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A SURVEY OF RESULTS INVOLVING TRANSFORMS AND CONVOLUTIONS IN FUNCTION SPACE

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A SURVEY OF RESULTS INVOLVING TRANSFORMS AND CONVOLUTIONS IN FUNCTION SPACE

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ABSTRACT. In this paper we survey various results involving Fourier-Wiener transforms, Fourier-Feynman transforms, integral transforms and convolution products of functionals over function space that have been established since Cameron and Martin first introduced Fourier-Wiener transforms in 1945.

1. Introduction. In a 1945 paper [**7**], Cameron defined a *transform* of a function which was somewhat analogous to the *Fourier transform* of a function. Since then, many results based on (or inspired by) this definition have appeared in the literature. In fact, research based on this definition is continuing at the present time; more than 55 years later. Our goal in this survey article is to discuss those results, of which we are aware, whose roots can be traced back to the pioneering work of Cameron and Martin [**7**–**10**].

Let $C_0[0,T]$ denote one-parameter Wiener space; that is, the space of $\mathbf{R}$-valued continuous functions $x(t)$ on $[0,T]$ with $x(0)=0$. Let $\mathcal{M}$ denote the class of all Wiener measurable subsets of $C_0[0,T]$, and let m denote Wiener measure. $(C_0[0,T],\mathcal{M},m)$ is a complete measure space, and we denote the Wiener integral of a Wiener integrable functional F by

$$E[F]=E_x[F(x)]=\int_{C_0[0,T]} F(x)m(dx).$$

Let $L_2(C_0[0,T])$ be the space of $\mathbf{C}$-valued functionals F satisfying

$$\int_{C_0[0,T]} |F(x)|^2 m(dx)<\infty.$$

Let $K=K_0[0,T]$ be the space of all $\mathbf{C}$-valued continuous functions defined on $[0,T]$ which vanish at $t=0$.

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Remark 1.1. In their earlier papers involving Wiener measure, Wiener, Cameron, Martin and others used the density function $(\pi t)^{-1/2} \times \exp\{-u^2/t\}$ to construct Wiener measure instead of the now standard normal density function $(2\pi t)^{-1/2} \exp\{-u^2/(2t)\}$. As a result, in [**7**–**10**], the Gaussian process $x(t)$ had mean zero and covariance function

$$E_x[x(s)x(t)] = \int_{C_0[0,T]} x(s)x(t)m(dx) = \frac{1}{2} \min\{s,t\},$$

whereas in the papers since 1955 or so, using the standard normal density function to construct m, the Gaussian process $x(t)$ has mean zero and covariance function

$$E_x[x(s)x(t)] = \min\{s,t\}.$$

We will always state all of the results for the Wiener space $C_0[0,T]$, although $C_0[0,1]$ was used in [**7**–**10, 36, 52**], $C_0[a,b]$ was used in [**3**–**5, 11, 14, 35**], etc.

2. Fourier transforms. In this section we give a very brief description of the Fourier transform of L_2-functions. Recall that a sequence $\{f_n\}_{n=1}^{\infty}$ of functions in $L_2(\mathbf{R})$ is said to converge to a function f as a limit in the mean if

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f(u) - f_n(u)|^2 \, du = 0,$$

and we write

$$f(u) = \underset{n\to\infty}{\mathrm{l.\,i.\,m.}}\, f_n(u).$$

For $f \in L_2(\mathbf{R})$,

$$\mathcal{F}(f)(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iuv} f(v)\, dv \tag{2.1}$$

and

$$\mathcal{F}^{-1}(f)(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-iuv} f(v)\, dv \tag{2.2}$$

are called the Fourier transform and the inverse Fourier transform of f, respectively. Some authors write a factor $(2\pi)^{-1}$ in front of the inverse transform only, while others write $2\pi i$ in place of i in the exponent. Also some authors define the Fourier transform with a minus sign in the exponent and the inverse transform with a plus sign in the exponent.

It is well known that, for $f \in L_2(\mathbf{R})$,

$$\mathcal{F}(f)(u) = \operatorname*{l.i.m.}_{A\to+\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{iuv} f(v)\, dv$$

exists and is an element of $L_2(\mathbf{R})$. In addition,

$$\mathcal{F}^{-1}(f)(u) = \operatorname*{l.i.m.}_{A\to+\infty} \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{-iuv} f(v)\, dv$$

exists and belongs to $L_2(\mathbf{R})$. In addition, the following properties hold

$$\mathcal{F} \quad \text{maps} \quad L_2(\mathbf{R}) \text{ onto } L_2(\mathbf{R}), \tag{2.3}$$

$$\mathcal{F}(\mathcal{F}(f))(u) = f(-u), \tag{2.4}$$

$$\mathcal{F}^{-1}(\mathcal{F}(f))(u) = f(u) = \mathcal{F}(\mathcal{F}^{-1}(f))(u), \tag{2.5}$$

Plancherel's relation holds in the form

$$\int_{-\infty}^{\infty} |f(u)|^2\, du = \int_{-\infty}^{\infty} |\mathcal{F}(f)(u)|^2\, du, \tag{2.6}$$

and Parseval's relation holds in the form

$$\int_{-\infty}^{\infty} f(u)g(-u)\, du = \int_{-\infty}^{\infty} \mathcal{F}(f)(u)\mathcal{F}(g)(u)\, du, \quad f, g \in L_2(\mathbf{R}). \tag{2.7}$$

3. Fourier-Wiener transforms—Cameron and Martin. In 1945, Cameron [**7**] introduced a transform of functionals defined on $K = K_0[0, T]$ which he called the *Fourier-Wiener transform* since it had many of the same properties as the Fourier transform of a function on the real line.

Definition 3.1. Let $\mathcal{F}$ be a functional which is defined throughout K and is such that $F(x+iy)$ is Wiener integrable in x over $C_0[0,T]$ for each fixed y in K. Then the functional

$$\mathcal{F}(F)(y) = \int_{C_0[0,T]} F(x+iy)m(dx), \quad y \in K \tag{3.1}$$

is called the Fourier-Wiener transform of F.

Remark 3.1. (i) The exponential corresponding to $\exp\{iuv\}$ in formula (2.1) for the Fourier transform doesn't appear in formula (3.1) because it is supplied by the exponentials which are inherent in the definition of the Wiener integral. (ii) Although Cameron didn't formally define the *inverse Fourier-Wiener transform* in [**7**], he used the concept in obtaining a reciprocal relation [**7**, p. 485]; hence we state the following definition.

Definition 3.2. Let F be a functional which is defined throughout K and is such that $F(x-iy)$ is Wiener integrable in x over $C_0[0,T]$ for each fixed y in K. Then the functional

$$\mathcal{F}^{-1}(F)(y) = \int_{C_0[0,T]} F(x-iy)m(dx) = \mathcal{F}(F)(-y), \quad y \in K \tag{3.2}$$

is called the inverse Fourier-Wiener transform of F.

