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INTEGRATING PATH-DEPENDENT FUNCTIONALS ON YEH-WIENER SPACE

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ABSTRACT. Denote by $C_{a,b}(Q)$ the generalized two-parameter Yeh-Wiener space with associated Gaussian measure. We investigate several scenarios in which integrals of functionals on this space can be reduced to integrals of related functionals over an appropriate single-parameter generalized Wiener space $C_{\hat{a},\hat{b}}[0, T]$. This extends some interesting results of R. H. Cameron and D. A. Storvick.

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1. Introduction

Let $C_0[0, T]$ denote one-parameter Wiener space (named after Norbert Wiener who did some of the earliest work in this area); that is the space of all continuous real-valued functions x on $[0, T]$ with $x(0) = 0$. During the past 75 years many people have made substantial contributions in studying this space with important applications to both physics and mathematics. In particular there is a considerable body of work relating to what we now call generalized Wiener spaces. Earlier discussions of these spaces can be found in [8, 22] and more recent developments can be found in [5, 6, 7, 9, 10]; the references listed in these papers led to a very large collection of other results.

The space which we will refer to as Yeh-Wiener space was introduced in [25] with further results in [23, 24]. In these papers, Yeh explored the structure and behavior of a Gaussian measure analogous to the classical Wiener measure but defined on the space of continuous functions of two variables. The associated stochastic process is often called the Brownian sheet. See [3, 12, 13, 14, 15, 16, 17, 18] for more information and examples.

The setting for this paper involves what we term a generalized Yeh-Wiener space. This function space extends the ordinary Yeh-Wiener space in a similar manner to

the single-parameter case. The main ideas and results of this paper follow and expand on those found in [2]. The primary goal is to relate certain integrals on a general two-parameter Wiener space with corresponding integrals on a general single-parameter space.

2. Definitions and Preliminaries

In [25], Yeh described the properties of a measure similar to Wiener measure on the space of continuous functions of two variables defined on the unit square. We are concerned with a family of similar measures.

Let Q denote the rectangle $[0, S] \times [0, T]$ in \mathbb{R}^2 and let \leq be the usual partial order on Q such that $s \leq t$ if and only if each $s_j \leq t_j$. Also let $a(s, t)$ be an absolutely continuous function with derivative $\frac{\partial^2 a}{\partial s \partial t} \in L^2(Q)$ and $a(0, 0) = 0$, and let $b(s, t)$ be an absolutely continuous function with a continuous derivative $\frac{\partial^2 b(s, t)}{\partial s \partial t} > 0$ on Q .

The functions a and b act to determine the center (or mean), and variance of the generalized Yeh-Wiener measure. We list several properties and useful basic results in this section; see Chapter 3 of [19] for a more detailed discussion of these matters.

A generalized Yeh-Wiener measure is a Gaussian measure on the space of continuous functions $C(Q)$. Therefore, the distribution of finite-dimensional projections of this space with respect to this measure are Gaussian, with the following basic form. For $0 < s_1 < \dots < s_m \leq S$ and $0 < t_1 < \dots < t_n \leq T$, the distribution of a finite-dimensional projection onto \mathbb{R}^{mn} with component projections $\{\delta_{(s_i, t_j)} : 1 \leq i \leq m, 1 \leq j \leq n\}$ (the generalized Yeh-Wiener kernel) is given by

$$W_{m,n}(\mathbf{u}, \mathbf{s}, \mathbf{t}) = \left(\prod_{i=1}^m \prod_{j=1}^n 2\pi \Delta_i \Delta_j b(s, t) \right)^{-\frac{1}{2}} \exp \left(-\frac{1}{2} \sum_{i=1}^m \sum_{j=1}^n \frac{(\Delta_i \Delta_j (u - a(s, t)))^2}{\Delta_i \Delta_j b(s, t)} \right),$$

where $\Delta_i \Delta_j u = u_{i,j} - u_{i-1,j} - u_{i,j-1} + u_{i-1,j-1}$ and $u_{i,0} = u_{0,j} = 0$ for all $i, j \geq 0$.

We will let \mathbf{m} denote the generalized Yeh-Wiener measure on $C_0(Q)$ determined by the functions a and b and will write $C_{a,b}(Q)$ for the resulting measure space.

A function f on Q is said to be of bounded variation in the sense of Hardy-Krause provided that $\sup_{P \in \mathcal{P}} \left\{ \sum_{R_j \in P} |\Delta_{R_j} f| \right\} < \infty$ over all finite partitions P of Q into non-degenerate rectangles $\{R_j\}$ and the restriction of f to any vertical or horizontal line in Q yields a single-variable function of bounded variation in the usual sense. We refer to the collection of such functions as $BV(Q)$. For a more detailed discussion of these functions and their properties see [1].

By $L^2_{a,b}(Q)$ we denote the collection of functions on Q that are square integrable with respect to the Lebesgue-Stieltjes measure induced by the functions a and b . That is,

$$L^2_{a,b}(Q) = \left\{ f : Q \rightarrow \mathbb{R} : \int_Q f^2(s, t) d(b(s, t) + |a|(s, t)) < \infty \right\}.$$

The space $L^2_{a,b}(Q)$ is in fact a Hilbert space (as our notation suggests), and has the obvious inner product

$$(f, g)_{a,b} = \int_Q f(s, t)g(s, t)d(b(s, t) + |a|(s, t)).$$

In addition, by $\|\cdot\|_b$ and $(\cdot, \cdot)_b$ we denote, respectively, the L^2 -norm of a function and the inner product with respect to the Lebesgue-Stieltjes measure induced by b ; that is

$$\|f\|_b^2 = \int_Q f^2(s, t)db(s, t).$$

and

$$(f, g)_b = \int_Q f(s, t)g(s, t)db(s, t).$$

Note that the conditions on $b(s, t)$ ensure that $L^2_b(Q)$ is equivalent to $L^2(Q)$ and further note that $BV(Q)$ is a subset of both $L^2_b(Q)$ and $L^2_{a,b}(Q)$.

