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The structure of free semigroup algebras

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A free semigroup algebra is the WOT-closed algebra generated by an n-tuple of isometries with pairwise orthogonal ranges. The interest in these algebras arises primarily from two of their interesting features. The first is that they provide useful information about unitary invariants of representations of the Cuntz-Toeplitz algebras. The second is that they form a class of nonself-adjoint operator algebras which are of interest in their own right. This class contains a distinguished representative, the "non-commutative Toeplitz algebra", which is generated by the left regular representation of the free semigroup on n letters and denoted $\mathfrak{L}_n$. This paper provides a general structure theorem for all free semigroup algebras, Theorem 2.6, which extends results for important special cases in the literature. The structure theorem highlights the importance of the *type* L *representations*, which are the representations which provide a free semigroup algebra isomorphic to $\mathfrak{L}_n$. Indeed, every free semigroup algebra has a 2×2 lower triangular form where the first column is a slice of a von Neumann algebra and the 22 entry is a type L algebra. We develop the structure of type L algebras in more detail. In particular, we show in Corollary 1.9 that every type L representation has a finite ampliation with a spanning set of wandering vectors. As an application of our structure theorem, we are immediately able to characterize the radical in Corollary 2.9. With additional work, we obtain Theorem 4.5 of Russo-Dye type showing that the convex hull of the isometries in any free semigroup algebra contains the whole open unit ball. Finally we obtain some information about invariant subspaces and hyper-reflexivity.

Background. The study of the non-commutative analytic Toeplitz algebra was initiated by Popescu [19], [20], [21] in the context of dilation theory. In particular, he obtains an analogue of Beurling's theorem for the structure of its invariant subspaces. A detailed analysis of this algebra is contained in the authors' papers [10], [11], [12] and Kribs [16] and Arias-Popescu [1], [2] which develop the analytic structure of these algebras. In particular, there is a natural map from the automorphism group onto the group of conformal automorphisms of the complex n-ball. The connection with dilation theory comes from a theo-

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rem of Frahzo, Bunce and Popescu [14], [15], [7], [18], which shows that every contractive n-tuple of operators has a unique minimal dilation to an n-tuple of isometries with pairwise orthogonal ranges. Popescu has a collection of papers pursuing analogues of the Sz. Nagy-Foiaş theory in this context.

In addition, [10] introduces the general class of free semigroup algebras, and demonstrates how they may be used to classify certain representations of the Cuntz-Toeplitz algebra. A representation on $\mathscr{H}$ of the Cuntz-Toeplitz algebra is *atomic* if there is an orthonormal basis for $\mathscr{H}$ which is permuted up to scalars by the generating n-tuple of isometries. Atomic representations are completely classified by making use of certain projections in the free semigroup algebra associated to the representation which are not apparent in the von Neumann algebra generated by the representation. A smaller class of representations had already been studied by Bratteli and Jorgensen [5] in connection with the construction of wavelets.

In [4], Bratteli and Jorgensen introduced a larger class of representations of the Cuntz algebra, the *finitely correlated representations*. The terminology arises because such representations have a finite dimensional cyclic subspace which is invariant for the adjoints of the isometric generators. They observe that the isometries are obtained from the Frahzo-Bunce-Popescu dilation of the finite rank n-tuple of operators obtained by compressing the isometries to this cyclic subspace. Recently in [9], Davidson, Kribs and Shpigel have completely classified all finitely correlated representations up to unitary equivalence, again by making use of spatial invariants evident in the associated free semigroup algebra.

Impetus for classifying representations of the Cuntz algebra comes from two different developments in C*-algebras. The work on wavelets has already been mentioned, but we add that a very recent paper by Bratteli and Jorgensen [6] shows how to systematically generate a huge family of wavelets by connecting them to a class of representations of the Cuntz algebra. The other direction of interest is the classification of WOT-continuous $*$-endomorphisms of $\mathscr{B}(\mathscr{H})$ initiated by Powers [22]. It is an easy result to relate these endomorphisms to representations of the Cuntz-Toeplitz algebra.

Our results. Take an arbitrary isometric n-tuple $S = [S_1 \dots S_n]$, which we may think of either as n isometries acting on a common Hilbert space $\mathscr{H}$ or as an isometry from $\mathscr{H}^{(n)}$ into $\mathscr{H}$. The free semigroup algebra $\mathfrak{S}$ is the WOT-closure of the algebra of (non-commuting) polynomials in the S_i. This is the span of the operators $S_w = w(S_1, \dots, S_n)$ for all words w in the free semigroup $\mathbb{F}_n^+$ on n letters. The ideal $\mathfrak{S}_0$ generated by $\{S_1, \dots, S_n\}$ plays a central role. If $\mathfrak{S}_0 = \mathfrak{S}$, then $\mathfrak{S}$ is a von Neumann algebra. Unfortunately we do not know if this situation ever occurs. Otherwise, $\mathfrak{S}_0$ is an ideal of codimension one.

When $\mathfrak{S}_0$ is a proper ideal, there is a canonical homomorphism of $\mathfrak{S}$ into $\mathfrak{L}_n$. We show that this map is always surjective, and the quotient of $\mathfrak{S}$ by the kernel is completely isometrically isomorphic and weak-$*$ homeomorphic to $\mathfrak{L}_n$. A free semigroup will be called *type* L if it is isomorphic to $\mathfrak{L}_n$. This result shows that these algebras all have the same operator algebra structure in the sense of operator spaces.

Moreover, we identify the kernel of the canonical homomorphism of $\mathfrak{S}$ onto $\mathfrak{L}_n$ as the ideal $\mathfrak{J} = \bigcap_{k \geqq 1} \mathfrak{S}_0^k$. This ideal contains a greatest projection P, and it is shown that $\mathfrak{J} = \mathfrak{W}P$ where $\mathfrak{W}$ is the von Neumann algebra generated by $\{S_1, \dots, S_n\}$. This leads to our central structure theorem which decomposes $\mathfrak{S}$ as a lower triangular 2×2 matrix with $\mathfrak{W}P$ occupying the first column, and the 22 entry is a type L algebra. This allows us to weaken the definition of type L from an isometric isomorphism to a strictly algebraic isomorphism. We also obtain a description of the radical of $\mathfrak{S}$.

Our structural results extend the structural results of [9], [10]: in those papers, the classes of atomic representations [10] and finitely correlated representations [9] are classified by obtaining a structure theorem of this type. In the more concrete contexts considered in [9], [10], complete classification up to unitary equivalence is possible because of additional information which can be read off from this structure.

We further delve into the structure of type L algebras. A *wandering vector* is a unit vector ξ such that $\{S_w\xi\colon w \in \mathbb{F}_n^+\}$ is orthonormal, whence the restriction of S to the subspace they span is unitarily equivalent to $\mathfrak{L}_n$. The most perplexing riddle about these algebras is whether they necessarily have wandering vectors. However, while we have been unable to determine whether wandering vectors exist for every type L algebra, we show that if $\mathfrak{S}$ is type L, then some finite ampliation of $\mathfrak{S}$ has wandering vectors. In fact, we show that there is a finite ampliation which has a spanning set of wandering vectors. We strongly suspect that every type L representation has wandering vectors. If so, the question of whether every free semigroup algebra has a wandering vector then reduces to whether one can be self-adjoint.

In section 3, we provide a variety of examples which exhibit some of the results in interesting contexts. A reader who is unfamiliar with the earlier literature may wish to examine some of the examples before going on. In particular, Example 3.9 provides many easy examples of type L representations. Example 3.5 provides some easily described examples which are not type L.

We show that every isometry in $\mathfrak{L}_n$ is mapped to an isometry in every type L representation. This lies much deeper than the fact that the map is completely isometric because it depends on the existence of sufficiently many wandering vectors. This allows us to study the geometry of the unit ball. We show that every element of the open ball in any free semigroup algebra is the convex combination of finitely many isometries. The key device is a factorization theorem which shows that whenever A lies in the open unit ball of $\mathfrak{S}$, there is another element B such that $A^*A + B^*B = I$, or equivalently that $\begin{bmatrix} A \\ B \end{bmatrix}$ is an isometry.

Finally, we examine invariant subspaces of these algebras. The results are not definitive, and serve to highlight certain problems which remain. However when the representation of $\mathcal{O}_n$ is type III, the algebra $\mathfrak{S}$ is hyper-reflexive with distance constant 3.

