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Minimizing the duration of isolated attosecond pulses

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We investigate theoretically how the duration of an isolated attosecond pulse (IAP) can be minimized by carefully selecting frequencies of high-order harmonic generation (HHG) spectra produced by ultrashort driving laser pulses. Based on numerical calculations of HHG by solving the time-dependent Schrödinger equation for a single H atom, we provide three strategies for generating shorter IAPs. First, when the high-frequency region of an HHG plateau is selected one should use frequencies below the cutoff. Second, for a wide HHG plateau the low-frequency region can produce shorter IAPs than the high-frequency region. Third, we propose a method of producing IAPs with transform-limited duration by selecting special frequency stripes across the entire HHG plateau. Analytic analyses show that how our strategies work is related to the (Fourier-transform) properties of an Airy function. We also carry out HHG calculations considering macroscopic effects by means of intensity averaging over the focal region. We find that our conclusions for single-atom calculations can still apply for macroscopic HHG spectra as long as they resemble single-atom spectra.

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I. INTRODUCTION

Owing to the rich potentials that shorter laser pulses can bring to ultrafast-process research in a variety of systems [1–6], obtaining ever-shorter attosecond pulses has been a challenging quest in the field of attosecond science [7–9]. Besides recent direct optical methods [10], the most common way of generating attosecond pulses is to spectrally filter part of a high-order harmonic generation (HHG) spectrum [1–3]. If the driving laser pulse is long with many cycles, one can select a group of frequencies in the HHG spectrum to generate a train of attosecond pulses [11,12]. When the driving laser pulse is short with very few cycles, isolated attosecond pulses (IAPs) can be obtained by filtering a single region of the HHG spectrum [7–9,13–16]. IAPs of a few hundred attosecond (as) duration can be obtained by filtering the cutoff HHG [14,17], while the current shortest IAPs of sub-hundred-as duration are generated by filtering plateau frequencies below HHG cutoffs [7–9,18].

Generating shorter and stronger attosecond pulses has been the goal for many studies of ultrafast processes. For attosecond pulses that are generated from high-harmonic generation, enhancing HHG yield and increasing its frequency bandwidth are good approaches for producing attosecond pulses of higher intensities and shorter duration [19]. Another important aspect in generating shorter attosecond pulses is to overcome the attochirp problem. An attochirp is an intrinsic chirp of the HHG process where high-harmonic photons of different frequencies are emitted at different times [7]. As a consequence, the attosecond pulse is not transform limited, i.e., the duration is longer than the Fourier transform of the HHG spectrum bandwidth [2,20]. Previous works have demonstrated that a

thin-film filter can partially compensate the attochirp for a limited frequency range [16,21,22], or a shaped pulse waveform can reduce the intrinsic chirp during the HHG processes by squeezing returning electron trajectories into a short time window [23].

In this paper, we investigate how the duration of an isolated attosecond pulse can be minimized by carefully selecting frequencies of an HHG spectrum produced by linearly polarized one-cycle driving laser pulses without compensating the attochirp. Our studies are based on HHG spectra from an H atom which are computed by numerically solving the time-dependent Schrödinger equation (TDSE) [24] as well as by employing an analytic description of HHG produced by few-cycle pulses [25]. Our results show that (a) when the high-frequency region of an HHG plateau is selected one should use frequencies below the cutoff; (b) for a wide HHG plateau the low-frequency region can produce shorter IAPs than the high-frequency region; and (c) IAPs with transform-limited duration can be produced by selecting special frequency stripes across the entire HHG plateau.

Theoretical formulations of how HHG spectra and attosecond pulses are calculated for a single H atom are presented in Sec. II. Features of HHG spectra produced by one-cycle driving pulses and the accompanied carrier-envelope phase (CEP) issue are discussed in Sec. III A. Results of minimizing IAP duration for three frequency-selection categories are presented in Sec. III B for a single spectral range near the cutoff, Sec. III C for a single spectral range between cutoffs, and Sec. III D for a striped-frequency range (sequential but separated frequency regions or “stripes”). Section IV discusses how our strategies based on single-atom calculations behave when considering macroscopic effects of HHG. Analytic analyses about how our frequency-selection strategies work are presented in Sec. V. Lastly, Sec. VI summarizes our results and gives some conclusions.

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II. THEORETICAL FORMULATION

A. Analytic calculation of HHG spectra

The analytically calculated HHG spectrum $\rho(\Omega)$ is obtained by coherently adding half-cycle spectral amplitudes that correspond to electron trajectories ionized during each of the few half cycles of a laser pulse. For a detailed formulation of calculations, see Refs. [25,26]. In short, $\rho(\Omega)$ can be written as

$$\rho(\Omega) = \sum_{j,k} s_{jk} \cos(\varphi_j - \varphi_k) \mathcal{A}_j(E) \mathcal{A}_k(E), \quad (1)$$

where $\hbar\Omega$ is the photon energy, j and k label individual spectral amplitudes that originate from the j th laser half cycle, $s_{jk} = \pm 1$ is a sign factor, φ_j is the phase of the j th amplitude, and E is the electron return energy. Note that $\hbar\Omega = E + |E_0|$ with E_0 being the ground-state energy of the electron, which equals $E_0 = -1/2$ a.u. (atomic units) for an H atom.

Each half-cycle spectral amplitude $\mathcal{A}_j(E)$ equals the square root of a product of three factors representing the three steps of HHG [27–30]: the ionization factor $\mathcal{I}_j$, the propagation factor $\mathcal{W}_j(E)$, and the recombination factor $\sigma^{(r)}(E)$:

$$\mathcal{A}_j(E) \equiv \sqrt{\mathcal{I}_j \mathcal{W}_j(E) \sigma^{(r)}(E)}. \quad (2)$$

The j th half-cycle spectral amplitude $\mathcal{A}_j(E)$ is determined by the electron trajectory that ionizes within laser half cycle j and returns with the maximum energy. The ionization factor $\mathcal{I}_j$ is determined by the electric field at the time of tunneling ionization. The propagation factor $\mathcal{W}_j(E)$ is related to the electron excursion time in the laser field and an Airy function of energy E which encodes the subcycle interference of short and long trajectories. The recombination factor $\sigma^{(r)}(E)$ is the photorecombination cross section of an electron to the 1s ground state of an H atom.

B. TDSE calculation of HHG spectra

We also calculate HHG spectra by solving the TDSE for an H atom interacting with a laser field $F(t)$ that is linearly polarized along the z axis:

$$i \frac{\partial}{\partial t} \Psi(\mathbf{r}, t) = \left[\frac{\mathbf{p}^2}{2} - \frac{1}{r} + zF(t) \right] \Psi(\mathbf{r}, t), \quad (3)$$

where m_e , $|e|$, and $\hbar$ are set to unity in atomic units. The TDSE is solved using a time-dependent generalized pseudospectral method [24], in which the wave function is expanded in Legendre polynomials and the time propagation is carried out using a second-order split-operator technique. The convergence of our TDSE calculations was monitored by increasing the basis size and the grid density in both space and time, where specific parameters are approximately $L \sim 200$ for the maximum angular momentum, $r_{\max} \sim 200$ a.u. for the maximum grid size, and $N \sim 1500$ for the number of radial grid points, and the time step is ≈ 0.125 a.u.