We next define the Paley-Wiener-Zygmund stochastic integral of a function $f \in L^2_{a,b}(Q)$, which is a basic tool in understanding how the measure works.

Definition 2.1. Let $\{\phi_j\}$ be a complete orthonormal set of functions of bounded variation in $L^2_{a,b}(Q)$. For $f \in L^2_{a,b}(Q)$, put

$$I_n f(x) = \sum_{j=1}^n (f, \phi_j)_{a,b} \int_Q \phi_j(u)dx(u),$$

Define the Paley-Wiener-Zygmund (PWZ) stochastic integral $If(x) = \lim_{n \rightarrow \infty} I_n f(x)$ for all $x \in C_{a,b}(Q)$ for which this limit exists.

The following theorem is fundamental in computing integrals over $C_{a,b}(Q)$. It gives some essential properties of the PWZ integral.

- Theorem 2.2.**
1. If $f \in L^2_{a,b}(Q)$, then the PWZ stochastic integral $If(x)$ exists for a.e. $x \in C_{a,b}(Q)$ and is essentially independent of the choice of orthonormal basis in Definition 2.1.
 2. If $f \in L^2_{a,b}(Q)$, then If is a Gaussian random variable with mean $If(a)$ and variance $\|f\|_{L^2_b(Q)}^2$.
 3. If f and g are in $L^2_{a,b}(Q)$, then the covariance of the random variables If and Ig is $(f, g)_{L^2_b(Q)}$.

Remark 2.3. We pause briefly to note that the order of measurability assumptions in the following theorems, where the Lebesgue measurability of f is assumed and the μ -measurability of F is a conclusion, is not actually necessary. By similar arguments to those found in [4, 11, 21], the hypothesis of measurability can be either that F is μ -measurable on $C_{a,b}(Q)$ or that f is Lebesgue measurable, and the measurability of one of these will imply the measurability of the other.

Theorem 2.4 (Tame Functionals). *Let $0 < s_1 < \dots < s_m \leq S$ and $0 < t_1 < \dots < t_n \leq T$ and let $f : \mathbb{R}^{mn} \rightarrow \mathbb{C}$ and $F : C_{a,b}(Q) \rightarrow \mathbb{C}$ be defined by $F(x) = f(x(s_1, t_1), \dots, x(s_m, t_n))$. Then F is measurable if and only if f is Lebesgue measurable, and in this case,*

$$\int_{C_{a,b}(Q)} F(x) \mathbf{m}(dx) \stackrel{*}{=} \int_{\mathbb{R}^{mn}} f(u_{1,1}, \dots, u_{m,n}) W_{m,n}(\mathbf{u}, \mathbf{s}, \mathbf{t}) d\mathbf{u}, \tag{2.1}$$

where the equality (*) is in the sense that if one of the integrals exists then the other exists and the equality holds.

Proof. Let $\phi_{i,j} = \chi_{[s_{i-1}, s_i] \times [t_{j-1}, t_j]}(u, v)$. It is easy to calculate

$$x(s_k, t_l) = \sum_{\substack{0 \leq i \leq k \\ 0 \leq j \leq l}} \Delta_i \Delta_j x(s, t)$$

for any (s_k, t_l) . It is not difficult to see that

$$\Delta_i \Delta_j x(s, t) = I\phi_{i,j}(x) = \int_Q \chi_{[s_{i-1}, s_i] \times [t_{j-1}, t_j]}(u, v) dx(u, v),$$

whence we have

$$\begin{aligned} F(x) &= f(x(s_1, t_1), \dots, x(s_m, t_n)) \\ &= f \left(I\phi_{1,1}(x), \dots, \sum_{\substack{0 \leq i \leq k \\ 0 \leq j \leq l}} I\phi_{i,j}(x), \dots, \sum_{\substack{0 \leq i \leq m \\ 0 \leq j \leq n}} I\phi_{i,j}(x) \right). \end{aligned}$$

As $\phi_{i,j} \in BV(Q)$, we can use Theorem 2.2 to see that

$$\int_{C_{a,b}(Q)} I\phi_{i,j}(x) \mu(dx) = I\phi_{i,j}(a) = \Delta_i \Delta_j a(s, t),$$

and also observe that

$$\begin{aligned} \int_{C_{a,b}(Q)} (I\phi_{i,j}(x) - I\phi_{i,j}(a)) (I\phi_{l,m}(x) - I\phi_{l,m}(a)) \mathbf{m}(dx) \\ &= \int_Q \phi_{i,j}(u, v) \phi_{l,m}(u, v) db(u, v) \\ &= \begin{cases} \Delta_i \Delta_j b(s, t) & \text{if } i = l, j = m \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Accordingly, we see that the covariance matrix M for the collection $\{\phi_{i,j}\}$ is a diagonal matrix whose nonzero entries are $\Delta_i \Delta_j b(s, t)$. Now we can apply Theorem 2.2 to complete the proof. □

3. Integrals Over Paths

We will first be concerned with integrating functionals defined in terms of certain paths in Q . We confine our discussion to paths $\phi : [0, S] \rightarrow Q$ for which $\phi(\tau) = (\phi_1(\tau), \phi_2(\tau))$ satisfies the condition that its component functions ϕ_1 and ϕ_2 are each piecewise continuously differentiable. We will say that ϕ is an increasing path in Q if $\phi' \cdot \phi' > 0$ on $[0, S]$ and $\phi(\tau_1) \leq \phi(\tau_2)$ whenever $\tau_1 \leq \tau_2$.

