Spring 3-19-2020

A Mathematical Model of Speeding

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Using Markov Chains to Model Speeding Interactions in Cities

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A thesis presented for the degree of
Bachelor of Science

Mathematics Department
University of Nebraska - Lincoln
USA

March 19th, 2020

Abstract

Crime is often regarded as nonsensical, impulsive, and irrational. These conjectures are pointed, though conversation about the pros and cons of crime does not happen often. People point to harsh fines, jail times, and life restrictions as their reason for judgement, stating that the trade-offs are far too unbalanced to participate in illicit activity. Yet, everyday people commit small crimes, sometimes based on hedonistic desires, other times based on a rational thought process. Speeding seems to be one of those that almost all people commit at least once during their life.

Our work hopes to make an incremental improvement on the question of "Is speeding rational?" The primary focus is to build a working, reasonable mathematical model. Once we have our model, we can begin gathering results that either verify expectations, provide insight into the workings of the system, or offer implications for speeders, officers, and their environments. From there we open the door to answering questions about the rationality of speeding.

Contents

1 Introduction

The questions of why people speed and how people get caught (or avoid getting caught) is one with many, many layers. As discussed below, there are a great deal of options present for modeling speeding using mathematics. Our approach included using a discrete time Markov Chain on a grid with a city center. Drivers speeds are changing, and Officer enforcement strategies can vary. Certainly this paper is not exhaustive of all possible modeling decisions.

The function of this paper is not entirely results driven, rather, it is a candid approach to modeling speeding using mathematics. We hope to guide the reader through our modeling process, first with drivers, then with officers, and then with the environment. After the construction of the model, we provide a few results that describe the structure the model gives to the problem. We then consider a more applied case study of the model that is intended to serve as a guide for the expectations of this model's behavior.

2 Driver Section

In this section, we explore the details behind drivers. We discuss how we might model them, how they might think and rationalize their decisions, and how they might move around in whatever environment they're placed in.

2.1 Building a Model

Balancing the motivations of this project with its mathematical efficacy was a key challenge. There were many variables identified that can affect the decisions drivers make when they speed. Variables such as

- Personal speed preferences
- Speed limits
- Speed of traffic (a driver is likely to mimic other drivers around them)
- Density of traffic (denser traffic makes speeding difficult)
- Familiarity with the locations of speed traps
- Perceived risk
- Payoff of speeding
- "Randomness" in how officers choose to pull people over
- The way officers choose to deploy themselves
- Traffic control structures
- Driving environment (an actual city, a a highway, etc.)
- Many, many more

For obvious reasons, managing this many variables is an unconscionable way to first approach the problem. Instead, our approach involved building a realistic model, adding variables that were interesting and manageable, proving results, and extending those results to situations that are more complicated.

Firstly, we decided to not consider variables that limited a driver. For instance, we intentionally left out variables such as traffic control structures, and the density of traffic. The reason being that our problem is more interested in questions about speeding, and not traffic. Traffic is well studied in PDEs, therefore it seemed likely that whatever results were achieved without traffic based on a simple enough model would be able to extend to situations with traffic.

Next, we eliminated driver knowledge bias. The only knowledge a driver could potentially have is of their own personal history with the police, or what is immediately around them at any time. The reason being similar to that above: the question is about speeding, and not of collecting information about officer deployment strategies. However, there are interesting questions to be asked about the game theoretical consequences of enforcement strategies (to include frequency of stops and physical location) against different speeding strategies. We will outline what this might look like later on.

2.2 Understanding Psychology

In order to best model drivers, we must attempt to discover the primary force that motivates speeding. Is it because it will lead to a faster arrival time? Is it to avoid hitting lights? If one sees a green light ahead they might speed up to make sure they make it. Is it because the driver simply feels comfortable going at a faster speed because it is enjoyable, if they know they're speeding at all? These reasons seem to summarize most of the arguments "casual" speeders make for their decision to speed. Other drivers going faster may be motivated by adrenaline, recklessness, or other reasons that extend beyond the desire to speed itself, i.e. the desire to "win" against power structures like police officers, or mental health issues.

Ultimately, we needed to make our model a mathematical one. In order to preserve uniformity across the environment, we figured every driver simply had a "preferred speed," which inherently factors in comfortability with the speed, and urgency to arrive at one's next location. The construction of driver speeds will be discussed in the next section.

2.3 Moving Around

2.3.1 Direction

In order for drivers to be speeding, they need to be moving. But how should they move? What influences a driver's decision to go one way versus another? Are the drivers trying to intentionally avoid "hot zones" where they know enforcement is more likely. These are all valid questions, but they detract from the core focus of the project, at least for now. The core focus lies in comparing *speeding* strategies and outcomes, while these questions involve a layer of *routing* or traffic flow. Perhaps our model could be incorporated with existing traffic models, but our core focus is elsewhere. Thus, we will assume the drivers choose their route randomly, following some probability distribution. Ultimately, the drivers will be performing a biased random walk on the vertex set $[-n, n]^2$ where $[-n, n] = \{-n, -n+1, \ldots, 0, 1, \ldots, n\}$. This random walk will be biased towards the city center, that is, the origin $(0, 0) \in [-n, n]^2$ in a way that we will specify later.

2.3.2 Speed

Furthermore, how fast should the drivers move? What are the major factors? Ultimately, we decided every driver factors in three speeds when driving around.

1. The speed limit
2. The speed of everyone around them
3. The speed they want to go

Thus, the speed of each driver d relies on a general weighting function $\mathcal{W}_{d,t}$ acting on a vector X_d whose components are different speeds. The weighting function utilizes a convex combination of three weights $w_{1,t}, w_{2,t}, w_{3,t}$ on the three different speed values x_1 (speed limit), x_2 (traffic speed), x_3 (target speed) to determine the speed of any driver at a specific time t. It is important to note that this weighting function is not fixed across drivers, or time. Target speeds are generally proportional to the speed limit by a factor of λ_d, i.e. $x_3 = \lambda_d \cdot x_1$. In general, the speed of a driver d at time t is

$$\mathcal{W}_{d,t}(X_d) = w_{1,t}x_1 + w_{2,t}x_2 + w_{3,t}(\lambda_d x_1)$$

The traffic speed is important, as not only do drivers want to avoid the appearance of going too fast, but also the speed of traffic might limit the ability of a driver to speed. For this reason, we will often assume that $w_{2,t}$ is constant for all t, writing w_2 instead. Otherwise, this weighting function may change with respect to time. If an officer pulls driver d over at time step t, as officers act in discrete time steps, then the weight of a driver's target speed may drop to zero, since drivers are acutely aware of the law. I.e., if a driver is pulled over at time t, then

$$w_{1,t+1} = (1 - w_2), \quad w_2 = w_2, \quad w_{3,t+1} = 0,$$

If a driver is not pulled over in a time step, then they decrease their inclination to follow the speed limit by a factor of ρ, given below

$$w_{1,t+1} = \rho w_{1,t}, \quad w_{2,t+1} = w_{2,t}, \quad w_{3,t+1} = (1 - \rho w_{1,t} - w_2)$$

3 Officer Section

In this section, we explore the details of Officers. What are their motivations for enforcing the law beyond earning a paycheck? How do they measure success, how do they deploy themselves, and how do they determine who gets a ticket?

3.1 Motivations

There are many incentives police have for pulling over a speeder. Whether it be for public safety, ticket income, or a door to a closer inspection of the driver, and whether they are guilty of more serious crimes. Each reason can shift the lens of our model. Naturally, we hope to first deliver a robust, adaptable model, where results are left generalized. In any case, it is helpful to identify precisely what metrics officers might find appropriate.