The HHG spectrum $S(\Omega)$ is obtained from the Fourier-transformed dipole acceleration along the z axis $\ddot{D}_z(\Omega)$:

$$S(\Omega) = \frac{1}{\hbar c^3} |\ddot{D}_z(\Omega)|^2, \quad (4)$$

where c is the speed of light, the spectral (frequency-dependent) dipole acceleration $\ddot{D}_z(\Omega)$ (or the spectral amplitude) is

$$\ddot{D}_z(\Omega) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \ddot{D}_z(t) e^{i\Omega t} dt, \quad (5)$$

and the time-dependent dipole acceleration $\ddot{D}_z(t)$ is [31]

$$\begin{aligned} \ddot{D}_z(t) &\equiv \langle \Psi(\mathbf{r}, t) | -\ddot{z} | \Psi(\mathbf{r}, t) \rangle \\ &= \langle \Psi(\mathbf{r}, t) | \frac{\partial V(r)}{\partial z} | \Psi(\mathbf{r}, t) \rangle + F(t), \end{aligned} \quad (6)$$

where $V(r) = -1/r$ is the Coulomb potential for an H atom. Note that both the analytic spectrum $\rho(\Omega)$ and the TDSE spectrum $S(\Omega)$ are dimensionless in our formulations [25,32].

C. Generating attosecond pulses

Attosecond pulses are generated by spectrally filtering HHG spectra, i.e., by Fourier transforming the HHG spectral amplitude through a window function $w(\Omega)$. For the TDSE calculations, we define a field amplitude $d(t)$ as the inverse Fourier transform of the spectral dipole acceleration $\ddot{D}_z(\Omega)$:

$$d(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \ddot{D}_z(\Omega) w(\Omega) e^{-i\Omega t} d\Omega. \quad (7)$$

For the analytic calculations, the spectrum $\rho(\Omega)$ can be written in a similar way as Eq. (4) [25], i.e.,

$$\rho(\Omega) = \frac{\Omega^4}{\hbar c^3} |D(\Omega)|^2, \quad (8)$$

where $D(\Omega)$ is the Fourier-transformed dipole moment. In this case, the field amplitude $d(t)$ is defined as

$$d(t) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Omega^2 D(\Omega) w(\Omega) e^{-i\Omega t} d\Omega. \quad (9)$$

The attosecond pulse intensity profile $I(t)$ is then proportional to the modulus square of the field amplitude:

$$I(t) = I_0 |d(t)|^2. \quad (10)$$

The pulse duration Δt is defined as the full width at half maximum (FWHM) of the intensity profile $I(t)$. Since the constant I_0 does not affect the pulse duration Δt or the comparison of relative pulse intensities, we set $I_0 = 1$ with dimensions $1/\hbar c^3 \omega_0^2$ so that $I(t)$ is treated dimensionlessly.

For ease of calculations, we use a rectangular window function $w(\Omega)$ to select HHG frequencies:

$$w(\Omega) = \begin{cases} 1 & \Omega \in \text{selected frequency range} \\ 0 & \text{otherwise} \end{cases}. \quad (11)$$

A smooth window function such as a super-Gaussian function with a flat top will not affect our attosecond pulse results much. Note that, unlike studies in Ref. [33], we do not compensate phases in HHG spectra during frequency selections using Eq. (11).

III. RESULTS FOR SINGLE-ATOM CALCULATIONS

A. HHG spectra produced by one-cycle laser

In order to generate isolated attosecond pulses instead of pulse trains, the effective driving pulse duration must be short

FIG. 1. (a) The electric field $F(t)$ of a one-cycle sine-shaped Gaussian pulse with a carrier wavelength of 1500 nm and a peak intensity of 2×10^{14} W/cm². (b) Calculated HHG spectra produced by the laser pulse in panel (a) from an H atom. The solid line is obtained by TDSE calculations and the dashed line is obtained by analytic calculations. (c) Same as panel (a) except for the CEP which gives a cosine-shaped pulse. (d) Calculated HHG spectra produced by the laser field in panel (c). See text for labeling explanations.

such that only one laser cycle contributes to the target HHG spectrum [2,15,16]. Therefore, we use a one-cycle Gaussian pulse as our driving laser field. Specifically, for a linearly polarized electric field, its vector potential magnitude $A(t)$ is described as

$$A(t) = -\frac{cF_0}{\omega_0} \exp\left(-\frac{2 \ln 2}{\tau^2} t^2\right) \sin(\omega_0 t + \phi_{\text{cep}}), \quad (12)$$

where F_0 is the peak amplitude, ω_0 is the carrier frequency, τ is the pulse duration, and ϕ_{cep} is the carrier-envelope phase. A one-cycle pulse has a FWHM in the intensity of $\tau = 2\pi/\omega_0$. The electric field is then obtained from the time derivative of the vector potential, i.e.,

$$F(t) = -\frac{1}{c} \frac{\partial A(t)}{\partial t}. \quad (13)$$

The spatial dependence of the laser field is neglected because the electric dipole approximation is applicable.

In order to generate short attosecond pulses, we would like broad HHG spectra with large cutoff energies. Based on the classical cutoff formula [27–29], two effective ways of obtaining large HHG cutoff energies are to use high intensities and long wavelengths of the driving laser field. For an H atom, we choose in our calculations a laser peak intensity of 2×10^{14} W/cm², which gives a peak amplitude $F_0 = 0.076$ a.u., and a carrier wavelength of 1500 nm ($= 2\pi c/\omega_0$).

For short driving pulses, the CEP is important [34–36]. Figure 1 shows the electric fields of a sine- and cosine-shaped one-cycle pulse and the HHG spectra produced from those two fields. One sees that, for a one-cycle driving laser, the cosine-shaped pulse in Fig. 1(c) gives a higher cutoff energy [25,34]: the 130-eV cutoff in Fig. 1(d) compared to the

110-eV cutoff in Fig. 1(b). Here, the cutoffs correspond to the maximum return energies of the electron plus the ionization potential. On the other side, the sine-shaped pulse in Fig. 1(a) gives a wider HHG plateau that comes from a single laser cycle: the “main” plateau in Fig. 1(b) has a bandwidth of 62 eV compared to that of 47 eV in Fig. 1(d). A wider single-cycle plateau is more favorable than the higher cutoff energy for generating short isolated attosecond pulses. Thus we use the sine-shaped pulse, i.e., $\phi_{\text{cep}} = 0.5\pi$ in Eq. (12), for our following studies on attosecond pulse generation.