The first theorem in this section establishes a special case of the tame functionals theorem in the case that one defines the functional in terms of a sequence of points lying on an increasing path.

Theorem 3.1. *Let $0 = s_0 < s_1 \leq \dots \leq s_n \leq S$ and $0 = t_0 < t_1 \leq \dots \leq t_n \leq T$ and let $f : \mathbb{R}^n \rightarrow \mathbb{C}$ be Lebesgue measurable. If $F : C_{a,b}(Q) \rightarrow \mathbb{C}$ is defined by $F(x) = f(x(s_1, t_1), x(s_2, t_2), \dots, x(s_n, t_n))$, then F is μ -measurable and*

$$\int_{C_{a,b}(Q)} F(x) \mathbf{m}(dx) \stackrel{*}{=} \left(\prod_{j=1}^n 2\pi(b(s_j, t_j) - b(s_{j-1}, t_{j-1})) \right)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(u_1, \dots, u_n) \exp \left(-\frac{1}{2} \sum_{j=1}^n \frac{(u_j - a(s_j, t_j) - u_{j-1} + a(s_{j-1}, t_{j-1}))^2}{b(s_j, t_j) - b(s_{j-1}, t_{j-1})} \right) du_1 \cdots du_n, \tag{3.1}$$

where the equality ($\stackrel{*}{=}$) is in the sense that if one of the integrals exists then the other exists and the equality holds.

Proof. The proof is by induction on n . The theorem is certainly true for $n = 1$ because by (2.1),

$$\begin{aligned} & \int_{C_{a,b}(Q)} f(x(s_1, t_1)) \mathbf{m}(dx) \\ &= \left(\prod_{i=1}^1 \prod_{j=1}^1 2\pi \Delta_i \Delta_j b(s, t) \right)^{-\frac{1}{2}} \int_{\mathbb{R}} f(u_{1,1}) \exp \left(-\frac{1}{2} \sum_{i=1}^1 \sum_{j=1}^1 \frac{(\Delta_i \Delta_j (u - a(s, t)))^2}{\Delta_i \Delta_j b(s, t)} \right) du_{1,1} \\ &= \frac{1}{\sqrt{2\pi b(s_1, t_1)}} \int_{\mathbb{R}} f(u_1) \exp \left(-\frac{1}{2} \sum_{j=1}^1 \frac{(u_j - a(s_j, t_j))^2}{b(s_j, t_j) - b(s_{j-1}, t_{j-1})} \right) du_1, \end{aligned}$$

and thus (3.1) holds.

Assume that the result holds for $n = k \geq 1$. Then for $n = k + 1$ we have

$$\begin{aligned} & \int_{C_{a,b}(Q)} f(x(s_1, t_1), \dots, x(s_n, t_n)) \mu(dx) = \\ & \left(\prod_{i=1}^{k+1} \prod_{j=1}^{k+1} 2\pi \Delta_i \Delta_j b(s, t) \right)^{-\frac{1}{2}} \int_{\mathbb{R}^{(k+1)^2}} f(u_{1,1}, \dots, u_{k+1,k+1}) \end{aligned}$$

$$\exp \left(-\frac{1}{2} \sum_{i=1}^{k+1} \sum_{j=1}^{k+1} \frac{(\Delta_i \Delta_j (u - a(s, t)))^2}{\Delta_i \Delta_j b(s, t)} \right) du_{1,1} du_{1,2} du_{2,1} \cdots du_{k+1,k+1}. \quad (3.2)$$

Note that for $j = 1, \dots, k$ the variables $u_{k+1,j}$ and $u_{j,k+1}$ appear in (3.2) only in the kernel as the functional $F(x)$ does not depend on the values of x at these points. Also observe that $b(s_{k+1}, t_1) - b(s_k, t_1) = \Delta_{k+1} \Delta_1 b(s, t)$, $b(s_{k+1}, t_2) - b(s_k, t_2) = \Delta_{k+1} \Delta_2 b(s, t) + \Delta_{k+1} \Delta_1 b(s, t)$, and eventually $b(s_{k+1}, t_k) - b(s_k, t_k) = \Delta_{k+1} \Delta_k b(s, t) + \cdots + \Delta_{k+1} \Delta_1 b(s, t)$. In addition observe that

$$\begin{aligned} u_{k+1,1} - a(s_{k+1}, t_1) - u_{k,1} + a(s_k, t_1) &= \Delta_{k+1} \Delta_1 (u - a(s, t)), \\ u_{k+1,2} - a(s_{k+1}, t_2) - u_{k,2} + a(s_k, t_2) &= \Delta_{k+1} \Delta_2 (u - a(s, t)) + \Delta_{k+1} \Delta_1 (u - a(s, t)), \\ &\vdots \\ u_{k+1,k} - a(s_{k+1}, t_k) - u_{k,k} + a(s_k, t_k) &= \Delta_{k+1} \Delta_k (u - a(s, t)) + \cdots + \Delta_{k+1} \Delta_1 (u - a(s, t)). \end{aligned}$$

