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RELATIONSHIPS AMONG THE FIRST VARIATION, THE CONVOLUTION PRODUCT, AND THE FOURIER-FEYNMAN TRANSFORM

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ABSTRACT. In this paper we examine the various relationships that exist among the first variation, the Fourier-Feynman transform, and the convolution product for functionals on Wiener space that belong to a Banach algebra $\mathcal{S}$.

1. Introduction. Let $C_0[0,T]$ denote one-parameter Wiener space; that is the space of **R**-valued continuous functions on $[0,T]$ with $x(0)=0$. The concept of an L_1 analytic Fourier-Feynman transform was introduced by Brue in [**1**]. In [**3**], Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform. In [**12**], Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \le p \le 2$ which extended the results in [**1, 3**] and gave various relationships between the L_1 and L_2 theories. In [**9**], Huffman, Park and Skoug defined a convolution product for functionals on Wiener space and in [**9, 10, 11**] obtained various results involving the Fourier-Feynman transform and the convolution product.

The class of functionals on $C_0[0,T]$ that we work with throughout this paper is the Banach algebra $\mathcal{S}$ introduced by Cameron and Storvick in [**4**]. Results in [**7, 8, 14, 15**] show that $\mathcal{S}$ contains many broad subclasses of functionals of interest in connection with Feynman integration theory and quantum mechanics.

In Section 3 of this paper we examine all relationships involving exactly two of the three concepts of "transform," "convolution product" and "first variation" of functionals in $\mathcal{S}$. In Section 4, we examine all relationships involving all three of these concepts where each concept is used exactly once. Study of these relationships yields many interesting formulas; see, for example, equations (3.7), (3.9), (4.1), (4.3) and (4.7).

2. Definitions and preliminaries. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$, and let m denote Wiener

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measure. $(C_0[0,T], \mathcal{M}, m)$ is a complete measure space, and we denote the Wiener integral of a functional F by $\int_{C_0[0,T]} F(x)m(dx)$.

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable [**6, 13**] provided $\rho E \in \mathcal{M}$ for all $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F \approx G$.

Let $\mathbf{C}_+ = \{\lambda \in \mathbf{C} : \mathrm{Re}\,\lambda > 0\}$ and $\mathbf{C}_+^\sim = \{\lambda \in \mathbf{C} : \lambda \neq 0$ and $\mathrm{Re}\ \lambda \geq 0\}$. Let F be a $\mathbf{C}$-valued scale-invariant measurable functional on $C_0[0,T]$ such that

$$J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-1/2}x)m(dx)$$

exists for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbf{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ and for $\lambda \in \mathbf{C}_+$ we write

$$\int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx) = J^*(\lambda).$$

Let $q \neq 0$ be a real number, and let F be a functional such that $\int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx)$ exists for all $\lambda \in \mathbf{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q, and we write

$$\int_{C_0[0,T]}^{anf_q} F(x)m(dx) = \lim_{\lambda \to -iq} \int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx)$$

where $\lambda \to -iq$ through $\mathbf{C}_+$.

Notation. (i) For $\lambda \in \mathbf{C}_+$ and $y \in C_0[0,T]$, let

$$T_\lambda(F)(y) = \int_{C_0[0,T]}^{anw_\lambda} F(x+y)m(dx).$$

(ii) Given a number p with $1 \leq p \leq +\infty$, p and p' will always be related by $1/p + 1/p' = 1$.

(iii) Let $1 < p \leq 2$, and let $\{H_n\}$ and H be scale-invariant measurable functions such that, for each $\rho > 0$,

$$\lim_{n\to\infty} \int_{C_0[0,T]} |H_n(\rho y) - H(\rho y)|^{p'} m(dy) = 0.$$

Then we write

$$\lim_{n\to\infty}(w_s^{p'})(H_n) \approx H,$$

and we call H the scale invariant limit in the mean of order p'. A similar definition is understood when n is replaced by the continuously varying parameter λ.

We are finally ready to state the definition of the L_p analytic Fourier-Feynman transform [**12**], the definition of the convolution product [**9**], and the definition of the first variation of a function [**2,5**].

Definition. Let $q \neq 0$ be a real number. For $1 < p \leq 2$ we define the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F by the formula ($\lambda \in \mathbf{C}_+$)

$$T_q^{(p)}(F) = \lim_{\lambda\to -iq}(w_s^{p'})T_\lambda(F),$$

whenever this limit exists. Also the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F is defined by ($\lambda \in \mathbf{C}_+$)

$$T_q^{(1)}(F) = \lim_{\lambda\to -iq} T_\lambda(F), \quad \text{s-a.e.}$$

We note that for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e. We also note that, if $T_q^{(p)}(F_1)$ exists and if $F_1 \approx F_2$, then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_1) \approx T_q^{(p)}(F_2)$.

Definition. Let F and G be functionals on $C_0[0,T]$. For $\lambda \in \mathbf{C}_+^\sim$ we define their *convolution product* (if it exists) by

$$(F*G)_\lambda(y) = \begin{cases} \int_{C_0[0,T]}^{anw_\lambda} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) m(dx) & \lambda \in \mathbf{C}_+ \\ \int_{C_0[0,T]}^{anf_q} F\left(\frac{y+x}{\sqrt{2}}\right) G\left(\frac{y-x}{\sqrt{2}}\right) m(dx), & \lambda = -iq,\ q \in \mathbf{R},\ q \neq 0. \end{cases}$$

Remarks. (i) When $\lambda = -iq$, we denote $(F*G)_\lambda$ by $(F*G)_q$.