In [**8**], Cameron and Martin defined three large classes of functionals on K and showed that if F is a member of any of these classes, then $\mathcal{F}(F)$ exists and belongs to the same class. In addition, they showed that

$$\mathcal{F}(\mathcal{F}(F))(y) = F(-y) \quad \text{for all } y \in K, \tag{3.3}$$

and that

$$\mathcal{F}^{-1}(\mathcal{F}(F))(y) = F(y) \quad \text{for all } y \in K. \tag{3.4}$$

Furthermore, they established the following form of Plancherel's relation,

$$\int_{C_0[0,T]} |F(x/\sqrt{2}\,)|^2 m(dx) = \int_{C_0[0,T]} |\mathcal{F}(F)(y/\sqrt{2}\,)|^2 m(dy), \tag{3.5}$$

as well as the following form of Parseval's relation,

$$\begin{aligned} &\int_{C_0[0,T]} F_1(x/\sqrt{2}\,)F_2(-x/\sqrt{2}\,)m(dx) \\ &\qquad = \int_{C_0[0,T]} \mathcal{F}(F_1)(y/\sqrt{2}\,)\mathcal{F}(F_2)(y/\sqrt{2}\,)m(dy). \end{aligned} \tag{3.6}$$

Next, we briefly describe two of the three classes of functionals considered in [**8**]. The first class E_m is the class of functionals $F : K \to \mathbf{C}$ which are mean continuous, i.e., continuous in the Hilbert topology, entire, and of mean exponential type. That is to say, E_m is the class of all functionals $F : K \to \mathbf{C}$ satisfying the three conditions

(3.7)

$$\|z_n - z\|_2 \to 0 \Longrightarrow F(z_n) \to F(z) \quad \text{where } \|z\|_2^2 = \int_0^T z(s)\bar{z}(s)\,ds,$$

$$F(z+\lambda y) \text{ is an entire function of } \lambda \text{ for all } (z,y) \in K \times K, \tag{3.8}$$

and there exist positive constants A and B only depending on F, such that

$$|F(z)| \le A\exp\{B\|z\|_2\} \quad \text{for all } z \in K. \tag{3.9}$$

The second class E_0 consists of functionals on K of the form

$$F(x) = \Phi\biggl(\int_0^T \alpha_1(t)\,dx(t), \dots, \int_0^T \alpha_n(t)\,dx(t)\biggr) \tag{3.10}$$

where $\Phi(z_1, \dots, z_n)$ is an entire function of exponential type satisfying

$$|\Phi(z_1, \dots, z_n)| \le Ae^{B(|z_1|+\cdots+|z_n|)}$$

and $\alpha_1(t), \dots, \alpha_n(t)$ are n linearly independent **R**-valued functions of bounded variation on $[0,T]$.

In [**8**], Cameron and Martin first showed that (3.3)–(3.6) hold for all functionals in E_0. Then, using the fact that the elements of E_0 are dense in E_m, they proceeded to show that (3.3)–(3.6) hold for all functionals in E_m.

In [**10**], using the Fourier-Hermite development from [**9**], Cameron and Martin showed that the functionals in E_m above are dense in $L_2(C_0[0,T])$. Then, in order to simplify Plancherel's relation and Parseval's relation, they modified their definition of the Fourier-Wiener transform slightly by letting

$$\mathcal{F}(F)(y) = \int_{C_0[0,T]} F(\sqrt{2}\,x + iy)m(dx), \quad y \in K. \tag{3.11}$$

They proceeded to show that, for $F \in L_2(C_0[0,T])$, the modified Fourier-Wiener transform $\mathcal{F}(F)$ exists and is an element of $L_2(C_0[0,T])$. Furthermore, they showed that equations (3.3) and (3.4) are valid provided that

$$\mathcal{F}^{-1}(F)(y) = \mathcal{F}(F)(-y) = \int_{C_0[0,T]} F(\sqrt{2}\,x - iy)m(dx).$$

In addition, they established Plancherel's relation

$$\int_{C_0[0,T]} |F(x)|^2 m(dx) = \int_{C_0[0,T]} |\mathcal{F}(F)(y)|^2 m(dy), \tag{3.12}$$

and Parseval's relation

$$\int_{C_0[0,T]} F_1(x)F_2(-x)m(dx) = \int_{C_0[0,T]} \mathcal{F}(F_1)(y)\mathcal{F}(F_2)(y)m(dy) \tag{3.13}$$

for all F, F_1 and F_2 in $L_2(C_0[0,T])$.

In Theorem 2 of [**51**], Segal wrote down the expression

$$F(y) = \int_{\mathcal{H}'} f(\sqrt{2}\,x + iy)\,dN(x)$$

which is an abstraction to a Hilbert space setting of equation (3.11) above. In fact, in footnote (5) [**51**, p. 120], Segal commented that Theorem 2 as well as preliminary work on "integration over Hilbert space" was an abstraction of work of Wiener and of Cameron and Martin relating to Brownian motion, and a rigorization of work of Feynman relating to quantum field theory. This seminal paper by Segal, influenced by [**10**], was very influential in later work by Leonard Gross, M. Ann Piech, Takeyuki Hida, Hui-Hsiung Kuo, Yuh-Jia Lee and many others concerning integration over Hilbert spaces, stochastic processes, abstract Wiener spaces, white noise and other related topics.

4. Further results involving the Fourier-Wiener transform. In 1965, Yeh [**52**] defined the convolution product of two functionals, F_1 and F_2, on K by the formula

$$(F_1 * F_2)(y) = \int_{C_0[0,T]} F_1\left(\frac{x+y}{\sqrt{2}}\right) F_2\left(\frac{x-y}{\sqrt{2}}\right) m(dx), \quad y \in K \tag{4.1}$$

whenever it exists. Yeh showed that, if F_1 and F_2 were both in E_m or if F_1 and F_2 were both in E_0, then $(F_1 * F_2)(y)$ exists for every $y \in K$ and satisfies the relationship

$$\mathcal{F}((F_1 * F_2))(y) = \mathcal{F}(F_1)(y/\sqrt{2}\,)\mathcal{F}(F_2)(-y/\sqrt{2}\,) \tag{4.2}$$

for all $y \in K$. As far as we know, this was the first result connecting the Fourier-Wiener transform and the convolution product in function space.

In 1995, Yoo [**53**] extended Yeh's results to abstract Wiener spaces. He also obtained Plancheral's relation and Parseval's relation corresponding to (3.5) and (3.6) for abstract Wiener spaces.

In his Ph.D. thesis [**3**], written under the direction of Cameron, Bridgeman extended the results in [**8**] to larger classes of functionals. In addition, for appropriate functionals F and G defined on K, he established the formula

$$\begin{aligned} &\int_{C_0[0,T]} F\left(\frac{x+iy}{\sqrt{2}}\right) G\left(\frac{x+iy}{\sqrt{2}}\right) m(dx) \\ &\qquad = \int_{C_0[0,T]} \mathcal{F}(F)\left(\frac{y-x}{\sqrt{2}}\right) \mathcal{F}(G)\left(\frac{y+x}{\sqrt{2}}\right) m(dx) \end{aligned} \tag{4.3}$$

for all $y \in K$, that is to say,

$$\mathcal{F}(F(\cdot/\sqrt{2}\,)G(\cdot/\sqrt{2}\,))(y) = (\mathcal{F}(F) * \mathcal{F}(G))(y), \quad y \in K \tag{4.4}$$

where the convolution product of the functionals F and G was defined by the formula

$$(F * G)(y) = \int_{C_0[0,T]} F\left(\frac{y-x}{\sqrt{2}}\right) G\left(\frac{y+x}{\sqrt{2}}\right) m(dx), \quad y \in K. \tag{4.5}$$

Note that, for the convolution product used by Bridgeman, $(F * G)(y) = (G * F)(y)$, whereas for the convolution product used by Yeh and Yoo, see equation (4.1), $(F * G)(y) = (G * F)(-y)$.