The electric field of a one-cycle sine-shaped pulse, plotted in Fig. 1(a), contains four half cycles, labeled as 1 through 4 in the figure. These four half cycles form three full cycles: the precycle (1 and 2), the main cycle (2 and 3), and the postcycle (3 and 4). HHG spectra produced by such a pulse are presented in Fig. 1(b), which are calculated from both analytic and TDSE methods. One sees that the analytic and TDSE spectra are very similar besides an overall intensity difference [26]. In general, HHG spectra obtained from the analytic method are accurate near the cutoff, and compare well with TDSE results over the entire plateau region. Moreover, this analytic method provides clear physical interpretations that we have previously utilized to explore how HHG can be enhanced by time delays [26] or chirps [32] in a two-color (e.g., ω - 2ω) setup. The same interpretations are used in the following to explain the trajectory correspondence of HHG spectra.

The HHG spectrum shown in Fig. 1(b) exhibits a two-plateau structure. The high-energy (main) plateau, above 48 eV with a cutoff of 110 eV, comes from electron motion during the main cycle of the laser pulse, i.e., the electron wave packet ionizes during the half cycle 2 and returns within one

cycle after ionization. The low-energy plateau, below 48 eV with a cutoff of 44 eV, is due to mainly electron motion during the postcycle of the laser field, i.e., the electron wave packet ionizes during the half cycle 3 and returns within one cycle after ionization. The more irregular structure of the lower-energy plateau compared to the higher-energy plateau is due to the fact that electron motions during both the main and postcycles contribute to this energy range and they interfere. Electron motion during the precycle contributes negligibly to the HHG spectrum owing to low ionization from the half cycle 1, which forms modulations around 120 eV with an intensity that is two orders of magnitude lower than the main plateau.

Similarly, the HHG spectrum in Fig. 1(d) produced by a cosine-shaped one-cycle pulse in Fig. 1(c) also presents two plateaus: a high-energy (main) plateau above 82 eV with a cutoff of 130 eV and a low-energy plateau below 82 eV with a cutoff of 75 eV. Both cutoff energies in Fig. 1(d) are higher than the cutoff energies in Fig. 1(b). This is because the electric fields of half cycles 2 and 3 in Fig. 1(c) are stronger than the electric fields of half cycles 3 and 4 in Fig. 1(b), and ionized electrons gain more energy during those returning half cycles with stronger electric fields. The bandwidth of the main plateau in Fig. 1(d) is narrower than that of Fig. 1(b). This is because the increase in cutoff energies for the postcycle is larger than that for the main cycle, which is due to the more pronounced field enhancement between half cycle 3 in Fig. 1(c) and half cycle 4 in Fig. 1(a) as compared to the field enhancement between half cycle 2 in Fig. 1(c) and half cycle 3 in Fig. 1(a).

We use the high-energy (main) plateau of the HHG spectrum in Fig. 1(b) to generate isolated attosecond pulses, since only one laser cycle contributes to this HHG spectrum range. Notice that for the high-energy plateau, there are large oscillations with a varying energy spacing between neighboring peaks. These oscillations originate from subcycle interference of short and long trajectories during the main cycle of the laser field, and they can be described analytically by squaring an Airy function [25,37]. Thus, we call these oscillations Airy peaks to distinguish them from harmonic peaks which come from intercycle interference and present an equal spacing of $2\omega_0$ between neighboring peaks. Airy peaks are observed in many HHG spectra produced by few-cycle driving lasers for single-atom calculations [25,33,34,38,39]. Note that the key to obtain Airy peaks is that only one laser cycle contributes dominantly to the HHG spectrum, which does not require strictly a single-cycle laser pulse. For example, our calculations show that a 20% longer driving pulse still gives a nice wide spectrum with many Airy peaks.

B. Results for a single spectral range near cutoff

One common way of generating isolated attosecond pulses is to filter a frequency bandwidth near the cutoff of an HHG spectrum [14,17]. In this section, we show that based on single-atom HHG calculations the selected frequency bandwidth should not expand across the cutoff in order to produce short IAPs. Figure 2 shows attosecond pulses generated when selecting a single spectral range the coverage of which is either below the cutoff, 90–106 and 90–110 eV, or across the cutoff, 90–120 eV, where the cutoff is 110 eV. One sees that

FIG. 2. Pulses generated from a single spectral range of the TDSE HHG spectrum in Fig. 1(b). The bandwidth range is either below (90–106 and 90–110 eV) or across (90–120 eV) the cutoff at 110 eV. The time scale is that of Fig. 1(a).

when the frequency range lies below the cutoff, a pair of short pulses is generated corresponding to short and long trajectory emissions. However, when the frequency bandwidth includes the cutoff peak, a long pulse is generated with subpeak structures. Therefore, in order to obtain short IAPs, the selected frequency range should lie below the cutoff.

The cross-cutoff behavior of attosecond pulses from single-atom HHG calculations, i.e., the sudden change of the pulse structure such that either a pair of short pulses or a long pulse is generated depending on whether the selected frequency range includes the cutoff, is due to the fact that the cutoff Airy peak is spectrally much wider than other Airy peaks. More discussions through analyses by manually varying the width of the cutoff Airy peak are presented in Sec. VB.

C. Results for a single spectral range between cutoffs

In this section, we investigate attosecond pulses generated by selecting harmonic frequencies of a single spectral range between HHG cutoffs. The HHG spectrum in Fig. 1(b) has two plateaus with two cutoffs: the lower-energy plateau with a cutoff of 44 eV and the higher-energy plateau with a cutoff of 110 eV. We select a range of frequencies between these two cutoffs, i.e., between 44 and 110 eV, to generate attosecond pulses.

It is known that the attosecond pulses are not transform limited, i.e., the duration is not as short as the Fourier transform of the HHG spectrum bandwidth, owing to the attochirp [20]. For example, when selecting a narrow frequency bandwidth that covers a portion of the plateau, i.e., 60–90 eV, a pair of short pulses is generated, as shown in Fig. 3(a). However, when selecting a wide frequency bandwidth that covers the whole plateau, i.e., 50–110 eV, pulses are much longer with subpeaks. The longer pulse is due to the attochirp which originates from the fact that electrons with different energies return at different times. Figure 3(b) shows the electron return time as a function of its return energy from classical calculations. The blue horizontal line indicates the division between short and long trajectories, where the cutoff trajectory returns at time $t_r(E_{\text{max}}) = 85.5$ a.u. with a maximum energy of 110 eV. Comparing the emission times of attosecond pulses in Fig. 3(a) with the return times from classical trajectories, it is clear that the earlier one of the pulse pair, located

FIG. 3. (a) Attosecond pulses generated from a single spectral range of the TDSE HHG spectrum in Fig. 1(b). The solid red line shows pulses generated from frequencies of 60–90 eV, and the dashed blue line shows pulses generated from a doubled frequency bandwidth of 50–110 eV. (b) Electron return time as a function of its return energy from classical calculations. Time scales are the same as that in Fig. 1(a). See text for labeling explanations.

around 60 a.u., corresponds to short trajectory emissions, and the later one, located around 110 a.u., corresponds to long trajectory emissions [11,38]. Short trajectory emissions are often selected in experiments [7,8,11], and thus we shall focus on the earlier pulse for the rest of this section.