Notation. The free semigroup on n letters $\mathbb{F}_n^+$ consists of all (non-commuting) words in n symbols $\{1, \dots, n\}$ including the empty word, which is the identity element. Use $|w|$ to denote the length of the word w. Form the Fock space $\mathcal{K}_n = \ell^2(\mathbb{F}_n^+)$ with orthonormal basis $\{\xi_w\colon w \in \mathbb{F}_n^+\}$. The left regular representation λ of $\mathbb{F}_n^+$ is given by $\lambda(v)\xi_w = \xi_{vw}$ for $v, w \in \mathbb{F}_n^+$. Set $L_i := \lambda(i)$. Then $\mathfrak{L}_n$ is the free semigroup algebra generated by $L = [L_1 \dots L_n]$. We write L_w for $w(L_1, \dots, L_n)$.

The C*-algebra $\mathcal{E}_n = \mathrm{C}^*(L_1, \dots, L_n)$ is the Cuntz-Toeplitz algebra. The C*-algebra generated by any n-tuple of isometries S such that $SS^* = \sum_{i=1}^{n} S_i S_i^* = I$ is uniquely determined up to $*$-isomorphism as the Cuntz algebra $\mathcal{O}_n$. There is a canonical quotient map of $\mathcal{E}_n$ onto $\mathcal{O}_n$ sending L_i to S_i whose kernel equals the compact operators $\mathfrak{K}$ on $\mathcal{K}_n$. Any representation of $\mathcal{E}_n$ therefore decomposes as a multiple of the identity representation plus a representation that factors through $\mathcal{O}_n$. This decomposes any isometric n-tuple S into a multiple of L plus a Cuntz n-tuple. In operator theoretic terms, this is known as the Wold decomposition.

Let $\mathfrak{A}_n$ denote the norm-closed unital nonself-adjoint algebra generated by L. The norm-closed unital nonself-adjoint algebra $\mathfrak{T}$ generated by an arbitrary isometry S is a subalgebra of either $\mathcal{E}_n$ or $\mathcal{O}_n$. However the quotient from $\mathcal{E}_n$ onto $\mathcal{O}_n$ is completely isometric on $\mathfrak{A}_n$. Thus there is a unique operator algebra structure on $\mathfrak{A}_n$, and $\mathfrak{T}$ is always completely isometrically isomorphic to $\mathfrak{A}_n$. Popescu calls $\mathfrak{A}_n$ the non-commutative disk algebra.

Every element A of $\mathfrak{L}_n$ has a Fourier series $A \sim \sum_{w \in \mathbb{F}_n^+} a_w L_w$ determined by

$$A\xi_\emptyset = \sum_{w \in \mathbb{F}_n^+} a_w \xi_w.$$

To make sense of this, we use the Cesaro means

$$\Sigma_j(A) = \sum_{|w|<j} \left(1 - \frac{|w|}{j}\right) a_w L_w.$$

These are completely positive unital maps from $\mathfrak{L}_n$ into $\mathfrak{A}_n$ such that $A = \text{SOT-}\lim_{j\to\infty} \Sigma_j(A)$ (see [10]).

The ideal $\mathfrak{L}_n^0$ of $\mathfrak{L}_n$ generated by $\{L_1, \dots, L_n\}$ has codimension one, and consists of all elements of the form $\sum_{i=1}^{n} L_i A_i$, where $A_i \in \mathfrak{L}_n$. The powers of this ideal, $\mathfrak{L}_n^{0,k}$, will play a useful role.

If $\mathcal{H}$ is a Hilbert space, $\mathcal{H}^{(k)}$ denotes the direct sum of k copies of $\mathcal{H}$. For an operator $T \in \mathcal{B}(\mathcal{H})$, we use $T^{(k)}$ to denote the direct sum of k copies of T acting on $\mathcal{H}^{(k)}$. Let $\mathfrak{M}_k$ denote the $k \times k$ scalar matrices. If $\mathfrak{A}$ is any operator algebra contained in $\mathcal{B}(\mathcal{H})$ and φ is a linear map of $\mathfrak{A}$ into another operator algebra $\mathfrak{B}$, then $\varphi^{(k)}$ denotes the map from $\mathfrak{M}_k \otimes \mathfrak{A}$ into $\mathfrak{M}_k \otimes \mathfrak{B}$ which acts as φ on each matrix entry. Recall that φ is completely bounded if $\|\varphi\|_{\text{cb}} := \sup_{k \geqq 1} \|\varphi^{(k)}\|$ is finite.

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1. Maps onto $\mathfrak{L}_n$

Let $\mathfrak{S}$ be an arbitrary free semigroup algebra acting on a Hilbert space $\mathcal{H}$ generated by isometries $S_1, \dots, S_n$. Our first result shows that in some sense, $\mathfrak{L}_n$ has a universal property.

Theorem 1.1. *Suppose* $\varphi: \mathfrak{S} \to \mathfrak{L}_n$ *is a homomorphism such that* $\varphi(S_i) = L_i$. *If* φ *is* WOT-*continuous, then* φ *is onto and* $\mathfrak{S}/\ker(\varphi)$ *is completely isometrically isomorphic to* $\mathfrak{L}_n$. *Moreover this map is a weak*∗*-weak*∗ *homeomorphism.*

Proof. Let $\mathfrak{A}_n \subseteqq \mathfrak{L}_n$ and $\mathfrak{T} \subseteqq \mathfrak{S}$ be the norm-closed unital algebras generated by $\{L_1, \dots, L_n\}$ and $\{S_1, \dots, S_n\}$ respectively. The norm-closed nonself-adjoint algebra generated by n isometries has a unique operator space structure as noted in the introduction. Consequently, $\varphi|_{\mathfrak{T}}$ is a complete isometry of $\mathfrak{T}$ onto $\mathfrak{A}$.

Let $A = \sum_{w \in \mathbb{F}_n^+} A_w \otimes L_w \in \mathfrak{M}_k \otimes \mathfrak{L}_n$ be given. Define an element X_j in $\mathfrak{M}_k \otimes \mathfrak{T}$ by

$$X_j = \sum_{|w|<j} A_w \left(1 - \frac{|w|}{j}\right) \otimes S_w.$$

Then $\varphi^{(k)}(X_j) = \Sigma_j^{(k)}(A)$. Thus

$$\|X_j\| = \|\Sigma_j^{(k)}(A)\| \leqq \|A\|.$$

Hence the bounded sequence X_j has a WOT-convergent subnet X_{j_α}. Let $X = \text{WOT-lim}\, X_{j_\alpha}$. It follows that

$$\varphi^{(k)}(X) = \text{WOT-lim}\, \varphi^{(k)}(X_{j_\alpha}) = \text{WOT-lim}\, \Sigma_{j_\alpha}^{(k)}(A) = A.$$

Therefore $\varphi^{(k)}$ is surjective for every $k \geqq 1$.

The equalities above show that

$$\|A\| \geqq \|X\| \geqq \|\varphi^{(k)}(X)\| = \|A\|.$$

So the map $\tilde{\varphi}$ of $\mathfrak{S}/\ker(\varphi)$ onto $\mathfrak{L}_n$ induced by φ is a complete isometry. This argument actually shows that $\ker(\varphi)$ is proximal in $\mathfrak{S}$.

Since $\mathfrak{S}$ and $\ker(\varphi)$ are WOT-closed, and thus weak∗ closed, the quotient $\mathfrak{S}/\ker(\varphi)$ inherits a weak∗ topology. Moreover the map φ is WOT-WOT continuous and thus is weak∗-WOT continuous. However the WOT and weak∗ topologies coincide on $\mathfrak{L}_n$ by [10], Corollary 2.12. Hence φ is weak∗-weak∗ continuous. Therefore the map $\tilde{\varphi}$ is weak∗-weak∗ continuous.

By general functional analytic arguments, $\tilde{\varphi}$ is the dual of a map ψ from the predual of $\mathfrak{L}_n$ to the predual of $\mathfrak{S}/\ker(\varphi)$. Moreover, since $\tilde{\varphi}$ is an isometric isomorphism, so is ψ. It now follows that $\tilde{\varphi}^{-1}$ is the dual of ψ^{-1}, and therefore is weak∗ continuous. Consequently, $\tilde{\varphi}$ is a weak∗-weak∗ homeomorphism. □

We shall call a free semigroup algebra on n generators *type* L if there is a WOT-continuous *injective* homomorphism of $\mathfrak{S}$ into $\mathfrak{L}_n$ taking each generator S_i to L_i. Theorem 1.1 shows that type L algebras are completely isometrically isomorphic and weak∗-weak∗ homeomorphic to $\mathfrak{L}_n$.

The following corollary is immediate and is our motivating example for the previous result. It demonstrates the central role of wandering vectors in this discussion. The second corollary extends this to the case in which $\mathfrak{S}$ acts on a space which is spanned by its wandering vectors.