3.1.1 Officer Enforcement Metrics

With primary goals in hand, we can begin to formulate how officers decide who to pull over or not. We can implement this by crafting metrics that measure an aspect of the system officers might want to control, and then modifying officer behavior to optimize the metric. We've identified a few metrics that represent clear outcomes for the police officer community. For notation, let D be the set of all drivers, and speed(d) be the speed of a driver $d \in D$. Let L_d be the speed limit of the edge driver $d \in D$ is on.

Optimal Metric 1. Officers try and minimize the largest deviation from the speed limit, that is, minimize the $\|\cdot\|_\infty$ of the driver speeds of the system. Formally, the goal for the officers is to minimize:

$$\max_{d \in D} \Big(\text{speed}(d) - L_d \Big)$$

Optimal Metric 2. Officers try and minimize the global speed of the system, that is, they try to minimize the $\|\cdot\|_p$ for some small p. We chose $p = 2$ as an intermediate case between $\|\cdot\|_1$ and $\|\cdot\|_\infty$. I.e., officers try to minimize

$$\sum_{d \in D} \|\text{speed}(d) - L_d\|_2$$

Optimal Metric 3. Officers try to maximize department revenue receiving the most money from tickets.

Optimal Metric 4. Officers try to keep the roads safe, and minimize the rate of accidents, i.e. officers try to minimize the number of car accidents each day. This metric has many factors, as speeding was a contributing factor of 26% of traffic fatalities in 2017 [1].

3.2 Moving Around... or Not?

We've established how drivers should move, but what about police officers? Police officers both patrol roads, and establish speed traps. As we'll see in the next section, we will be working in an environment similar to a city, where speed traps are more common and intentional. Thus, officers will be stationary, and they will be placed at intersections of roads. Recall that we model our city as a graph on a vertex set $[-n, n]^2$, and the city center is the origin $(0, 0)$.

3.2.1 Officer Position Strategies

Our goal is to optimize different enforcement goals across a variety of strategies employed by officers. Since officers are static, under a fixed enforcement strategy, the only thing that may change is their deployment across the city.

Deployment Strategy 1. Officers all deploy to the center of the city.

Deployment Strategy 2. Officers are deployed uniformly on $[-n, n]^2$, i.e. every intersection receives the same number of officers.

Deployment Strategy 3. Officers are deployed proportionally to the steady state distribution of the drivers. Meaning, after the system has been running for a long time we expect the ratio of between officers and drivers to be around the same for every intersection.

Deployment Strategy 1 offers high interaction with all drivers, as the City Center will be the most visited location. Deployment Strategy 2 seeks to equally deploy the officers across all states, and Deployment Strategy 3 hopes to engage the same fraction of drivers at each vertex.

3.3 Who Gets Pulled Over?

Ideally, we would like to determine a strategy that tells officers who to pull over and when such that this strategy optimizes one of the given metrics. However, in some components of our analysis, we focus on driver behavior. In these situations, we think about how officers behave in real life.

3.3.1 How Officers Seem to Behave

Officers do not always pull over drivers who are speeding, for whatever reason that may be. Perhaps they aren't speeding "enough," or the officer's primary concern is not to distribute tickets to low-significance violators. Thus, we assume that officers always pull over speeders, that is, the speed limit in our environment is the effective speed limit from enforcement. For instance, the speed limit may be 35mph, but officers might not pull those over going less than 42mph, therefore the effective speed limit of the system is 42mph, and we can adjust the system to be centered around this "effective" speed limit.

Furthermore, once an officer does decide to pull someone over, it takes time to complete the stop. The officer selects a driver, runs their plates through a database query, approaches the vehicle, runs the license, insurance information, and registration through a database, issues their ticket(s), sends the car on its way, and finishes any remaining paperwork as they prepare themselves for their next interaction. During this, the officer may question the driver, and perform other tasks not mentioned above. Thus, there is some time that each officer should spend on each traffic stop, but the amount of time can change. Thus, it is natural to assume that this waiting time until the officer is ready to stop a new driver is exponentially distributed (if time is continuous, geometric if discrete), as we assume it to be a memoriless process. Moreover, this waiting time is independent from driver to driver. Then a Poisson point process allows us to capture this behavior, where each event in the process is the officer "waking up" from stop, signalling they are ready to pull someone over. Even if an officer "wakes up" multiple times before a traffic stop, they simply remain awake. Thus, we can model officers being "awake," or ready to pull someone over, as a Poisson process with rate μ, which is inversely proportional to the expected amount of time between when an officer begins a traffic stop, and is ready to to begin another traffic stop.

Furthermore, if drivers' inter-arrival time is exponentially distributed, then we can combine these two processes into one, where some events are drivers arriving, and others are officers waking up, seen below:

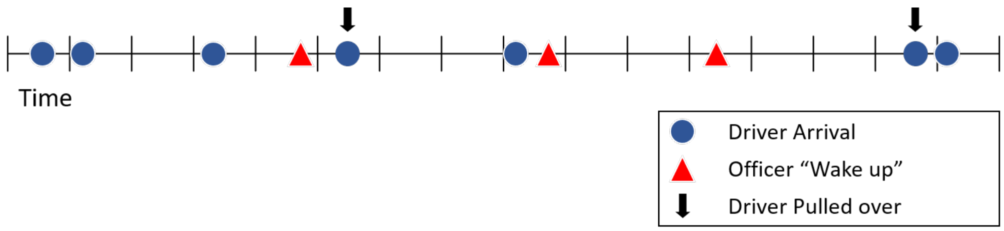

Figure 1: The combined Poisson Point process with arriving drivers, and officers waking up, with officers pulling over the first driver they see after they wake up, marked by the black arrow.

4 Environment/Basics

Finally, after fleshing out the subjects of our study, we turn towards constructing an environment where they can interact. Once an environment is built, we can start to gain results and a better understanding of the problem at hand. Among the results are some that are trivial and not new, and some that illuminate interesting mathematical facts that may have more applications than the problem at hand.

4.1 Choosing the Model

When first building the model, it became apparent that there are multiple ways in which the environment dictates how drivers behave. For instance, those making a long road trip may have more to gain by going a few miles over the speed limit than those commuting across town for work. For those commuting across town, they are "competing" against many other drivers to not be pulled over, while highway traffic is more sparse. Furthermore, when only travelling in one dimension on a less crowded highway, one can build an intuition for where officers may be positioned, and may be able to see an officer before the officer is able to measure their speed. Additionally, a major concern of officers on highways is trafficking, and thus it is hard to know with certainty how they decide how to pull drivers over. This dichotomy presented us with two options: either focus on how the driver can avoid detection from an officer (the highway case), or focus on how the driver should behave when there are many other cars present (city case). We decided to focus on the situation where there are many drivers, as this would enable us to use more arguments from asymptotics.

4.1.1 A City with a Center

Ultimately, we want our results to be fore some kind of grid with a center vertex attracting the drivers. What we will create is a $2n+1 \times 2n+1$ grid where drivers choose whether to move horizontally or vertically uniformly, moving towards the city center with probability p, and away with probability $q = 1 - p$, where $p > q$, pictured in Figure 2.

4.1.2 Restricting the Scope

Thus, taking this decision in mind, and the information from the above sections, we have limited the scope of or model to one that factors in driving preference, the speed limit, speed of traffic, a city environment, and different enforcement strategies. From this, we built our model that initially incorporates a few adjustments in order to make results more pronounced.