For a wide HHG plateau such that only part of the spectrum can be used to generate short attosecond pulses due to the attochirp, we investigate which part of the HHG spectrum one should use to minimize the duration of IAPs. For this purpose, we perform a bandwidth scan to see how the pulse duration behaves as the frequency bandwidth increases.

We set a single spectral range of $[\Omega_1, \Omega_2]$ ($\Omega_1 < \Omega_2$) with a frequency bandwidth $\Delta\Omega \equiv \Omega_2 - \Omega_1$. In this case, the window function $w(\Omega)$ is

$$w(\Omega) = \begin{cases} 1 & \Omega_1 \leq \Omega \leq \Omega_2 \\ 0 & \text{otherwise} \end{cases}. \quad (14)$$

In order to reveal properties in different energy regions of the spectrum, we carry out the bandwidth scan starting from two opposite sides. First, we fix Ω_1 at the lowest-energy end of the plateau ($\Omega_1 = 48$ eV) while Ω_2 is scanned from low energies to high energies in the spectrum until Ω_2 reaches 106 eV. Then, we reverse the scan, i.e., fixing $\Omega_2 = 106$ eV and scanning Ω_1 from high energies to low energies. For both cases, we record the pulse duration Δt from the short trajectory emission as the frequency bandwidth $\Delta\Omega$ increases, and the results are plotted in Fig. 4(a). Note that as the frequency bandwidth gets wider, the attosecond pulses exhibit subpeak structures, as shown in Fig. 3(a) for the wider bandwidth. For

FIG. 4. (a) Plots of attosecond pulse duration Δt as a function of the frequency bandwidth $\Delta\Omega$ for a single spectral range between cutoffs. Dash-dot symbols are for the bandwidth scan when Ω_1 is fixed at 48 eV and line-square symbols are for the bandwidth scan when Ω_2 is fixed at 106 eV. The minimum pulse duration for each case is indicated by arrows. (b) Minimum pulses as indicated by arrows in panel (a) that are generated from low- and high-frequency regions and are plotted on the same time scale as Fig. 1(a).

these pulses with subpeak structures, the pulse duration Δt is the width of the entire single pulse.

A common feature for the two cases in Fig. 4(a) is that, as the frequency bandwidth $\Delta\Omega$ increases, the pulse duration Δt first decreases, then reaches a limit and rises up rapidly. However, when Ω_1 is fixed on the low-energy end, the minimum pulse duration is 124 as with a bandwidth of 33 eV, whereas when Ω_2 is fixed on the high-energy end, the minimum pulse duration is 139 as with a bandwidth of 29 eV. Thus, the low-frequency region of the spectrum can produce shorter attosecond pulses than the high-frequency region [40].

The minimum pulses generated from low-frequency and high-frequency regions are shown in Fig. 4(b). One sees that besides the shorter pulse duration, the minimum pulse obtained from the low-frequency region also has a higher peak intensity than the minimum pulse obtained from the high-frequency region. Emission times of the pulse pair generated from low frequencies are farther apart compared to that of high frequencies [21]. This can be understood by the electron return times shown in Fig. 3(b): Electrons with lower return energies have a bigger difference in return times between short and long trajectories.

D. Results for a striped-frequency range

In this section, we present a way of producing short isolated attosecond pulses. This method selects sequential but separated frequency regions (stripes) across the entire HHG spectrum and generates isolated attosecond pulses with

transform-limited pulse duration without compensating the attochirp. Specifically, for the HHG spectrum in Fig. 1(b), we select every other Airy peak in the high-energy plateau and generate a single short pulse from these frequency stripes, or a striped-frequency pulse (SFP).

The high-energy plateau of the TDSE HHG spectrum in Fig. 1(b) is replotted in Fig. 5(a) on a linear scale. There are 19 Airy peaks, labeled 1 starting from the cutoff peak at 110 eV, through 19 for the lowest-energy peak at 49.5 eV. As an example, we select all the odd Airy peaks (1, 3, 5, etc., through 19) and the generated pulse is plotted in Fig. 5(c). The pulse duration Δt , defined as the FWHM of the pulse intensity profile described in Eq. (10), is indicated in the figure. One sees that a single pulse is obtained at the center, i.e., at time $t_r(E_{\max}) = 85.5$ a.u., with negligible sidebands. One can also select even Airy peaks (2, 4, 6, etc., through 18) to generate a SFP as shown in Fig. 5(d). The even-peak SFP has a longer duration of 57 as compared to 45 as of the odd-peak pulse because the frequency bandwidth of even Airy peaks is smaller than that of odd Airy peaks. Both SFPs, i.e., from either odd or even Airy peaks, are shorter than the minimum pulse generated by selecting a single spectral range, the pulse duration of which is 124 as. The odd-peak SFP is also more intense than the minimum pulse generated by selecting a single spectral range.

The duration of striped-frequency pulses is very close to that of the transform-limited pulse from a square spectrum, as shown in Fig. 5(b). A square spectrum can be obtained by simply setting $\tilde{D}_z(\Omega) = 1$ in Eq. (7) and taking a rectangular window function like Eq. (14). The bandwidth-duration product of the transform-limited pulse from a square spectrum is $\Delta\Omega\Delta t = 3664$ eV as = 5.566 a.u. For a striped-frequency range of odd Airy peaks, we start with peak 1 and gradually add additional lower-energy peaks. The frequency bandwidth and corresponding pulse duration are recorded and plotted in Fig. 5(b). The bandwidth for a striped-frequency range is defined as its frequency coverage, i.e., the energy difference between the left and right ends of selected peaks. For example, the frequency bandwidth $\Delta\Omega$ for selecting all the odd peaks is $\Delta\Omega = 120 - 48 = 72$ eV.

One sees from Fig. 5(b) that the duration of striped-frequency pulses produced from odd Airy peaks is very close to the transform-limited pulse duration for a square spectrum. The discrepancy of the small bandwidth point is an artifact of the bandwidth definition for a striped-frequency range: The spectral shape is no longer like a square for a small bandwidth. The duration of SFPs produced from only even Airy peaks (not shown in the figure) is also very close to the transform-limited pulse duration for a square spectrum.

The mechanism of generating SFPs can be understood by the phase structure of the HHG spectrum. Adjacent peaks in the spectrum have an approximate π difference in the spectral phase. Selecting every other peak means selecting only the positive (or negative) parts of the spectral amplitude. This results in a pulse peak generated at time $t_r(E_{\max})$. Moreover, the attochirp in the HHG spectrum suppresses sidebands that are also generated from selecting every other peak. Detailed discussions through coherent summation of field amplitudes generated from individual Airy peaks are presented in Sec. V C.