(ii) Our definition of the convolution product is different than the definition given by Yeh in [**16**] and used by Yoo in [**17**]. In [**16**] and [**17**] Yeh and Yoo study relationships between their convolution product and Fourier-Wiener transforms.

We finish this section by giving the definition of the first variation δF of a functional F.

Definition. Let F be a *Wiener measurable functional* on $C_0[0,T]$, and let $w \in C_0[0,T]$. Then

$$\delta F(x \mid w) = \frac{\partial}{\partial h} F(x+hw)\Big|_{h=0}$$

(if it exists) is called the first variation of $F(x)$.

3. Relationships involving exactly two of the three concepts of transform, convolution and variation.

Notation. For $u, v \in L_2[0,T]$ and $x \in C_0[0,T]$, we let $(u,v) \equiv \int_0^T u(t)v(t)dt$ and $\langle u, x\rangle \equiv \int_0^T u(t)dx(t)$, the Paley-Wiener-Zygmund stochastic integral. Also, let

$$A \equiv \{y \in C_0[0,T] : y \text{ is absolutely continuous on } [0,T] \text{ with } y' \in L_2[0,T]\}.$$

The following analytic Feynman integration formula is used throughout this paper.

$$\int_{C_0[0,T]}^{anf_q} \exp\left\{\frac{i}{a}\langle u,x\rangle\right\} m(dx) = \exp\left\{-\frac{i(u,u)}{2a^2q}\right\}$$

for $a > 0$ and $u \in L_2[0,T]$.

Definition. The Banach algebra $\mathcal{S}$ consists of functionals on $C_0[0,T]$ expressible in the form

$$F(y) = \int_{L_2[0,T]} \exp\{i\langle u,y\rangle\}\, df(u) \tag{3.1}$$

for s-a.e. $y \in C_0[0,T]$ where f is an element of $M(L_2[0,T])$, the space of $\mathbf{C}$-valued countably additive (and hence finite) Borel measures on $L_2[0,T]$.

In our first lemma we obtain a formula for the first variation of functionals in $\mathcal{S}$.

Lemma 3.1. *Let $F \in \mathcal{S}$ be given by* (3.1) *with $\int_{L_2[0,T]} \|u\|_2\, |df(u)| < \infty$. Then, for each $w \in A$ and s-a.e. $y \in C_0[0,T]$,*

$$\delta F(y \mid w) = \int_{L_2[0,T]} i\langle u,w\rangle \exp\{i\langle u,y\rangle\}\, df(u). \tag{3.2}$$

Furthermore, as a function of y, $\delta F(y \mid w)$ is an element of $\mathcal{S}$.

Proof. By use of [**5**, Lemma 2], which justifies taking the partial derivative under the integral sign, we see that for s-a.e. $y \in C_0[0,T]$,

$$\begin{aligned}\delta F(y \mid w) &= \frac{\partial}{\partial h}\bigg(\int_{L_2[0,T]} \exp\{i\langle u,y\rangle + ih\langle u,w\rangle\}\, df(u)\bigg)\bigg|_{h=0} \\ &= \int_{L_2[0,T]} i\langle u,w\rangle \exp\{i\langle u,y\rangle\}\, df(u).\end{aligned}$$

Next let $\phi_w(E) = \int_E i\langle u, w\rangle \, df(u)$ for each set $E \in \mathcal{B}(L_2[0,T])$. But then, $\delta F(y \mid w) = \int_{L_2[0,T]} \exp\{i\langle u, y\rangle\} \, d\phi_w(u)$ is an element of $\mathcal{S}$ since $\|\phi_w\| \le \|w'\|_2 \int_{L_2[0,T]} \|u\|_2 \, |df(u)| < \infty$. $\square$

Now let G in $\mathcal{S}$ be given by

$$G(y) = \int_{L_2[0,T]} \exp\{i\langle v, y\rangle\} \, dg(v) \tag{3.3}$$

for s-a.e. $y \in C_0[0,T]$ where $g \in M(L_2[0,T])$. In [**11**], Huffman, Park and Skoug show that, for all $p \in [1,2]$ and all $q \in \mathbf{R} - \{0\}$, the Fourier-Feynman transform of $F \in \mathcal{S}$ is given by the formula

$$T_q^{(p)}(F)(y) = \int_{L_2[0,T]} \exp\left\{ i\langle u, y\rangle - i\frac{(u,u)}{2q} \right\} df(u) \tag{3.4}$$

for s-a.e. $y \in C_0[0,T]$, while the convolution product $(F * G)_q$ is given by the formula

$$(F * G)_q(y) = \int_{L_2^2[0,T]} \exp\left\{ \frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{i(u-v, u-v)}{4q} \right\} df(u) \, dg(v)$$

for s-a.e. $y \in C_0[0,T]$.

In the three subsections below, we establish the various relationships that hold involving exactly two of the three concepts "the Fourier-Feynman transform," "the convolution product," and "the first variation" of functionals in the Banach algebra $\mathcal{S}$ where each operation is used only once.

3.1. *Relationships involving Fourier-Feynman transforms and convolution.*

3.1.1. *The transform of the convolution product.* In [**11**, Theorem 3.3] it was shown that, for all $p \in [1,2]$ and all $q \in \mathbf{R} - \{0\}$,

$$T_q^{(p)}((F * G)_q)(y) = T_q^{(p)}(F)(y/\sqrt{2}) T_q^{(p)}(G)(y/\sqrt{2}) \tag{3.5}$$

for s-a.e. $y \in C_0[0,T]$.

