Summer 6-2013

Thickness-shear and thickness-twist vibrations of circular AT-cut quartz resonators

Huijing He

University of Nebraska-Lincoln, he.hui.jing@hotmail.com

Jiashi Yang

University of Nebraska-Lincoln, jyang1@unl.edu

Qing Jiang

Sun Yat-sen University

Follow this and additional works at: <http://digitalcommons.unl.edu/mechengfacpub>

Part of the [Mechanics of Materials Commons](#), [Nanoscience and Nanotechnology Commons](#), [Other Engineering Science and Materials Commons](#), and the [Other Mechanical Engineering Commons](#)

He, Huijing; Yang, Jiashi; and Jiang, Qing, "Thickness-shear and thickness-twist vibrations of circular AT-cut quartz resonators" (2013). *Mechanical & Materials Engineering Faculty Publications*. 219.
<http://digitalcommons.unl.edu/mechengfacpub/219>

This Article is brought to you for free and open access by the Mechanical & Materials Engineering, Department of at DigitalCommons@University of Nebraska - Lincoln. It has been accepted for inclusion in Mechanical & Materials Engineering Faculty Publications by an authorized administrator of DigitalCommons@University of Nebraska - Lincoln.

THICKNESS-SHEAR AND THICKNESS-TWIST VIBRATIONS OF CIRCULAR AT-CUT QUARTZ RESONATORS**

Huijing He¹ Jiashi Yang^{1*} Qing Jiang^{2,3}

(¹*Department of Mechanical and Materials Engineering, University of Nebraska-Lincoln,
Lincoln, NE 68588-0526, USA*)

(²*School of Engineering, Sun Yat-Sen University, Guangzhou 510275, China*)

(³*Guangzhou Ande Biotechnology Co. Ltd., Guangzhou 510663, China*)

Received 29 June 2012, revision received 4 September 2012

ABSTRACT Exact solutions for free vibration frequencies and modes are obtained for thickness-shear and thickness-twist vibrations of unelectroded circular AT-cut quartz plates governed by the two-dimensional scalar differential equation derived by Tiersten and Smythe. Comparisons are made with experimental results and the widely-used perturbation solution by Tiersten and Smythe under the assumption of weak in-plane anisotropy. Our solution is found to be much closer to the experimental results than the perturbation solution. For the frequency of the fundamental thickness-shear mode, the error of the perturbation method is 0.4549%, significant in resonator applications.

KEY WORDS plate, linear vibration, resonance

I. INTRODUCTION

Piezoelectric crystals are widely used to make acoustic wave resonators for time-keeping, frequency generation and operation, telecommunication, and sensing. Quartz is the most widely used crystal for resonator applications. A large portion of quartz resonators operate with the so-called thickness-shear vibration modes of a plate^[1,2]. Owing to material anisotropy and electromechanical coupling, theoretical modeling of crystal resonators using the equations of anisotropic elasticity or piezoelectricity is very complicated. Approximations of the governing equations or approximate solution techniques are usually necessary. For analyzing crystal resonators, Mindlin and his coworkers developed two-dimensional equations for motions of elastic and piezoelectric plates^[3–5]. The plate equations make resonator theoretical analysis possible in many practically useful cases^[6–11]. More references along this line can be found in a review article^[12]. Mindlin's plate equations are most effective for the frequency range from zero up to slightly above the fundamental thickness-shear frequency. For higher-order modes

* Corresponding author. E-mail: jyang1@unl.edu

** The work was supported by the Phase III Construction of the '985' Project of Sun Yat-Sen University. We acknowledge the support of the Introduction of Innovative R&D Team Project of Guangdong Province. We also acknowledge the support of the Science and Technology Planning Project of Guangdong Province (No. 2011A060901013) and the Science and Technology Planning Project of Guangzhou (No. 2011Y1-00029). And this work was also supported by the Industry-Universities-Research Cooperation Project of Guangdong Province and Ministry of Education of China (No. 2011A090200123). Additional support from the US Army Research Laboratory/US Army Research Office under agreement number W911NF-10-1-0293 is also acknowledged.