Corollary 1.2. *Suppose $\zeta \in \mathscr{H}$ is a wandering vector for $\mathfrak{S}$, and let $\mathscr{M} = \mathfrak{S}[\zeta]$ be the cyclic invariant subspace for $\mathfrak{S}$ generated by ζ. Let $U: \mathscr{M} \to \mathscr{K}_n$ be the unitary which maps $w(S)\zeta$ to ξ_w for $w \in \mathbb{F}_n^+$. For $T \in \mathfrak{S}$, define $\varphi(T) = U(T|_{\mathscr{M}})U^*$. Then $\varphi: \mathfrak{S} \to \mathfrak{L}_n$ is a* WOT-*continuous, completely contractive epimorphism. Moreover, if $\mathfrak{J}$ is the kernel of the restriction map from $\mathfrak{S}$ onto $\mathfrak{S}|_{\mathscr{M}}$, then $\mathfrak{S}/\mathfrak{J}$ is completely isometrically isomorphic and weak$*$-weak$*$ homeomorphic to $\mathfrak{L}_n$.*

Corollary 1.3. *Suppose that ζ_j for $j \in J$ are wandering vectors for $\mathfrak{S}$. Put $\mathscr{M}_j = \mathfrak{S}[\zeta_j]$ and suppose that $\mathscr{H} = \bigvee_{j \in J} \mathscr{M}_j$. Then $\mathfrak{S}$ is completely isometrically isomorphic and weak$*$-weak$*$ homeomorphic to $\mathfrak{L}_n$.*

Proof. Suppose that $T \in \mathfrak{S}$ and set $T_j := T|_{\mathscr{M}_j}$. The unitary equivalence between each algebra $\mathfrak{S}|_{\mathscr{M}_j}$ and $\mathfrak{L}_n$ shows that $T_j \simeq T_1$ for all $j \in J$. Thus if $T_1 = 0$, then $T_j = 0$ for every j. Since the span of the subspaces $\mathscr{M}_i$ are dense in $\mathscr{H}$, we then conclude that $T = 0$. Hence the restriction map $\varphi: \mathfrak{S} \to \mathscr{L}_n$ obtained by restricting T to $\mathscr{M}_1$ is injective, and the result follows from the Corollary 1.2. $\square$

Next we wish to determine when there exist homomorphisms of $\mathfrak{S}$ onto $\mathfrak{L}_n$. The answer is contained in the WOT-closed ideal

$$\mathfrak{S}_0 = \langle S_1, \ldots, S_n \rangle = \text{WOT-span}\{S_w : |w| \geqq 1\}.$$

It is evident that either this ideal has codimension 1 or it equals $\mathfrak{S}$. We conjecture that if $\mathfrak{S}_0$ is a proper ideal in $\mathfrak{S}$, then $\mathfrak{S}$ has a wandering vector. While we cannot show this, we will come close.

Lemma 1.4. *Suppose that $\mathfrak{S}_0$ is a proper ideal in $\mathfrak{S}$. Then there is an integer p such that $\mathfrak{S}^{(p)}$ has wandering vectors.*

Proof. Let ρ_1 be the WOT-continuous linear functional on $\mathfrak{S}$ which annihilates $\mathfrak{S}_0$ and $\rho_1(I) = 1$. By extending ρ_1 to a WOT-continuous functional on all of $\mathscr{B}(\mathscr{H})$, we may find vectors $\zeta_1, \ldots, \zeta_p$ and $\eta_1, \ldots, \eta_p$ such that $\rho_1(T) = \sum_{j=1}^{p} \langle T\zeta_j, \eta_j \rangle$. Let $\zeta = \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_p \end{pmatrix}$ and $\eta = \begin{pmatrix} \eta_1 \\ \vdots \\ \eta_p \end{pmatrix}$ be vectors in $\mathscr{H}^{(p)}$. Then $\rho_1(T) = (T^{(p)}\zeta, \eta)$ for all $T \in \mathscr{B}(\mathscr{H})$. Since ρ_1 annihilates $\mathfrak{S}_0$, it follows that η is a non-zero vector orthogonal to the cyclic subspace $\mathfrak{S}_0^{(p)}[\zeta]$ but not orthogonal to $\mathfrak{S}^{(p)}[\zeta]$. So we may choose a unit vector ξ in $\mathfrak{S}^{(p)}[\zeta] \ominus \mathfrak{S}_0^{(p)}[\zeta]$. Then $\mathfrak{S}_0^{(p)}[\xi]$ is contained in $\mathfrak{S}_0^{(p)}[\zeta]$. So $\rho(T) := (T^{(p)}\xi, \xi)$ is a WOT-continuous state on $\mathscr{B}(\mathscr{H})$ which annihilates $\mathfrak{S}_0$.

We claim that ξ is a wandering vector for $\mathfrak{S}^{(p)}$. Indeed, if v and w are distinct words in $\mathbb{F}_n^+$ with $|v| \geqq |w|$, there is a word u so that $v = uv'$ and $w = uw'$ where either $w' = \emptyset$ or $v' = iv''$ and $w' = jw''$ for $i \neq j$. In the first case,

$$(S_v\xi, S_w\xi) = (S_{v'}\xi, \xi) = \rho(S_v) = 0.$$

In the latter case, $S_{v'}\xi \in S_i\mathscr{H}$ while $S_{w'}\xi \in S_j\mathscr{H}$. So

$$(S_v\xi, S_w\xi) = (S_{v'}\xi, S_{w'}\xi) = 0.$$

Thus $\{S_v\xi\}$ is orthonormal, and ξ is wandering. $\square$

This allows us to make a sharp dichotomy between the two possibilities for $\mathfrak{S}_0$. Recall that an operator algebra $\mathfrak{A}$ is a *complete quotient* of $\mathfrak{B}$ if $\mathfrak{B}$ has an ideal $\mathfrak{J}$ such that $\mathfrak{B}/\mathfrak{J}$ is completely isometrically isomorphic to $\mathfrak{A}$.

Theorem 1.5. *Let $\mathfrak{S}$ be a free semigroup algebra. Then either*

(1) *$\mathfrak{S}_0 = \mathfrak{S}$ and $\mathfrak{S}$ is a von Neumann algebra, or*

(2) *$\mathfrak{S}_0$ is a proper ideal, and there is a canonical homomorphism of $\mathfrak{S}$ onto $\mathfrak{L}_n$, making $\mathfrak{L}_n$ a complete quotient of $\mathfrak{S}$.*

Proof. If $\mathfrak{S}_0 = \mathfrak{S}$, then there is a net A_α of polynomials in $\mathfrak{S}_0$ WOT-converging to the identity. Since $S_i^*S_j = \delta_{ij}I$ always belongs to $\mathfrak{S}$, it follows that $S_i^*A_\alpha$ lies in $\mathfrak{S}$ and converges weakly to S_i^*. Hence $\mathfrak{S}$ is self-adjoint.

Conversely suppose that $\mathfrak{S}_0$ is proper. Let ξ be the wandering vector for $\mathfrak{S}^{(p)}$ constructed in Lemma 1.4. The restrictions of $S_i^{(p)}$ to the invariant subspace $\mathscr{P} := \mathfrak{S}^{(p)}[\xi]$ are simultaneously unitarily equivalent to the generators L_i of $\mathfrak{L}_n$. Let φ denote the homomorphism of $\mathfrak{S}$ into $\mathfrak{L}_n$ obtained by first ampliating p-fold, restricting to $\mathscr{P}$, and then making the natural unitary identification with $\mathfrak{L}_n$. This is a WOT-continuous homomorphism which carries the generators S_i onto the generators L_i of $\mathfrak{L}_n$. By Theorem 1.1, $\mathfrak{S}/\ker\varphi$ is completely isometrically isomorphic and weak$*$ homeomorphic to $\mathfrak{L}_n$. $\square$

We now push the argument of Lemma 1.4 harder to show that in the type L case, some ampliation of $\mathfrak{S}$ is spanned by its wandering vectors. See Corollary 1.9.

Theorem 1.6. *Suppose that $S = (S_1, \dots, S_n)$ are isometries on $\mathscr{H}_s$ which generate a type* L *free semigroup algebra, and let $\mathfrak{A}$ be the free semigroup algebra on $\mathscr{H}_s \oplus \mathscr{K}_n$ generated by*

$$S \oplus L := (S_1 \oplus L_1, \dots, S_n \oplus L_n).$$

Let $\xi \in \mathscr{H}_s$ be any unit vector, and let $\varepsilon > 0$ be given. Then there exists a vector $\zeta \in \mathscr{K}_n$ with $\|\zeta\| < \varepsilon$ such that the restriction of $\mathfrak{A}$ to the cyclic subspace $\mathscr{N} := \mathfrak{A}[\xi \oplus \zeta]$ is unitarily equivalent to $\mathfrak{L}_n$.