3.1.2. *The convolution product of transforms.*

Theorem 3.1. *Let F and G be given by* (3.1) *and* (3.3), *respectively. Let $p \in [1,2]$ and $q \in \mathbf{R} - \{0\}$ be given. Then, for s-a.e. $y \in C_0[0,T]$,*

$$(T_q^{(p)}(F) * T_q^{(p)}(G))_{-q}(y) = T_q^{(p)}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2}))(y).$$

Proof. Fix $p \in [1,2]$ and $q \in \mathbf{R} - \{0\}$. Then, for s-a.e. $y \in C_0[0,T]$,

$$
\begin{aligned}
&(T_q^{(p)}(F) * T_q^{(p)}(G))_{-q}(y) \\
&\quad = \int_{C_0[0,T]}^{anf_{-q}} T_q^{(p)}(F)\bigg(\frac{y+x}{\sqrt{2}}\bigg) T_q^{(p)}(G)\bigg(\frac{y-x}{\sqrt{2}}\bigg) m(dx) \\
&\quad = \int_{C_0[0,T]}^{anf_{-q}} \int_{L_2[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle u,y\rangle + \frac{i}{\sqrt{2}}\langle u,x\rangle - \frac{i(u,u)}{2q}\bigg\}\, df(u) \\
&\qquad \cdot \int_{L_2[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle v,y\rangle - \frac{i}{\sqrt{2}}\langle v,x\rangle - \frac{i(v,v)}{2q}\bigg\}\, dg(v) m(dx) \\
&\quad = \int_{L_2^2[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{2q}[(u,u)+(v,v)]\bigg\} \\
&\qquad \cdot \int_{C_0[0,T]}^{anf_{-q}} \exp\bigg\{\frac{i}{\sqrt{2}}\langle u-v,x\rangle\bigg\} m(dx)\, df(u)\, dg(v) \qquad (3.6) \\
&\quad = \int_{L_2^2[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i}{2q}[(u,u)+(v,v)] \\
&\qquad\qquad + \frac{i(u-v,u-v)}{4q}\bigg\}\, df(u)\, dg(v) \\
&\quad = \int_{L_2^2[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i(u+v,u+v)}{4q}\bigg\}\, df(u)\, dg(v)
\end{aligned}
$$

$$= \int_{L_2^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle u+v, y\rangle\right\}$$

$$\cdot \int_{C_0[0,T]}^{anf_q} \exp\left\{\frac{i}{\sqrt{2}}\langle u+v, x\rangle\right\} m(dx)\, df(u)\, dg(v)$$

$$= \int_{C_0[0,T]}^{anf_q} \int_{L_2^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle u+v, x+y\rangle\right\} df(u)\, dg(v) m(dx)$$

$$= \int_{C_0[0,T]}^{anf_q} F\left(\frac{x+y}{\sqrt{2}}\right) G\left(\frac{x+y}{\sqrt{2}}\right) m(dx)$$

$$= T_q^{(p)}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2}))(y). \qquad \square$$

3.2. *Relationships involving Fourier-Feynman transforms and the first variation.*

3.2.1. *The transform with respect to the first argument of the variation equals the variation of the transform.*

Theorem 3.2. *Let F be as in Lemma 3.1, $w \in A$, $p \in [1,2]$ and $q \in \mathbf{R} - \{0\}$ be given. Then, for s-a.e. $y \in C_0[0,T]$,*

$$T_q^{(p)}(\delta F(\cdot \mid w))(y) = \delta T_q^{(p)}(F)(y \mid w). \tag{3.7}$$

Also, both of the expressions in (3.7) are given by the expression

$$\int_{L_2[0,T]} i\langle u, w\rangle \exp\left\{i\langle u, y\rangle - \frac{i(u,u)}{2q}\right\} df(u). \tag{3.8}$$

Proof. We first note that, by use of equation (3.2),

$$
\begin{aligned}
T_q^{(p)}(\delta F(\cdot \mid w))(y) &= \int_{C_0[0,T]}^{anf_q} \delta F(x+y \mid w) m(dx) \\
&= \int_{C_0[0,T]}^{anf_q} \int_{L_2[0,T]} i\langle u, w\rangle \exp\{i\langle u, x+y\rangle\}\, df(u) m(dx) \\
&= \int_{L_2[0,T]} i\langle u, w\rangle \exp\{i\langle u, y\rangle\} \\
&\quad \cdot \int_{C_0[0,T]}^{anf_q} \exp\{i\langle u, x\rangle\} m(dx)\, df(u) \\
&= \int_{L_2[0,T]} i\langle u, w\rangle \exp\left\{i\langle u, y\rangle - \frac{i(u,u)}{2q}\right\} df(u)
\end{aligned}
$$

for s-a.e. $y \in C_0[0,T]$. On the other hand, using (3.4) we obtain that

$$
\begin{aligned}
\delta T_q^{(p)}(F)(y \mid w) &= \frac{\partial}{\partial h}(T_q^{(p)}(F)(y+hw))\Big|_{h=0} \\
&= \int_{L_2[0,T]} i\langle u, w\rangle \exp\left\{i\langle u, y\rangle - \frac{i(u,u)}{2q}\right\} df(u)
\end{aligned}
$$

for s-a.e. $y \in C_0[0,T]$ as desired. □

3.2.2. *The transform with respect to the second argument of the variation equals the variation of the functional.*