1. Officers may pull over more drivers than normal

2. Officers may appear more often than normal

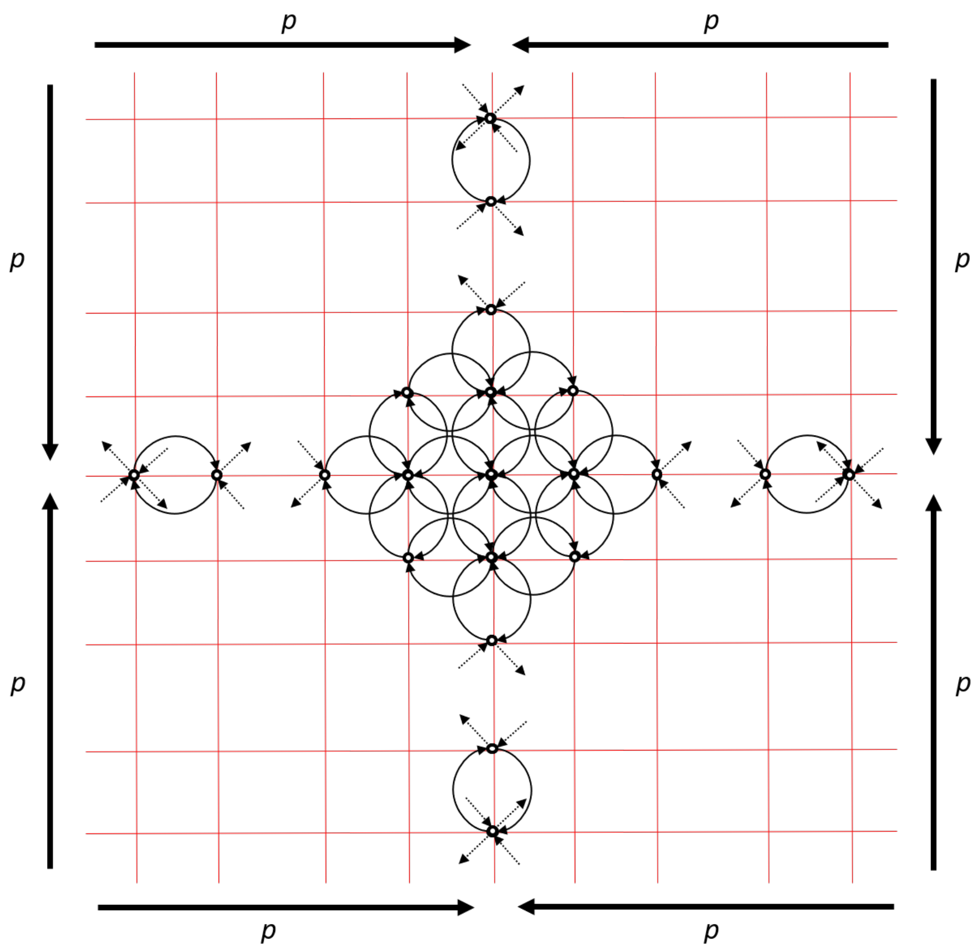

Figure 2: The Markov chain on a grid, moving towards the center with probability p, and away with probability q. (This graph hides the fact that at the origin, a driver can remain at 0 with probability p, and on the boundary, a driver's position can be fixed.)

3. Officers' positions are fixed
4. Drivers choose direction randomly
5. Everyone travels at least the speed limit

The reason for 1 and 2 is it appears being pulled over is a rare event. What we focused on during the beginning of our project is the relative likelihood of being pulled over based on habitual speed. In reality, the probability of being pulled over is quite small. Just consider the ratio of how many times you've been pulled over to how many miles you've driven. Thus, any small deviations in this may render insignificant to those who are asked. An increase from 0.00001% to 0.00005% is nothing. However, when driving past an officer, being five times as likely to be pulled over is quite unnerving. Thus, the likelihood of being pulled over only truly matters when interacting with police officers, hence the desire to make those interactions less rare. There are, however, side effects of this decision, which we will discuss later.

The reason for 3 is that with mobile officers, one can have a hard time managing who is truly "eligible" to be pulled over. In addition, it makes managing the model more difficult. Simplification 4 was added to avoid issues of traffic and route decisions for drivers. We begin our investigation eliminating any bias that a particular route may give, and treat each vertex as the same locally.

Simplification 5 is to make our functions a bit nicer. With drivers going under the speed limit, the model can get complicated trying to avoid pulling over those that aren't speeding. Of course, we can think of this speeding network as a subnetwork of a city with drivers of all speeds.

4.2 Building the Environment

4.2.1 Setup

Constructing an environment for this problem was difficult. We wanted to be able to use a discrete model on a connected set of nodes, with the nodes representing intersections in the city. We will first work on a flat network of nodes with a uniform speed, and then extend our results to a product of these networks obtain a grid, and a quotient to extend our results to a torus.

Thus, ignoring speeds, we can model the movement of each driver as an independent random walk on the grid by taking discrete time steps. The vertex set is $[-n,n]^2$ where $[-n,n]=\{-n,-n+1,\dots,0,1,\dots,n\}$. The edge set $E=\{\big((i,j),(k,l)\big)\mid i=k \text{ and } |j-l|=1, \text{ or } j=l \text{ and } |i-k|=1\}$. That is, the edges are those between horizontally or vertically adjacent coordinate pairs on the two dimensional integer lattice. Drivers will then move vertically or horizontally uniformly at random, selecting to move towards the city center (the vertex (0, 0)) with probability p, and away with probability q. If at the center, drivers can remain at the center with probability p, and leave with probability q, while on the boundary of the grid, drivers can remain on the boundary with probability q (if the boundary restricts movement in that direction, i.e. trying to move left from $(-n,0)$). The grid in reference can be found in Figure 2.

The above setup gives us the skeleton of the idea of our model. However the problem is not interesting if everyone is traveling the same speed, which is what a discretized set of intersections would require for easy analysis. We addressed this issue by further discretizing the edges between intersections. We can then take the limit as the number of discretizations on the edges goes to infinity, and construct a continuous time Markov chain, where the states are the intersections (nodes) and traveling between intersections (edges).

So, keeping in mind what the environment *will* be, we first work on the half (discrete) interval ($[0,n]$), extend to the full (discrete) interval ($[-n,n]$), and then extend our results to the product, which yields a $2n+1\times 2n+1$ grid. Then, we will discretize the edges to incorporate speeding.

4.2.2 Construction of Base Chain

We first build a base Markov Chain on $n+1$ states, $\{0,1,\dots,n\}$, where $n\in\mathbb{N}$. Let X_t be the position at time t with transitional probabilities given below, (with $p+q=1$).

$$\begin{cases} P(X_t=0|X_{t-1}=0) & =p \\ P(X_t=n|X_{t-1}=n) & =q \\ P(X_t=j|X_{t-1}=j+1) & =p \\ P(X_t=j|X_{t-1}=j-1) & =q \\ P(X_t=j||X_{t-1}-j|>1) & =0 \end{cases} \tag{1}$$

I.e., we have a finite irreducible Markov chain where the position moves to the right with probability q, and to the left with probability p, with ends that can travel to themselves. Furthermore, this chain is aperiodic, as the self-reflecting ends remove any periodicity from the system. The Markov chain can be pictured below in Figure 3.

Technically, the whole system with all drivers is a product of these chains, however since drivers behave independently, we can restrict our analysis to a single driver. So, we can simply say a driver takes a biased random walk on $N=\{0,\dots n\}$.

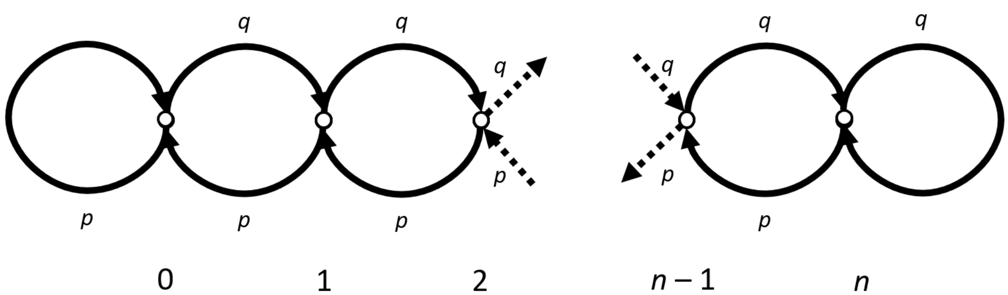

Figure 3: The random walk on the grid graph moving towards zero with probability p, and away from 0 with probability q.