Mindlin's equations become complicated. For higher-order modes, Tiersten and his coworkers developed a scalar equation approach. Specifically, by studying the propagation of long thickness-shear waves in a crystal plate, from the approximate dispersion relations of these waves accurate to the second order of the small in-plane wave numbers, a single two-dimensional scalar differential equation governing the thickness-shear displacement was obtained for the widely used quartz crystal plates of AT-cut^[13] and SC-cut^[14, 15]. This paper is on higher-order modes in AT-cut quartz plate resonators using the scalar equation in Ref.[13].

For most quartz resonators, the shapes of the crystal plates or the electrodes on the top and bottom of the plates are either rectangular or circular. Since the scalar differential equation derived in Ref.[13] is separable in Cartesian coordinates, free vibration frequencies and modes of unelectroded rectangular AT-cut plates can be readily obtained^[13]. Solutions for unbounded contoured resonators made from plates with nonuniform thickness have also been obtained in Cartesian coordinates for AT-cut^[16] and SC-cut^[14]. These modes have further been extensively used in frequency stability analysis of crystal resonators due to a temperature change^[17] and normal^[18–20] or in-plane^[21–23] accelerations.

For circular resonators, however, the situation is different. This is because the scalar differential equation obtained in Ref.[13] is not separable in polar coordinates due to material anisotropy. Hence the analysis of circular resonators is mathematically more challenging than rectangular resonators. Tiersten and Smythe^[13] introduced a perturbation procedure by writing the coefficients of the scalar differential equation as the sum of a mean or isotropic part plus a deviation due to anisotropy. This perturbation procedure was used in a series of later analyses^[24–28] for circular resonators with various complications like contouring, beveled edges, and partial electrodes, etc.

The perturbation procedure proposed in Ref.[13] is valid for resonators with weak anisotropy. For the most widely used fundamental mode AT-cut quartz resonator, the ratio between the deviation and the mean of the coefficients of the scalar differential equation is 0.231^[24] which is not very small. Then it is natural to wonder what the error or inaccuracy of the perturbation procedure is. This can be best answered by a direct comparison with the exact solution of the scalar differential equation, which is obtained in the present paper. We also make comparisons with available experimental results.

II. GOVERNING EQUATION

The n th-order thickness-shear displacement $u_1^{(n)}(x_1, x_3, t)$ of an unelectroded AT-cut quartz resonator with thickness $2h$ is defined by^[13]

$$u_1(x_1, x_2, x_3, t) = \sum_{n=1,3,5,\dots}^{\infty} u_1^{(n)}(x_1, x_3, t) \sin(\eta_n x_2) \quad (1)$$

where

$$\eta_n \cong \frac{n\pi}{2h} \quad (n \text{ is odd}) \quad (2)$$

By studying the propagation of long u_1 waves in the (x_1, x_3) plane, from the approximate dispersion relations of these waves accurate to the second order of the small in-plane wave numbers, it was shown^[13] that $u_1^{(n)}$ is governed by the following two-dimensional scalar partial differential equation:

$$M_n \frac{\partial^2 u_1^{(n)}}{\partial x_1^2} + c_{55} \frac{\partial^2 u_1^{(n)}}{\partial x_3^2} - \frac{n^2 \pi^2}{4h^2} \bar{c}_{66} u_1^{(n)} - \rho \ddot{u}_1^{(n)} = 0, \quad (3)$$

where

$$M_n = c_{11} + (c_{12} + c_{66})r + \frac{4(r\bar{c}_{66} - c_{66})(rc_{22} + c_{12})}{c_{22}n\pi\kappa} \cot \frac{\kappa n\pi}{2} \quad (4)$$

$$\bar{c}_{66} = c_{66} + \frac{e_{26}^2}{\varepsilon_{22}}, \quad \kappa = \left(\frac{\bar{c}_{66}}{c_{22}} \right)^{1/2}, \quad r = \frac{c_{12} + c_{66}}{\bar{c}_{66} - c_{22}} \quad (5)$$

c_{pq} , e_{ip} and ε_{ij} are the usual elastic, piezoelectric and dielectric constants. We consider time-harmonic free vibrations for which all fields have the same $\exp(i\omega t)$ factor which is dropped for simplicity. For

harmonic motions, Eq.(3) is reduced to:

$$M_n \frac{\partial^2 u_1^{(n)}}{\partial x_1^2} + c_{55} \frac{\partial^2 u_1^{(n)}}{\partial x_3^2} - \frac{n^2 \pi^2}{4h^2} \bar{c}_{66} u_1^{(n)} + \rho \omega^2 u_1^{(n)} = 0 \quad (6)$$