Theorem 3.3. *Let F, p and q be as in Theorem 3.2, and let $w \in A$. Then for s-a.e. $y \in C_0[0,T]$,*

$$
T_q^{(p)}(\delta F(y \mid \cdot))(w) = \delta F(y \mid w). \tag{3.9}
$$

Proof. Using (3.2) we obtain

$$\begin{aligned}
T_q^{(p)}(\delta F(y\mid\cdot))(w) &= \int_{C_0[0,T]}^{anf_q} \delta F(y\mid x+w)m(dx)\\
&= \int_{C_0[0,T]}^{anf_q}\int_{L_2[0,T]} i\langle u, x+w\rangle \exp\{i\langle u,y\rangle\}\,df(u)m(dx)\\
&= i\int_{L_2[0,T]} \exp\{i\langle u,y\rangle\}\int_{C_0[0,T]}^{anf_q}\langle u,x+w\rangle m(dx)\,df(u)\\
&= i\int_{L_2[0,T]} \exp\{i\langle u,y\rangle\}\langle u,w\rangle\,df(u)\\
&= \int_{L_2[0,T]} i\langle u,w\rangle \exp\{i\langle u,y\rangle\}\,df(u)\\
&= \delta F(y\mid w)
\end{aligned}$$

for s-a.e. $y \in C_0[0,T]$ as desired. □

3.3. *Relationships involving the first variation and convolution.*

3.3.1. *A formula for the first variation of the convolution product.* Let F, G, p, q and w be as in Theorem 3.2, and let G be given by (3.3) with $\int_{L_2[0,T]}\|v\|_2\,|dg(v)| < \infty$. Then for s-a.e. $y \in C_0[0,T]$ we have the formula

$$\begin{aligned}
\delta(F*G)_q(y\mid w) = \frac{i}{\sqrt{2}}\int_{L_2^2[0,T]} \langle u+v, w\rangle \exp\Bigg\{&\frac{i}{\sqrt{2}}\langle u+v, y\rangle\\
&- \frac{i(u-v,u-v)}{4q}\Bigg\}\,df(u)\,dg(v).
\end{aligned}$$

The proof of formula (3.10) parallels that of Lemma 3.1 above and so is omitted.

3.3.2. *Formulas for the convolution product of the first variation of functionals.* As in subsection 3.2 above, there are two cases; in formula (3.11) below we take the convolution with respect to the first argument of the variations while in formula (3.12) we take the convolution with respect to the second argument of the variations.

Case 1. Let F, G, p and q be as in subsection 3.3.1 above, and let $w \in A$. Then for s-a.e. $y \in C_o[0,T]$ we have the formula

$$(3.11)\quad (\delta F(\cdot \mid w) * \delta G(\cdot \mid w))_q(y) = -\int_{L_2^2[0,T]} \langle u,w\rangle\langle v,w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i(u-v,u-v)}{4q}\right\} df(u)\,dg(v).$$

Proof of formula (3.11). Using the definition of the convolution product and equation (3.2), we see that, for s-a.e. $y \in C_0[0,T]$,

$$\begin{aligned}
(\delta F(\cdot|w) * \delta G(\cdot|w))_q(y) &= \int_{C_0[0,T]}^{anf_q} \delta F\left(\frac{y+x}{\sqrt{2}}\middle| w\right)\delta G\left(\frac{y-x}{\sqrt{2}}\middle| w\right) m(dx) \\
&= \int_{C_0[0,T]}^{anf_q} \int_{L_2[0,T]} i\langle u,w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u,y+x\rangle\right\} df(u) \\
&\quad \cdot \int_{L_2[0,T]} i\langle v,w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle v,y-x\rangle\right\} dg(v) m(dx) \\
&= -\int_{L_2^2[0,T]} \langle u,w\rangle\langle v,w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle\right\} \\
&\quad \cdot \int_{C_0[0,T]}^{anf_q} \exp\left\{\frac{i}{\sqrt{2}}\langle u-v,x\rangle\right\} m(dx)\,df(u)\,dg(v)
\end{aligned}$$

$$= -\int_{L_2[0,T]} \langle u,w\rangle\langle v,w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u+v,y\rangle - \frac{i(u-v,u-v)}{4q}\right\} df(u)\,dg(v),$$

as desired. □

Case 2. Let F, G, p and q be as in Case 1 above, and let $w \in A$. Then, for s-a.e. $y \in C_0[0,T]$, we have the formula

$$(\delta F(y \mid \cdot) * \delta G(y \mid \cdot))_q(w) = \delta F(y \mid w/\sqrt{2})\delta G(y \mid w/\sqrt{2}) + \frac{i}{2q}\int_{L_2^2[0,T]} (u,v)\exp\{i\langle u+v,y\rangle\}\,df(u)\,dg(v). \tag{3.12}$$