In [**3**], Bridgeman also obtained equations (3.3) through (3.6) for his extended classes of functionals on K.

In his 1977 Ph.D. thesis at the University of Minnesota written under the direction of Cameron and Storvick, Caldwell [**5**] worked with the modified Fourier-Wiener transform [**10**] given by equation (3.11) above and obtained several interesting results. He established a translation formula for the Fourier-Wiener transform and used it to solve a difference equation. He also obtained some results in which he combined the concept of the "Fourier-Wiener transform of a functional" with the concept of the "first variation of a functional" [**6**]. In addition, Caldwell extended various results of [**3, 8, 10**] for functionals of several Wiener variables. He also studied a class, E_u, of functionals based on the uniform topology on $C_0^n[0,T]$; recall that E_m, discussed in Section 3 above, was based on the Hilbert topology on $C_0[0,T]$. Bridgeman [**3**] had earlier studied the class E_u for the case $n = 1$.

In Chapter 9 of [**26**], Hida gave an informative discussion of some of the properties of the modified Fourier-Wiener transform given by Cameron and Martin in [**10**]. He then used this transform to do Fourier analysis on various Hilbert spaces of functionals.

In papers [**41**] and [**42**], Lee presented a number of applications of the Fourier-Wiener transform to the study of differential equations on infinite dimensional spaces. He defined the Fourier-Wiener transform on the class of exponential type analytic functions in the setting of abstract Wiener space and obtained theorems giving existence, uniqueness and regularity of solutions for the Cauchy problem associated with several equations.

In a general setting Kuo [**40**] established several results involving the Fourier-Wiener transform of Brownian functionals. He then used these results to solve a differential equation. The basic idea, also used by Lee [**41, 42**] is to take the Fourier-Wiener transform of the differential equation and then solve the resulting equation.

5. Fourier-Feynman transforms through 1985. In his Ph.D. thesis [**4**], written under the direction of Cameron, Brue introduced the concept of an L_1 analytic Fourier-Feynman transform. In [**11**], Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform. In [**35**], Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \leq p \leq 2$ which extended the results in [**4, 11**] and gave various relationships between the L_1 and the L_2 theories.

Because of the measurability problems, in [**11**] and in [**35**] all of the functionals F on Wiener space and all of the functions f on $\mathbf{R}^n$ were assumed to be Borel measurable. Unfortunately, one cannot avoid all scale change pathologies by restricting attention to Borel measurable functionals F. In [**11**, pp. 5–7], Cameron and Storvick exhibit two Borel measurable functionals F and G which agree except on a Wiener null set and yet their Fourier-Feynman transforms are unequal almost everywhere on Wiener space. In [**36**, p. 170] (incidentally, much of the motivation for writing the manuscript [**36**] was the measurability problems encountered in [**11, 35**]), Johnson and Skoug pointed out that the concept of scale-invariant measurability in Wiener space together with Lebesgue measurability in $\mathbf{R}^n$ is precisely correct for the analytic Fourier-Feynman transform theory. Thus, in this survey, we will phrase the results of [**11, 35**] in the context of scale-invariant measurability.

A subset E of $C_0[0,T]$ is said to be scale-invariant measurable, s.i.m., [**36**] provided $\rho E \in \mathcal{M}$ for each $\rho > 0$, and an s.i.m. set N is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.s.). If two functionals F and G are equal s-a.e., we write $F \approx G$.

Let $\mathbf{C}, \mathbf{C}_+$ and $\mathbf{C}_+^\sim$ denote respectively the complex numbers, the complex numbers with positive real part and the nonzero complex numbers with nonnegative real part. Let F be a $\mathbf{C}$-valued scale-invariant measurable functional on $C_0[0,T]$ such that

$$J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-1/2}x)m(dx)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbf{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0,T]$ with parameter λ and for $\lambda \in \mathbf{C}_+$ we write

$$\int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx) = J^*(\lambda).$$

Let $q \neq 0$ be a real number, and let F be a functional such that

$$\int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx)$$

exists for all $\lambda \in \mathbf{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q and we write

$$\int_{C_0[0,T]}^{anf_q} F(x)m(dx) = \lim_{\lambda \to -iq} \int_{C_0[0,T]}^{anw_\lambda} F(x)m(dx)$$

where $\lambda \to -iq$ through $\mathbf{C}_+$.

Notation. (i) For $\lambda \in \mathbf{C}_+$ and $y \in C_0[0,T]$, let

$$T_\lambda(F)(y) = \int_{C_0[0,T]}^{anw_\lambda} F(x+y)m(dx). \tag{5.1}$$

(ii) Given a number p with $1 \leq p \leq +\infty$, p and p' will always be related by $1/p + 1/p' = 1$.

(iii) Let $1 < p \leq 2$, and let $\{H_n\}$ and H be scale-invariant measurable functionals such that, for each $\rho > 0$,

$$\lim_{n\to\infty} \int_{C_0[0,T]} |H_n(\rho y) - H(\rho y)|^{p'} m(dy) = 0. \tag{5.2}$$

Then we write

$$\operatorname*{l.\,i.\,m.}_{n\to\infty}(w_s^{p'})(H_n) \approx H \tag{5.3}$$

and we call H the scale invariant limit in the mean of order p'. A similar definition is understood when n is replaced by the continuously varying parameter λ. Next we state the definition of the L_p analytic Fourier-Feynman transform [**35**].

Let $q \neq 0$ be a real number. For $1 < p \leq 2$ we define the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F by the formula ($\lambda \in \mathbf{C}_+$)

$$T_q^{(p)}(F)(y) = \operatorname*{l.\,i.\,m.}_{\lambda\to -iq}(w_s^{p'})(T_\lambda(F)(y)) \tag{5.4}$$

whenever this limit exists. We define the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F by the formula

$$T_q^{(1)}(F)(y) = \lim_{\lambda\to -iq}(T_\lambda(F))(y) \tag{5.5}$$

for s-a.e. y. We note that, for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e. We also note that if $T_q^{(p)}(F_1)$ exists and, if $F_1 \approx F_2$, then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_2) \approx T_q^{(p)}(F_1)$.

Remarks. (i) In view of (5.4) it would seem natural and desirable to define $T_q^{(1)}(F)$ by requiring that, for each $\rho > 0$,

$$\lim_{\lambda\to -iq}\Big[\operatorname{ess}\sup_{y\in C_0[0,T]}|T_\lambda(F)(\rho y) - T_q^{(1)}(F)(\rho y)|\Big] = 0. \tag{5.6}$$

Unfortunately, (5.6) doesn't even hold for any $\rho > 0$ for a functional as simple as $F(x) = \chi_{[-1,1]}(x(T))$.

(ii) $T_q^{(2)}(F)$ agrees with the L_2 analytic Fourier-Feynman transform as given by Cameron and Storvick in [**11**].

(iii) The definition of $T_q^{(1)}(F)$ given above by (5.5) is more restrictive than that given by Brue in [**4**] in that (5.5) must hold s-a.e. rather than just a.e. However all of Brue's results actually hold in this stronger sense.