III. CIRCULAR RESONATOR

Consider the AT-cut circular quartz resonator in Fig.1. Introduce the following coordinate transformation from (x_1, x_3) to (y_1, y_3) :

$$x_1 = \lambda y_1, \quad x_3 = \mu y_3 \quad (7)$$

Fig. 1. A circular AT-cut quartz plate and coordinate system.

where λ and μ are parameters to be specified later. Equation (7) transforms Eq.(6) into:

$$\frac{M_n}{\lambda^2} \frac{\partial^2 u_1^{(n)}}{\partial y_1^2} + \frac{c_{55}}{\mu^2} \frac{\partial^2 u_1^{(n)}}{\partial y_3^2} - \frac{n^2 \pi^2}{4h^2} \bar{c}_{66} u_1^{(n)} + \rho \omega^2 u_1^{(n)} = 0 \quad (8)$$

We choose

$$\lambda = \frac{1}{\omega_0} \sqrt{\frac{M_n}{\rho}}, \quad \mu = \frac{1}{\omega_0} \sqrt{\frac{c_{55}}{\rho}} \quad (9)$$

where

$$\omega_0 = \frac{n\pi}{2h} \sqrt{\frac{\bar{c}_{66}}{\rho}} \quad (10)$$

Then Eq.(8) becomes the following Helmholtz equation:

$$\frac{\partial^2 u_1^{(n)}}{\partial y_1^2} + \frac{\partial^2 u_1^{(n)}}{\partial y_3^2} + \left(\frac{\omega^2}{\omega_0^2} - 1 \right) u_1^{(n)} = 0 \quad (11)$$

Let the boundary of the circular resonator be defined by

$$x_1^2 + x_3^2 = R^2 \quad (12)$$

In the transformed plane the boundary of the resonator is

$$\lambda^2 y_1^2 + \mu^2 y_3^2 = R^2 \quad (13)$$

which represents an ellipse. For $n = 1$ and $n = 3$, we have $\lambda > \mu$. Therefore the semi-major and semi-minor axes are R/μ and R/λ , respectively.

We introduce elliptical coordinates (ξ, η) defined by^[29]

$$y_1 = c \sinh \xi \sin \eta, \quad y_3 = c \cosh \xi \cos \eta \quad (14)$$

where c is a parameter and $2c$ is the focal distance of the ellipses associated with Eq.(14). Then Eq.(11) takes the following form:

$$\frac{\partial^2 u_1^{(n)}}{\partial \xi^2} + \frac{\partial^2 u_1^{(n)}}{\partial \eta^2} + 2q[\cosh(2\xi) - \cos(2\eta)]u_1^{(n)} = 0 \quad (15)$$

where

$$q = \frac{c^2}{4} \left(\frac{\omega^2}{\omega_0^2} - 1 \right), \quad J = \frac{\sqrt{2}}{2} \sqrt{\cosh(2\xi) - \cos(2\eta)} \quad (16)$$

IV. SOLUTIONS OF THE HELMHOLTZ EQUATION

By the standard method of separation of variables in partial differential equations, we write $u_1^{(n)}$ as

$$u_1^{(n)}(\xi, \eta) = U(\xi) V(\eta) \quad (17)$$

Substitution of Eq.(17) into Eq.(15) results in the following two ordinary differential equations:

$$\frac{d^2 V(\eta)}{d\eta^2} + [\alpha - 2q \cos(2\eta)] V(\eta) = 0 \quad (18)$$

$$\frac{d^2 U(\xi)}{d\xi^2} + [2q \cosh(2\xi) - \alpha] U(\xi) = 0 \quad (19)$$

where α is a separation constant. Equations (17) and (18) are known as the angular and radial Mathieu equations, respectively.