Proof of formula (3.12). Using the definition of the convolution product and equation (3.2), we see that, for s-a.e. $y \in C_0[0,T]$,

$$\begin{aligned}
&(\delta F(y \mid \cdot) * \delta G(y \mid \cdot))_q(w)\\
&= \int_{C_0[0,T]}^{anf_q} \delta F\left(y\middle|\frac{w+x}{\sqrt{2}}\right)\delta G\left(y\middle|\frac{w-x}{\sqrt{2}}\right) m(dx)\\
&= \int_{C_0[0,T]}^{anf_q} \int_{L_2[0,T]} \frac{i}{\sqrt{2}}\langle v,w+x\rangle \exp\{i\langle u,y\rangle\}\,df(u)\\
&\quad\cdot \int_{L_2[0,T]} \frac{i}{\sqrt{2}}\langle v,w-x\rangle \exp\{i\langle v,y\rangle\}\,dg(v) m(dx)\\
&= -\frac{1}{2}\int_{L_2^2[0,T]} \exp\{i\langle u+v,y\rangle\}\\
&\quad\cdot \int_{C_0[0,T]}^{anf_q} \langle u,w+x\rangle\langle v,w-x\rangle m(dx)\,df(u)\,dg(v)
\end{aligned}$$

$$= -\frac{1}{2} \int\limits_{L_2^2[0,T]} \exp\{i\langle u+v, y\rangle\} \cdot \left[\langle u,w\rangle\langle v,w\rangle - \frac{i}{q}(u,v)\right] df(u)\, dg(v)$$

since $\int_{C_0[0,T]}^{anf_q} \langle u,x\rangle\langle v,x\rangle m(dx) = (i/q)(u,v)$ and $\int_{C_0[0,T]}^{anf_q} \langle u,x\rangle m(dx) = 0$. Hence

$$\begin{aligned}(\delta F(y \mid \cdot) * \delta G(y \mid \cdot))_q(w) &= \int\limits_{L_2[0,T]} \frac{i}{\sqrt{2}}\langle u,w\rangle \exp\{i\langle u,y\rangle\}\, df(u) \\ &\quad \cdot \int\limits_{L_2[0,T]} \frac{i}{\sqrt{2}}\langle v,w\rangle \exp\{i\langle v,y\rangle\}\, dg(u) \\ &\quad + \frac{i}{2q} \int\limits_{L_2^2[0,T]} (u,v) \exp\{i\langle u+v,y\rangle\}\, df(u)\, dg(v) \\ &= \delta F(y \mid w/\sqrt{2})\delta G(y \mid w/\sqrt{2}) \\ &\quad + \frac{i}{2q} \int\limits_{L_2^2[0,T]} (u,v) \exp\{i\langle u+v,y\rangle\}\, df(u)\, dg(v)\end{aligned}$$

for s-a.e. $y \in C_0[0,T]$ as desired. □

4. Relationships involving all three concepts. In this section we look at all the relationships involving the"transform," the "convolution" and the "variation" where each operation is used exactly once; i.e., we will consider the variation of the transform of the convolution product, but not the transform of the variation of the transform. There are more than six possibilities since one can take the transform or the convolution with respect to the first or the second argument of the variation. However, there are some repetitions because, as we observed above in subsection 3.2, the transform with respect to the first argument of the variation equals the variation of the transform, while the transform with respect to the second argument of the variation equals the variation of the functional. We need to take the "transform" of the expressions in subsection 3.3, the "variation" of the expressions in subsection 3.1, and the "convolution product" of the expressions in subsection 3.2. In fact, there are nine distinct possibilities; these are given in the nine equations (4.1) through (4.9) below. We formally state some of these nine results as theorems and others as formulas; however, all nine of these equations hold for s-a.e. $y \in C_0[0,T]$.

4.1. *A formula for the transform with respect to the first argument of the variation of the convolution product which equals the variation of the transform of the convolution product.*

Theorem 4.1. *Let F, G, p, and q be as in Subsection* 3.3.1, *and let $w \in A$. Then for s-a.e. $y \in C_0[0,T]$,*

$$\begin{aligned} (4.1) \qquad & T_q^{(p)}(\delta(F*G)_q(\cdot \mid w))(y) \\ &= T_q^{(p)}(F)(y/\sqrt{2})T_q^{(p)}(\delta G(\cdot \mid w/\sqrt{2}))(y/\sqrt{2}) \\ &\quad + T_q^{(p)}(\delta F(\cdot \mid w/\sqrt{2}))(y/\sqrt{2})T_q^{(p)}(G)(y/\sqrt{2}). \end{aligned}$$

Proof. By Theorem 3.2 we have that

$$T_q^{(p)}(\delta(F*G)_q(\cdot \mid w))(y) = \delta T_q^{(p)}((F*G)_q)(y \mid w).$$

But, using equations (3.5) and (3.4), we obtain that

$$\begin{aligned} &\delta T_q^{(p)}((F*G)_q)(y \mid w) \\ &\qquad = \frac{\partial}{\partial h}(T_q^{(p)}((F*G)_q)(y+hw))\Big|_{h=0} \\ &\qquad = \frac{\partial}{\partial h}\bigg(T_q^{(p)}(F)\bigg(\frac{y+hw}{\sqrt{2}}\bigg)T_q^{(p)}(G)\bigg(\frac{y+hw}{\sqrt{2}}\bigg)\bigg)\bigg|_{h=0} \\ &\qquad = \frac{\partial}{\partial h}\bigg[\int_{L_2[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle u,y\rangle + \frac{ih}{\sqrt{2}}\langle u,w\rangle - \frac{i(u,u)}{2q}\bigg\}\, df(u) \\ &\qquad\quad \cdot \int_{L_2[0,T]} \exp\bigg\{\frac{i}{\sqrt{2}}\langle v,y\rangle + \frac{ih}{\sqrt{2}}\langle v,w\rangle - \frac{i(v,v)}{2q}\bigg\}\, dg(v)\bigg]\bigg|_{h=0} \end{aligned}$$