In [**4**] Brue showed the existence of $T_q^{(1)}(F)$ for F in several classes of functionals on Wiener space. He also showed that $T_{-q}^{(1)}(T_q^{(1)}(F))(y) = F(y)$ for a.e. y in Wiener space. Actually he only considered the case $q = 1$, but clearly his results are valid for all real $q \neq 0$. At the end of his thesis, Brue includes a nice collection of examples.

In [**11**] Cameron and Storvick obtained the existence of $T_q^{(2)}(F)$ for several large classes of functionals F on Wiener space. In particular, they showed that if $\Phi(z) = \sum_{n=0}^{\infty} a_n z^n$ is an entire function of order less than four and if

$$F(x) = \Phi\left[\int_0^T \theta(s, x(s))\, ds\right] \tag{5.7}$$

s-a.e. with $\|\theta(t,\cdot)\|_2 \in L_2[0,T]$, then $T_q^{(2)}(F)$ exists and $T_{-q}^{(2)}(T_q^{(2)}(F)) \approx F$ for all real $q \neq 0$. In particular, note that

$$F(x) = \exp\left\{\int_0^T \theta(s, x(s))\, ds\right\}$$

is of the desired form (5.7).

In [**14**] Cameron and Storvick used the definition of the sequential Feynman integral [**13**] to define a sequential Fourier-Feynman transform.

6. Fourier-Feynman transforms and convolution products since 1990. Except for paper [**44**] which we will discuss in Section 7 below, we aren't aware of any papers in the literature which mention the Fourier-Feynman transform after paper [**14**] which appeared in 1985 and before paper [**28**] which was written in 1993 and appeared in 1995.

A major goal of the authors in [**28**] was to define a convolution product of functionals on Wiener space in such a way that the Fourier-Feynman transform of the convolution product was equal to the product of the Fourier-Feynman transforms, i.e., to define $(F * G)q$ in such a way that the equation

$$T_q^{(p)}((F * G)_q)(y) = T_q^{(p)}(F)(y/\sqrt{2}\,)T_q^{(p)}(G)(y/\sqrt{2}\,) \tag{6.1}$$

would hold under reasonable restrictions on F and G.

Another goal in [**28**], as well as in [**29**–**31**] was to find useful classes of functionals satisfying equation (6.1). In particular, one needs to find conditions on F and G guaranteeing the existence of both $(F*G)_q$ and $T_q^{(p)}((F*G)_q)$. Note that once one knows that equation (6.1) holds for all functionals F and G in some class A, then equation (6.1) allows us to find $T_q^{(p)}((F*G)_q)$ for all F and G in A without actually calculating $(F*G)_q$. In practice, $T_q^{(p)}(F)$ and $T_q^{(p)}(G)$ are usually much easier to calculate than $(F*G)_q$ and $T_q^{(p)}((F*G)_q)$.

In [**28**], Huffman, Park and Skoug define the convolution product, if it exists, of functionals F and G on $C_0[0,T]$ for $\lambda \in \mathbf{C}_+^\sim$ by the formulas

$$(6.2) \qquad (F*G)_\lambda(y) = \int_{C_0[0,T]}^{anw_\lambda} F\biggl(\frac{y+x}{\sqrt{2}}\biggr) G\biggl(\frac{y-x}{\sqrt{2}}\biggr) m(dx)$$

for $\lambda \in \mathbf{C}_+$ and

(6.3)
$$(F*G)_q(y) \equiv (F*G)_{-iq}(y) = \int_{C_0[0,T]}^{anf_q} F\biggl(\frac{y+x}{\sqrt{2}}\biggr) G\biggl(\frac{y-x}{\sqrt{2}}\biggr) m(dx)$$

for $\lambda = -iq \in \mathbf{C}_+^\sim$. When $\lambda = 1$, this definition agrees with the definition used by Bridgeman [**3**]. It is different than the definition used by Yeh [**52**] and Yoo [**53**] where, for $\lambda = 1$, they let

$$(F*G)(y) = \int_{C_0[0,T]} F\biggl(\frac{x+y}{\sqrt{2}}\biggr) F\biggl(\frac{x-y}{\sqrt{2}}\biggr) m(dx) = (G*F)(-y).$$

Next we describe the class of functionals $A_n^{(p)}$ studied in [**28**]. Let $\{\alpha_1, \dots, \alpha_n\}$ be an orthonormal set of functions in $L_2[0,T]$. For $1 \le p \le \infty$, let $A_n^{(p)}$ be the set of all functionals F on $C_0[0,T]$ of the form

$$(6.4) \qquad F(x) = f\biggl(\int_0^T \alpha_1(s)\,dx(s), \dots, \int_0^T \alpha_n(s)\,dx(s)\biggr)$$

s-a.e. where $f : \mathbf{R}^n \to \mathbf{R}$ is in $L_p(\mathbf{R}^n)$. Let $A_n^{(\infty)}$ be the space of all functionals of the form (6.4) with $f \in C_0(\mathbf{R}^n)$, the space of all bounded continuous functionals on $\mathbf{R}^n$ that vanish at infinity. Note that $F \in A_n^{(p)}$ implies that F is s.i.m..

Following are some of the results established in [**28**]:

1. If $F \in A_n^{(p)}$, then $T_{-q}^{(p)}(T_q^{(p)}(F)) \approx F$ for all real $q \neq 0$.

2. Let F and G be elements of $\cup_{1 \le p \le \infty} A_n^{(p)}$. Then, for all $\lambda \in \mathbf{C}_+$, see equation (5.1) above,

$$T_\lambda((F * G)_\lambda)(y) = T_\lambda(F)(y/\sqrt{2}\,)T_\lambda(G)(y/\sqrt{2}\,) \qquad \text{for s-a.e. } y \in C_0[0,T].$$

3. Let F and G be elements of $A_n^{(1)}$. Then, for all real $q \neq 0$,

$$T_q^{(1)}((F * G)_q)(y) = T_q^{(1)}(F)(y/\sqrt{2}\,)T_q^{(1)}(G)(y/\sqrt{2}\,) \qquad \text{for s-a.e. } y \in C_0[0,T].$$

4. Let $F \in A_n^{(1)}$ and $G \in A_n^{(2)}$. Then, for all real $q \neq 0$,

$$T_q^{(2)}((F * G)_q)(y) = T_q^{(1)}(F)(y/\sqrt{2}\,)T_q^{(2)}(G)(y/\sqrt{2}\,) \qquad \text{for s-a.e. } y \in C_0[0,T].$$

In [**39**], Kim, Chang and Yoo generalized the results in [**28**] to a class of cylinder functionals on an abstract Wiener space.

Next we describe the class of functionals $A = A_{pr}$ that Huffman, Park and Skoug worked with in [**29**]. For $p \in [1,2]$ and $r \in ((2p/2p-1), +\infty]$, let $L_{pr}([0,T] \times \mathbf{R})$ be the space of all $\mathbf{C}$-valued Lebesgue measurable functions f on $[0,T] \times \mathbf{R}$ such that $f(t,\cdot)$ is in $L_p(\mathbf{R})$ for almost all $t \in [0,T]$ and, as a function of t, $\|f(t,\cdot)\|_p$ is in $L_r([0,T])$. Then $A = L_{pr}$ is the class of all functionals F such that, for some $f \in L_{pr}([0,T] \times \mathbf{R})$,

$$F(x) = \exp\bigg\{ \int_0^T f(t, x(t))\, dt \bigg\}, \tag{6.5}$$

for s-a.e. x in $C_0[0,T]$. Then F is defined s-a.e. and is s.i.m..