Lemma 1. The chain above is a time-reversible Markov chain with steady state distribution π, when $p > q$, given by

$$\begin{aligned} \pi_0 &= \frac{1 - q/p}{1 - (q/p)^{n+1}} \\ \pi_i &= \left(\frac{q}{p}\right)^i \pi_0 \text{ for } 0 \leq i \leq n \end{aligned} \tag{2}$$

Proof. For all $j \in \{1, .., n\}$. For notational purposes, let the state -1 be another label for state 0, and $n+1$ another label for state n. We check to see that it is normalized.

$$\begin{aligned} \sum_{i=0}^{n} \pi_i &= \sum_{i=0}^{n} \left(\frac{q}{p}\right)^i \pi_0 \\ &= \pi_0 \sum_{i=0}^{n} \left(\frac{q}{p}\right)^i \\ &= \pi_0 \frac{1 - \left(\frac{q}{p}\right)^{n+1}}{1 - \frac{q}{p}} \\ &= 1 \end{aligned} \tag{3}$$

We next verify that the distribution is time-reversible, so then it is a steady state [2]. I.e., for all $i, j \in N$,

$$\pi_i P(X_t = j | X_{t-1} = i) = \pi_j P(X_t = i | X_{t-1} = j) \tag{4}$$

If $|j - i| > 1$, then

$$\pi_i P(X_t = j | X_{t-1} = i) = 0 = \pi_j P(X_t = i | X_{t-1} = j) \tag{5}$$

Else, W.L.O.G. assume $j = i + 1$

$$\begin{aligned} \pi_i P(X_t = j | X_{t-1} = i) &= \pi_i P(X_t = j | X_{t-1} = j - 1) \\ &= (q) \left(\frac{q}{p}\right)^i \pi_0 \\ &= p \left(\frac{q}{p}\right)^{i+1} \pi_0 \\ &= p \left(\frac{q}{p}\right)^j \pi_0 \\ &= \pi_j P(X_t = i | X_{t-1} = j) \end{aligned} \tag{6}$$

Thus the claim is true. ∎

4.2.3 Sign Extensions of the Base Chain

The current chain currently views 0 as an "endpoint" to the chain, but we would like to think of the origin as a "city center" that attracts traffic. Thus, we first define the Sign Extension $\widetilde{X}_t$ of the Markov chain X_t given above to be a Markov chain on the state space $\Big((N \times \{+,-\})/\sim$ where $(0,+) \sim (0,-)$. For notational purposes, let $(j,+) = j^+$, and $0 = 0^+ = 0^-$. Furthermore, assume for notation that when j^+, j^- is seen, that $j^+ \neq 0, j^- \neq 0$, We define the transitional probabilities as

$$\begin{cases} P(\widetilde{X}_t = j^+ | \widetilde{X}_{t-1} = i^-) & = 0 \\ P(\widetilde{X}_t = j^+ | \widetilde{X}_{t-1} = i^+) = P(\widetilde{X}_t = j^- | \widetilde{X}_{t-1} = i^-) & = P(X_t = j | X_{t-1} = i) \\ P(\widetilde{X}_t = 0 | \widetilde{X}_{t-1} = i^+) = P(\widetilde{X}_t = 0 | \widetilde{X}_{t-1} = i^-) & = P(X_t = 0 | X_{t-1} = i) \\ P(\widetilde{X}_t = j^+ | \widetilde{X}_{t-1} = 0) = P(\widetilde{X}_t = j^- | \widetilde{X}_{t-1} = 0) & = \frac{1}{2} P(X_t = 1 | X_{t-1} = 0) \end{cases} \tag{7}$$

In other words, the Sign Extension $\widetilde{X}_t$ can be viewed simply as X_t, choosing a sign uniformly at random whenever the chain leaves the 0 vertex. Note then that the steady state distribution of the Sign Extension is simply

$$\begin{aligned} \widetilde{\pi}_0 &= \pi_0 \\ \widetilde{\pi}_{j^+} = \widetilde{\pi}_{j^-} &= \frac{1}{2}\pi_j \end{aligned} \tag{8}$$

This is true, as we have a finite state, irreducible, aperiodic Markov chain that, when restricting to the positives or negatives, must match π. Furthermore, we could make the Sign Extension a closed loop by modding by the equivalence $\{n^+ \sim n^-\}$, and this would yield a very similar Markov chain and distribution, only with the change that we can switch signs at n as well, and the steady state itself is exactly the same with the added adjustment $\widetilde{\pi}_n = \pi_n$. One can view the sign extension as the chain on $[-n, n]$, and whenever a driver is at 0, they choose to go left or right uniformly, and then flipping a coin to move away from zero, or remain at zero, pictured below:

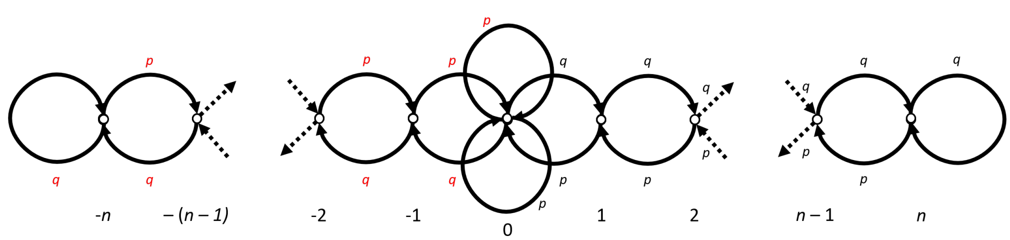

Figure 4: The sign extension of our original base chain in Figure 3. Note that we are still able to transition from 0 to 0 in each time step.

Thus, we have three main base chains that can be extended to chains with discretized edges, or continuous edges:

- The chain on states $\{0, \ldots, n\}$
- The chain on states $\{-n, \ldots, 0, \ldots, n\}$
- The chain on states $\{-n, \ldots, 0, \ldots, n\}/\{n \sim -n\}$

4.2.4 Extending to Products

The construction above is for a one dimension Markov Chain. Thus, we construct a grid of "streets" by taking a product of such Markov Chains, and flipping a coin when deciding to travel vertically or horizontally, rendering the two chains independent.

Theorem 1. Suppose there are Markov processes on a collection of finite state spaces $\{X_n\}_{n=1}^N$, $N \in \mathbb{N}$ with corresponding stationary distributions $\{\pi_n\}_{n=1}^N$. Consider a Markov process on $X = \prod_{n=1}^N X_n$, where at each time step, the Markov process changes one coordinate, selecting which coordinate to change uniformly at random. Then the steady state distribution for X is $\pi = \prod_{n=1}^N \pi_n$, i.e. for all $(a_1, a_2, .., a_N) \in X$,

$$\pi((a_1, ..., a_N)) = \pi_1(a_1) \cdot \pi_2(a_2) \cdot ... \cdot \pi_N(a_N)$$