FIG. 5. (a) Linear replot of the TDSE HHG spectrum in Fig. 1(b). Numbers label Airy peaks. (b) Plots of pulse duration Δt as a function of the frequency bandwidth $\Delta\Omega$ for the transform-limited case of a square spectrum (solid line), the striped-frequency pulse (SFP) case of odd Airy peaks (dot-line symbols), and the SFP case of all peaks (square-dash symbols). (c–e) Striped-frequency pulses from all odd peaks, all even peaks, and all peaks of the HHG spectrum. The time scale is that of Fig. 1(a).

Since adjacent Airy peaks in the HHG spectrum have an approximate π difference in the spectral phases, we can improve our SFP method by introducing a phase π to even Airy peaks and select both odd and even peaks to generate stronger isolated attosecond pulses. Figure 5(e) shows an improved

FIG. 6. Illustration of a dispersion-free optical setup for selecting frequency stripes of every other Airy peak in the HHG spectrum. G1 and G2 are soft-x-ray gratings. M1 and M2 are parabolic mirrors. SM is a spatial mask to block unwanted frequencies. Gratings are placed at the focuses of mirrors.

SFP when all Airy peaks are included and a phase π added to even peaks. Owing to constructive interference of both odd and even Airy peaks, the improved SFP exhibits a much stronger peak intensity compared to regular SFPs from either odd or even Airy peaks. The pulse duration of improved SFPs remains at the transform-limited level, as shown in Fig. 5(b).

One possible way to select frequency stripes of every other Airy peak in the HHG spectrum is a dispersion-free $4f$ setup illustrated in Fig. 6, which resembles the configuration of a zero-dispersion pulse compressor and a pulse shaping mask used in femtosecond pulse shaping for infrared and visible light [41]. The soft-x-ray beam produced from HHG is first dispersed by a grating G1 (e.g., Shimadzu, 1200 1/mm [9]), and then filtered by a structured spatial mask SM so that unwanted frequencies are blocked (or alternatively selected frequencies are changed in phase by π), and finally a second grating G2 combines remaining Airy-peak frequencies and generates a SFP. Parabolic mirrors are located at one focal length from the gratings and from the spatial mask. This setup ensures no dispersion, i.e., equal paths for different frequencies. For good reflectivity of a broad bandwidth, one may use diamondlike carbon coated mirrors with grazing incidence [9] which is not illustrated in Fig. 6.

IV. RESULTS FOR FOCAL AVERAGING

We have shown how the duration of isolated attosecond pulses can be minimized by carefully selecting frequencies of a one-cycle HHG spectrum based on single-atom HHG calculations. Such spectra exhibit Airy peaks, which are large oscillations with uneven widths [25,33,34,38,39]. Experimental HHG spectra from few-cycle laser pulses can exhibit a supercontinuum structure [8,9]. Large scale modulated structures around the cutoff accompanied with fine oscillations in lower energies are also observed in many experiments [42–44]. In this section, we investigate how macroscopic effects of HHG play a role in minimizing the duration of IAPs by taking account of the longitudinal and transverse averaging of the driving laser spatial profile.

We take the integral approach to calculate the macroscopic effects where local high-harmonic radiations within the gas medium are integrated over the interaction region with proper phases [45]. We consider a dilute gas and low ionization where absorption of high-harmonic radiation and distortion of the driving pulse can be ignored [11]. We also consider a gently focused laser beam with a narrow gas target (compared to the laser Rayleigh length) so that the laser intensity does not vary much along the gas medium. We estimate the propagation effects for two representative situations.

In the first case, we consider the interaction region as one long thin tube, where the laser intensity variance is small so that we ignore key phase factors coming from Gouy phase (induced by focusing of the driving laser) and dipole phase (linearly dependent on laser intensity) [46]. Thus, three-dimensional (3D) propagation is reduced to a one-dimensional (1D) averaging with good phase matching in the forward direction. Results for 1D intensity averaging are presented in Sec. IV A. Note that this averaging still takes into account coherence of short and long trajectories.

In the second case, we consider the interaction region as a bunch of many long thin tubes where coherence inside tubes is considered but coherence between tubes is ignored. We test the plausibility of this 3D modeling by comparing our calculation with results in Ref. [47] where fully 3D propagation is considered, and we see good qualitative agreements. Results for 3D intensity averaging are presented in Sec. IV B, where we estimate HHG spectra at the medium exit by integrating over a cylindrical volume, which consists of a 1D *coherent* intensity averaging along the propagating axis [48,49] and a two-dimensional *incoherent* intensity averaging perpendicular to the axis [11].

A. One-dimensional intensity averaging

We consider a gently focused Gaussian laser beam with a narrow gas target (compared to the laser Rayleigh length) so that the laser intensity does not vary much along the gas medium. For example, considering a driving wavelength of 1500 nm and a beam waist of 50 μm , the laser intensity drop is less than 4% for a 1-mm-long gas medium near the focus. For a uniform gas density, we estimate phase-matched high-harmonic radiation at the medium exit by coherently summing HHG amplitudes produced by a range of laser peak intensities [48,49]:

$$S_{1D}(\Omega) = \frac{1}{\hbar c^3} \left| \sum_i \ddot{D}_z^{(i)}(\Omega) \right|^2, \quad (15)$$

where $\ddot{D}_z^{(i)}(\Omega)$ is the spectral amplitude corresponding to the i th laser intensity. Specifically, we choose the peak intensity range from 1.96 to $2.04 \times 10^{14} \text{ W/cm}^2$, and the number of i 's is chosen such that the HHG spectrum is converged. Attosecond pulses are generated by Fourier transforming the total HHG spectral amplitudes through a window function.

The calculated TDSE HHG spectrum with 1D intensity averaging is shown in Fig. 7. Compared with the single-atom HHG spectrum in Figs. 1(b) and 5(a), the ordinary HHG harmonics with an equal spacing of $2\omega_0$ in the low-energy plateau are well resolved (see the spectrum below 50 eV in the inset of

FIG. 7. TDSE HHG spectrum with 1D intensity averaging calculated using Eq. (15). Numbers label Airy peaks. The inset shows the same spectrum for a broader energy range with a log scale.

Fig. 7), while the Airy peaks in the high-energy plateau above 50 eV have more complex structures due to the interference of spectral amplitudes from different laser intensities. The cutoff range corresponding to the laser peak intensities is 108.7 to 112.5 eV, which is indicated in the figure.

For a single spectral range near the cutoff, we select a frequency bandwidth that is either below the cutoff region, 90–109 eV, or across the cutoff region, 90–120 eV, and the generated attosecond pulses are plotted in Fig. 8. Unlike the attosecond pulses generated from the single-atom HHG spectrum in Fig. 2 where both short and long trajectory emissions are significant, the long trajectory emission in Fig. 8 around time 100 a.u. is negligible compared to the short trajectory emission around time 70 a.u. due to the intensity averaging [50]. When the bandwidth is below the cutoff range, the duration of the attosecond pulse is 193.5 as; when the bandwidth includes the cutoff range, the duration of the attosecond pulse is longer with 205.6 as. Since the HHG yield of the cutoff range is much lower than that below the cutoff range, the effect of including the cutoff range is less pronounced compared to that for the single-atom case.