A. Solutions of the Angular Mathieu Equation

Equation (18) has four kinds of periodic solutions with period 2π ^[29]:

$$se_{2m+1}(\eta, q) = \sum_{r=0}^{\infty} A_{2r+1}^{(2m+1)}(q) \sin[(2r+1)\eta] \quad (20)$$

$$se_{2m+2}(\eta, q) = \sum_{r=0}^{\infty} A_{2r+2}^{(2m+2)}(q) \sin[(2r+2)\eta] \quad (21)$$

$$ce_{2m}(\eta, q) = \sum_{r=0}^{\infty} B_{2r}^{(2m)}(q) \cos(2r\eta) \quad (22)$$

$$ce_{2m+1}(\eta, q) = \sum_{r=0}^{\infty} B_{2r+1}^{(2m+1)}(q) \cos[(2r+1)\eta] \quad (23)$$

where $m=0, 1, 2, 3, \dots$. $A_r^{(m)}(q)$ and $B_r^{(m)}(q)$ are coefficients for the series of η and they depend on q . They are solutions of the following equations obtained by substituting Eqs.(20)-(23) into Eq.(18):

$$\begin{bmatrix} 1-q-\alpha & q & 0 & 0 & 0 & 0 & \dots \\ q & 3^2-\alpha & q & 0 & 0 & 0 & \dots \\ 0 & q & 5^2-\alpha & q & 0 & 0 & \dots \\ 0 & 0 & q & 7^2-\alpha & q & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{Bmatrix} A_1^{(2m+1)} \\ A_3^{(2m+1)} \\ A_5^{(2m+1)} \\ A_7^{(2m+1)} \\ \dots \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dots \end{Bmatrix} \quad (24)$$

$$\begin{bmatrix} 2^2-\alpha & q & 0 & 0 & 0 & 0 & \dots \\ q & 4^2-\alpha & q & 0 & 0 & 0 & \dots \\ 0 & q & 6^2-\alpha & q & 0 & 0 & \dots \\ 0 & 0 & q & 8^2-\alpha & q & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{Bmatrix} A_2^{(2m+2)} \\ A_4^{(2m+2)} \\ A_6^{(2m+2)} \\ A_8^{(2m+2)} \\ \dots \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dots \end{Bmatrix} \quad (25)$$

$$\begin{bmatrix} -\alpha & q & 0 & 0 & 0 & 0 & \dots \\ 2q & 2^2-\alpha & q & 0 & 0 & 0 & \dots \\ 0 & q & 4^2-\alpha & q & 0 & 0 & \dots \\ 0 & 0 & q & 6^2-\alpha & q & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \begin{Bmatrix} B_0^{(2m)} \\ B_2^{(2m)} \\ B_4^{(2m)} \\ B_6^{(2m)} \\ \dots \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \dots \end{Bmatrix} \quad (26)$$

$$\begin{bmatrix} 1+q-\alpha & q & 0 & 0 & 0 & 0 & \cdots \\ q & 3^2-\alpha & q & 0 & 0 & 0 & \cdots \\ 0 & q & 5^2-\alpha & q & 0 & 0 & \cdots \\ 0 & 0 & q & 7^2-\alpha & q & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{Bmatrix} B_1^{(2m+1)} \\ B_3^{(2m+1)} \\ B_5^{(2m+1)} \\ B_7^{(2m+1)} \\ \cdots \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \cdots \end{Bmatrix} \quad (27)$$

a_{2m+1} , a_{2m+2} , b_{2m} , and b_{2m+1} are the roots of the polynomial equations of α obtained by setting the determinants of the coefficient matrices of Eqs.(24)-(27) to zero. $se_{2m+1}(\eta, q)$ and $se_{2m+2}(\eta, q)$ correspond to modes odd in x_1 (called antisymmetric in this paper). $ce_{2m}(\eta, q)$ and $ce_{2m+1}(\eta, q)$ correspond to symmetric modes. a_{2m+1} and a_{2m+2} are associated with the $(2m+1)$ th and $(2m+2)$ th antisymmetric modes, and b_{2m} and b_{2m+1} the $2m$ th and $(2m+1)$ th symmetric modes. $ce_m(\eta, q)$ and $se_m(\eta, q)$ are orthogonal functions satisfying the following condition:

$$\int_0^{2\pi} ce_m(\eta, q) ce_n(\eta, q) d\eta = \int_0^{2\pi} se_m(\eta, q) se_n(\eta, q) d\eta = \begin{cases} \pi, & \text{if } m = n \\ 0, & \text{if } m \neq n \end{cases} \quad (28)$$