The main result in [**29**] is that equation (6.1) holds for all functionals F and G in A_{pr}, a class of functionals which arise naturally in quantum mechanics.

In [**30**], Huffman, Park and Skoug considered functionals of the form

$$F(x) = \exp\left\{ \int_0^T \int_0^T f(s,t,x(s),x(t))\,ds\,dt \right\}. \tag{6.6}$$

Feynman [**25**] obtained such functionals by formally integrating out the oscillator coordinates in a system involving a harmonic oscillator interacting with a particle moving in a potential. Moreover, functionals similar to those in (6.6) but involving multiple integrals of more time dimensions than two arise when more particles are involved. In [**30**], the authors showed that equation (6.1) holds for such a class of functionals where $f : [0,T]^n \times \mathbf{R}^n \to \mathbf{C}$ is quadratic in the space variables.

The Banach algebra S of functionals on $C_0[0,T]$, each of which is a type of stochastic Fourier transform of a bounded $\mathbf{C}$-valued Borel measure, was introduced by Cameron and Storvick in [**12**]. The Banach algebra S consists of functionals of the form

$$F(x) = \int_{L_2[0,T]} \exp\{i\langle v,x\rangle\}\,df(v) \tag{6.7}$$

for s-a.e. $x \in C_0[0,T]$, where f is an element of $M(L_2[0,T])$, the space of $\mathbf{C}$-valued countably additive Borel measures on $L_2[0,T]$ and $\langle v,x\rangle$ denotes the Paley-Wiener-Zygmund stochastic integral $\int_0^T v(s)\,dx(s)$.

Let $L_{1\infty}([0,T]^2 \times \mathbf{R}^2)$ be the space of all $\mathbf{C}$-valued Lebesgue measurable functions f on $[0,T]^2 \times \mathbf{R}^2$ such that $f(s,t,\cdot,\cdot)$ is in $L_1(\mathbf{R}^2)$ for a.e. $(s,t) \in [0,T]^2$ and as a function of s,t, $\|f(s,t,\cdot,\cdot)\|_1$ is in $L_\infty([0,T]^2)$. We define $\mathcal{A}$ to be the class of all functionals F such that for some $f \in L_{1\infty}([0,T]^2 \times \mathbf{R}^2)$, $F(x)$ is given by equation (6.6) for s-a.e. $x \in C_0[0,T]$.

In Section 4 of [**30**], Huffman, Park and Skoug showed that equation (6.1) holds for all F and G in $\mathcal{A}$. In Section 3 they showed that (6.1) holds for all F and G in S. In addition, they established Parseval's relation

$$\begin{aligned} \int_{C_0[0,T]}^{anf_{-q}} & T_q^{(p)}(F)(y/\sqrt{2})T_q^{(p)}(G)(y/\sqrt{2})m(dy) \\ &= \int_{C_0[0,T]}^{anf_q} F(y/\sqrt{2})G(-y/\sqrt{2})m(dy). \end{aligned} \tag{6.8}$$

In [**48**], Park and Skoug for functionals on $C_0[0,T]$ of the form

$$F_s(x) = \exp\left\{ \int_s^T f(t, x(t))\, dt \right\} \varphi(x(T))$$

established various relationships involving Fourier-Feynman transforms and convolutions. In addition, they showed that the L_1 analytic Fourier-Feynman transform of F_s satisfies a Feynman integral equation formally equivalent to the Schroedinger equation.

Let $Q = \{(s,t) : 0 \le s \le b, 0 \le t \le \beta\}$ and let $C_2[Q]$, often called Yeh-Wiener space, denote the Wiener space of functionals of two variables over Q, that is to say,

$$C_2[Q] = \{x(s,t) : x(0,t) = x(s,0) = 0 \text{ and } x(s,t) \text{ is continuous on } Q\}.$$

In his Ph.D. thesis [**27**], written under the direction of Skoug, Huffman defined an L_p analytic Yeh-Feynman-Fourier transform $T_q^p(F)$ and a convolution product $(F * G)_q$ for s.i.m. functionals on $C_2[Q]$. For several large classes of functionals on $C_2[Q]$, Huffman showed that $T_q^p(F)$ exists, that $T_{\bar\lambda}(T_\lambda(F))(y) \to F(y)$ s-a.e. on $C_2[Q]$ as $\lambda \to -iq$ through values in $\mathbf{C}^+$, that the Yeh-Feynman-Fourier transform of the convolution product is the product of their Yeh-Feynman-Fourier transforms and that Parseval's relation holds in the form

$$\int_{C_2[Q]}^{anf_{-q}} T_q^p(F)(x/\sqrt{2}\,) T_q^p(G)(-x/\sqrt{2}\,) dm(x) = \int_{C_2[Q]}^{anf_q} F(x/\sqrt{2}\,) G(x/\sqrt{2}\,) dm(x).$$

In [**31**], using ideas from [**24**], Huffman, Park and Skoug defined a generalized Fourier-Feynman transform (also denoted by $T_q^{(p)}(F)$) and a generalized convolution product by replacing equations (5.1) and (6.2) above with the equations

$$T_\lambda(F)(y) = \int_{C_0[0,T]}^{anw_\lambda} F(y + z(x,\cdot)) m(dx), \tag{6.9}$$

and

$$(F * G)_\lambda(y) = \int_{C_0[0,T]}^{anw_\lambda} F\left(\frac{y + z(x,\cdot)}{\sqrt{2}} \right) g\left(\frac{y - z(x,\cdot)}{\sqrt{2}} \right) m(dx), \tag{6.10}$$

respectively, where, for $h \in L_2[0,T]$, the Gaussian process

$$z(x,t) \equiv \int_0^t h(s)\,dx(s)$$

has mean zero and covariance function $a(\min\{s,t\})$ with

$$a(t) \equiv \int_0^t h^2(u)\,du. \tag{6.11}$$

Then for two classes of functionals on $C_0[0,T]$, they showed that the generalized transform of the generalized convolution product is a product of their generalized transforms. In addition, they obtained Parseval's relation

$$\begin{aligned}\int_{C_0[0,T]}^{anf_{-q}} T_q^{(p)}(F)\Big(\frac{z(x,\cdot)}{\sqrt{2}}\Big)T_q^{(p)}(G)\Big(\frac{z(x,\cdot)}{\sqrt{2}}\Big)m(dx)\\ = \int_{C_0[0,T]}^{anf_q} F\Big(\frac{z(x,\cdot)}{\sqrt{2}}\Big)G\Big(\frac{-z(x,\cdot)}{\sqrt{2}}\Big)m(dx).\end{aligned}$$

In [**33**], Huffman, Skoug and Storvick, using a general Fubini theorem from [**32**], established several Feynman integration formulas involving Fourier-Feynman transforms. The conditions they put on each functional F are very minimal; namely, that

(i) $F : C_0[0,T] \to \mathbf{C}$ is defined s-a.e. and is s.i.m.,

(ii) $\int_{C_0[0,T]} |F(\rho x)|m(dx) < \infty$ for each $\rho > 0$, and that

(iii) $\int_{C_0[0,T]}^{anf_q} F(x)m(dx)$ exists for all real $q \neq 0$.