For a single spectral range between cutoffs, we carry out similar bandwidth scans as in Sec. III C with either Ω_1 fixed at the low energy end of 49 eV or Ω_2 fixed at the high energy end below the cutoff region of 109 eV. One sees in Fig. 9(a) that the pulse duration of the high-frequency region starts to rise when the bandwidth reaches 27 eV, while the pulse dura-

FIG. 8. Attosecond pulses for a single spectral range that either lies below (90–109 eV) the cutoff range or includes (90–120 eV) the cutoff range.

FIG. 9. (a) Plots of attosecond pulse duration Δt as a function of the frequency bandwidth $\Delta\Omega$ for two bandwidth scans. Dash-dot symbols are for the case when Ω_1 is fixed at 49 eV. Line-square symbols are for the case when Ω_2 is fixed at 109 eV. The minimum pulse duration for each case is indicated by arrows. (b) Minimum pulses that are generated from low- and high-frequency regions and are plotted on the same time scale as Fig. 1(a).

tion of the low-frequency region rises at a larger bandwidth of 33 eV. The minimum attosecond pulses generated from both regions are plotted in Fig. 9(b) where the low-frequency pulse has a bandwidth of 49–82 eV and the high-frequency pulse has a bandwidth of 82–109 eV. Besides the shorter duration of 127 as as compared to 148 as, the minimum pulse from the low-frequency region is also more intense than that from the high-frequency region due to the higher HHG yield in the low-frequency region.

For a striped-frequency range, we first select eight odd peaks (1, 3, 5, etc., through 15) and the generated attosecond pulse is plotted in Fig. 10(a). One sees that a short pulse is produced around time 86 a.u. but with significant side pulses at earlier times. To suppress those side pulses, we employ the improved SFP method where both odd and even peaks are selected with a phase π added to even peaks. One sees in Fig. 10(b) that the improved SFP of all 15 peaks produces a single short pulse of 63.5 as with significantly suppressed sidebands. Compared to the SFP of odd peaks, the improved SFP produces weaker sidebands and a stronger main pulse. Figure 10(c) plots the pulse duration of the improved SFPs when the number of peaks included increases from 1 to 15. Clearly, the pulse duration of improved SFPs still remains at the transform-limited level. Note that when more than 15 peaks are included, the pulse duration of improved SFPs increases slightly and this is the reason we include only up to 15 peaks.

Striped-frequency pulses need Airy peaks of one-cycle HHG spectra which originate from subcycle interference of

FIG. 10. (a) The striped-frequency pulse from eight odd peaks of the spectrum in Fig. 7. (b) The striped-frequency pulse from 15 odd and even peaks. (c) Plots of pulse duration Δt as a function of the frequency bandwidth $\Delta\Omega$ for the transform-limited case of a square spectrum (solid line) and the SFP case of all peaks (dot-dash symbols).

short and long trajectories [25,37]. In general, high-harmonic radiation from long trajectories diverges more than that from short trajectories due to the more rapid variation of the dipole phase with the laser intensity which leads to a stronger curvature of the phase front [51,52]. We consider a gently focused beam such that the laser intensity variance within the gas medium is small. One can also adjust the gas target position [11,53] and collect HHG radiation off axis [54] so that both short and long trajectory radiations are phase matched. Moreover, our results show that in cases when short trajectory contributions are larger than long trajectory contributions (see, e.g., Fig. 9), one can still obtain SFPs.

B. Three-dimensional intensity averaging

When transverse averaging of the laser intensity cannot be ignored (e.g., the HHG detection area is large), we perform a 3D intensity averaging that includes both coherent longitudinal averaging and incoherent transverse averaging. In addition to summing spectral amplitudes along the laser propagating axis, we also integrate spectral intensities of the cross section at the medium exit:

$$S_{3D}(\Omega) = \int_0^{h_{\max}} S_{1D}(\Omega, h) 2\pi h dh, \quad (16)$$

FIG. 11. TDSE HHG spectrum with 3D intensity averaging calculated using Eq. (16). The inset shows the same spectrum for a broader energy range on a log scale.

where h is the radius of the cross section. The 1D spectrum S_{1D} is evaluated as in Eq. (15) with a 4% variance in the laser intensity. $h_{\max}$ is chosen such that the laser intensity of a Gaussian beam drops to half of the center intensity, i.e., about 1×10^{14} W/cm². Integration beyond such $h_{\max}$ does not change the HHG spectrum very much. Attosecond pulses are obtained by integrating 1D pulse intensities [see, e.g., Eq. (3) in Ref. [11]].

The TDSE HHG spectrum with 3D intensity averaging is shown in Fig. 11. The ordinary HHG harmonics with an equal spacing of $2\omega_0$ in the low-energy plateau are still well resolved (see the spectrum below 50 eV in the inset of Fig. 11). For the high-energy plateau above 50 eV, unlike the single-atom spectrum, Airy peaks are no longer visible and a supercontinuum structure appears. There is no definite cutoff for the high-energy plateau due to the wide range of laser intensities. For these reasons, we shall not present results for striped-frequency pulses or a single spectral range near the cutoff. Note that Airy peaks in HHG spectra gradually disappear as we increase $h_{\max}$ in Eq. (16) and results shown in this section represent the worst averaging case.

We perform a two-dimensional attopulse duration scan for the supercontinuum spectrum in Fig. 11. The frequency bandwidth is chosen between Ω_1 and Ω_2 with $\Omega_2 > \Omega_1$ where Ω_1 is scanned from 48 to 99 eV and Ω_2 is scanned from 58 to 110 eV. The pulse duration is plotted in Fig. 12. There are three valleys in the contour plot and the lower left one in the low-frequency region exhibits the shortest pulse duration. The two edges of the triangle correspond to the 1D duration scan we have shown in Figs. 4(a) and 9(a): The left edge represents when Ω_1 is fixed at the low-energy end 48 eV and the top edge represents when Ω_2 is fixed at the high-energy end 110 eV. Clearly, the minimum pulse generated from the low-frequency region (the valley on the lower left side) has a shorter duration than that from the high-frequency region (the valley on the upper right side).

The minimum attosecond pulses generated from low-frequency and high-frequency regions are plotted in Fig. 13 where the low-frequency pulse has a bandwidth of 49–73 eV and the high-frequency pulse has a bandwidth of 87–109 eV. Besides the shorter duration of 236 as as compared to 266 as, the minimum pulse generated from the low-frequency region is also more intense than that from the high-frequency region due to the higher HHG yield in the low-frequency region.

FIG. 12. Two-dimensional scan of attopulse duration Δt . Ω_1 varies from 48 to 99 eV and Ω_2 varies from 58 to 110 eV.