In accordance with Eq.(28), we have the following normalization conditions:

$$\begin{aligned} \sum_{r=0}^{\infty} \left(A_{2r+1}^{(2m+1)} \right)^2 &= 1, & \sum_{r=0}^{\infty} \left(A_{2r+2}^{(2m+2)} \right)^2 &= 1 \\ 2(B_0^{(2m)})^2 + \sum_{r=1}^{\infty} \left(B_{2r}^{(2m)} \right)^2 &= 1, & \sum_{r=0}^{\infty} \left(B_{2r+1}^{(2m+1)} \right)^2 &= 1 \end{aligned} \quad (29)$$

B. Solutions of the Radial Mathieu Equation

The four kinds of radial Mathieu functions corresponding to the above angular Mathieu functions are:

$$\begin{aligned} Se_{2m+1}(\xi, q) &= \frac{se'_{2m+1}(0, q) se_{2m+1}(\frac{\pi}{2}, q)}{\sqrt{q} \left(A_1^{(2m+1)} \right)^2} \\ &\times \sum_{r=0}^{\infty} (-1)^m A_{2r+1}^{(2m+1)} [J_m(v_1) J_{m+1}(v_2) - J_m(v_2) J_{m+1}(v_1)] \end{aligned} \quad (30)$$

$$\begin{aligned} Se_{2m+2}(\xi, q) &= -\frac{se'_{2m+2}(0, q) se'_{2m+2}(\frac{\pi}{2}, q)}{q \left(A_2^{(2m+2)} \right)^2} \\ &\times \sum_{r=0}^{\infty} (-1)^m A_{2r+2}^{(2m+2)} [J_m(v_1) J_{m+2}(v_2) - J_m(v_2) J_{m+2}(v_1)] \end{aligned} \quad (31)$$

$$Ce_{2m}(\xi, q) = \frac{ce_{2m}(0, q) ce_{2m}(\frac{\pi}{2}, q)}{\left(B_0^{(2m)} \right)^2} \sum_{r=0}^{\infty} (-1)^m B_{2r}^{(2m)}(q) J_m(v_1) J_m(v_2) \quad (32)$$

$$\begin{aligned} Ce_{2m+1}(\xi, q) &= -\frac{ce_{2m+1}(0, q) ce_{2m+1}(\frac{\pi}{2}, q)}{\sqrt{q} \left(B_1^{(2m+1)} \right)^2} \\ &\times \sum_{r=0}^{\infty} (-1)^m B_{2r+1}^{(2m+1)}(q) [J_m(v_1) J_{m+1}(v_2) + J_m(v_2) J_{m+1}(v_1)] \end{aligned} \quad (33)$$

where $v_1 = \sqrt{q} \exp(-\xi)$, $v_2 = \sqrt{q} \exp(\xi)$, and J_m are Bessel functions of the first kind.

V. THICKNESS-SHEAR AND THICKNESS-TWIST MODES

We discuss antisymmetric and symmetric modes separately.

A. Antisymmetric Modes

The displacement field of these modes is

$$u_1^{(n)}(\xi, \eta) = \sum_{m=1}^{\infty} S_m S e_m(\xi, q) s e_m(\eta, q) \quad (34)$$

where S_m are undetermined constants. Let the elliptical boundary of the plate in the transformed plane be at $\xi = \xi_0$. The boundary condition is^[13]

$$u_1^{(n)} = \sum_{m=1}^{\infty} S_m S e_m(\xi_0, q) s e_m(\eta, q) = 0 \quad (35)$$

or

$$S e_m(\xi_0, q) = 0 \quad (36)$$

Equation (36) determines q . Then ω can be determined from Eq.(16).