Proof. Let $\varphi\colon \mathfrak{S} \to \mathfrak{L}_n$ be the canonical homomorphism sending each S_j to L_j. Note then that $\mathfrak{A}$ consists of all operators on $\mathscr{H}_s \oplus \mathscr{K}_n$ of the form $A \oplus \varphi(A)$ for some $A \in \mathfrak{S}$.

Consider the vector state on $\mathfrak{S}$ given by $\psi(A) = \langle A\xi, \xi\rangle$ for $A \in \mathfrak{S}$. Since $\mathfrak{S}$ is type L, we may consider this as a state on $\mathfrak{L}_n$. By applying [10], Theorem 2.10, one obtains vectors ζ and η in $\mathscr{K}_n$ such that $\psi(A) = \langle\varphi(A)\zeta, \eta\rangle$. As $\psi(A) = \langle\varphi(A)\, t\zeta, t^{-1}\eta\rangle$ for any non-zero scalar t, we may assume that $\|\zeta\| < \varepsilon$. We then compute for all $A \in \mathfrak{S}$,

$$\langle (A \oplus \varphi(A))(\xi \oplus \zeta), \xi \oplus -\eta \rangle = \langle A\xi, \xi \rangle - \langle \varphi(A)\zeta, \eta \rangle = 0.$$

So $\xi \oplus -\eta$ is orthogonal to the cyclic subspace $\mathscr{N} := \mathfrak{A}[\xi \oplus \zeta]$.

By the Wold decomposition for $\mathfrak{A}$ restricted to $\mathscr{N}$, we obtain an *orthogonal* decomposition $\mathscr{N} = \mathscr{L} + \mathscr{M}$ where $\mathscr{L}$ is the pure part

$$\mathscr{L} := \bigvee_{k \geqq 0} \mathfrak{A}_0^k[\xi \oplus \zeta] \ominus \mathfrak{A}_0^{k+1}[\xi \oplus \zeta]$$

and $\mathscr{M}$ is the Cuntz part

$$\mathscr{M} := \bigcap_{k \geqq 0} \mathfrak{A}_0^k[\xi \oplus \zeta].$$

Now $\mathfrak{A}_0^k(\mathscr{H}_s \oplus \mathscr{K}_n) \subset \mathscr{H}_s \oplus Q_k \mathscr{K}_n$ where Q_k projects onto the span of $\{\xi_w : |w| \geqq k\}$. These spaces have intersection $\mathscr{H}_s \oplus 0$. Thus $\mathscr{M}$ is contained in $\mathscr{H}_s \oplus 0$.

Since $\xi \oplus \zeta$ belongs to $\mathscr{N} = \mathscr{M} + \mathscr{L}$, we may write $\xi \oplus \zeta = \mu + \lambda$ where $\mu = \alpha \oplus 0 \in \mathscr{M}$ and $\lambda = (\xi - \alpha) \oplus \zeta$ lies in $\mathscr{L}$. Moreover, μ and λ are orthogonal, whence $0 = \langle \alpha, \xi - \alpha \rangle$. We saw above that $\xi \oplus -\eta$ is orthogonal to $\mathscr{N} = \mathscr{M} + \mathscr{L}$, and, in particular, is orthogonal to $\alpha \oplus 0$. Therefore we also have $\langle \xi, \alpha \rangle = 0$. As $\xi = \alpha + (\xi - \alpha)$ is a sum of orthogonal vectors, it follows that $\|\xi\|^2 \geqq \|\xi - \alpha\|^2 = \|\xi\|^2 + \|\alpha\|^2$, whence $\alpha = 0$.

Therefore, $\xi \oplus \zeta$ is contained in $\mathscr{L}$. But $\mathscr{L}$ is invariant for $\mathfrak{A}$, and $\xi \oplus \zeta$ generates $\mathscr{N}$, hence $\mathscr{N} = \mathscr{L}$ and $\mathscr{M} = 0$. Thus, since $\mathscr{N}$ is a cyclic subspace for $\mathfrak{A}$, the dimension of $\mathfrak{A}\mathscr{N} \ominus \mathfrak{A}_0\mathscr{N}$ is one. Therefore, $\mathfrak{A}|_{\mathscr{N}}$ is unitarily equivalent to $\mathfrak{L}_n$. □

We immediately obtain:

Corollary 1.7. *Suppose that $S = (S_1, \ldots, S_n)$ generates a type* L *algebra. Then $S \oplus L = (S_1 \oplus L_1, \ldots, S_n \oplus L_n)$ generates a type* L *algebra which is spanned by its wandering vectors.*

Proof. Theorem 1.6 shows that the span of all wandering vectors contains $\mathscr{H}_s \oplus 0$. Clearly $0 \oplus \mathscr{K}_n$ is also spanned by wandering vectors, and the result follows. □

This can now be extended to a more general situation.

Theorem 1.8. *Suppose that $S = (S_1, \ldots, S_n)$ and $T = (T_1, \ldots, T_n)$ are isometries on $\mathscr{H}_s$ and $\mathscr{H}_t$ generating type* L *algebras $\mathfrak{S}$ and $\mathfrak{T}$ respectively. Suppose that $\mathfrak{T}$ has a wandering vector. Then $\mathscr{H}_s \oplus 0$ is contained in the span of the wandering vectors of $S \oplus T$.*

Proof. Let $\xi \oplus 0$ be any vector in $\mathscr{H}_s \oplus 0$. Since $\mathfrak{T}$ has a wandering vector γ, we may restrict $S \oplus T$ to the invariant subspace $\mathscr{M} = \mathscr{H}_s \oplus \mathfrak{T}[\gamma]$. Then $(S \oplus T)|_{\mathscr{M}}$ is unitarily equivalent to $S \oplus L$ acting on $\mathscr{H}_s \oplus \mathscr{K}_n$. Thus without loss of generality, we may assume $T = L$. Now the result follows from Corollary 1.7. □

The converse of the following corollary is trivially true. Thus we obtain a spatial characterization of type L algebras as those which have a finite ampliation spanned by wandering vectors.

Corollary 1.9. *Suppose that $\mathfrak{S}$ is type* L. *Then for p sufficiently large, the space $\mathscr{H}^{(p)}$ is spanned by wandering vectors of $\mathfrak{S}^{(p)}$.*

Proof. Let p be chosen so that $\mathfrak{S}^{(p-1)}$ has wandering vectors, which follows from Lemma 1.4. Then by Theorem 1.8, the span of the wandering vectors contains $\mathscr{H} \oplus 0^{(p-1)}$. By symmetry considerations, it follows that the wandering vectors span all of $\mathscr{H}^{(p)}$. $\square$

2. The Structure Theorem

Our goal in this section is to obtain a structure theorem for a general free semigroup algebra. This result does not say much in the case when $\mathfrak{S}$ equals its enveloping von Neumann algebra $\mathfrak{W}$. So by Theorem 1.5, we may as well assume that $\mathfrak{S}_0$ is a proper ideal. In this case, there is a canonical homomorphism from $\mathfrak{S}$ onto $\mathfrak{L}_n$. The first challenge is to compute the kernel $\mathfrak{J}$ of this map.

First we need some factorization results about ideals which parallel the known results for $\mathfrak{L}_n$ [11].

Lemma 2.1. *Suppose that $\mathfrak{J}$ is a* WOT*-closed right ideal in $\mathfrak{S}$ generated by isometries $T_1, \dots, T_m$ with pairwise orthogonal ranges. Then every element $A \in \mathfrak{J}$ can be uniquely written as $A = \sum_{i=1}^{m} T_i A_i$, where $A_i \in \mathfrak{S}$.*

Proof. There is a net A_α in the algebraic right ideal generated by T WOT-converging to A. Hence $A_\alpha = \sum_{i=1}^{m} T_i A_{\alpha,i}$. It follows that $T_i^* A_\alpha = A_{\alpha,i}$. Thus

$$A_i := T_i^* A = \text{WOT-}\lim_{\alpha} T_i^* A_\alpha = \text{WOT-}\lim_{\alpha} A_{\alpha,i}$$

belongs to $\mathfrak{S}$. Whence $A = \sum_{i=1}^{m} T_i T_i^* A = \sum_{i=1}^{m} T_i A_i$. $\square$

Corollary 2.2. *The ideal $\mathfrak{S}_0^k$ consists of the elements $\sum_{|w|=k} S_w A_w$, and each element of $\mathfrak{S}_0^k$ has a unique expression of this form.*

Proof. We may assume that $\mathfrak{S}_0$ is proper, as this result is trivial in the von Neumann algebra $\mathfrak{W}$. $\mathfrak{S}_0^k$ is the WOT-closure of the algebraic span of all monomials S_w for $|w| \geqq k$. This algebraic ideal is easily seen to be the right ideal generated by $\{S_w\colon |w| = k\}$. Thus the previous lemma applies. $\square$

The next lemma requires that $\mathfrak{S}_0$ be a proper ideal, for if $\mathfrak{S}_0 = \mathfrak{S} = \mathfrak{W}$, then one may express the identity operator in two distinct ways in $\mathfrak{S}$ by $I = \sum_{i=1}^{n} S_i S_i^*$.