Following are some special cases of results from [**33**] ($q_1 + q_2 \neq 0$):

$$\begin{aligned}\int_{C_0[0,T]}^{anf_{q_1}} T_{q_2}^{(1)}(F)(y)m(dy) &= \int_{C_0[0,T]}^{anf_{q_2}} T_{q_1}^{(1)}(F)y(m)(dy),\\ T_{q_2}^{(1)}(T_{q_1}^{(1)}(F))(y) &= T^{(1)}_{q_1q_2/q_1+q_2}(F)(y)\\ &= T_{q_1}^{(1)}(T_{q_1}^{(1)}(F))(y) \quad \text{s-a.e.},\\ T_q^{(1)}(T_q^{(1)}(F))(y) &= T_{q/2}^{(1)}(F)(y)\\ &= \int_{C_0[0,T]}^{anf_q} F(y+\sqrt{2}x)m(dx) \quad \text{s-a.e.},\end{aligned}$$

and, under the additional assumption that F is a continuous functional,

$$T_{-q}^{(1)}(T_q^{(1)}(F))(y) = F(y) \quad \text{s-a.e.}$$

In [**1**], Ahn defined the L_1 analytic Fourier-Feynman transform for functionals in the Fresnel class $F(B)$ of an abstract Wiener space B and, for these functionals established equation (6.1) in the case $p = 1$. In [**16**], Chang, Kim and Yoo generalized the results of [**1, 27**] to a larger class of functionals than the Fresnel class $F(B)$.

On pages 609–637 of [**34**], Johnson and Lapidus give a detailed discussion of the Fresnel integral and various Fresnel classes of functionals. In particular, their discussion includes many references.

7. Integral transforms. We begin this section with a very brief description of an abstract Wiener space.

Let H be a real separable infinite-dimensional Hilbert space with inner product $\langle \cdot , \cdot \rangle$ and norm $\| \cdot \|$. Let $||| \cdot |||$ be a measurable norm on H with respect to the Gaussian cylinder set measure σ on H, i.e.,

$$\sigma(E) = (2\pi)^{-n/2} \int_E \exp\left\{ - \frac{|x|^2}{2} \right\} dx.$$

Let B denote the completion of H with respect to $||| \cdot |||$. Let i denote the natural injection from H into B. The adjoint operator i^* of i is one-to-one and maps B^* continuously onto a dense subset of H^*. By identifying H with H^* and B^* with $i^* B^*$, we have a triple $B^* \subset H^* \equiv H \subset B$ and $\langle y, x \rangle = (y, x)$ for all $y \in H$ and x in B^* where $(\cdot, \cdot)$ denotes the natural dual pairing between B and B^*. It is well known that $\sigma \circ i^{-1}$ has a unique countably additive extension v to the Borel σ-algebra $\mathcal{B}(B)$ of B. The triple (H, B, v) is called an abstract Wiener space and the Hilbert space H is called the generator of (H, B, v). For more detail, see [**1, 16–18, 23, 34, 39–44, 53**] and the references in these papers which specifically refer to abstract Wiener spaces.

In a unifying paper [**43**], Lee defined an integral transform $\mathcal{F}_{\alpha,\beta}$ of analytic functionals on abstract Wiener spaces. For certain values of the parameters α and β and for certain classes of functionals, the

Fourier-Wiener transform, the Fourier-Feynman transform and the Gauss transform are special cases of Lee's integral transform $\mathcal{F}_{\alpha,\beta}$.

Let (H, B, v) be an abstract Wiener space. Let $\mathcal{E}_a$ be the class of functionals F defined on the complexification $[B]$ of B with

$$|F(z)| \leq c\exp\{d\sqrt{|\mathrm{Re}\, z|^2 + |\mathrm{Im}\, z|^2}\,\} \tag{7.1}$$

for some positive constants c and d only depending on F, and

(7.2) $F(x + \lambda y)$ is an entire function of λ throughout $\mathbf{C}$ for all $(x, y) \in [B] \times [B]$.

Then, for each pair of nonzero complex numbers α and β, Lee defines his integral transform $\mathcal{F}_{\alpha,\beta}$ on $[B]$ by

$$\mathcal{F}_{\alpha,\beta}F(y) = \int_B F(\alpha x + \beta y)v(dx), \quad y \in [B] \tag{7.3}$$

if it exists. When $\alpha = i$ and $\beta = 1$, $\mathcal{F}_{i,1}F$ is the Gauss transform of F. When $B = C_0[0, T]$, $\alpha = 1$ and $\beta = i$, $\mathcal{F}_{1,i}F$ is the Fourier-Wiener transform [**8**] given by equation (3.1) above, $\mathcal{F}_{\sqrt{2},i}F$ is the modified Fourier-Wiener transform [**10**] given by equation (3.11) above and, when $\alpha = (-iq)^{-1/2}$, $q > 0$ and $\beta = 1$, $\mathcal{F}_{\alpha,\beta}F$ is the Fourier-Feynman transform $T_q^{(2)}(F)$ [**11**].

Among several results in [**43**], Lee showed that $\mathcal{F}_{\alpha,\beta}(\mathcal{E}_a) = \mathcal{E}_a$ and that

$$\mathcal{F}_{\alpha',\beta'}(\mathcal{F}_{\alpha,\beta}F)(z) = F(z) \tag{7.4}$$

for all F in $\mathcal{E}_a$ if and only if

$$\beta\beta' = 1 \quad \text{and} \quad (\beta\alpha')^2 + \alpha^2 = 0. \tag{7.5}$$

In [**53**], Yoo defined a convolution product for the case $\beta = i$, $\beta' = -i$ and $\alpha = 1 = \alpha'$, see equation (4.1) above, for functionals on $[B]$ and showed that, for $y \in [B]$,

$$\mathcal{F}_{1,i}(F * G)(y) = \mathcal{F}_{1,i}F(2/\sqrt{2}\,)\mathcal{F}_{1,i}G(-y/\sqrt{2}\,). \tag{7.6}$$

In [**16**], Chang, Kim and Yoo using the integral transform (7.3) and the convolution product, see equation (4.5) above, defined for $y \in [B]$ by the formula

$$(F * G)_\alpha(y) = \int_B F\left(\frac{y+\alpha x}{\sqrt{2}}\right) G\left(\frac{y-\alpha x}{\sqrt{2}}\right) v(dx), \tag{7.7}$$

generalized the results of Yeh [**52**] and Yoo [**53**]. In particular, for F and G in $\mathcal{E}_a$, they showed that

$$\mathcal{F}_{\alpha,\beta}(F * G)_\alpha(y) = \mathcal{F}_{\alpha,\beta}F(y/\sqrt{2}\,)\mathcal{F}_{\alpha,\beta}G(y/\sqrt{2}\,), \quad y \in [B]. \tag{7.8}$$

Among several results in [**44**], Lee showed that Plancherel's relation

$$\int_B |\mathcal{F}_{\alpha,\beta}F(y)|^2 v(dy) = \int_B |F(y)|^2 v(dy)$$

holds if and only if $\alpha^2 + \beta^2 = 1$ and $|\beta| = 1$.

In [**37**], Kim and Skoug obtained a necessary and sufficient condition that a functional F in $L_2(C_0[0,T])$ has an integral transform

$$\mathcal{F}_{\alpha,\beta}F(y) = \int_{C_0[0,T]} F(\alpha x + \beta y) m(dx)$$

also belonging to $L_2(C_0[0,T])$.