V. ANALYTIC ANALYSES OF SINGLE-ATOM RESULTS

In this section, we analyze and explain our results about minimizing IAP duration through frequency selections of a one-cycle HHG spectrum based on single-atom calculations. Specifically, Sec. V A provides an approximation of the one-cycle HHG spectral amplitude with an Airy function, which serves as a foundation for later analyses. Section V B analyzes results for a single spectral range near the cutoff by manually adjusting the cutoff Airy peak width. Section V C discusses striped-frequency pulses by coherently summing field amplitudes from individual Airy peaks at special times.

A. Approximating HHG spectral amplitude with the Airy function

First, we show that the spectral amplitude of a one-cycle HHG spectrum (meaning that only one laser cycle contributes to the HHG spectrum) can be approximated by an Airy function. For our analytic calculations of HHG, the field amplitude of the attosecond pulse is obtained by Fourier transforming the spectral dipole moment $D(\Omega)$ as given in Eq. (9). In general,

FIG. 13. Minimum attosecond pulses generated from low- and high-frequency regions of the HHG spectrum in Fig. 11.

the spectral dipole moment $D(\Omega)$ is a coherent sum of half-cycle spectral amplitudes $\mathcal{A}_j(E)$ from each of the few cycles of a laser pulse [25,26]. For the main plateaus of the HHG spectra in Fig. 1, only one laser cycle contributes dominantly. In this case, the spectral dipole moment $D(\Omega)$ can be simply written as

$$D(\Omega) = \eta \sqrt{\sigma^{(r)}} \text{Ai}(x) e^{i(\phi + \Omega t_r)}, \quad (17)$$

where the factor η and the phase ϕ are trajectory-related constants independent of Ω , and t_r is the return time of the cutoff trajectory, i.e., the trajectory with the maximum return energy. For the 1500-nm one-cycle pulse in our calculations, $t_r = 85.5$ a.u. The argument x of the Airy function $\text{Ai}(x)$ is a scaled dimensionless “energy”:

$$x = \frac{\hbar\Omega - E_{\text{cut}}}{\epsilon E_{\text{au}}} - 1.019, \quad (18)$$

where E_{cut} is the cutoff energy and $E_{\text{au}} = |e|^2/a_0 = 27.21$ eV (a_0 is the Bohr radius). Clearly, when the frequency Ω lies below the cutoff, the spectral dipole moment $D(\Omega)$ exhibits the oscillatory features of an Airy function which leads to the Airy peaks in the HHG spectrum. When Ω lies beyond the cutoff, the spectral dipole moment decays exponentially. ϵ is a dimensionless trajectory parameter which can be expressed as [25,26]

$$\epsilon = \zeta^{1/3}, \quad \zeta = \frac{F(t_r)^2}{2F_{\text{au}}^2} \left[\frac{F(t_r)}{F(t_i)} - \frac{\dot{F}(t_r)}{F(t_r)}(t_r - t_i) - 1 \right]. \quad (19)$$

Here $F(t)$ is the electric field in Eq. (13), t_i is the ionization time of the cutoff trajectory, and $F_{\text{au}} = |e|/a_0^2 = 5.142$ GV/cm. For the one-cycle pulse we use in our calculations, $\epsilon = 0.132$. For simpler formulation, we also define a scaled dimensionless “time” y :

$$y = \frac{E_{\text{au}}}{\hbar} \epsilon (t - t_r). \quad (20)$$

Substituting Eqs. (18) and (20) into Eqs. (17) and (9), we can approximate the attosecond-pulse field amplitude $d(t)$ as the Fourier transform of an Airy function $\tilde{d}(y)$:

$$d(t) \sim \epsilon \tilde{d}(y) = \epsilon \int_{-\infty}^{\infty} \text{Ai}(x) \tilde{w}(x) e^{-ixy} dx. \quad (21)$$

Notice that besides constant factors and overall phases, we also ignored the factor $\Omega^2 \sqrt{\sigma^{(r)}}$ which has less pronounced variation with Ω compared to the oscillating Airy function in the HHG spectral range. Moreover, the phase factor $e^{i\Omega t_r}$ in Eq. (17) simply shifts the pulse in time. Therefore, in order to analyze attosecond pulses we can just focus on the Fourier transform of an Airy function $\tilde{d}(y)$ in Eq. (21).

To demonstrate that our analytic approximations are appropriate, we compare attosecond pulses generated from the TDSE spectrum [the solid line in Fig. 1(b)], the analytic spectrum [the dashed line in Fig. 1(b)], and the Airy function, which are calculated using Eqs. (7), (9), and (21), respectively. Figure 14(a) shows attosecond pulses obtained from a frequency bandwidth that includes the cutoff, 90–120 eV, of the TDSE and analytic spectra, and from the first four oscillations of an Airy peak. Pulses are scaled in intensity and the time of the Airy pulse is shifted according to Eq. (20). The TDSE pulse and the analytic pulse agree qualitatively that they

FIG. 14. Attosecond pulses generated from the TDSE spectrum (dotted lines), the analytic spectrum (dashed lines), and the Airy function (solid lines). Panel (a) is for a bandwidth of 90–120 eV and panel (b) is for the SFP of odd peaks.

both produce one long pulse when the cutoff is included. The analytic pulse and the Airy pulse agree very well. Striped-frequency pulses of odd peaks are shown in Fig. 14(b). In order to cover about the same bandwidth, the TDSE pulse is obtained from nine odd Airy peaks (1 through 17) of the TDSE spectrum, the analytic pulse is obtained from eight odd peaks (1 through 15) of the analytic spectrum, and the Airy pulse is obtained from the first eight odd oscillations of an Airy function. One sees that all three pulses agree very well.

B. Cross-cutoff behavior and the cutoff Airy peak

The cross-cutoff behavior of attosecond pulses that we see in Sec. III B where the pulse shape changes when including the cutoff is due to the fact that the cutoff Airy peak is spectrally much wider than other Airy peaks. This can be shown by manually adjusting the width of the cutoff peak. For example, one can approximate the cutoff peak of the spectral amplitude with a half sine wave f_1 :

$$f_1(x) = a_1 \sin\left(-b_1 x + \frac{\pi}{4}\right) \text{ for } x \in \left[-\frac{3\pi}{4b_1}, \frac{\pi}{4b_1}\right], \quad (22)$$

where x is given by Eq. (18). The width of the cutoff peak can be then adjusted by varying the value of b_1 . One sees in Figs. 15(a) and 15(b) that when $b_1 = 0.85$ the approximated cutoff peak f_1 can produce almost the exact attosecond pulse as that from the analytic HHG spectrum. When b_1 is increased such that the cutoff peak width is decreased to a comparable width as (or a smaller width than) the other Airy peaks, the cross-cutoff behavior disappears. For example, in Figs. 15(c) and 15(d), when b_1 is increased to 1.35 and the cutoff peak has a comparable width as its neighbor, a pulse pair is obtained as the case for a single spectral range between cutoffs. On the other hand, when b_1 is decreased such that the cutoff peak is significantly much wider than the other Airy peaks,