B. Symmetric Modes

Similarly, the symmetric modes are given by

$$u_1^{(n)} = \sum_{m=0}^{\infty} C_m C e_m(\xi_0, q) c e_m(\eta, q) = 0 \quad (37)$$

or

$$C e_m(\xi_0, q) = 0 \quad (38)$$

VI. NUMERICAL RESULTS AND DISCUSSION

For AT-cut quartz the material constants can be found in Ref.[30]. As an example consider a circular plate resonator whose diameter $2R = 7.874$ mm and thickness $2h = 1.10236$ mm. This determines $\omega_0 = 9.467803 \times 10^7$ rad/s or $f_0 = 1.506848 \times 10^7$ Hz, and $\xi_0 = 0.10033$. We use the first thirty terms of each series to calculate the displacement field. For the most widely used fundamental thickness-shear mode, the number of significant figures for its frequency is sixteen when thirty terms are kept in the series.

Figures 2(a) and (b) show the distributions of $u_1^{(1)}$ for the fundamental thickness-shear mode with $n = 1$ and $u_1^{(3)}$ for the third overtone thickness-shear mode with $n = 3$, respectively. Darker areas have larger $u_1^{(1)}$ or $u_1^{(3)}$. The fundamental mode is the most widely used operating mode in devices. When the plate is vibrating in this mode, all particles at the plate top surface vibrate in phase. The plate

Fig. 2. (a) Fundamental thickness-shear mode. $n = 1$. 1st mode of $C e_0 = 0$. $f = 1.5079038 \times 10^7$ Hz. $f_T = 1.5010392 \times 10^7$ Hz ($n = 1$, $m = 0$, $p = 0$ in Ref.[13]). (b) Third overtone thickness-shear mode. $n = 3$. 1st mode of $C e_0 = 0$. $f = 4.5208283 \times 10^7$ Hz. $f_T = 4.5002345 \times 10^7$ Hz ($n = 3$, $m = 0$, $p = 0$ in Ref.[13]).

Table 1. Frequencies of the fundamental ($n = 1$) and the third overtone ($n = 3$)

	$n = 1$	$n = 3$
Experiment	1.50677×10^7 Hz	4.51776×10^7 Hz
Perturbation ^[13] (f_T)	1.50104×10^7 Hz	4.50023×10^7 Hz
Present paper (f)	1.50790×10^7 Hz	4.52083×10^7 Hz

bottom surface has a 180° phase difference from the top surface. The plate middle plane is a nodal plane without motion. The third overtone mode has three nodal planes parallel to the plate surfaces. We note that, because of the in-plane material anisotropy of the plate, the vibration distributions are not exactly circular in Fig.2. Rather, they are slightly longer along x_1 and narrower along x_3 . The distribution in (b) is more confined to the center than that in (a). In the figure caption, the frequencies predicted by our solution are denoted by f , and those by the perturbation solution in Ref.[13] by f_T . The frequency of the third overtone is roughly three times that of the fundamental.

We compare theoretical results with experimental measurements in Table 1, where the experimental results are the averages of ten repeated measurements for each of the two cases of $n = 1$ and 3^[13]. In both cases, our solution is closer to the experimental result than the perturbation solution. For $n = 1$ and 3, the differences between our solution and the perturbation solution are 68600 Hz and 206000 Hz,

Fig. 3. Other thickness-shear modes ($n = 1$). (a) 1st mode of $Se_1 = 0$. $f = 1.5098337 \times 10^7$ Hz. $f_T = 1.5029467 \times 10^7$ Hz ($n = 1$, $m = 1$, $p = 0$ in Ref.[13]). (b) 2nd mode of $Ce_0 = 0$. $f = 1.5129323 \times 10^7$ Hz. $f_T = 1.5047554 \times 10^7$ Hz ($n = 1$, $m = 2$, $p = 0$ in Ref.[13]). (c) 2nd mode of $Se_1 = 0$. $f = 1.5171146 \times 10^7$ Hz. $f_T = 1.5098956 \times 10^7$ Hz ($n = 1$, $m = 1$, $p = 1$ in Ref.[13]).