Lemma 2.3. *If $\mathfrak{S} \neq \mathfrak{S}_0$, then every $A \in \mathfrak{S}$ can be expressed uniquely as $A = \sum_{|v|<k} a_v S_v + \sum_{|w|=k} S_w A_w$ for certain scalars a_v and elements $A_w \in \mathfrak{S}$.*

Proof. Since $\mathfrak{S}_0$ is a proper ideal, there is a canonical WOT-continuous representation φ of $\mathfrak{S}$ onto $\mathfrak{L}_n$ taking S_i to L_i for $1 \leqq i \leqq n$. It is clear that $\varphi^{-1}(\mathfrak{L}_n^0) = \mathfrak{S}_0$ since $\mathfrak{S}_0$ is contained in this ideal and has codimension 1. Let a_v be the unique scalars (Fourier coefficients) such that $\varphi(A)\xi_\emptyset = \sum_{v \in \mathbb{F}_n^+} a_v \xi_v$. Then in particular, $a_\emptyset$ is the unique scalar such that $A - a_\emptyset I$ belongs to $\mathfrak{S}_0$. Hence by the previous corollary, $A - a_\emptyset I = \sum_{i=1}^n S_i A_i$, and this decomposition is also unique.

This same decomposition may be applied to each A_i. Proceeding by induction, we obtain the desired decomposition

$$A = \sum_{|v|<k} b_v S_v + \sum_{|w|=k} S_w A_w$$

for certain scalars b_v and elements $A_w \in \mathfrak{S}$. Applying φ to this decomposition, it follows that $b_v = a_v$ are uniquely determined. Applying the previous corollary to $A - \sum_{|v|<k} a_v S_v$ shows that the operator coefficients A_w are also uniquely determined. $\square$

Corollary 2.4. *Suppose that $\mathfrak{S}_0 \neq \mathfrak{S}$ and φ is the canonical map of $\mathfrak{S}$ onto $\mathfrak{L}_n$ taking S_i to L_i for $1 \leqq i \leqq n$. Then $\varphi^{-1}(\mathfrak{L}_n^{0,k}) = \mathfrak{S}_0^k$.*

Proof. From the previous corollary, each $A \in \mathfrak{S}$ may be written as

$$A = \sum_{|v|<k} a_v S_v + \sum_{|w|=k} S_w A_w.$$

So $\varphi(A) \in \sum_{|v|<k} a_v L_v + \mathfrak{L}_n^{0,k}$. Therefore $\varphi(A)$ belongs to $\mathfrak{L}_n^{0,k}$ precisely when $a_v = 0$ for all $|v| < k$. Thus $\varphi^{-1}(\mathfrak{L}_n^{0,k}) = \sum_{|w|=k} S_w \mathfrak{S} = \mathfrak{S}_0^k$ by Corollary 2.2. $\square$

Now we are able to compute the kernel $\mathfrak{J}$ of the canonical map φ of $\mathfrak{S}$ onto $\mathfrak{L}_n$.

Theorem 2.5. *Suppose that $\mathfrak{S}_0$ is a proper ideal. Let φ be the* WOT*-continuous homomorphism of $\mathfrak{S}$ onto $\mathfrak{L}_n$ taking S_i to L_i for $1 \leqq i \leqq n$. Then $\mathfrak{J} := \ker\varphi = \bigcap_{k \geqq 1} \mathfrak{S}_0^k$. There is a projection $P \in \mathfrak{S}$ such that $\mathfrak{J} = \mathfrak{W}P$. The range of $P^\perp$ is invariant for $\mathfrak{S}$.*

Proof. By the previous corollary, $\varphi^{-1}(\mathfrak{L}_n^{0,k}) = \mathfrak{S}_0^k$. Thus

$$\ker\varphi = \bigcap_{k \geqq 1} \varphi^{-1}(\mathfrak{L}_n^{0,k}) = \bigcap_{k \geqq 1} \mathfrak{S}_0^k.$$

Let $A \in \mathfrak{J}$ be factored as $A = \sum_{i=1}^n S_i A_i$. Then

$$0 = L_j^* \varphi(A) = L_j^* \sum_{i=1}^n L_i \varphi(A_i) = \varphi(A_j).$$

Thus $A_j = S_j^* A$ also belongs to $\mathfrak{J}$. Since $\mathfrak{J}$ is WOT-closed, it follows that $\mathfrak{J}$ is a left ideal in the von Neumann algebra $\mathfrak{W}$. Therefore there is a unique projection P in $\mathfrak{W}$ such that $\mathfrak{J} = \mathfrak{W}P$.

Since $P\mathscr{H} = \mathfrak{J}^*\mathscr{H}$, it follows that

$$S_i^* P\mathscr{H} = S_i^* \mathfrak{J}^* \mathscr{H} = (\mathfrak{J}S_i)^* \mathscr{H} \subset \mathfrak{J}^*\mathscr{H} = P\mathscr{H}.$$

So $P\mathscr{H}$ is invariant for $\mathfrak{S}^*$, whence $P^\perp\mathscr{H}$ lies in $\operatorname{Lat}\mathfrak{S}$. $\square$

Now that the ideal $\mathfrak{J}$ is understood, we obtain a canonical decomposition of the algebra $\mathfrak{S}$ into a lower triangular form. The description of P only in terms of $\mathfrak{S}$ implies that P is an invariant of the algebra, and is not dependent on a choice of generators.

Structure Theorem 2.6. *Let $\mathfrak{S}$ be a free semigroup algebra, and let $\mathfrak{W}$ be the enveloping von Neumann algebra. Then there is a largest projection P in $\mathfrak{S}$ such that $P\mathfrak{S}P$ is self-adjoint. It has the following properties*:

(i) $\mathfrak{W}P = \bigcap_{k \geqq 1} \mathfrak{S}_0^k$.

(ii) *$P^\perp \mathscr{H}$ is invariant for $\mathfrak{S}$.*

(iii) *If $P \neq I$, then $\mathfrak{S}P^\perp$ is completely isometrically isomorphic and weak$*$ homeomorphic to $\mathfrak{L}_n$ by a canonical* WOT*-continuous homomorphism φ with $\varphi(S_i) = L_i$ for $1 \leqq i \leqq n$.*

(iv) $\mathfrak{S} = \mathfrak{W}P + P^\perp \mathfrak{S} P^\perp$.

Proof. If $\mathfrak{S}_0 = \mathfrak{S}$, then $\mathfrak{S} = \mathfrak{W}$ and $P = I$, and the various properties are clearly true. So we may assume that $\mathfrak{S}_0 \neq \mathfrak{S}$.

By Theorem 1.5, there is a canonical WOT-continuous homomorphism φ onto $\mathfrak{L}_n$ such that $\varphi(S_i) = L_i$ for $1 \leqq i \leqq n$ and $\mathfrak{S}/\ker\varphi$ is completely isometrically isomorphic and weak$*$ homeomorphic to $\mathfrak{L}_n$. By Theorem 2.5, the kernel of this map is $\mathfrak{J} = \bigcap_{k \geqq 1} \mathfrak{S}_0^k = \mathfrak{W}P$ for a certain projection $P \in \mathfrak{S}$. Moreover $P^\perp \mathscr{H}$ is invariant for $\mathfrak{S}$.

Consider the WOT-continuous homomorphism ψ of restriction of $\mathfrak{S}$ to the invariant subspace $P^\perp \mathscr{H}$. Since P belongs to $\mathfrak{S}$, it is evident that $\ker\psi = \mathfrak{S}P$. It follows that $\psi(\mathfrak{S})$ may be identified with $\mathfrak{S}/\mathfrak{J}$ completely isometrically. Hence the map φ onto $\mathfrak{L}_n$ factors through $\psi(\mathfrak{S})$, and the connecting map $\varphi\psi^{-1}$ is injective. By Theorem 1.1, $\psi(\mathfrak{S})$ is completely isometrically isomorphic and weak$*$ homeomorphic to $\mathfrak{L}_n$.

Now $\mathfrak{S} = \mathfrak{S}P + \mathfrak{S}P^\perp = \mathfrak{W}P + P^\perp \mathfrak{S} P^\perp$ yields the desired structure for the algebra $\mathfrak{S}$. In particular, $P\mathfrak{S}P = P\mathfrak{W}P$ is self-adjoint.