FIG. 15. (a, c, e) Solid (red) lines plot the approximated spectral amplitudes of the first three Airy peaks: the cutoff peak is described by f_1 in Eq. (22) and the second and third peaks are described by the Airy function $\text{Ai}(x)$. The cutoff peak described by $\text{Ai}(x)$ is plotted as dashed (green) lines for comparison. (b, d, f) Solid (red) lines plot attosecond pulses corresponding to spectral amplitudes (the solid lines) shown in panels (a), (c), and (e), respectively. The dashed (blue) line in panel (b) is the attosecond pulse obtained from the first three Airy peaks of the analytic HHG spectrum, which is then shifted in time according to Eq. (20).

the cross-cutoff behavior sustains. For example, in Figs. 15(e) and 15(f) when b_1 is decreased to 0.65 and the cutoff peak is significantly wider one obtains a single long pulse.

C. Striped-frequency pulses

We have seen that a half sine wave can describe the peak of the spectral amplitude very well. Indeed, the n th peak (or valley) of the spectral amplitude can be approximated by a half sine wave f_n :

$$f_n(x) = a_n \sin\left(-b_n x + \frac{\pi}{4}\right), \quad (23)$$

where $x \in [-\pi/b_n(n-1/4), -\pi/b_n(n-5/4)]$ and x is given by Eq. (18). Values of a_n and b_n can be obtained from, e.g., the Airy function:

$$a_n = \frac{1}{\sqrt{\pi} \sqrt{-\alpha_n}} \text{ and } b_n = \frac{2}{3} \sqrt{-\alpha_n}, \quad (24)$$

where α_n are the zeros of $\text{Ai}'(x)$, i.e., $\text{Ai}'(\alpha_n) = 0$.

Now the Fourier transform of each peak or valley of the spectral amplitude is simply the Fourier transform of a half sine wave, which yields

$$\hat{f}_n(y) = \frac{a_n b_n}{y^2 - b_n^2} \left[e^{i \frac{\pi}{b_n} (n-1/4)y} + e^{i \frac{\pi}{b_n} (n-5/4)y} \right] e^{in\pi}, \quad (25)$$

where y is given in Eq. (20). Each $\hat{f}_n$ is the individual field amplitude originated from the n th Airy peak in the HHG spectrum. We can obtain an attosecond-pulse field amplitude $\tilde{d}(y)$ by summing corresponding $\hat{f}_n$'s:

$$\tilde{d}(y) \approx \sum_n \hat{f}_n(y). \quad (26)$$

FIG. 16. Diagram of individual field amplitudes $\hat{f}_n$ at times of integer multiples of b_n with locked phases [see Eq. (25)]. The height of the line-dot (dash-square) symbol is scaled to the magnitude of $\hat{f}_n$. For a striped-frequency pulse of odd (even) Airy peaks, one should add multiple copies of the odd-peak (even-peak) amplitude diagram [see Eq. (26)].

For example, to produce SFPs from odd or even Airy peaks, one can simply sum odd or even terms of $\hat{f}_n$.

The mechanism of generating SFPs can be understood by the phase structure of individual field amplitudes $\hat{f}_n$. In order for individual field amplitudes to add up constructively, they must have locked phases at a certain time, meaning that the phases of $\hat{f}_n$ are independent of n at a certain time. One sees from Eq. (25) that at times of integer multiples of b_n , i.e., $y = mb_n$ where $m \in \mathbb{Z}$, the phases of odd (even) amplitudes, i.e., $\hat{f}_n$ with odd (even) n 's, are the same. Features of phase-locked individual field amplitudes are illustrated in Fig. 16, where the m -dependent phases are not shown.

At time $y = 0$, i.e., at time $t = t_r$, individual field amplitudes $\hat{f}_n$ from odd (even) Airy peaks have exactly the same phase. Therefore, when selecting every other Airy peak in the HHG spectrum, individual field amplitudes add up constructively at time t_r and produce a pulse peak with transform-limited duration.

Sidebands appear at times of integer multiples of b_n , i.e., $y = mb_n$, when individual field amplitudes also have lock phases at those times. However, owing to the attochirp in the HHG spectrum b_n has different values for different Airy peaks n . Therefore, unlike the pulse peak at time $y = 0$ which is inherently immune to the attochirp, sidebands from different Airy peaks are generated at different times and do not add up constructively when selecting many odd or even Airy peaks.

As a result of the above two aspects, i.e., the locked phase at time t_r and the suppressed sidebands, a single attosecond pulse is produced with transform-limited duration when selecting every other Airy peak across the entire HHG spectrum.

VI. SUMMARY AND CONCLUSION

Based on TDSE and analytic calculations of one-cycle HHG spectra (meaning that only one laser cycle contributes to the spectra) from an H atom produced by linearly polarized short laser pulses, we investigate ways to minimize the duration of an isolated attosecond pulse by selectively choosing frequencies in the HHG spectra without compensating the attochirp. One important feature of one-cycle HHG spectra is that they exhibit Airy peaks that come from subcycle interference of short and long trajectories, which is different from harmonic peaks with a $2\omega_0$ spacing that come from intercycle interference.

From single-atom calculation results, we present strategies of producing shorter IAPs by careful frequency selections for three scenarios. When the high-frequency region of an HHG plateau is selected, one should use frequencies below the cutoff to produce shorter IAPs. For a wide HHG plateau where one cannot use the entire plateau bandwidth to produce short IAPs due to the attochirp, the low-frequency region can produce shorter IAPs than the high-frequency region. Lastly, one can obtain IAPs with transform-limited duration by selecting frequency stripes of every other Airy peak across the entire HHG plateau, or selecting all Airy peaks with a phase π added to every other Airy peak. These striped-frequency pulses are the shortest (and strongest) attosecond pulses that can be obtained among the three scenarios. Analytic analyses show that how our frequency-selection strategies work is related to the (Fourier-transform) properties of an Airy function.

We also carry out HHG calculations considering macroscopic effects by means of intensity averaging over the focal region. We find that when laser intensity variance in the focal region is small such that macroscopic HHG spectra resemble key features of single-atom spectra, i.e., the Airy peaks, our strategies for minimizing IAPs can still apply. When macroscopic HHG spectra do not exhibit Airy peaks (but a supercontinuum), only one strategy is applicable that the low-frequency region can produce shorter IAPs than the high-frequency region.

Our studies provide methods to minimize IAP duration especially for a broad one-cycle HHG spectrum. Since the HHG spectrum bandwidth scales roughly quadratically with the driving laser wavelength [28,29], application of emerging long-wavelength driving lasers can generate HHG spectra with broader bandwidths. Our methods take advantage of new laser systems with ultrashort pulse duration and long driving wavelengths in generating short and bright attosecond pulses effectively.

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