Proof. Clearly, π is a distribution. We need only check if it is a steady-state distribution. Let $a = (a_1, \ldots, a_N) \in X$ and $j \in \{1, ..., n\}$ be given. Let $b_j : X \to X$ be given by $b_j(c) = b$, where $b_i = a_i$ if $i \neq j$, and $b_i = c$ if $i = j$. I.e., b is a, but with the jth coordinate of a replaced with c. Let $P(s,t)$ be the probability that after one time step we are at state s given in the previous time step we were at state t. Then

$$\begin{aligned}
\sum_{x \in X} \pi(x) P(x,a) &= \sum_{j=1}^{N} \sum_{x \in X} \pi(b_j(x_j)) P(b_j(x_j), a) \\
&= \sum_{j=1}^{N} \sum_{x \in X} \pi(b_j(x_j)) \frac{1}{N} P(x_j, a_j) \\
&= \sum_{j=1}^{N} \sum_{x \in X} \left(\prod_{k \neq j} \Big(\pi_k(a_k)\Big) \cdot \pi_j(x_j) \cdot \frac{1}{N} \pi_j(x_j) P(x_j, a_j) \right) \\
&= \sum_{j=1}^{N} \left(\frac{1}{N} \prod_{k \neq j} \Big(\pi_k(a_k)\Big) \sum_{x \in X} \pi_j(x_j) P(x_j, a_j) \right) \\
&= \sum_{j=1}^{N} \left(\frac{1}{N} \prod_{k \neq j} \Big(\pi_k(a_k)\Big) \sum_{x'_j \in X_j} \pi_j(x'_j) P(x'_j, a_j) \right) \\
&= \sum_{j=1}^{N} \left(\frac{1}{N} \prod_{k \neq j} \Big(\pi_k(a_k)\Big) \pi_j(a_j) \right) \\
&= \sum_{j=1}^{N} \left(\frac{1}{N} \prod_{k=1}^{N} \Big(\pi_k(a_k)\Big) \right) \\
&= \pi(a)
\end{aligned} \tag{9}$$

Thus, π is a steady-state distribution. □

As a note, this theorem allows us to extend our results to a $2n+1 \times 2n+1$ grid with a center attractor that can reflect to itself, and boundary that can reflect to itself, and a torus with one attractor and one repeller, both which can reflect to themselves. These extensions can be derived from our chains at the end of Section 4.2.3

Corollary 1. The steady state distribution of the grid is

$$\pi^*_{(i,j)} = \widetilde{\pi}_i \cdot \widetilde{\pi}_j \tag{10}$$

Proof. An easy application of the above theorem. □

4.3 Discretizing the Edges

4.3.1 The Idea

Now we return to the original Markov chain on the half interval $[0, n]$, where we will assign speeds to edges. We then slice up each edge inversely proportional to the speed on that edge so that lower speed edges have more discretizations, or segments, making the edge "longer" so that drivers move "slower" on that edge. The discretization given below extends easily to the sign extension chain, and similarly to the grid, as the only real mechanism at work is the finite amount of edges, which is preserved through the extensions.

The idea is simple, though the construction can be a bit hectic. For every edge assign a speed. Speed truly only matters relative to eachother, so we can take our timestep to be a convenient one - one that makes sense for every speed. If we take l to be the least common multiple of all the speeds, then we let the time step be $1/l$. Therefore, we can discretize the edges proportionally to l. I.e., if the speed along a certain edge is $l/5$, then there will be $\frac{l}{l/5} = 5$ discretizations along the edge (adding four new "states" on the edge). Meaning, the faster the speed on an edge, the less discretizations there are, meaning drivers move through the edge faster.

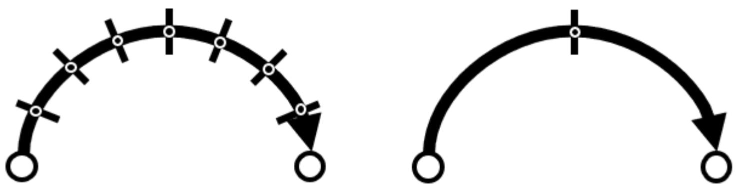

Figure 5: The edge on the left has 8 segments (discretizations), while the edge on the right has 2 segments. Thus, the speed of the edge on the right is 4x that of the speed on the left. We could say that the speed on the right is 40mph, while the speed on the left is 10mph.

Furthermore, we can scale l by a factor of m, and when we take the limit as m goes to infinity. This will yield a continuous time Markov chain, where each edge retains its proper length proportions to the other.

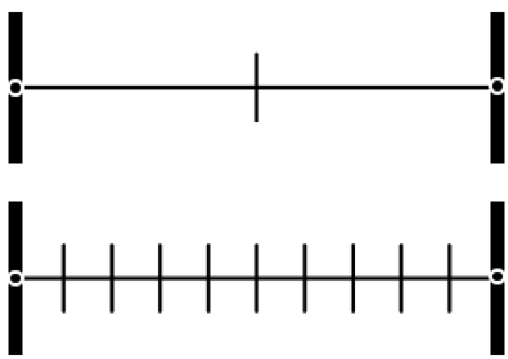

Figure 6: The below edges give a close up of what the scaling would look like for when $m = 2$, and when $m = 10$ between two states on a discretized edge.

4.3.2 The Construction

For notational purposes, let node -1 be node 0, and node $n+1$ be node n, and let $[k]$ denote the let $\{0, 1, ..., k\}$. Let $\{v_i^+\}_{i=0}^n$ represent the speeds on the edge going from intersection i to $i+1$.

Let $\{v_i^-\}_{i=0}^n$ represent the speeds going from intersection i to $i-1$ Let $V = \{v_i^+\}_{i=0}^n \cup \{v_i^-\}_{i=0}^n$ be the set of all speeds. Let l be the least common multiple of all speeds, i.e. $l = \text{lcm}(\{v_i^* \mid i \in [n], * \in \{+, -\}\})$ and let $m \in \mathbb{N}$.

Let e_j^+ be an edge from state j to $j+1$, and let $|e_j^+|$ be the number of nodes on edge e_j^+. Define $|e_j^+| = \frac{l \cdot m}{v_j^+} - 1$ to be the number of intermediate nodes along edge e_j^+. Similarly, define e_j^- to be an edge from i to $i-1$ with $|e_j^-| = \frac{l \cdot m}{v_j^-} - 1$. Let $E = \{e_i^+\}_{i=0}^n \cup \{e_i^-\}_{i=0}^n$ be the set of all edges. Let $e_j(i)$ denote the ith node on an edge $e_j \in E$.

We can modify the Markov chain given above by allowing decisions to be made at $0, 1, .., n$ in the same way, but forcing the states to move on to edges, and traverse those edges before we reach the next "intersection". i.e. if X' is the state-space, if $X'_{t-1} = i$ is an intersection, then

$$\begin{cases} P(X'_t = e_0^-(1)|X'_{t-1} = 0) & = p \\ P(X'_t = e_n^+(1)|X'_{t-1} = n) & = q \\ P(X'_t = e_j^-(1)|X'_{t-1} = j) & = p \\ P(X'_t = e_j^+(1)|X'_{t-1} = j) & = q \\ P(X'_t = e_j^-(1)||X'_{t-1} - j| > 1) & = 0 \\ P(X'_t = e_j^-(1)||X'_{t-1} - j| > 1) & = 0 \end{cases} \tag{11}$$

Meaning, we still maintain the structure of moving away from 0 with probability q, and moving towards 0 with probability p. If $X_{t-1} = e_j(k)$ is a node on an edge $e_j \in E$, then

$$\begin{cases} P(X'_t = e_j(k)|X_{t-1} = e_j(k-1), 1 < k) = 1 \\ P(X'_t = j|X_{t-1} = e_{j+1}^-(k), k = |e_{j+1}^-|) = 1 \\ P(X'_t = j|X_{t-1} = e_{j-1}^+(k), k = |e_{j-1}^+|) = 1 \\ P(X'_t = e_j(k)|X'_{t-1} \neq e_j(k-1), 1 < k < |e_j|) = 0 \end{cases} \tag{12}$$

Meaning, once a driver is on an edge, they only have one option for movement, which is furhter along the edge. For notational purposes, we allow

$$\begin{aligned} e_j^-(|e_j^-| + 1) &= j - 1 \\ e_j^+(|e_j^+| + 1) &= j + 1 \\ e_j^+(0) = e_j^-(0) &= j \end{aligned} \tag{13}$$

.

We will use functional notation for the steady state with discretized edges, as this is much more convenient. I.e. $\pi'(k)$ is the proportion of time spent at state k in the steady state.