Applying the Chapman-Kolmogorov equation $2k - 2$ times to the right side of (3.2) yields

$$\begin{aligned} &\left(\prod_{i=1}^k \prod_{j=1}^k 2\pi \Delta_i \Delta_j b(s, t) \right)^{-\frac{1}{2}} \quad (3.3) \\ &\int_{\mathbb{R}^{k^2+1}} f(u_{1,1}, \dots, u_{k,k}) \exp \left(-\frac{1}{2} \sum_{i=1}^k \sum_{j=1}^k \frac{(\Delta_i \Delta_j (u - a(s, t)))^2}{\Delta_i \Delta_j b(s, t)} \right) \\ &\left(\frac{1}{\sqrt{2\pi \Delta_{k+1} \Delta_{k+1} b(s, t)}} \frac{1}{\sqrt{2\pi (b(s_k, t_{k+1}) - b(s_k, t_k))}} \frac{1}{\sqrt{2\pi (b(s_k, t_{k+1}) - b(s_k, t_k))}} \right) \\ &\int_{\mathbb{R}^2} \exp \left(\frac{(\Delta_{k+1} \Delta_{k+1} (u - a(a, t)))^2}{-2\Delta_{k+1} \Delta_{k+1} b(s, t)} \right) \exp \left(\frac{(u_{k+1,k} - a(s_{k+1}, t_k) - u_{k,k} + a(s_k, t_k))^2}{-(b(s_{k+1}, t_k) - b(s_k, t_k))} \right) \\ &\exp \left(\frac{(u_{k,k+1} - a(s_k, t_{k+1}) - u_{k,k} + a(s_k, t_k))^2}{-(b(s_k, t_{k+1}) - b(s_k, t_k))} \right) du_{k+1,k} du_{k,k+1} du_{k+1,k+1} du_{k,k} \dots du_{1,1} \end{aligned}$$

Now notice that

$$\begin{aligned} &\Delta_{k+1} \Delta_{k+1} (\mathbf{u} - a(s, t)) \\ &= u_{k+1,k+1} - a(s_{k+1}, t_{k+1}) - u_{k+1,k} + a(s_{k+1}, t_k) \\ &\quad - u_{k,k+1} + a(s_k, t_{k+1}) + u_{k,k} - a(s_k, t_k) \\ &= [(u_{k+1,k+1} - a(s_{k+1}, t_{k+1})) - (u_{k,k} - a(s_k, t_k))] \\ &\quad - [(u_{k,k+1} - a(s_k, t_{k+1})) - (u_{k,k} - a(s_k, t_k))] \\ &\quad - [(u_{k+1,k} - a(s_{k+1}, t_k)) - (u_{k,k} - a(s_k, t_k))], \end{aligned}$$

and also that

$$\begin{aligned} \Delta_{k+1}\Delta_{k+1}b(s, t) &= b(s_{k+1}, t_{k+1}) - b(s_k, t_{k+1}) - b(s_{k+1}, t_k) + b(s_k, t_k) \\ &= [b(s_{k+1}, t_{k+1}) - b(s_k, t_k)] - [b(s_k, t_{k+1}) - b(s_k, t_k)] - [b(s_{k+1}, t_k) - b(s_k, t_k)], \end{aligned}$$

and apply the Chapman-Kolmogorov equation twice to the inner double integral in (3.3),

$$\begin{aligned} &\left(\frac{1}{\sqrt{2\pi\Delta_{k+1}\Delta_{k+1}b(s, t)}} \frac{1}{\sqrt{2\pi(b(s_k, t_{k+1}) - b(s_k, t_k))}} \frac{1}{\sqrt{2\pi(b(s_k, t_{k+1}) - b(s_k, t_k))}} \right) \\ &\int_{\mathbb{R}^2} \exp\left(-\frac{(\Delta_{k+1}\Delta_{k+1}(u - a(s, t)))^2}{2\Delta_{k+1}\Delta_{k+1}b(s, t)}\right) \\ &\exp\left(-\frac{(u_{k+1,k} - a(s_{k+1}, t_k) - u_{k,k} + a(s_k, t_k))^2}{b(s_{k+1}, t_k) - b(s_k, t_k)}\right) \\ &\exp\left(-\frac{(u_{k,k+1} - a(s_k, t_{k+1}) - u_{k,k} + a(s_k, t_k))^2}{b(s_k, t_{k+1}) - b(s_k, t_k)}\right) du_{k+1,k} du_{k,k+1}. \end{aligned}$$

This yields

$$\left(\frac{1}{2\pi(b(s_{k+1}, t_{k+1}) - b(s_k, t_k))}\right)^{\frac{1}{2}} \exp\left(\frac{(u_{k+1,k+1} - a(s_{k+1}, t_{k+1}) - u_{k,k} + a(s_k, t_k))^2}{-2(b(s_{k+1}, t_{k+1}) - b(s_k, t_k))}\right)$$

Thus (3.2) becomes

$$\begin{aligned} &\int_{C_{a,b}(Q)} f(x(s_1, t_1), \dots, x(s_n, t_n)) \mathbf{m}(dx) \\ &= \left(\prod_{i=1}^k \prod_{j=1}^k 2\pi\Delta_i\Delta_jb(s, t)\right)^{-\frac{1}{2}} \left(\frac{1}{2\pi(b(s_{k+1}, t_{k+1}) - b(s_k, t_k))}\right)^{\frac{1}{2}} \\ &\int_{\mathbb{R}^{k^2}} \int_{\mathbb{R}} f(u_{1,1}, \dots, u_{k+1,k+1}) \\ &\exp\left(-\frac{(u_{k+1,k+1} - a(s_{k+1}, t_{k+1}) - u_{k,k} + a(s_k, t_k))^2}{2(b(s_{k+1}, t_{k+1}) - b(s_k, t_k))}\right) du_{k+1,k+1} \\ &\exp\left(-\frac{1}{2} \sum_{i=1}^k \sum_{j=1}^k \frac{(\Delta_i\Delta_j(u - a(s, t)))^2}{\Delta_i\Delta_jb(s, t)}\right) \prod_{\substack{i=1 \\ j=1}}^k du_{i,j}. \end{aligned} \tag{3.4}$$