We finish this section by noting that the conditions placed on F are quite different in [**43, 52**] and [**53**] than they are in papers [**11, 28, 29**] and [**35**]. In the notation of [**11, 28, 29, 35**], Lee [**43**], Yeh [**52**] and Yoo [**53**] require $F(x + \lambda y)$ to be an entire function of λ over $\mathbf{C}$ for each x and y in $C_0[0,T]$, whereas in [**11, 28, 29, 35**], F isn't even required to be a continuous function. But, on the other hand, in the Fourier-Feynman theory, [**11, 28, 29, 35**], the expression

$$\int_{C_0[0,T]} F(\lambda^{-1/2}x + y) m(dx)$$

is required to be an analytic function of λ over $\mathbf{C}_+$. Thus, in both approaches, an analyticity condition is required.

8. Transforms, convolutions and first variations. In Sections 8 and 9 we will simplify matters somewhat by writing $T_q(F)$ in place of $T_q^{(p)}(F)$, $T_q^{(2)}(F)$ and $T_q^{(1)}(F)$ for all $q \in \mathbf{R} - \{0\}$. Also, all of the functionals involved are assumed to be s.i.m. and all of the formulas involving y are assumed to hold for s-a.e. $y \in C_0[0,T]$.

We first give the definition of the first variation of a functional on Wiener space [**6, 15**]. Let F be a Wiener measurable functional on $C_0[0,T]$ and let $w \in C_0[0,T]$. Then

$$\delta F(x|w) = \frac{\partial}{\partial h} F(x+hw)|_{h=0}, \tag{8.1}$$

if it exists, is called the first variation of $F(x)$. Also let

$$A = \{w \in C_0[0,T] : w \text{ is absolutely continuous on } [0,T] \text{ with } w' \in L_2[0,T]\}. \tag{8.2}$$

In [**50**], for the space S, see equation (6.7) above, Park, Skoug and Storvick examined the various relationships that occur among the first variation $\delta F(x|w)$, the Fourier-Feynman transform $T_q(F)$ and the convolution product $(F * G)_q$. In Section 3 they studied the various relationships involving exactly two of the three concepts of *transform, convolution* and *first variation*. In Section 4 they examined the relationship involving all three concepts but where each concept is used exactly once. These are more than six possibilities since one can take the *transform*, and the *convolution* with respect to either the first or the second argument of the variation. It turns out that there are nine distinct possibilities. below we give three of the many formulas from [**50**]. For $w \in A$,

$$T_q(\delta F(\cdot|w))(y) = \delta T_q(F)(y|w), \tag{8.3}$$

$$T_q((\delta F(\cdot|w) * \delta G(\cdot|w))_q)(y) = \delta T_q(F)(y/\sqrt{2}\,|w)\delta T_q(G)(y/\sqrt{2}\,|w), \tag{8.4}$$

and

$$T_q((\delta F(y|\cdot) * \delta G(y|\cdot))_q)(w) = \delta F(y|w/\sqrt{2}\,)\delta G(y|w\sqrt{2}\,). \tag{8.5}$$

Included among the results of [**45**] is the following integration by parts formula for Fourier-Feynman transforms:

$$\int_{C_0[0,T]}^{anf_q} [T_q(F)(x)\delta T_q(G)(x|w) + \delta T_q(F)(x|w)T_q(G)(x)]m(dx)$$
$$= -iq \int_{C_0[0,T]}^{anf_q} T_q(F)(x)T_q(G)(x)\langle w', x\rangle m(dx)$$

for $w \in A$ and appropriate F and G.

In [**49**], the authors found formulas for $T_q(F_n)$ where

$$F_n(x) = F(x)\prod_{j=1}^{n}\langle w_j', x\rangle$$

where $F \in S$ and where $w_1, \dots, w_n$ are elements of A. In [**23**], Chang, Song and Yoo generalized the results of [**49**] to the Fresnel class $F(B)$ on an abstract Wiener space (H, B, v). In [**22**], Chang and Skoug extended these results to a very general function space $C_{a,b}[0,T]$ and Banach algebra $\mathcal{S}(L^2_{a,b}[0,T])$.

In [**38**], Kim, Ko, Park and Skoug studied the relationships that occur among transforms, convolutions and first variations for functionals on $C_0[0,T]$ of the form

$$F(x) = f(\langle \alpha_1, x\rangle, \dots, \langle \alpha_n, x\rangle) \tag{8.6}$$

for s-a.e. $x \in C_0[0,T]$ where $\{\alpha_1, \dots, \alpha_n\}$ is an orthonormal set of functionals in $L_2[0,T]$. While some of the results in [**38**] are quite similar to those in [**50**], many are quite different. For example, for F, and G, of the form (8.6) with appropriate f, and g,

$$\delta F(x|w) = \sum_{j=1}^{n}\langle \alpha_j, w\rangle f_j(\langle \alpha_1, x\rangle, \dots, \langle \alpha_n, x\rangle), \quad w \in A,$$
$$\delta^3 F(\cdot|w_1)(\cdot|w_2)(x|w_3) = \sum_{j_1=1}^{n}\sum_{j_2=1}^{n}\sum_{j_3=1}^{n}\Big[\prod_{k=1}^{3}\langle \alpha_{jk}, w_k\rangle\Big]$$
$$f_{j1,j2,j3}(\langle \alpha_1, x\rangle, \dots, \langle \alpha_n, x\rangle), w_k \in A,$$

and

$$T_q^{(r)}((F*G)_q)(y) = T_q^{(p_1)}(F)(y/\sqrt{2}\,)T_q^{(p_2)}(G)(y/\sqrt{2}\,)$$

where $1/r = 1/p_1 + 1/p_2 - 1$.

In [**21**], Chang and Skoug examined the effects that a "drift $b(t)$" has on the various relationships that occur among the Fourier-Feynman transform, the convolution product and the first variation for various functionals on Wiener space.

Let $b = b(t)$ be an $\mathbf{R}$-valued function on $[0,T]$, and let $h \in L_2[0,T]$ with $\|h\|_2 > 0$. Let $X_b : C_0[0,T] \times [0,T] \to \mathbf{R}$ be the Gaussian process

$$X_b(x,t) = Z(x,t) + b(t)$$

with $Z(x,t)$ given by equation (6.9) above. Note that the Wiener process $W(x,t) = x(t)$ is free of drift and is stationary in time, the process $Z(x,t)$ is free of drift and is nonstationary in time, while the process $X_b(x,t)$ is subject to the drift b and is nonstationary in time. Of course, if $h(t) \equiv 1$ and $b(t) \equiv 0$, then

$$X_b(x,t) = Z(x,t) = W(x,t) = x(t).$$

The analytic Fourier-Feynman transform of F with drift b, ${}_bT_q(F)$, is defined by simply replacing equation (5.1) with

$${}_bT_\lambda(F)(y) = \int_{C_0[0,T]}^{anw_\lambda} F(y + X_b(x,\cdot))m(dx),$$

and the convolution product with drift b is defined by the formula

$${}_b(F*G)_q(y) = \int_{C_0[0,T]}^{anf_q} F\bigg(\frac{y + X_b(x,\cdot)}{\sqrt{2}}\bigg)G\bigg(\frac{y - X_b(x,\cdot)}{\sqrt{2}}\bigg)m(dx).$$