$$= \int_{L_2[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle u, y\rangle - \frac{i(u,u)}{2q}\right\} df(u)$$
$$\cdot \int_{L_2[0,T]} \frac{i}{\sqrt{2}}\langle v, w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle v, y\rangle - \frac{i(v,v)}{2q}\right\} dg(v)$$
$$+ \int_{L_2[0,T]} \frac{i}{\sqrt{2}}\langle u, w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u, y\rangle - \frac{i(u,u)}{2q}\right\} df(u)$$
$$\cdot \int_{L_2[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle v, y\rangle - \frac{i(v,v)}{2q}\right\} dg(v)$$
$$= T_q^{(p)}(F)(y/\sqrt{2})T_q^{(p)}(\delta G(\cdot \mid w/\sqrt{2}))(y/\sqrt{2})$$
$$+ T_q^{(p)}(\delta F(\cdot \mid w/\sqrt{2}))(y/\sqrt{2})T_q^{(p)}(G)(y/\sqrt{2})$$

for s-a.e. $y \in C_0[0,T]$. □

4.2. *The transform with respect to the second argument of the variation of the convolution product equals the variation of the convolution product.* Let F, G, p, q and w be as in Theorem 4.1. Then by Theorem 3.3 and equation (3.10) we see that

$$\begin{aligned} T_q^{(p)}&(\delta(F*G)_q(y \mid \cdot))(w) \\ &= \delta(F*G)_q(y \mid w) \\ &= \int_{L_2^2[0,T]} \frac{i}{\sqrt{2}}\langle u+v, w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u+v, y\rangle \right. \\ &\qquad \left. - \frac{i(u-v, u-v)}{4q}\right\} df(u)\, dg(v). \end{aligned} \tag{4.2}$$

4.3. *Formulas for the transforms of the convolution product with respect to the first argument of the variations.* Here again there are two cases; namely, we can take the transform of the expressions in equation (3.11) either with respect to y or else with respect to w.

Case 1. Let F, G, p, q, and w be as in Theorem 4.1. Then using equations (3.5) and (3.7) we get that

$$\begin{aligned}(4.3)\quad T_q^{(p)}((\delta F(\cdot \mid w) * \delta G(\cdot \mid w))_q)(y) &= T_q^{(p)}(\delta F(\cdot \mid w))(y/\sqrt{2}) T_q^{(p)}(\delta G(\cdot \mid w))(y/\sqrt{2}) \\ &= \delta T_q^{(p)}(F)(y/\sqrt{2} \mid w) \delta T_q^{(p)}(G)(y/\sqrt{2} \mid w).\end{aligned}$$

Case 2. Let F, G, p, q, and w be as in Theorem 4.1. Then, proceeding as in the proof of equation (3.12) and using (3.11), we obtain

$$\begin{aligned}&\int_{C_0[0,T]}^{anf_q} (\delta F(\cdot \mid w+x) * \delta G(\cdot \mid w+x))_q(y) m(dx) \\ &= \int_{C_0[0,T]}^{anf_q} \int_{L_2^2[0,T]} i\langle u, w+x\rangle i\langle v, w+x\rangle \\ &\quad \cdot \exp\left\{\frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{i(u-v,u-v)}{4q}\right\} df(u)\, dg(v) m(dx) \\ &= -\int_{L_2^2[0,T]} \exp\left\{\frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{i(u-v,u-v)}{4q}\right\} \\ &\qquad \cdot \left[\langle u, w\rangle\langle v, w\rangle + \frac{i}{q}(u,v)\right] df(u)\, dg(v) \\ &= (\delta F(\cdot \mid w) * \delta G(\cdot \mid w))_q(y) \\ (4.4)\quad &\quad - \frac{i}{q} \int_{L_2^2[0,T]} (u,v) \exp\left\{\frac{i}{\sqrt{2}}\langle u+v, y\rangle \right. \\ &\qquad \left. - \frac{i(u-v,u-v)}{4q}\right\} df(u)\, dg(v).\end{aligned}$$

4.4 *The transforms of the convolution product with respect to the second argument of the variations.* Again there are two cases since we can take the transforms of equation (3.12) either with respect to w or with respect to y.

Case 1. Let F, G, p, q, and w be as in Theorem 4.1. Then, using equations (3.5) and (3.9), we obtain

$$\begin{aligned} &T_q^{(p)}((\delta F(y\mid\cdot)*\delta G(y\mid\cdot))_q)(w) \\ &\qquad = T_q^{(p)}(\delta F(y\mid\cdot))(w/\sqrt{2})T_q^{(p)}(\delta G(y\mid\cdot))(w/\sqrt{2}) \\ &\qquad = \delta F(y\mid w/\sqrt{2})\delta G(y\mid w/\sqrt{2}). \end{aligned} \tag{4.5}$$