Define the function $g : \mathbb{R}^{k^2} \rightarrow \mathbb{C}$ so that $g(u_{1,1}, \dots, u_{k,k})$ is equal to

$$\begin{aligned} &\int_{\mathbb{R}} f(u_{1,1}, \dots, u_{k+1,k+1}) \\ &\exp\left(-\frac{(u_{k+1,k+1} - a(s_{k+1}, t_{k+1}) - u_{k,k} + a(s_k, t_k))^2}{2(b(s_{k+1}, t_{k+1}) - b(s_k, t_k))}\right) du_{k+1,k+1} \end{aligned} \tag{3.5}$$

and define a tame functional $G(x) : C_{a,b}(Q) \rightarrow \mathbb{R}$ by

$$G(x) = g(x(s_1, t_1), \dots, x(s_k, t_k)). \tag{3.6}$$

Combine (3.4) and (3.6) to obtain

$$\begin{aligned}
 \int_{C_{a,b}(Q)} f(x(s_1, t_1), \dots, x(s_n, t_n)) \mathbf{m}(dx) &= \\
 & \left(\prod_{i=1}^k \prod_{j=1}^k 2\pi \Delta_i \Delta_j b(s, t) \right)^{-\frac{1}{2}} \int_{\mathbb{R}^{k^2}} g(u_{1,1}, \dots, u_{k,k}) \\
 & \exp \left(-\frac{1}{2} \sum_{i=1}^k \sum_{j=1}^k \frac{(\Delta_i \Delta_j (u - a(s, t)))^2}{\Delta_i \Delta_j b(s, t)} \right) \prod_{i=1}^k \prod_{j=1}^k du_{i,j} \\
 &= \int_{C_{a,b}(Q)} G(x) \mathbf{m}(x). \tag{3.7}
 \end{aligned}$$

Apply the induction hypothesis to the functional G . Put $u_{k+1,k+1} = u_{k+1}$ and $u_{k,k} = u_k$ in equation (3.5) and then use (3.7) to obtain

$$\begin{aligned}
 \int_{C_{a,b}(Q)} f(x(s_1, t_1), \dots, x(s_n, t_n)) \mathbf{m}(dx) &= \\
 & \left(\prod_{j=1}^k 2\pi (b(s_j, t_j) - b(s_{j-1}, t_{j-1})) \right)^{-\frac{1}{2}} \int_{\mathbb{R}^k} g(u_1, \dots, u_k) \\
 & \exp \left(-\frac{1}{2} \sum_{j=1}^k \frac{(u_j - a(s_j, t_j) - u_{j-1} + a(s_{j-1}, t_{j-1}))^2}{b(s_j, t_j) - b(s_{j-1}, t_{j-1})} \right) du_k \cdots du_1 \\
 &= \left(\prod_{j=1}^{k+1} 2\pi (b(s_j, t_j) - b(s_{j-1}, t_{j-1})) \right)^{-\frac{1}{2}} \int_{\mathbb{R}^{k+1}} f(u_1, \dots, u_{k+1}) \\
 & \exp \left(-\frac{1}{2} \sum_{j=1}^{k+1} \frac{(u_j - a(s_j, t_j) - u_{j-1} + a(s_{j-1}, t_{j-1}))^2}{b(s_j, t_j) - b(s_{j-1}, t_{j-1})} \right) du_{k+1} \cdots du_1,
 \end{aligned}$$

and so for $n = k + 1$, equation (3.1) holds by induction. □

4. One-line Theorems

We are now equipped to investigate formulas for the integration of functionals depending only on the values of x on certain well-behaved paths in Q . The following theorem permits reduction of certain integrals over $C_{a,b}(Q)$ to integrals over an appropriately chosen single-parameter function space $C_{\tilde{a},\tilde{b}}[0, S]$.

Theorem 4.1. *Let $\phi : [0, S] \rightarrow Q$ be an increasing path. Let $a_\phi(\tau) = a(\phi(\tau)) - a(\phi(0))$ and $b_\phi(\tau) = b(\phi(\tau)) - b(\phi(0))$, and let \mathbf{m}_ϕ be the Gaussian measure on $C_0[0, S]$ subordinate to a_ϕ and b_ϕ . If $F(x) = f(x(\phi(\cdot)))$ is a measurable functional on $C_{a,b}(Q)$,*

then

$$\int_{C_{a,b}(Q)} F(x) \mathbf{m}(dx) \stackrel{*}{=} \int_{C_{a_\phi, b_\phi}[0, S]} f(w) \mathbf{m}_\phi(dw), \tag{4.1}$$

where the equality ($\stackrel{*}{=}$) is in the sense that if one of the integrals exists then the other exists and the equality holds.

Proof. Let $0 = \tau_0 < \tau_1 < \dots < \tau_j < \dots < \tau_n \leq S$ and let $I = \{x \in C_{a,b}(Q) : \alpha_j < x(\phi(\tau_j)) < \beta_j\}$ and $J = \{w \in C_{a_\phi, b_\phi}[0, S] : \alpha_j < w(\tau_j) < \beta_j\}$. Note that by the conditions on γ we have $\phi_1(0) \leq \phi_1(\tau_1) \leq \dots \leq \phi_1(\tau_n)$ and $\phi_2(0) \leq \phi_2(\tau_1) \leq \dots \leq \phi_2(\tau_n)$. Then by Theorem 3.1,