Suppose that Q is any projection in $\mathfrak{S}$ such that $Q\mathfrak{S}Q$ is self-adjoint. Then decomposing $\mathscr{H} = P\mathscr{H} \oplus P^\perp \mathscr{H}$, we obtain $Q = \begin{bmatrix} Q_{11} & 0 \\ Q_{21} & Q_{22} \end{bmatrix}$. Then $Q_{21} = 0$ and Q_{22} is a projection in $\mathfrak{S}P^\perp$, which is isomorphic to $\mathfrak{L}_n$. But the only idempotents in $\mathfrak{L}_n$ are 0 and I by [10], Corollary 1.8. If $Q_{22} = I$, then $Q\mathfrak{S}Q|_{P^\perp \mathscr{H}}$ is type L, and thus is not self-adjoint. So $Q_{22} = 0$ and $Q \leqq P$. Hence P is the largest projection with this property. □

One immediate consequence is the following:

Corollary 2.7. *If $\mathfrak{S}_0 \neq \mathfrak{S}$, then there is an invariant projection Q such that $\mathfrak{S}|_{Q\mathscr{H}}$ is type* L. *If P is given by Theorem* 2.6, *then the largest such projection is $P^\perp$.*

Proof. If $\mathfrak{S}_0 \neq \mathfrak{S}$, then by Theorem 1.5, $\mathfrak{S}$ is not self-adjoint. Thus the projection P of Theorem 2.6 satisfies $P < I$. It follows from Theorem 2.6(iii) that $P^\perp$ has the desired property.

Suppose that Q is an invariant projection for $\mathfrak{S}$. Then Q is invariant for $P\mathfrak{S}P$, and in particular $PQ = QP$. If $PQ \neq 0$, then $PQ\mathfrak{S}PQ$ is self-adjoint and thus the restriction to $Q\mathscr{H}$ is not type L. Hence $Q \leqq P^{\perp}$. $\square$

Our next corollary allows us to weaken the definition of a type L algebra to be merely (algebraically) isomorphic to $\mathfrak{L}_n$ or even a subalgebra of $\mathfrak{L}_n$.

Corollary 2.8. *Suppose that $\mathfrak{S}$ is a free semigroup algebra on n generators which is algebraically isomorphic to a subalgebra of $\mathfrak{L}_n$. Then $\mathfrak{S}$ is type* L.

Proof. Since $\mathfrak{L}_n$ contains no non-scalar idempotents, it follows that $\mathfrak{S}$ contains no non-scalar idempotents either. Thus the projection P in the Structure Theorem must be 0. (It cannot be I, for then $\mathfrak{S} = \mathfrak{W}$ contains many projections.) Consequently $\mathfrak{S}$ is type L, and there is a canonical isomorphism of $\mathfrak{S}$ onto $\mathfrak{L}_n$ taking the standard generators to standard generators which is completely isometric and a weak$*$ homeomorphism. $\square$

As an immediate corollary, we can characterize the radical.

Corollary 2.9. *With $n \geqq 2$ and notation as above, the radical of $\mathfrak{S}$ is $P^{\perp}\mathfrak{S}P$. Thus the following are equivalent*:

(i) *$\mathfrak{S}$ is semisimple and is not self-adjoint.*

(ii) *$\mathfrak{S}$ is type* L.

(iii) *$\mathfrak{S}$ has no non-scalar idempotents.*

(iv) *$\mathfrak{S}$ has no non-zero quasinilpotent elements.*

Proof. It is evident that $P^{\perp}\mathfrak{S}P$ is a nil ideal, and hence is contained in the radical. The quotient by this ideal is (completely isometrically) isomorphic to $P\mathfrak{W}P \oplus P^{\perp}\mathfrak{S}P^{\perp}$ which is the direct sum of two semisimple algebras, a von Neumann algebra and an isomorphism of $\mathfrak{L}_n$. See [10], Corollary 1.9. Hence $P^{\perp}\mathfrak{S}P$ is the radical.

Consequently, if $\mathfrak{S}$ is semisimple and is not a von Neumann algebra, it follows that $P = 0$ and $\mathfrak{S}$ is type L. Conversely, type L algebras are non-self-adjoint and semisimple. Type L in turn implies that there are no non-trivial idempotents or quasinilpotent elements [10], Corollary 1.8. Conversely, if $P \neq 0$, then P is a non-scalar idempotent and $P^{\perp}\mathfrak{S}P$ is non-zero and nilpotent. $\square$

We conclude this section with an easy result which identifies $\mathfrak{W}$ as a commutant when S is of Cuntz type, meaning that $\sum_{i=1}^{n} S_i S_i^* = I$. This result is not true for $\mathfrak{L}_n$, as the commutant is the right regular representation algebra $\mathfrak{R}_n$ which is not self-adjoint.

Proposition 2.10. *Suppose that $\mathfrak{S}$ is of Cuntz type. If T belongs to the commutant $\mathfrak{S}'$ of $\mathfrak{S}$, then $T^* \in \mathfrak{S}'$. Hence $\mathfrak{W} = \mathfrak{S}''$. In particular, if $\mathfrak{S}$ is irreducible, then $\mathfrak{S}' = \mathbb{C}I$.*

Proof. We have $T = \sum_i TS_iS_i^* = \sum_i S_iTS_i^*$. Hence $S_j^*T = TS_j^*$, as desired. Thus $\mathfrak{S}'$

is a von Neumann algebra, and so by the Double Commutant Theorem, $\mathfrak{S}''$ must be the von Neumann algebra generated by $\mathfrak{S}$. $\square$

3. Examples

In this section, we examine various examples to demonstrate the previous results.

Example 3.1. Consider the case $n = 1$. We are given an isometry S, which decomposes using the Wold decomposition and the spectral theory of unitary operators as

$$S \simeq U_+^{(\alpha)} \oplus U_a \oplus U_s$$

where U_+ is the unilateral shift, U_a is a unitary with spectral measure absolutely continuous with respect to Lebesgue measure m, and U_s is a singular unitary. Let m_a and m_s denote scalar measures equivalent to the spectral measures of U_a and U_s respectively. If $\alpha > 0$ or if $m_a = m$, then by [24]

$$\mathfrak{S} = W(S) \simeq H^\infty(U_+^{(\alpha)} \oplus U_a) \oplus L^\infty(m_s)(U_s).$$

The von Neumann algebra it generates is

$$\mathfrak{W} = W^*(S) = \mathscr{B}(\mathscr{H})^{(\alpha)} \oplus L^\infty(U_a) \oplus L^\infty(m_s)(U_s).$$

The projection P of the Structure Theorem is just the projection onto the singular part. In this case, it is always a direct summand.

If $\alpha = 0$ and the essential support of U_a is a proper measurable subset of the circle, then $\mathfrak{S} = \mathfrak{W}$ is self-adjoint. Note that this may occur even though the spectrum of S is the whole circle.

The $n = 1$ case exhibits two phenomena which we cannot seem to replicate in the non-commutative case, and remain important open questions.

The first is the situation just noted that $\mathfrak{S}$ can be self-adjoint. The second is that there are isometries S_1 and S_2 such that $W(S_1 \oplus S_2) \simeq H^\infty$, the type L case, yet neither $W(S_i)$ are type L, and in fact are von Neumann algebras. One simply takes S_i to be multiplication by z on the upper and lower half circles respectively.

Example 3.2. This example relates the Structure Theorem 2.6 to the finitely correlated representations studied in [9]. Start with a contractive n-tuple A acting on a finite dimensional space $\mathscr{V}$. By the Frahzo-Bunce-Popescu Dilation Theorem, there is a unique minimal isometric dilation S. Let $\mathfrak{S}$ be the free semigroup algebra generated by S. Denote by $\mathfrak{F}$ the set of all subspaces $\mathscr{M}$ of $\mathscr{V}$ which are invariant for $\{A_i^*: 1 \leqq i \leqq n\}$ and satisfy $\sum_{i=1}^n A_i A_i^*|_{\mathscr{M}} = I_{\mathscr{M}}$, and are minimal with respect to this property. Let $\tilde{\mathscr{V}}$ be the span of the subspaces in $\mathfrak{F}$.

The results of [9] show that the projection P onto $\tilde{\mathscr{V}}$ belongs to $\mathfrak{S}$, that $P\mathfrak{S}$ is a finite dimensional C*-algebra, and that $\mathfrak{S}|_{P^\perp\mathscr{H}}$ is unitarily equivalent to an ampliation of $\mathfrak{L}_n$ and thus of type L. So this projection P is the same projection identified in the Structure Theorem. It was not explicitly shown in [9] that $\mathfrak{S}P = \mathfrak{W}P$, thus this fact is new (although it can be deduced without difficulty from the results of [9]).