Case 2. Let F, G, p, q, and w be as in Theorem 4.1. Then

$$\begin{aligned} &\int_{C_0[0,T]}^{anf_q} (\delta F(y+x\mid\cdot)*\delta G(y+x\mid\cdot))_q(w)m(dx) \\ &= \int_{L_2^2[0,T]} \frac{i}{\sqrt{2}}\langle u,w\rangle\frac{i}{\sqrt{2}}\langle v,w\rangle \exp\{i\langle u+v,y\rangle\}) \\ &\qquad \cdot \int_{C_0[0,T]}^{anf_q} \exp\{i\langle u+v,x\rangle\}m(dx)\,df(u)\,dg(v) \\ &\quad + \frac{i}{2q}\int_{L_2^2[0,T]} (u,v)\exp\{i\langle u+v,y\rangle\} \\ &\qquad \cdot \int_{C_0[0,T]}^{anf_q} \exp\{i\langle u+v,x\rangle\}m(dx)\,df(u)\,dg(v) \\ &= -\frac{1}{2}\int_{L_2^2[0,T]} \langle u,w\rangle\langle v,w\rangle \\ &\qquad \cdot \exp\left\{i\langle u+v,y\rangle - \frac{i(u+v,u+v)}{2q}\right\}df(u)\,dg(v) \\ &\quad + \frac{i}{2q}\int_{L_2^2[0,T]} (u,v) \\ &\qquad \cdot \exp\left\{i\langle u+v,y\rangle - \frac{i(u+v,u+v)}{2q}\right\}df(u)\,dg(v). \end{aligned} \tag{4.6}$$

4.5 *The variation of the convolution product of transforms.*

Theorem 4.2. *Let F, G, p, q, and w be as in Theorem 4.1. Then for s-a.e. $y \in C_0[0,T]$,*

$$\begin{aligned}
&\delta(T_q^{(p)}(F) * T_q^{(p)}(G))_{-q}(y \mid w) \\
&\qquad = \delta T_q^{(p)}(F(\cdot/\sqrt{2})G(\cdot/\sqrt{2}))(y \mid w) \\
&\qquad = \int_{L_2^2[0,T]} \frac{i}{\sqrt{2}}\langle u+v, w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u+v, y\rangle \right. \\
&\qquad\qquad \left. - \frac{i(u+v,u+v)}{4q}\right\} df(u)\,dg(v).
\end{aligned} \tag{4.7}$$

Proof. A direct calculation shows that the variation of the sixth expression in equation (3.6) is the third expression in (4.7), and thus (4.7) follows from (3.6). □

4.6 *Formulas for the convolution product of the transform of the variation.* Again there are two cases; namely, we can take the convolution with respect to the first argument or the second argument of the variation. However, in both cases, the transform is taken with respect to the first argument of the variation.

Theorem 4.3. *Let F, G, p, q, and w be as in Theorem 4.1. Then for s-a.e. $y \in C_0[0,T]$,*

$$\begin{aligned}
&(\delta T_q^{(p)}(F)(\cdot \mid w)) * \delta T_q^{(p)}(G)(\cdot \mid w)))_{-q}(y) \\
&\qquad \equiv \int_{C_0[0,T]}^{anf_{-q}} \delta T_q^{(p)}(F)\left(\frac{y+x}{\sqrt{2}}\middle| w\right)\delta T_q^{(p)}(G)\left(\frac{y-x}{\sqrt{2}}\middle| w\right) m(dx) \\
&\qquad = -\int_{L_2^2[0,T]} \langle u, w\rangle\langle v, w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u+v, y\rangle \right. \\
&\qquad\qquad \left. - \frac{i(u+v,u+v)}{4q}\right\} df(u)\,dg(v).
\end{aligned} \tag{4.8}$$

Proof. Using equations (3.7) and (3.8), we see that

$$\int_{C_0[0,T]}^{anf_{-q}} \delta T_q^{(p)}(F)\left(\frac{y+x}{\sqrt{2}}\middle| w\right)\delta T_q^{(p)}(G)\left(\frac{y-x}{\sqrt{2}}\middle| w\right)m(dx)$$

$$= \int_{C_0[0,T]}^{anf_{-q}} \int_{L_2[0,T]} i\langle u,w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u, y+x\rangle - \frac{i(u,u)}{2q}\right\} df(u)$$

$$\cdot \int_{L_2[0,T]} i\langle v,w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle v, y-x\rangle - \frac{i(v,v)}{2q}\right\} dg(v)m(dx)$$

$$= -\int_{L_2^2[0,T]} \langle u,w\rangle\langle v,w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{i[(u,u)+(v,v)]}{2q} + \frac{i(u-v,u-v)}{4q}\right\} df(u)\,dg(v)$$

$$= -\int_{L_2^2[0,T]} \langle u,w\rangle\langle v,w\rangle \exp\left\{\frac{i}{\sqrt{2}}\langle u+v, y\rangle - \frac{i(u+v,u+v)}{4q}\right\} df(u)\,dg(v),$$

as desired. □

Theorem 4.4. *Let F, G, p, q, and w be as in Theorem 4.1. Then for s-a.e. $y \in C_0[0,T]$,*

(4.9)

$$(\delta T_q^{(p)}(F)(y\mid\cdot) * \delta T_q^{(p)}(G)(y\mid\cdot))_{-q}(w)$$

$$\equiv \int_{C_0[0,T]}^{anf_{-q}} \delta T_q^{(p)}(F)\left(y \mid \frac{w+x}{\sqrt{2}}\right)\delta T_q^{(p)}(G)\left(y \mid \frac{w-x}{\sqrt{2}}\right)m(dx)$$

$$= \delta T_q^{(p)}(F)(y\mid w/\sqrt{2})\delta T_q^{(p)}(G)(y\mid w/\sqrt{2})$$

$$= T_q^{(p)}(\delta F(\cdot\mid w/\sqrt{2}))(y)T_q^{(p)}(\delta G(\cdot\mid w/\sqrt{2}))(y)$$

$$- \int_{L_2^2[0,T]} \frac{i}{2q}(u,v) \exp\left\{ i\langle u+v, y\rangle - \frac{i[(u,u)+(v,v)]}{2q} \right\} df(u)\, dg(v).$$