Theorem 2. Let $C = \sum_{j=0}^n ((p|e_j^-| + q|e_j^+| + 1)\pi_j)$. Then the steady state distribution for the Markov system above is

$$\begin{cases} \pi'(j) & = \frac{\pi_j}{C} \text{ for all } k \in [n] \\ \pi'(e_j^+(j_k)) & = q\frac{\pi_j}{C} \text{ for all } k \in [n],\ j_k \in \{1, ..., |e_j^+|\}. \\ \pi'(e_j^-(j_k)) & = p\frac{\pi_j}{C} \text{ for all } k \in [n],\ j_k \in \{1, ..., |e_j^-|\}. \end{cases} \tag{14}$$

Where π is the distribution given in Lemma 1.

Proof. Note that necessarily, $\pi'(e_j^+(k)) = q \cdot \pi'(j)$ for all $1 \leq j \leq |e_j^+|$ since since we move from j to $e_j^+(1)$ with probability q, and from each node on the edge to the next with probability 1.

Similarly, $\pi'(e_j^-(k)) = p \cdot \pi'(j)$ for all $1 \leq k \leq |e_j^-|$. Then

$$\begin{aligned}\sum_{x\in X} \pi'(x) &= \sum_{j\in[n]} \pi'(j) + \sum_{j\in[n]} \sum_{k=1}^{|e_j^+|} \pi'(e_j^+(k)) + \sum_{j\in[n]} \sum_{k=1}^{|e_j^-|} \pi'(e_j^-(k)) \\ &= \sum_{j\in[n]} \pi'(j) + \sum_{j\in[n]} q|e_j^+|\pi'(j) + \sum_{j\in[n]} p|e_j^-|\pi'(j) \\ &= \frac{1}{C}\sum_{j=0}^{n}((p|e_j^-| + q|e_j^+| + 1)\pi_j) \\ &= \frac{C}{C} \\ &= 1\end{aligned} \tag{15}$$

So, certainly π' is a distribution. Next, let $y \in X$. If y is an intersection, then

$$\begin{aligned}\sum_{x\in X} \pi'(x)P(x,y) &= q\pi'(y-1) + p\pi'(y+1) \\ &= \frac{1}{C}(q\pi(y-1) + p\pi(y+1)) \quad = \frac{1}{C}\pi(y) \\ &= \pi'(y)\end{aligned} \tag{16}$$

Else, if $y = e_j^-(1)$ for some j, then

$$\sum_{x\in X} \pi'(x)P(x,y) = p\pi'(j) \quad = \pi'(e_j^-(1)) \tag{17}$$

Else, if $y = e_j^-(k)$ for some $k \in \{2..., |e_j^-|\}$,

$$\begin{aligned}\sum_{x\in X} \pi'(x)P(x,y) &= 1\pi'(e_j^-(k-1)) \\ &= p\pi'(j) \\ &= \pi'(e_j^-(1))\end{aligned} \tag{18}$$

Similarly, the condition holds for states on e_j^+ edges. Thus, π' is a steady state distribution on X. □

Corollary 2. Taking the Markov chain as above, and letting m, the number scaling of the discretizations, approach infinity ($m \to \infty$, then the steady state (where we are either at an intersection or on an edge)

$$\begin{aligned}\pi'(j) &= 0 \text{ if } j \in [n] \text{ an intersection} \\ \pi'(e_j^+) &= q \cdot \frac{1}{C' v_j^+}\pi(j) \\ \pi'(e_j^-) &= p \cdot \frac{1}{C' v_j^-}\pi(j)\end{aligned} \tag{19}$$

Where $C' = \sum_{j=0}^{n}((p\frac{1}{v_j^-} + q\frac{1}{v_j^+})\pi(j))$, the sum of weighted inverse speeds, is the normalizing constant, and with drivers uniformly distributed along the edge.

Proof. Consider

$$
\begin{aligned}
C &= \sum_{j=0}^{n}((p(\frac{l \cdot m}{v_j^-} - 1) + q(\frac{l \cdot m}{v_j^+} - 1) + 1)\pi_j) \\
&= lm\sum_{j=0}^{n}((p(\frac{1}{v_j^-} - \frac{1}{lm}) + q(\frac{1}{v_j^+} - \frac{1}{lm}) + \frac{1}{lm})\pi_j) \\
&= lm\sum_{j=0}^{n}((p(\frac{1}{v_j^-}) + q(\frac{1}{v_j^+}))\pi_j)
\end{aligned}
\tag{20}
$$

Since everything but m is fixed, then as $m \to \infty$, $C \to \infty$. Thus, if j an intersection, and $m \to \infty$, $\pi'(j) = \frac{\pi(j)}{C} = 0$.

Well, the probability of being on edge e_j^- for all $j \in [n]$ is

$$
\begin{aligned}
\frac{1}{C}\sum_{k=1}^{|e_j^-|} p\pi(j) &= p\pi(j)\frac{1}{C}(\frac{l \cdot m}{v_j^-} - 1) \\
&= p\pi(j)\frac{lm - v_j^-}{v_j^-(lmC')} \\
&= p\frac{1}{C'v_j^-}\pi(j) - p\pi(j)\frac{1}{lmC'}
\end{aligned}
\tag{21}
$$

Then,

$$
\lim_{m \to \infty} p\frac{1}{C'v_j^-}\pi(j) - p\pi(j)\frac{1}{lmC'} = p\frac{1}{C'v_j^-}\pi(j) \tag{22}
$$

A similar argument applies to the e_j^+ edges, thus the claim is true. □

5 Case Study

5.1 Background

Corollary 3. Consider a Markov chain with with d drivers on the state space N, moving to the left with probability p, and to the right with probability q, with the ability to remain at 0 and n with probabilities p, q respectively. Assume $q < p$. Assume that we are in the steady-state distribution above. Then the number of drivers ξ_i at a position $i \in N$ is binomially distributed, i.e. it is distributed as $\text{Bin}(d, (q/p)^i\pi_0)$, which has mean $d(q/p)^i\pi_0$. Binomial random variables are concentrated around their mean [3], so in particular with probability tending to 1 as $d \to \infty$, $\xi_i \sim d(q/p)^i\pi_0$ (i.e. $\xi_i = (1 + o(1))d(q/p)^i\pi_0$). The steady state distribution for this system as n goes to infinity is $\xi = (\xi_0, \xi_1, ..., \xi_n, ...)$ where

$$
\xi_i = d\Big(\frac{q}{p}\Big)^{i+1}
$$

In this case study, we wanted to analyze how the average of driver weights might behave under different deployment strategies and enforcement. This would be most practical to do in Optimal Metric 2, where officers try to minimize the global speed of the system.

A key backbone of the results below is the existence of a steady state distribution for the average of each weight at each vertex under an enforcement and deployment strategy by officers. I.e., as the system operates, as drivers adjust their speeds to enforcement, would we expect that the global speed of the system stabilizes? Specifically, would we expect the speed at each *vertex* to stabilize? To be clear, we did not find results that indicate one way or another, and we hypothesize that there are both cases where the system does stabilize, and does not stabilize, where the case we are in is determined by:

- The metric officers are trying to optimize, which guides how officers decide who to pull over and when
- The deployment strategy of the officers (and whether there are enough of them for enforcement to have a significant impact on average speed)
- The distribution of the preferred speed of drivers (i.e. how the parameter λ is chosen for each driver)
- The rarity of officers pulling drivers over

A remark about the fourth bullet: Because we artificially arranged our system so that drivers interact with officers more, and more officers are deployed, this can introduce a degree of volatility into our model, as there are going to be more "resets" to the speed limit. With reduced interactions, this effect would be more limited, enabling there to be more support for convergence. Furthermore, it could be the case that by restricting the distribution of preferred speeds we could bound the maximal change in average weight from one step to another, which could lead to convergence. However, our results do not stretch that far.