Decomposing the finite dimensional C*-algebra $P\mathfrak{S}$ into its direct summands enables one to decompose the representation determined by S into a direct sum of irreducible summands. One can then read off complete unitary invariants from the finite rank n-tuple A.

Next we examine some of the atomic representations classified in [10].

Example 3.3. To construct a wandering vector in $\mathfrak{S}$, one needs to find an invariant subspace $\mathscr{M}$ of $\mathfrak{S}$ such that $\mathscr{M} \ominus \sum S_i\mathscr{M} \neq 0$. The following example shows that it is entirely possible for a free semigroup algebra $\mathfrak{S}$ with generators of Cuntz type ($\mathrm{C}^*(S) = \mathcal{O}_n$) to be type L.

Let $x = i_1 i_2 \dots$ be an infinite word in the alphabet $\{1, \dots, n\}$. We construct a representation as in [10], §3. Let $x_m = i_1 i_2 \dots i_m$ for $m \geqq 0$. Define $\mathbb{F}_n^+ x^{-1}$ denote the collection of words in the free group on n generators of the form $v = u x_m^{-1}$ for u in $\mathbb{F}_n^+$ and $m \geqq 0$. Identify words which are the same after cancellation, namely $u x_m^{-1} = (u i_{m+1}) x_{m+1}^{-1}$. Let $\mathscr{H}_x$ be the Hilbert space with orthonormal basis $\{\xi_v : v \in \mathbb{F}_n^+ x^{-1}\}$. A representation π_x of the free semigroup $\mathbb{F}_n^+$ is defined on $\mathscr{H}_x$ by

$$\pi_x(i)\xi_v = \xi_{iv} \quad \text{for } v \in \mathbb{F}_n^+ x^{-1}.$$

Let $\mathfrak{S}_x$ be the free semigroup algebra with generators $S_i = \pi_x(i)$.

The subspace $\mathscr{H}_m = \ell^2(\mathbb{F}_n^+ x_m^{-1})$ is generated as an $\mathfrak{S}_x$ invariant subspace by the wandering vector $\xi_{x_m^{-1}}$. Thus the compression of $\mathfrak{S}_x$ to $\mathscr{H}_j$ is unitarily equivalent to $\mathfrak{L}_n$. The subspaces $\mathscr{H}_m$ form a nested sequence with union dense in $\mathscr{H}_x$. Hence π_x is an inductive limit of representations equivalent to λ, yet is evidently of Cuntz type. Moreover the restriction of $\mathfrak{S}$ to any $\mathscr{H}_m$ is a completely isometric isomorphism. Therefore $\mathfrak{S}_x$ is type L.

Motivated by this example, we identify type L representations of *inductive* type as those which act on a Hilbert space which is the increasing union of subspaces on which $\mathfrak{S}$ is unitarily equivalent to a multiple of $\mathfrak{L}_n$. We will discuss this notion again in other examples.

Example 3.4. This is an example to show that certain Cuntz representations may be expressed in terms of the span of two invariant subspaces, each of which is determined by a wandering vector.

Let x denote the infinite periodic word 1^∞, and consider the representation π_x associated to this infinite tail. Let ω denote a unimodular function in L^2 such that $H^2 \cap \omega H^2 = \{0\}$ and $H^2 \vee \omega H^2 = L^2$. This is the generic situation. A straightforward example is $\omega(z) = \operatorname{sgn}(\operatorname{Im}(z))$. Denote the Fourier expansion of $\omega(z)$ by $\sum_n a_n z^n$.

Let $\xi_n = \xi_{1^n}$ for $n \in \mathbb{Z}$ denote the vectors along the 'spine' of the representation space of π_x; and let $\mathscr{K} = \operatorname{span}\{\xi_n : n \in \mathbb{Z}\}$. Set $\xi = \xi_0$ and $\eta = \sum_n a_n \xi_n$. Evidently ξ is wandering.

This is also the case for η. Indeed, if u is a word which is not a power of 1, then $(S_u\eta, \eta) = 0$ is clear. While $(S_1^k\eta, \eta) = (z^k\omega, \omega)_{L^2} = 0$ because $|\omega| = 1$ a.e. Let $\mathscr{M} = \mathfrak{S}[\xi]$ and $\mathscr{N} = \mathfrak{S}[\eta]$.

Let U denote the set of words u in $\mathbb{F}_n^+$ which do not terminate with a 1, including the identity $\emptyset$. For $u \in U$, the subspaces $S_u\mathscr{K}$ are pairwise orthogonal and span $\mathscr{H}$. Let us identify H^2 and ωH^2 with the corresponding subspaces $\mathscr{K}_0$ and $\mathscr{K}_\omega$ of $\mathscr{K}$. Now one may readily verify that $\mathscr{M} = \bigvee_{u \in U} S_u\mathscr{K}_0$ and $\mathscr{N} = \bigvee_{u \in \mathscr{U}} S_u\mathscr{K}_\omega$. From this, it is immediate that $\mathscr{M} \cap \mathscr{N} = \{0\}$ and $\mathscr{M} \vee \mathscr{N} = \mathscr{H}$.

This argument may be modified to work for any infinite periodic word.

Example 3.5. Now consider the general atomic representation classified in [10]. Such representations are characterized by the existence of an orthonormal basis which is permuted up to scalars by each isometry S_i. In addition to the left regular representation λ and the 'infinite tail' representations of Example 3.3, there is another class of 'ring' representations $\sigma_{u,\lambda}$ where u is a primitive word (not a power of a smaller word) and $|\lambda| = 1$.

The details of this representation may be seen in the reference. We wish to point out that these particular representations are a special case of Example 3.2. Let e_k for $1 \leqq k \leqq m$ be an orthonormal basis for $\mathbb{C}^m$. If $u = j_1 \dots j_m$, then define partial isometries A_i on $\mathbb{C}^m$ by

$$A_i e_k = \begin{cases} e_{k+1} & \text{if } i = j_k \text{ and } 1 \leqq k < m, \\ \lambda e_1 & \text{if } i = j_m \text{ and } k = m, \\ 0 & \text{otherwise.} \end{cases}$$

The representation $\sigma_{u,\lambda}$ is obtained from the minimal isometric dilation of this n-tuple.

Since u is primitive, this representation is irreducible and the algebra

$$\mathfrak{S} = \mathscr{B}(\mathscr{H})P_u + (0_m \oplus \mathfrak{L}_n^{((n-1)m)})$$

acting on $\mathbb{C}^m \oplus \mathscr{K}_n^{((n-1)m)}$, where P_u projects onto $\mathbb{C}^m$. In this example, the projection P of the structure theorem is just the projection P_u and $\mathfrak{W} = \mathscr{B}(\mathscr{H})$.

The most general atomic representation is the direct sum of copies of these three types of atomic representations. The tail representations are inductive, and thus of type L, as is the left regular representation. However the ring representations are not. It follows from [10] that the projection P of the Structure Theorem is just the direct sum of the projections obtained in the previous paragraph. Indeed, there are explicit polynomials constructed in [10] which converge weakly to P_u in $\mathfrak{S}$. The projection onto the smallest invariant subspace of $\mathfrak{S}$ containing the range of P_u picks out the multiple of $\sigma_{u,\lambda}$ in the representation and vanishes on all other summands. The fact that there is such an explicit formula allows a very detailed decomposition theory with precise invariants.

Proposition 3.6. *Suppose that $\mathfrak{S}$ is type* L *and $\mathscr{M}$ is an invariant subspace of $\mathfrak{S}$ containing a wandering vector. Then $\mathfrak{S}|_{\mathscr{M}}$ is type* L.

Proof. Let $\mathfrak{T}$ denote the WOT-closure of $\mathfrak{S}|_{\mathscr{M}}$; and let $\mathscr{N}$ be the invariant subspace of $\mathscr{M}$ generated by a wandering vector. Note that $\mathfrak{A} := \mathfrak{T}|_{\mathscr{N}} = \mathfrak{S}|_{\mathscr{N}}$ is unitarily equivalent to $\mathfrak{L}_n$. Let φ be the restriction map of $\mathfrak{S}$ into $\mathfrak{T}$ and ψ the restriction map of $\mathfrak{T}$ onto $\mathfrak{A}$. Both maps are completely contractive and WOT-continuous, and the composition σ is a complete isometry and WOT-homeomorphism. Thus φ is also completely isometric.