Fig. 4. Thickness-twist modes ($n = 1$). (a) 1st mode of $Ce_1 = 0$. $f = 1.5092173 \times 10^7$ Hz. $f_T = 1.5023269 \times 10^7$ Hz ($n = 1, m = 1, p = 0$ in Ref.[13]). (b) 1st mode of $Se_2 = 0$. $f = 1.5116458 \times 10^7$ Hz. $f_T = 1.5047660 \times 10^7$ Hz ($n = 1, m = 2, p = 0$ in Ref.[13]). (c) 2nd mode of $Ce_1 = 0$. $f = 1.5154326 \times 10^7$ Hz. $f_T = 1.5078266 \times 10^7$ Hz ($n = 1, m = 1, p = 1$ in Ref.[13]). (d) 2nd mode of $Se_2 = 0$. $f = 1.5202202 \times 10^7$ Hz. $f_T = 1.5128068 \times 10^7$ Hz ($n = 1, m = 2, p = 1$ in Ref.[13]).

respectively. The relative difference is 0.4547% for $n = 1$ and 0.4557 for $n = 3$. This is quite significant because resonator frequency requirements are often in terms of ppm (parts per million).

When $n = 1$, in addition to the mode in Fig.2(a), there are other modes whose frequencies are slightly higher than the mode in Fig.2(a), with nodal lines roughly parallel to the x_3 axis in the (x_1, x_3) plane. Three of these modes are shown in Fig.3. Thickness-shear modes are usually excited electrically by electrodes on the plate top and bottom surfaces. If a nodal line appears under the electrodes, charge cancellation on the electrodes will occur which is usually undesirable. Therefore the mode in Fig.2(a) without nodal lines in the (x_1, x_3) plane is the most useful mode. Still for the case of $n = 1$, there also exist modes with nodal lines roughly parallel to the x_1 axis. One of such modes is shown in Fig.4(a). There are also more complicated modes with nodal lines roughly along both x_1 and x_3 . These modes are called thickness-twist modes. Their frequencies are only slightly different, increasing slowly when there are more nodal lines.

VII. CONCLUSION

Exact thickness-shear and thickness-twist modes are obtained from the scalar differential equation by Tiersten and Smythe for a circular plate crystal resonator of AT-cut quartz. Our solution is closer to experimental results than the perturbation solution by Tiersten and Smyth. For the frequency of

the fundamental thickness-shear mode, the error of the perturbation method is 0.4549%. The results obtained are of fundamental importance in the understanding and design of crystal resonators.