Therefore, the results below rely on the assumption that $\overline{\mathcal{W}_j}$, the steady state of the average weights at vertex j for some Officer strategy, exists. We let $\overline{w_{k,j}}$ be the average of weight k at vertex j. In general, we assume that $w_{2,t}$ is the same for all drivers and does not change over time. Therefore, $\overline{w_{2,j}} = w_{2,t}$ for any driver at anytime.

It should be noted, furthermore, that our results rely on the fact that the distribution of drivers remains to be ξ, and does not change. We do have some indication that this might be true, as our results above from Section 4.3 with the steady state distribution π' indicate to us that the number of drivers that appear at a vertex is still proportional to the the number of drivers on the base Markov chain, regardless of what the speed on the edge is. However, we didn't extend that to this case in fully generality, for we just discussed how the average speeds might *not* converge. It is most likely the case that this assumption would follow from the previous assumption.

So, let $p = 2/3$, $q = 1/3$, and $w_{2,t} = 1/2$. Let the number of drivers, $d = c2^n$, where c is the ratio of drivers to officers, and let n to infinity. These conditions make computation simple, and allow us to gain intuition towards the general case of the problem.

Fact: By proposition 5, the number of drivers at node i is $c2^{n-i-1}$.

Proposition 1. Assume $\overline{\mathcal{W}_j}$, the steady state of the average weights at vertex j for Deployment Strategy 1, exists. Assume the driver position distribution does not change significantly under enforcement, $\overline{w_{2,j}}$ given. Then the steady state for the average weights for this strategy as n goes to infinity is the solution to the below recursion:

$$\begin{aligned} \overline{w}_{1,i} &= \begin{cases} \frac{1}{3}\left(\frac{2}{c} + 2\rho(1-\frac{2}{c})\overline{w}_{1,0} + \rho\overline{w}_{1,1}\right) & i=0 \\ \frac{1}{3}\left(\frac{2}{c} + 2\rho(1-\frac{2}{c})\overline{w}_{1,0} + \rho\overline{w}_{1,2}\right) & i=1 \\ \frac{1}{3}\rho(2\overline{w_{1,i-1}} + \overline{w_{1,i-1}}) & i \geq 2 \end{cases} \\ \overline{w}_{2,i} &= \frac{1}{2} \\ \overline{w}_{3,i} &= 1 - \overline{w}_{1,i} \end{aligned} \tag{23}$$

Proof. For all drivers, w_2 and w_3 are defined as $1/2$, and $1 - w_1$ respectively, so clearly those two solutions hold. We now derive $\overline{w}_{1,i}$.

The recursion can be broken up into four components:

$A :=$	The average weight of the people coming from $i-1$, times the number of those type (pulled over)
$B :=$	The average weight of the people coming from $i-1$, times the number of those type (not pulled over)
$C :=$	The average weight of the people coming from $i+1$, times the number of those type (pulled over)
$D :=$	The average weight of the people coming from $i+1$, times the number of those type (not pulled over)

We see that

$$\overline{w}_{1,i} = \frac{A+B+C+D}{d} = \frac{A+B+C+D}{c2^{n-i-1}} \tag{24}$$

For each part, we see that it can be made up of number of drivers at that node, multiplied by the probability they get pulled over or not, multiplied by the weight that comes with being pulled over (or not) and then multiplied by the probability they move to node i.

For $i = 0$, we recall that drivers can remain at 0 over a time step. Therefore, for notational reasons, let $\overline{w}_{1,-1} := \overline{w}_{1,0}$. Then

Part	Num. Drivers at node	Prob. Pulled Over (or not) at node	Weight	Prob Move to i
A	$c2^{n-1}$	$\frac{2}{c}$	$\frac{1}{2}$	$\frac{2}{3}$
B	$c2^{n-1}$	$1-\frac{2}{c}$	$\rho\overline{w}_{1,0}$	$\frac{2}{3}$
C	$c2^{n-2}$	0	$\frac{1}{2}$	$\frac{2}{3}$
D	$c2^{n-2}$	1	$\rho\overline{w}_{1,1}$	$\frac{2}{3}$

So

$$\begin{aligned} A &= c2^{n-1}\frac{2}{3}\frac{2}{c}\frac{1}{2} \\ B &= c2^{n-1}\frac{2}{3}(1-\frac{2}{c})\rho\overline{w}_{1,0} \\ C &= 0 \\ D &= c2^{n-2}\rho\overline{w}_{1,1}\frac{2}{3} = c2^{n-1}\frac{1}{3}\rho\overline{w}_{1,1} \end{aligned} \tag{25}$$

Therefore

$$\overline{w}_{1,0} = \frac{A+B+D}{c2^{n-1}} = \frac{1}{3}\Big(\frac{2}{c} + 2\rho(1-\frac{2}{c})\overline{w}_{1,0} + \rho\overline{w}_{1,1}\Big) \tag{26}$$

For $i = 1$, we have

Part	Num. Drivers at node	Prob. Pulled Over (or not) at node	Weight	Prob Move to i
A	$c2^{n-1}$	$\frac{2}{c}$	$\frac{1}{2}$	$\frac{1}{3}$
B	$c2^{n-1}$	$1-\frac{2}{c}$	$\rho\overline{w}_{1,0}$	$\frac{1}{3}$
C	$c2^{n-3}$	0	$\frac{1}{2}$	$\frac{2}{3}$
D	$c2^{n-3}$	1	$\rho\overline{w}_{1,2}$	$\frac{2}{3}$

So

$$\begin{aligned} A &= c2^{n-1}\frac{1}{3}\frac{2}{c}\frac{1}{2} = c2^{n-2}\frac{2}{3}\frac{2}{c}\frac{1}{2} \\ B &= c2^{n-1}\frac{1}{3}(1-\frac{2}{c})\rho\overline{w}_{1,i-1} = c2^{n-2}\frac{2}{3}(1-\frac{2}{c})\rho\overline{w}_{1,i-1} \\ C &= 0 \\ D &= c2^{n-3}\rho\overline{w}_{1,2}\frac{2}{3} = c2^{n-2}\frac{1}{3}\rho\overline{w}_{1,2} \end{aligned} \tag{27}$$

Therefore

$$\overline{w}_{1,1} = \frac{A+B+D}{c2^{n-2}} = \frac{1}{3}\Big(\frac{2}{c} + 2\rho(1-\frac{2}{c})\overline{w}_{1,0} + \rho\overline{w}_{1,2}\Big) \tag{28}$$

For $i \geq 2$, we have

Part	Num. Drivers at node	Prob. Pulled Over (or not) at node	Weight	Prob Move to i
A	$c2^{n-i}$	0	$\frac{1}{2}$	$\frac{1}{3}$
B	$c2^{n-i}$	1	$\rho\overline{w}_{1,i-1}$	$\frac{1}{3}$
C	$c2^{n-i-2}$	0	$\frac{1}{2}$	$\frac{2}{3}$
D	$c2^{n-i-2}$	1	$\rho\overline{w}_{1,i+1}$	$\frac{2}{3}$

So

$$
\begin{aligned}
A &= 0 \\
B &= c2^{n-i}\rho\overline{w}_{1,i-1}\frac{1}{3} = c2^{n-i-1}\frac{2}{3}\rho\overline{w}_{1,i-1} \\
C &= 0 \\
D &= c2^{n-i-2}\rho\overline{w}_{1,i+1}\frac{2}{3} = c2^{n-i-1}\frac{1}{3}\rho\overline{w}_{1,i+1}
\end{aligned} \tag{29}
$$

Therefore

$$\overline{w}_{1,i} = \frac{B+D}{c2^{n-i-1}} = \frac{1}{3}\rho(2\overline{w}_{1,i-1} + \overline{w}_{1,i+1}) \tag{30}$$