$$\begin{aligned} \mathbf{m}(I) &= \int_{C_{a,b}(Q)} \chi_I(x)(dx) \\ &= \left(\prod_{j=1}^n 2\pi(b(\phi(\tau_j)) - b(\phi(\tau_{j-1}))) \right)^{-\frac{1}{2}} \\ &\quad \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} \sum_{j=1}^n \frac{(u_j - a(\phi(\tau_j)) - u_{j-1} + a(\phi(\tau_{j-1})))^2}{b(\phi(\tau_j)) - b(\phi(\tau_{j-1}))}\right) du_n \cdots du_1 \\ &= \left(\prod_{j=1}^n 2\pi(b_\phi(\tau_j) - b_\phi(\tau_{j-1})) \right)^{-\frac{1}{2}} \\ &\quad \int_{\mathbb{R}^n} \exp\left(-\frac{1}{2} \sum_{j=1}^n \frac{(u_j - a_\phi(\tau_j) - u_{j-1} + a_\phi(\tau_{j-1}))^2}{b_\phi(\tau_j) - b_\phi(\tau_{j-1})}\right) du_n \cdots du_1 \\ &= \int_{C_{a_\phi, b_\phi}[0, S]} \chi_J(y) \mathbf{m}_\phi(dy) \\ &= \mathbf{m}_\phi(J). \end{aligned}$$

Hence the result holds for characteristic functions of sets of the form $\{x \in C_{a,b}(Q) : \alpha_j < x(\phi(\tau_j)) < \beta_j\}$. The theorem follows by taking the function f to successively be the characteristic function of a Borel set, and then to be a simple function. From here, by monotone convergence the theorem holds for positive functions, and hence for general functions by taking positive and negative and real and imaginary parts. \square

As a corollary to Theorem 4.1 we have the following one-line theorem of Cameron and Storvick from [2].

Corollary 4.2. *Let $0 < \beta \leq T$ and let $f(\cdot)$ be a real or complex valued functional defined on $C_0[0, S]$ such that $f(\sqrt{\beta}w)$ is a Wiener measurable functional on $C_0[0, S]$. Then $f(x(\cdot, \beta))$ is a Yeh-Wiener measurable functional of x on $C_0(Q)$ and*

$$\int_{C_0(Q)} f(x(\cdot, \beta))(dx) \stackrel{*}{=} \int_{C_0[0, S]} f(\sqrt{\beta} w) \mathbf{w}(dw), \tag{4.2}$$

where \mathbf{w} denotes the ordinary Wiener measure and the equality ($\stackrel{*}{=}$) is in the sense that if one of the integrals exists then the other exists and the equality holds.

Proof. Take $\phi : [0, S] \rightarrow Q$ to be $\phi(\tau) = (\tau, \beta)$ and note that $a(s, t) = 0$ and $b(s, t) = st$. Applying Theorem 4.1 to any tame functional $F(x) = f(x(s_1, \beta), \dots, x(s_n, \beta))$ we obtain

$$\begin{aligned} & \int_{C_0(Q)} F(x) \mathbf{m}(dx) \\ &= \left(\prod_{j=1}^n 2\pi(\beta s_j - \beta s_{j-1}) \right)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(u_1, \dots, u_n) \exp \left(-\frac{1}{2} \sum_{j=1}^n \frac{(u_j - u_{j-1})^2}{\beta s_j - \beta s_{j-1}} \right) \mathbf{d}u \\ &= \left(\prod_{j=1}^n 2\pi(s_j - s_{j-1}) \right)^{-\frac{1}{2}} \int_{\mathbb{R}^n} f(\sqrt{\beta} w_1, \dots, \sqrt{\beta} w_n) \\ & \quad \exp \left(-\frac{1}{2} \sum_{j=1}^n \frac{(w_j - w_{j-1})^2}{s_j - s_{j-1}} \right) \mathbf{d}w \\ &= \int_{C_0[0, S]} f(\sqrt{\beta} y(s_1), \dots, \sqrt{\beta} y(s_n)) \mathbf{w}(dw). \end{aligned}$$

The theorem holds in the general case by the same argument used to finish the proof of Theorem 4.1. □

5. *n*-line Theorem

We can use Theorem 4.1 to extend the *n*-line theorem of Cameron and Storvick from [2].

Theorem 5.1. *Let $0 < \beta_1 < \dots < \beta_n \leq T$ and let $F(x) = f(x(\cdot, \beta_1), \dots, x(\cdot, \beta_n))$ be μ -measurable. Put $a_1(s) = a(s, \beta_1)$ and $a_k(s) = a(s, \beta_k) - a(s, \beta_{k-1})$ and put $b_1(s) = b(s, \beta_1)$ and $b_k(s) = b(s, \beta_k) - b(s, \beta_{k-1})$ for $k = 2, \dots, n$. Let $\mathbf{m}_1, \dots, \mathbf{m}_n$ be Gaussian measures on $C_0[0, S]$, each subordinate to the corresponding a_k and b_k . Then*

$$\begin{aligned} & \int_{C_{a,b}(Q)} F(x) \mathbf{m}(dx) \\ & \stackrel{*}{=} \int_{C_{a_n, b_n}[0, S]} \dots \int_{C_{a_1, b_1}[0, S]} f(y_1, y_1 + y_2, \dots, y_1 + y_2 + \dots + y_n) \mathbf{m}_1(dy_1) \dots \mathbf{m}_n(dy_n), \end{aligned}$$

where the equality ($\stackrel{*}{=}$) is in the sense that if one of the integrals exists then the other exists and the equality holds.