Proof. Using equations (3.7) and (3.8) and then proceeding as in the proof of equation (3.12), we obtain

$$\int_{C_0[0,T]}^{anf_{-q}} \delta T_q^{(p)}(F)\left(y \middle| \frac{w+x}{\sqrt{2}}\right) \delta T_q^{(p)}(G)\left(y \middle| \frac{w-x}{\sqrt{2}}\right) m(dx)$$

$$= \int_{C_0[0,T]}^{anf_{-q}} \int_{L_2[0,T]} \frac{i}{\sqrt{2}}\langle u, w+x\rangle \exp\left\{ i\langle u,y\rangle - \frac{i(u,u)}{2q} \right\} df(u)$$

$$\cdot \int_{L_2[0,T]} \frac{i}{\sqrt{2}}\langle v, w-x\rangle \exp\left\{ i\langle v,y\rangle - \frac{i(v,v)}{2q} \right\} dg(v) m(dx)$$

$$= -\frac{1}{2} \int_{L_2^2[0,T]} \exp\left\{ i\langle u+v, y\rangle - \frac{i}{2q}[(u,u)+(v,v)] \right\}$$

$$\cdot \int_{C_0[0,T]}^{anf_{-q}} [\langle u,w\rangle + \langle u,x\rangle][\langle v,w\rangle - \langle v,x\rangle] m(dx)\, df(u)\, dg(v)$$

$$= -\frac{1}{2} \int_{L_2^2[0,T]} \exp\left\{ i\langle u+v, y\rangle - \frac{i}{2q}[(u,u)+(v,v)] \right\}$$

$$\cdot \left[\langle u,w\rangle\langle v,w\rangle + \frac{i}{q}(u,v) \right] df(u)\, dg(v)$$

$$= \int_{L_2^2[0,T]} \frac{i}{\sqrt{2}}\langle u,w\rangle \exp\left\{ i\langle u,y\rangle - \frac{i(u,u)}{2q} \right\} \frac{i}{\sqrt{2}}\langle v,w\rangle$$

$$\cdot \exp\left\{ i\langle v,y\rangle - \frac{i(v,v)}{2q} \right\} df(u)\, dg(v)$$

$$- \frac{i}{2q} \int_{L_2^2[0,T]} (u,v)$$

$$
\begin{aligned}
&\qquad\qquad \cdot \exp\left\{ i\langle u+v, y\rangle - \frac{i}{2q}[(u,u)+(v,v)]\right\} df(u)\, dg(v) \\
&= \delta T_q^{(p)}(F)(y \mid w/\sqrt{2})\delta T_q^{(p)}(G)(y \mid w/\sqrt{2}) \\
&\quad - \int_{L_2^2[0,T]} \frac{i}{2q}(u,v) \\
&\qquad\qquad \cdot \exp\left\{ i\langle u+v, y\rangle - \frac{i}{2q}[(u,u)+(v,v)]\right\} df(u)\, dg(v)
\end{aligned}
$$

REFERENCES

1. M.D. Brue, *A functional transform for Feynman integrals similar to the Fourier transform*, Thesis, University of Minnesota, 1972.

2. R.H. Cameron, *The first variation of an indefinite Wiener integral*, Proc. Amer. Math. Soc. **2** (1951), 914–924.

3. R.H. Cameron and D.A. Storvick, *An L_2 analytic Fourier-Feynman transform*, Michigan Math. J. **23** (1976), 1–30.

4. ———, *Some Banach algebras of analytic Feynman integrable functionals*, in *Analytic functions*, Lecture Notes in Math. **798** (1980), 18–67.

5. ———, *Feynman integral of variations of functions*, in *Gaussian random fields*, World Scientific, Singapore, 1991, 144–157.

6. K.S. Chang, *Scale-invariant measurability in Yeh-Wiener space*, J. Korean Math. Soc. **19** (1982), 61–67.

7. K.S. Chang, G.W. Johnson and D.L. Skoug, *The Feynman integral of quadratic potentials depending on two time variables*, Pacific J. Math. **122** (1986), 11–33.

8. ———, *Functions in the Banach algebra $\mathcal{S}(v)$*, J. Korean Math. Soc. **24** (1987), 151–158.

9. T. Huffman, C. Park and D. Skoug, *Analytic Fourier-Feynman transforms and convolution*, Trans. Amer. Math. Soc. **347** (1995), 661–673.

10. ———, *Convolution and Fourier-Feynman transforms*, Rocky Mountain J. Math., to appear.

11. ———, *Convolutions and Fourier-Feynman transforms of functionals involving multiple integrals*, Michigan Math. J. **43** (1996), 247–261.

12. G.W. Johnson and D.L. Skoug, *An L_p analytic Fourier-Feynman transform*, Michigan Math. J. **26** (1979), 103–127.

13. ———, *Scale-invariant measurability in Wiener space*, Pacific J. Math. **83** (1979), 157–176.

14. ———, *Notes on the Feynman integral*, II, J. Funct. Anal. **41** (1981), 277–289.

15. C. Park and D. Skoug, *The Feynman integral of quadratic potentials depending on n time variables*, Nagoya Math. J. **110** (1988), 151–162.

16. J. Yeh, *Convolution in Fourier-Wiener transform*, Pacific J. Math. **15** (1965), 731–738.

17. I. Yoo, *Convolution in Fourier-Wiener transform on abstract Wiener space*, Rocky Mountain J. Math. **25** (1995), 1577–1587.

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