References

- [1] Koga, I., Thickness vibrations of piezoelectric oscillating crystals. *Physics*, 1932, 3(2): 70-80.
- [2] Tiersten, H.F., Thickness vibrations of piezoelectric plates. *Journal of the Acoustical Society of America*, 1963, 35(1): 53-58.
- [3] Mindlin, R.D., High frequency vibrations of crystal plates. *Quarterly of Applied Mathematics*, 1961, 19(1): 51-61.
- [4] Tiersten, H.F. and Mindlin, R.D., Forced vibrations of piezoelectric crystal plates. *Quarterly of Applied Mathematics*, 1962, 20(2): 107-119.
- [5] Mindlin, R.D., High frequency vibrations of piezoelectric crystal plates. *International Journal of Solids and Structures*, 1972, 8(7): 895-906.
- [6] Mindlin, R.D. and Lee, P.C.Y., Thickness-shear and flexural vibrations of partially plated, crystal plates. *International Journal of Solids and Structures*, 1966, 2(1): 125-139.
- [7] Wang, J. and Zhao, W.H., The determination of the optimal length of crystal blanks in quartz crystal resonators. *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control*, 2005, 52(11): 2023-2030.
- [8] Wang, J., Zhao, W.H. and Du, J.K., The determination of electrical parameters of quartz crystal resonators with the consideration of dissipation. *Ultrasonics*, 2006, 44(Suppl.1): E869-E873.
- [9] Zhang, C.L., Chen, W.Q. and Yang, J.S., Electrically forced vibration of a rectangular piezoelectric plate of monoclinic crystals. *International Journal of Applied Electromagnetics and mechanics*, 2009, 31(4): 207-218.
- [10] Wang, J.N., Hu, Y.T. and Yang, J.S., Frequency spectra of at-cut quartz plates with electrodes of unequal thickness. *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control*, 2010, 57(5): 1146-1151.
- [11] Chen, G.J., Wu, R.X., Wang, J., Du, J.K. and Yang, J.S., Five-mode frequency spectra of x_3 -dependent modes in AT-cut quartz resonators. *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control*, 2012, 59(4): 811-816.
- [12] Wang, J. and Yang, J.S., Higher-order theories of piezoelectric plates and applications. *Applied Mechanics Reviews*, 2000, 53(4): 87-99.
- [13] Tiersten, H.F. and Smythe, R.C., Coupled thickness-shear and thickness-twist vibrations of unelectroded AT-cut quartz plates. *Journal of the Acoustical Society of America*, 1985, 78(5): 1684-1689.
- [14] Stevens, D.S. and Tiersten, H.F., An analysis of doubly rotated quartz resonators utilizing essentially thickness modes with transverse variation. *Journal of the Acoustical Society of America*, 1986, 79(6): 1811-1826.
- [15] EerNisse, E.P., Analysis of thickness modes of contoured, doubly rotated, quartz resonators. *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control*, 2001, 48(5): 1351-1361.
- [16] Tiersten, H.F. and Smythe, R.C., An analysis of contoured crystal resonators operating in overtones of coupled thickness shear and thickness twist. *Journal of the Acoustical Society of America*, 1979, 65(6): 1455-1460.
- [17] Stevens, D.S., Tiersten, H.F. and Sinha, B.K., Temperature dependence of the resonance frequency of electroded contoured AT-cut quartz crystal resonators. *Journal of Applied Physics*, 1983, 54(4): 1704-1716.
- [18] Tiersten, H.F. and Shick, D.V., On the normal acceleration sensitivity of contoured quartz resonators rigidly supported along rectangular edges. *Journal of Applied Physics*, 1990, 67(1): 60-67.
- [19] Tiersten, H.F. and Zhou, Y.S., On the normal acceleration sensitivity of contoured quartz resonators with the mode shape displaced with respect to rectangular supports. *Journal of Applied Physics*, 1991, 69(5): 2862-2970.
- [20] Tiersten, H.F. and Zhou, Y.S., On the normal acceleration sensitivity of contoured quartz resonators stiffened by quartz cover plates supported by clips. *Journal of Applied Physics*, 1992, 72(4): 1244-1254.
- [21] Tiersten, H.F. and Zhou, Y.S., On the in-plane acceleration sensitivity of contoured quartz resonators supported along rectangular edges. *Journal of Applied Physics*, 1991, 70(9): 4708-4714.
- [22] Tiersten, H.F. and Zhou, Y.S., The increase in the in-plane acceleration sensitivity of the plano-convex resonator resulting from its thickness asymmetry. *Journal of Applied Physics*, 1992, 71(10): 4684-4692.
- [23] Zhou, Y.S. and Tiersten, H.F., In-plane acceleration sensitivity of contoured quartz resonators stiffened by quartz cover plates supported by clips. *Journal of Applied Physics*, 1993, 74(12): 7067-7077.
- [24] Tiersten, H.F. and Zhou, Y.S., Transversely varying thickness modes in quartz resonators with beveled cylindrical edges. *Journal of Applied Physics*, 1994, 76(11): 7201-7208.
- [25] Yang, J.S. and Tiersten, H.F., An analysis of contoured quartz resonators with beveled cylindrical edges. In: *Proceedings of the IEEE International Frequency Control Symposium 1995*, Institute of Electrical and Electronics Engineers, 1995: 727-739.

- [26] Yang, J.S. and Tiersten, H.F., The influence of the free edge on the vibration characteristics of a contoured, beveled cylindrical quartz resonator. In: Proceedings of the IEEE International Frequency Control Symposium 1996, Institute of Electrical and Electronics Engineers, 1996: 657-664.
- [27] Huang, L.D., Tiersten, H.F. and Yang, J.S., An analysis of contoured quartz resonator with beveled cylindrical edges using the correct variation of thickness. In: Proceedings of the IEEE International Frequency Control Symposium 1997, Institute of Electrical and Electronics Engineers, 1997: 668-676.
- [28] Yang, J.S., An analysis of partially electroded, contoured quartz resonators with beveled cylindrical edges. *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control*, 2007, 54(11): 2407-2409.
- [29] McLachlan, N.W., Theory and Application of Mathieu Functions. Oxford: Oxford Press, 1951.
- [30] Tiersten, H.F., Linear Piezoelectric Plate Vibrations. New York: Plenum, 1969.