□

Proposition 2. Let $\overline{\mathcal{W}_j}$ to be the steady state of the average weights at vertex j for Deployment Strategy 2. Assume the driver position distribution does not change significantly under enforcement, $\overline{w_{2,j}}$ given. Then the steady state for the average weights for this strategy as n goes to infinity is the solution to the below recursion:

$$
\begin{aligned}
\overline{w}_{1,i} &= \begin{cases} \frac{1}{3}\Big(\frac{4}{cn} + 2\rho(1-\frac{2}{cn})\overline{w}_{1,0} + \rho(1-\frac{4}{cn})\overline{w}_{1,1}\Big) & i = 0 \\ \frac{2}{3}\frac{2^i}{cn}\frac{1}{2} + \frac{1}{3}\frac{2^{i+2}}{cn}\frac{1}{2} + \frac{1}{3}\rho(1-\frac{2}{cn})(2\overline{w}_{1,i-1} + \overline{w}_{i+1}) & 1 \le i < \log_2(cn) - 1 \\ 1 & \log_2(cn) - 1 \le i \end{cases} \\
\overline{w}_{2,i} &= \frac{1}{2} \\
\overline{w}_{3,i} &= 1 - \overline{w}_{1,i}
\end{aligned} \tag{31}
$$

Proof. The style of the proof mirrors that of the previous. For all drivers, w_2 and w_3 are defined as 1/2, and $1 - w_1$ respectively, so clearly those two solutions hold. We now derive $\overline{w}_{1,i}$.

The recursion can be broken up into four components:

$A :=$ The average weight of the people coming from $i-1$, times the number of those type (pulled over)
$B :=$ The average weight of the people coming from $i-1$, times the number of those type (not pulled over)
$C :=$ The average weight of the people coming from $i+1$, times the number of those type (pulled over)
$D :=$ The average weight of the people coming from $i+1$, times the number of those type (not pulled over)

We see that

$$\overline{w}_{1,i} = \frac{A+B+C+D}{d} = \frac{A+B+C+D}{c2^{n-i-1}} \tag{32}$$

For each part, we see that it can be made up of number of drivers at that node, multiplied by the probability they get pulled over or not, multiplied by the weight that comes with being pulled over (or not) and then multiplied by the probability they move to node i. For each node, the probability of being pulled over is the minimum of 1 and the following value:

$$\frac{\text{Num of Officers at } i}{\text{Num of Drivers at } i} = \frac{2^n/n}{c2^{n-i-1}} = \frac{2^{i+1}}{cn} \tag{33}$$

We see that $\frac{2^{i+1}}{cn} \le 1$ as long as $i \le \log_2(cn) - 1$. If it is larger than 1, it means there are more officers than drivers, and they will all be pulled over. Recall that drivers can remain at 0 over a time step. Therefore, for notational reasons, let $\overline{w}_{1,-1} := \overline{w}_{1,0}$. For $i = 0$, we have

Part	Num. Drivers at node	Prob. Pulled Over (or not) at node	Weight	Prob Move to i
A	$c2^{n-1}$	$\frac{2}{cn}$	$\frac{1}{2}$	$\frac{2}{3}$
B	$c2^{n-1}$	$1-\frac{2}{cn}$	$\rho\overline{w}_{1,0}$	$\frac{2}{3}$
C	$c2^{n-2}$	$\frac{4}{cn}$	$\frac{1}{2}$	$\frac{2}{3}$
D	$c2^{n-2}$	$1-\frac{4}{cn}$	$\rho\overline{w}_{1,1}$	$\frac{2}{3}$

So

$$\begin{aligned}
A &= c2^{n-1}\frac{2}{3}\frac{2}{cn}\frac{1}{2} \\
B &= c2^{n-1}\frac{2}{3}(1-\frac{2}{cn})\rho\overline{w}_{1,0} \\
C &= c2^{n-2}\frac{2}{3}\frac{4}{cn}\frac{1}{2} = c2^{n-1}\frac{1}{3}\frac{2}{cn} \\
D &= c2^{n-2}\rho(1-\frac{4}{cn})\overline{w}_{1,1}\frac{2}{3} = c2^{n-1}\frac{1}{3}\rho(1-\frac{4}{cn})\overline{w}_{1,1}
\end{aligned} \tag{34}$$

Therefore

$$\overline{w}_{1,0} = \frac{A+B+D}{c2^{n-1}} = \frac{1}{3}\Big(\frac{4}{cn} + 2\rho(1-\frac{2}{cn})\overline{w}_{1,0} + \rho(1-\frac{4}{cn})\overline{w}_{1,1}\Big) \tag{35}$$

Then for $i < \log_2(cn) - 1$, we have □

Proposition 3. Let $\overline{\mathcal{W}_j}$ to be the steady state of the average weights at vertex j for Deployment Strategy 3. Assume the driver position distribution does not change significantly under enforcement, $\overline{w_{2,j}}$ given. Then the steady state for the average weights for this strategy as n goes to infinity is the solution to the below recursion:

$$\begin{aligned}
\overline{w}_{1,i} &= \begin{cases} \frac{1}{2c} + \frac{1}{3}\rho(1-\frac{1}{c})(2\overline{w}_{1,i} + \overline{w}_{1,i+1}) & i = 0 \\ \frac{1}{2c} + \frac{1}{3}\rho(1-\frac{1}{c})(2\overline{w}_{1,i-1} + \overline{w}_{1,i+1}) & i \geq 1 \end{cases} \\
\overline{w}_{2,i} &= \frac{1}{2} \\
\overline{w}_{3,i} &= 1 - \overline{w}_{1,i}
\end{aligned} \tag{36}$$

Proof. The proof is similar to the two above. □

6 Discussion

Our model on $2n+1 \times 2n+1$ grid has some nice properties. We know from Theorem 2 and Corollary 2 that the number of drivers an officer sees at an intersection is directly proportional to the steady state of drivers moving at uniform speed on the grid, regardless of how many discretizations are on the edges, or even if time is continuous. This provides really nice structure to our model, and is a critical result. However, these results hold for a model on where the speed on each edge may be different. It currently does not allow for different speeds among that edge. However, a natural solution has already been prepared.

The key element of discretizing the edges was allowing for a finite number of speeds. Recall from Section 2.3.2 that drivers modify their weight towards their prefered speed by a factor of ρ at each discrete time step. Thus, the weight of the speed limit is always $1, \rho, \rho^2, \dots, \rho^n, \dots$. If we limit this reduction to a very large N, meaning that if a driver is not pulled over after N steps, their speed remains ρ^N, then each driver has a finite amount of speeds. Thus, if there are finitely many drivers, each with finitely many weights for speeds, and a finite amount of speed limits in the system, then there exists a least common multiple of all of these speeds (we can scale every speed to be an integer). We can then perform the same kind of discretization in section 4.3 to have a system that allows for drivers of varying speeds.

We would expect then that the steady state distribution in the above case still remains proportional to that of the base chain. That is, even with drivers moving around faster, it doesn't change where we expect them to be. It simply means there will be more movement. Thus, if a driver is moving significantly fast, and officers are pulling over the first speeder they see, our results indicate that fast drivers crossing a single intersection do not face more immediate risk of being pulled over compared to other drivers from speeding. However, these drivers *do* have a higher chance of being pulled over by virtue of crossing *more* intersections as a result of their heightened speed.

7 Concluding Remarks

Our work established a solid mathematical model, and several methods, with which we can view speeding. Extending our work to the situation where all drivers travel at their own speed is a small step away, and our intuition and current work clues us into those results quite neatly. Thus, for anyone who reads this paper, there is a working, rigorously justified mathematical model with which to test questions of speeding on, as well as discussion on the modelling process to enable any modifications. Future work could include modifying the model, testing theories on the motivations of speeding, and attempting to optimize enforcement strategies based on a given officer metric.

References

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