Below for F and G in S, b and w in A and $h \in L_\infty[0,T]$, we list a few of the formulas from [**20**] relating the *first variation*, the *transform with drift* and the *convolution product with drift*. Note that all of the transforms and convolutions that appear on the lefthand sides of the equations below involve the drift, b, while all of the transforms which appear on the righthand sides are transforms without drift:

$$\begin{aligned} {}_bT_q(F)(y) &= T_q(F)(y+b), \\ {}_bT_q({}_b(F*G)_q)(y) &= T_q(F)\bigg(\frac{y+2b}{\sqrt{2}}\bigg)T_q(G)\bigg(\frac{y}{\sqrt{2}}\bigg), \end{aligned}$$

$$\begin{aligned}&{}_b\big({}_bT_q(F) *_b T_q(G)\big)_{-q}(y)\\&\quad = T_q\Big(F\big(\frac{\cdot}{\sqrt{2}} + \big(1+\frac{1}{\sqrt{2}}\big)b\big)G\big(\frac{\cdot}{\sqrt{2}} + \big(1-\frac{1}{\sqrt{2}}\big)b\big)\Big)(y),\end{aligned}$$

$${}_bT_q(\delta F(\cdot|w))(y) = \delta T_q(F)(y+b|w),$$

and

$${}_bT_q\big({}_b(\delta F(\cdot|w)*\delta G(\cdot|w))_{-q}\big)(y) = \delta T_q(F)\left(\frac{y+2b}{\sqrt{2}}\Big|\, w\right)\delta T_q(G)\left(\frac{y}{\sqrt{2}}\Big|\, w\right).$$

On page 636 of [**34**], Johnson and Lapidus give a brief discussion of the Fourier-Feynman transform, the convolution product and the first variation for functionals in the space S; see equation (6.7) above.

9. Conditional transforms and convolutions. In [**47**], Park and Skoug, using the conditioning function

$$X(x) = Z(x,T) = \int_0^T h(s)\,dx(x), \quad h \in L_2[0,T], \tag{9.1}$$

and using ideas from [**24, 46**] and [**31**], defined the concept of a generalized (see equations (6.9) and (6.10) above) conditional Fourier-Feynman transform $T_q(F|X)(y,\eta)$, and the concept of a generalized conditional convolution product $((F*G)_q|X)(y,\eta)$. Then, under appropriate conditions on F and G, they showed that

$$\begin{aligned}&T_q\big(((F*G)_q|X)(\cdot,\eta_1)|X\big)(y,\eta_2)\\&\qquad = T_q(F|X)\left(\frac{y}{\sqrt{2}},\frac{\eta_1+\eta_2}{\sqrt{2}}\right)T_q(G|X)\left(\frac{y}{\sqrt{2}},\frac{\eta_2-\eta_1}{\sqrt{2}}\right).\end{aligned}$$

Furthermore, they showed that if $\{\eta_1,\eta_2,\eta_3,\eta\}$ is in the solution set of the system

$$\begin{cases}\eta - \sqrt{2}\,\eta_1 - \eta_3 = 0\\ \eta - \sqrt{2}\,\eta_2 + \eta_3 = 0,\end{cases}$$

then

$$\big((T_q(F|X)(\cdot,\eta_1) * T_q(G|X)(\cdot,\eta_2))_{-q}|X\big)(y,\eta_3) \\ = T_q\Big(F\big(\frac{\cdot}{\sqrt{2}}\big)G\big(\frac{\cdot}{\sqrt{2}}\big)\big|X\Big)(y,\eta).$$

In [**19**], Chang, Park and Skoug using ideas from [**47**] and using the conditioning function (9.1), obtained the following translation formula for generalized conditional Fourier-Feynman transforms

$$\begin{aligned} &T_q(F|X)(y+x_0,\eta) \\ &\quad = \exp\left\{iq\Big\langle\frac{g}{h^2},y\Big\rangle + \frac{iq}{q}\,\|g/h\|_2^2 + \frac{iqx_0(T)}{a(T)}\Big(\eta+\frac{x_0(T)}{2}\Big)\right\} \\ &\qquad \times T_q(F^*|X)(y,\eta+x_0(T)) \end{aligned}$$

for appropriate $h \in L_\infty[0,T]$ and $g \in L_2[0,T]$ where $x_0(t) = \int_0^t g(s)\,ds$, $F^*(z(x,\cdot)) = \exp\{-iq\int_0^T (g(s)/h^2(s))\,dz(x,s)\}F(z(x,\cdot))$ and where $a(t)$ is given by equation (6.11) above.

In Section 3 of [**21**], Chang and Skoug obtained 20 formulas listing all of the effects that a drift $b = b(t)$ has on conditional Fourier-Feynman transforms, on conditional convolution products and on the conditional Fourier-Feynman transform of the conditional convolution product. For example, for $b \in A$ given by equation (8.2) above and for the conditioning functions $X(x) = Z(x,T)$ and $X_b(x) = Z(x,T)+b(T)$, they showed that

$${}_bT_q(F|X_b)(y,\eta) = T_q(F|X)(y+b,\eta-b(T)),$$

and that

$$\begin{aligned} &{}_bT_q(({}_b(F*G)_q|X_b)(\cdot,\eta_1)|X_b)(y,\eta_2) \\ &\quad = T_q(F|X)\Big(\frac{y+2b}{\sqrt{2}},\frac{\eta_2+\eta_1-2b(T)}{\sqrt{2}}\Big)T_q(G|X)\Big(\frac{y}{\sqrt{2}},\frac{\eta_2-\eta_1}{\sqrt{2}}\Big). \end{aligned}$$

In Section 4 of [**21**], Chang and Skoug studied the effects that a drift $b = b(t)$ has on the conditional convolution product of conditional transforms. Note that the expression

$$\big({}_b({}_bT_q(F|X_b)(\cdot,\eta_1) *_b T_q(G|X_b)(\cdot,\eta_2))_{-q}|X_b\big)(y,\eta_3)$$

involves the drift $b = b(t)$ in six different places. In each place where a "b" occurs, one could either include it or not. Thus there are $2^6 = 64$ possible cases of which 63 include the drift b. Below we list 2 of these 63 formulas:

$$\begin{aligned}\big({}_b({}_bT_q(F|X_b)(\cdot,\eta_1) *_b T_q(G|X_b)(\cdot,\eta_2))_{-q}|X_b\big)(y,\eta_3)&\\ = \big((T_q(F|X)(\cdot + b + \sqrt{2}\,b, \eta_1 - b(T)) * T_q((G|X)&\\ \times (\cdot - b + \sqrt{2}\,b, \eta_2 - b(T)))_{-q}|X\big)(y, \eta_3 - b(T))&\end{aligned}$$

and

$$\begin{aligned}\big({}_b({}_bT_q(F|X)(\cdot,\eta_1) *_b T_q(G|X)(\cdot,\eta_2))_{-q}|X_b\big)(y,\eta_3)&\\ = \big((T_q(F|X)(\cdot + b + \sqrt{2}\,b, \eta_1) * T_q(G|X)&\\ \times (\cdot - b + \sqrt{2}\,b, \eta_2))_{-q}|X\big)(y, \eta_3 - b(T)).&\end{aligned}$$

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