Proof. Let $0 = s_0 < s_1 < \dots < s_m \leq S$ and $t_k = \beta_k$ for $k = 1, \dots, n$ and let $p_{j,k} < q_{j,k}$ for all $j = 1, \dots, m$ and $k = 1, \dots, n$. Define

$$\begin{aligned} I_j &= \{x \in C_{a,b}(Q) : p_{j,k} < x(s_j, \beta_k) \leq q_{j,k} \text{ for } k = 1, \dots, n\}, \\ E_j &= \{(u_{j,1}, \dots, u_{j,n}) \in \mathbb{R}^n : p_{j,k} < u_{j,k} \leq q_{j,k} \text{ for } k = 1, \dots, n\}, \\ J_j &= \{(y_1, \dots, y_n) \in \times_{k=1}^n C_{a_k, b_k}[0, S] : p_{j,k} < \sum_{l=1}^k y_l(s_j) \leq q_{j,k} \text{ for } k = 1, \dots, n\}. \end{aligned}$$

Notice that measurability of E_j in \mathbb{R}^n assures the measurability of I_j and J_j in their respective spaces. Moreover, for a cylinder set $I(p_{1,1}, \dots, p_{m,n}, q_{1,1}, \dots, q_{m,n}) \subseteq C_{a,b}(Q)$ determined solely by the values of $x(\cdot, \cdot)$ at the points (s_j, β_k) for $j = 1, \dots, m$ and $k = 1, \dots, n$, we have

$$\begin{aligned} &I(p_{1,1}, \dots, p_{m,n}, q_{1,1}, \dots, q_{m,n}) \\ &= \{x \in C_{a,b}(Q) : p_{j,k} < x(s_j, \beta_k) \leq q_{j,k} \text{ for } j = 1, \dots, m; k = 1, \dots, n\} \\ &= \bigcap_{j=1}^m I_j. \end{aligned}$$

Begin by considering the case in which

$$F(x) = \chi_I(x) = \prod_{j=1}^m \chi_{I_j}(x) = \prod_{j=1}^m \chi_{E_j}(x(\cdot, \beta_1), \dots, x(\cdot, \beta_n)).$$

By Theorem 2.4,

$$\begin{aligned} \int_{C_{a,b}(Q)} F(x) \mathbf{m}(dx) &= \int_{C_{a,b}(Q)} \prod_{j=1}^m \chi_{E_j}(x(s_j, \beta_1), \dots, x(s_j, \beta_n)) \mathbf{m}(dx) \\ &= \left(\prod_{k=1}^n \prod_{j=1}^m 2\pi \Delta_k \Delta_j b(s, \beta) \right)^{-\frac{1}{2}} \int_{\mathbb{R}^{mn}} \prod_{j=1}^m \chi_{E_j}(u_{j,1}, \dots, u_{j,n}) \quad (5.1) \\ &\quad \prod_{k=1}^n \exp\left(-\frac{1}{2} \sum_{j=1}^m \frac{(\Delta_k \Delta_j (u - a(s, t)))^2}{\Delta_k \Delta_j b(s, t)}\right) du_{1,1} \cdots du_{m,n}. \end{aligned}$$

Note that

$$\begin{aligned} \Delta_k \Delta_j (u - a(s, t)) &= u_{j,k} - u_{j,k-1} - a(s_j, \beta_k) + a(s_j, \beta_{k-1}) \\ &\quad - u_{j-1,k} + u_{j-1,k-1} + a(s_{j-1}, \beta_k) - a(s_{j-1}, \beta_{k-1}) \quad (5.2) \end{aligned}$$

$$\begin{aligned} &= [u_{j,k} - u_{j,k-1}] - [a(s_j, \beta_k) - a(s_j, \beta_{k-1})] \quad (5.3) \\ &\quad - [u_{j-1,k} - u_{j-1,k-1}] + [a(s_{j-1}, \beta_k) - a(s_{j-1}, \beta_{k-1})], \end{aligned}$$

and also that

$$\begin{aligned} \Delta_k \Delta_j b(s, t) &= b(s_j, \beta_k) - b(s_{j-1}, \beta_k) - b(s_j, \beta_{k-1}) + b(s_{j-1}, \beta_{k-1}) \\ &= [b(s_j, \beta_k) - b(s_j, \beta_{k-1})] - [b(s_{j-1}, \beta_k) - b(s_{j-1}, \beta_{k-1})]. \quad (5.4) \end{aligned}$$

Take $b_1(\cdot) = b(\cdot, \beta_1)$, $b_k(\cdot) = b(\cdot, \beta_k) - b(\cdot, \beta_{k-1})$, $a_1(\cdot) = a(\cdot, \beta_1)$, and $a(\cdot) = b(\cdot, \beta_k) - b(\cdot, \beta_{k-1})$ for $k = 2, \dots, n$ as in the statement of the theorem. Let

$$v_{j,k} = u_{j,k} - u_{j,k-1}, \quad (5.5)$$

and observe that $dv_{j,k} = du_{j,k}$ under this change of variables, and that

$$u_{j,k} = v_{j,k} + u_{j,k-1} = v_{j,k} + v_{j,k-1} + \dots + v_{j,1} \quad (5.6)$$

for $1 \leq k \leq n$. Substitute (5.2), (5.4), (5.5), and (5.6) in (5.1) to obtain

$$\begin{aligned} & \prod_{k=1}^n \left(\prod_{j=1}^m 2\pi \Delta_j b_k(s) \right)^{-\frac{1}{2}} \int_{\mathbb{R}^{mn}} \prod_{j=1}^m \chi_{E_j}(v_{j,1}, v_{j,1} + v_{j,2}, \dots, v_{j,1} + \dots + v_{j,n}) \\ & \quad \prod_{k=1}^n \exp \left(-\frac{1}{2} \sum_{j=1}^m \frac{(\Delta_j(v_j - a_j(s)) - v_{j-1} + a_{j-1}(s))^2}{\Delta_j b_k(s)} \right) dv_{1,1} \dots dv_{m,n} \\ &= \int_{C_{a_1, b_1}[0, S]} \dots \int_{C_{a_n, b_n}[0, S]} \prod_{j=1}^m \chi_{E_j}(y_1(s_j), \dots, y_1(s_j) + \dots + y_n(s_j)) \mathbf{m}_n(dy_n) \dots \mathbf{m}_1(dy_1) \\ &= \int_{C_{a_1, b_1}[0, S]} \dots \int_{C_{a_n, b_n}[0, S]} \prod_{j=1}^m \chi_{E_j}(y_1(\cdot), \dots, y_1(\cdot) + \dots + y_n(\cdot)) \mathbf{m}_n(dy_n) \dots \mathbf{m}_1(dy_1). \end{aligned}$$

