2)^2 - (a^2 - b^2) = 0$ is the lemniscate $r^2 = \cos(2\theta)$. Hence $\text{Re}[1 - (1/\beta^2)] > 0$ if and only if the point $(a, b) \in \mathbf{R}^2$ lies outside the lemniscate $(a^2 + b^2)^2 - (a^2 - b^2) = 0$.

Theorem 11. *Let F, β and α be as in Theorem 6, and assume that (3.7) holds. Furthermore, assume that β satisfies inequality (4.3). Then, for $\alpha' \equiv \sqrt{1 - 1/\beta^2}$ and $\beta' = \pm 1/\beta$, we have that*

$$(4.4) \quad \mathcal{F}_{\alpha',\beta'}\mathcal{F}_{\alpha,\beta}F(y) = F(\beta\beta'y), \quad y \in C_0[0, T].$$

That is to say,

$$\mathcal{F}_{\alpha',1/\beta}\mathcal{F}_{\alpha,\beta}F(y) = F(y), \quad y \in C_0[0, T],$$

and

$$\mathcal{F}_{\alpha',-1/\beta}\mathcal{F}_{\alpha,\beta}F(y) = F(-y), \quad y \in C_0[0, T].$$

Proof. Since $\mathcal{F}_{\alpha,\beta}F$ exists, the Fourier-Hermite expression of it is given by

$$\mathcal{F}_{\alpha,\beta}F(y) = \text{l.i.m.}_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N} \Psi_{n_1, \dots, n_N}(y).$$

Now

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N |A_{n_1, \dots, n_N}^F \beta^{n_1 + \dots + n_N} (\beta')^{n_1 + \dots + n_N}|^2 \\ = \lim_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N |A_{n_1, \dots, n_N}^F|^2 = \|F\|_2^2 < \infty. \end{aligned}$$

Hence, by Theorem 6, $\mathcal{F}_{\alpha',\beta'}\mathcal{F}_{\alpha,\beta}F$ exists and is given by

$$\begin{aligned}\mathcal{F}_{\alpha',\beta'}\mathcal{F}_{\alpha,\beta}F(y) &= \text{l.i.m.}_{N \rightarrow \infty} \sum_{n_1, \dots, n_N=0}^N A_{n_1, \dots, n_N}^F (\beta\beta')^{n_1 + \dots + n_N} \Psi_{n_1, \dots, n_N}(y) \\ &= F(\beta\beta'y)\end{aligned}$$

which completes the proof of Theorem 11. \square

Remark 4. Some special cases of Theorem 11:

$$\begin{aligned}\mathcal{F}_{\sqrt{2},-i}\mathcal{F}_{\sqrt{2},i}F(y) &= F(y), \\ \mathcal{F}_{\sqrt{2},i}\mathcal{F}_{\sqrt{2},i}F(y) &= F(-y), \\ \mathcal{F}_{(7-4\sqrt{2}i)^{1/2}/3, (2+\sqrt{2}i)/3}\mathcal{F}_{1+i/\sqrt{2}, 1-i/\sqrt{2}}F(y) &= F(y).\end{aligned}$$

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