Therefore the theorem is true for characteristic functions of cylinder sets that are dependent only on the value of $x(\cdot, \cdot)$ at the points $\{(s_j, \beta_k)$ for $j = 1, \dots, m$; $k = 1, \dots, n\}$. In the usual manner we can prove the theorem for characteristic functions of measurable sets depending only on the values of $x(\cdot, \beta_k)$ for $k = 1, \dots, n$. The proof is then completed in the same fashion as the proof of Theorem 4.1. \square

6. Examples

Example 6.1. This first example demonstrates the use of Theorem 5.1. Let $a(s, t) = b(s, t) = st$ on $[0, S] \times [0, 2T]$ and put $F(x) = \int_0^S x(s, T)x(s, 2T)ds$. We find the value of $\int_{C_{a,b}(Q)} F(x)\mu(dx)$. Note that $a_1(s) = b_1(s) = sT$ and $a_2(s) = b_2(s) = 2sT - sT = sT$, and thus

$$\begin{aligned} \int_{C_{a,b}(Q)} F(x)\mathbf{m}(dx) &= \int_{C_{a_2, b_2}[0, S]} \int_{C_{a_1, b_1}[0, S]} \int_0^S y_1(s)(y_1(s) + y_2(s))ds \mathbf{m}_1(dy_1)\mathbf{m}_2(dy_2) \\ &= \int_0^S \int_{C_{a_2, b_2}[0, S]} \int_{C_{a_1, b_1}[0, S]} (y_1^2(s) + y_1(s)y_2(s)) \mathbf{m}_1(dy_1)\mathbf{m}_2(dy_2)ds \\ &= \int_0^S \int_{C_{a_2, b_2}[0, S]} (sT + s^2T^2 + sTy_2(s)) \mathbf{m}_2(dy_2)ds \\ &= \int_0^S (sT + s^2T^2 + s^2T^2) ds \end{aligned}$$

$$= \frac{1}{2}S^2T + \frac{2}{3}S^3T^2,$$

where Fubini's theorem justifies the change in order of integration. In this example, we can easily verify our result without using Theorem 5.1, for

$$\begin{aligned} \int_{C_{a,b}(Q)} F(x)\mathbf{m}(dx) &= \int_{C_{a,b}(Q)} \int_0^S x(s,T)x(s,2T)ds\mathbf{m}(dx) \\ &= \int_0^S \int_{C_{a,b}(Q)} x(s,T)x(s,2T)\mathbf{m}(dx)ds \\ &= \int_0^S (sT + 2s^2T^2) ds \\ &= \frac{1}{2}S^2T + \frac{2}{3}S^3T^2. \end{aligned}$$

Example 6.2. We next demonstrate the use of Theorem 4.1. Let $Q = [0, S]^2$, $a(s, t) = st$, $b(s, t) = s^2t^2$, and $F(x) = \exp\left(\int_0^S x(s, s)ds\right)$. Note that $\phi : [0, S] \rightarrow Q$ defined by $\phi(s) = (s, s)$ is increasing. Then

$$\int_{C_{a,b}(Q)} F(x)\mathbf{m}(dx) = \int_{C_{a_1,b_1}[0,S]} \exp\left(\int_0^S y(s)ds\right) \mathbf{m}_\phi(dy), \quad (6.1)$$

where $a_1(s) = a(\phi(s)) - a(\phi(0)) = a(s, s) - a(0, 0) = s^2$ and $b_1(s) = b(\phi(s)) - b(\phi(0)) = s^4$. Integrating by parts we obtain that

$$\int_0^S y(s)ds = Sy(S) - \int_0^S sdy(s) = \langle S, y \rangle - \langle s, y \rangle = \langle S - s, y \rangle,$$

for \mathbf{m} a.e. $y \in C_{a_\phi, b_\phi}[0, S]$. In this case we let $\langle f, y \rangle$ denote the single-variable PWZ stochastic integral of the function $f \in L^2_{a_\phi, b_\phi}[0, S]$. Compute the values $A = \int_0^S (S - s)da_\phi(s) = \frac{1}{3}S^3$ and $B = \int_0^S (S - s)^2db_\phi(s) = \frac{1}{15}S^6$ and make use of a theorem from [20] to compute the right-hand side of (6.1) to obtain

$$\begin{aligned} &\int_{C_{a_1,b_1}[0,S]} \exp\left(\int_0^S y(s)ds\right) \mathbf{m}_\phi(dy) \\ &= \frac{1}{\sqrt{2\pi B}} \int_{-\infty}^{\infty} \exp(u) \exp\left(-\frac{(u-A)^2}{2B}\right) du \\ &= \frac{1}{\sqrt{2\pi B}} \int_{-\infty}^{\infty} \exp\left(-\frac{1}{2B}[u^2 - 2Au - 2Bu + A^2]\right) du \\ &= \exp\left(-\frac{A^2}{2B}\right) \exp\left(\frac{(A+B)^2}{2B}\right) \int_{-\infty}^{\infty} \exp\left(-\frac{[u-(A+B)]^2}{2B}\right) du \\ &= \exp\left(\frac{2AB + B^2}{2B}\right) \\ &= \exp\left(\frac{1}{3}S^3 + \frac{1}{30}S^6\right). \end{aligned}$$

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