Define $\Psi = \varphi\sigma^{-1}\psi$. This is a completely contractive WOT-continuous map of $\mathfrak{T}$ into itself. Moreover, it is easily shown to be idempotent, and thus is an expectation onto a subalgebra completely isometrically isomorphic to $\mathfrak{L}_n$. We will show that Ψ is the identity map. Since the range of Ψ is precisely $\mathfrak{S}|_{\mathscr{M}}$, it suffices to show that this is WOT-closed. Let T_α be a net of operators in $\mathfrak{S}|_{\mathscr{M}}$ WOT-converging to an element $T \in \mathfrak{T}$. Then

$$T = \text{WOT-}\lim_{\alpha} T_\alpha = \text{WOT-}\lim_{\alpha} \Psi(T_\alpha) = \Psi(T).$$

So T belongs to $\mathfrak{S}|_{\mathscr{M}}$. Thus $\mathfrak{T}$ is type L. $\square$

Example 3.7. Consider the representation π_{1^∞}. This is an atomic representation of inductive type, and hence is of type L. By [10], it is also the direct integral $\pi_{1^\infty} \simeq \int_{\mathbb{T}} \sigma_{1,\lambda}\, d\lambda$. Indeed, let $\mathscr{K} = \mathbb{C} \oplus \mathscr{K}_n$; and the representation $\sigma_{1,\lambda}$ is determined by generators

$$S_1^\lambda = \begin{bmatrix} \lambda & 0 \\ 0 & L_1 \end{bmatrix} \quad \text{and} \quad S_2^\lambda = \begin{bmatrix} 0 & 0 \\ \xi_\emptyset & L_2 \end{bmatrix}.$$

Thus the representation π_{1^∞} may be represented on $\mathscr{H}_T := L^2(\mathbb{T}) \otimes \mathscr{K}$ by

$$S_1 = \begin{bmatrix} U & 0 \\ 0 & I \otimes L_1 \end{bmatrix} \quad \text{and} \quad S_2 = \begin{bmatrix} 0 & 0 \\ I \otimes \xi_\emptyset & I \otimes L_2 \end{bmatrix}$$

where U is multiplication by z on $L^2(\mathbb{T})$.

Let E be a measurable subset of $\mathbb{T}$ with positive measure. Let V denote $U|_{L^2(E)}$ and $J = I_{L^2(E)}$. Now consider the representation ρ_E on $\mathscr{H}_E = L^2(E) \otimes \mathscr{K}$ by

$$\begin{bmatrix} V & 0 \\ 0 & J \otimes L_i \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ J \otimes \xi_\emptyset & J \otimes L_2 \end{bmatrix}.$$

It is evident that any vector of the form $0 \oplus (f \otimes \xi_\emptyset)$ is a wandering vector. Hence $\mathscr{H}_E$ is naturally identified with a subspace of $\mathscr{H}_T$. By the previous proposition, it follows that this representation is of type L.

Split E into two disjoint sets E_1 and E_2 of positive measure. Then $\mathscr{H}_E = \mathscr{H}_{E_1} \oplus \mathscr{H}_{E_2}$ yields an orthogonal decomposition into reducing subspaces. The restriction to each subspace is type L and contains wandering vectors by the previous paragraph. By Theorem 1.8, it follows that the wandering vectors in $\mathscr{H}_E$ span both $\mathscr{H}_{E_1} \oplus 0$ and $0 \oplus \mathscr{H}_{E_2}$. Thus $\mathscr{H}_E$ is spanned by wandering vectors.

Consider the representation $\pi_{1^\infty} \oplus \rho_E$ on $\mathscr{H}_T \oplus \mathscr{H}_E$. This is again of type L. It is easy to see that the restriction to the subspaces $\mathscr{H}_T \oplus 0$ and $\mathscr{H}_{E^c} \oplus \mathscr{H}_E$ are both unitarily equivalent to π_{1^∞}. Thus $\mathscr{H}_T \oplus \mathscr{H}_E$ is the span of two inductive type reducing subspaces.

Questions. Is $\mathscr{H}_E$ of inductive type? Is $\mathscr{H}_E^{(\infty)}$ of inductive type? Is $\mathscr{H}_T \oplus \mathscr{H}_E$ of inductive type?

Example 3.8. This example is to show that there is a large natural class of representations of $\mathcal{O}_n$ which are inductive limits of (multiples of) the left regular representation.

Let $\mathscr{U}_n$ denote the canonical copy of the n^∞ UHF algebra contained in $\mathcal{O}_n$ spanned by the words $\{s_\alpha s_\beta^*\colon |\alpha| = |\beta|, \alpha, \beta \in \mathbb{F}_n^+\}$ (we use s_β^* as a shorthand for $(s_\beta)^*$). Let E denote the expectation of $\mathcal{O}_n$ onto $\mathscr{U}_n$. If φ_0 is any state on $\mathscr{U}_n$, let $\varphi = \varphi_0 E$. This is a state on $\mathcal{O}_n$. Let $(\pi_\varphi, \mathscr{H}_\varphi, \xi_\varphi)$ denote the GNS representation associated to φ.

We claim that π_φ is always an inductive limit of multiples of λ. Indeed, let $x_{\alpha,\beta} = [s_\alpha s_\beta^*]$ be the image of $s_\alpha s_\beta^*$ in $\mathscr{H}_\varphi$. This is a spanning set. It is easy to calculate that

$$\langle x_{\alpha,\beta}, x_{\gamma,\delta}\rangle = \begin{cases} \varphi_0(s_\delta s_{\beta\varepsilon}^*) & \text{if } \gamma = \alpha\varepsilon,\ |\delta| = |\beta| + |\varepsilon|, \\ \varphi_0(s_{\delta\varepsilon} s_\beta^*) & \text{if } \alpha = \gamma\varepsilon,\ |\beta| = |\delta| + |\varepsilon|, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, each vector $x_{\alpha,\beta}$ is wandering (or zero). Thus for each $k \geqq 0$, $\mathscr{W}_k = \operatorname{span}\{x_{\emptyset,\beta}\colon |\beta| = k\}$ is a wandering space generating the subspace

$$\mathscr{M}_k = \operatorname{span}\{x_{\alpha,\beta}\colon |\beta| = k, \alpha \in \mathbb{F}_n^+\}.$$

Now we show that $\mathscr{W}_k$ is contained in $\mathscr{M}_{k+1}$. From this it follows that $\mathscr{M}_k \subset \mathscr{M}_{k+1}$ for $k \geqq 0$; whence $\mathscr{H}_\varphi$ is spanned by the union of the $\mathscr{M}_k$s. To this end, compute

$$\langle x_{\emptyset,\beta}, x_{i,\beta i}\rangle = \varphi_0(s_{\beta i} s_{\beta i}^*) = \|x_{i,\beta i}\|^2.$$

Hence

$$\begin{aligned} \left\langle x_{\emptyset,\beta}, \sum_{i=1}^n x_{i,\beta i}\right\rangle &= \sum_{i=1}^n \varphi_0(s_{\beta i} s_{\beta i}^*) = \sum_{i=1}^n \|x_{i,\beta i}\|^2 \\ &= \varphi_0\left(\sum_{i=1}^n s_\beta s_i s_i^* s_\beta^*\right) = \varphi_0(s_\beta s_\beta^*) = \|x_{\emptyset,\beta}\|^2. \end{aligned}$$

Hence it follows that

$$\left\| x_{\emptyset,\beta} - \sum_{i=1}^n x_{i,\beta i}\right\|^2 = \|x_{\emptyset,\beta}\|^2 + \left\|\sum_{i=1}^n x_{i,\beta i}\right\|^2 - 2\operatorname{Re}\left\langle x_{\emptyset,\beta}, \sum_{i=1}^n x_{i,\beta i}\right\rangle = 0.$$

This establishes the claim.

The restriction of $\mathfrak{S}$ to each $\mathscr{M}_k$ is a multiple of the left regular representation. Since the $\mathscr{M}_k$ are nested and span $\mathscr{H}$, it follows that π_φ is inductive.

Example 3.9. We conclude with one final example which yields a large class of inductive type L representations. Let $S = (S_1, \dots, S_n)$ be any n-tuple of isometries with orthogonal ranges acting on $\mathscr{H}$. Let U be the unitary of multiplication by z on $L^2(\mathbb{T})$. Define a new n-tuple of isometries on $\mathscr{H} \otimes L^2(\mathbb{T})$ by $S \otimes U = (S_1 \otimes U, \dots, S_n \otimes U)$. Evidently

$$\sum_{i=1}^{n} (S_i \otimes U)(S_i \otimes U)^* = \sum_{i=1}^{n} S_i S_i^* \otimes I.$$

So these isometries have orthogonal ranges. In addition, if S is of Cuntz type, then so is the tensored n-tuple.

However, this new representation has a spanning set of wandering vectors of the form $\xi \otimes z^k$ for any $\xi \in \mathscr{H}$ and $k \in \mathbb{Z}$, as a simple calculation shows. By Corollary 1.3, this representation has type L.

4. The geometry of the ball

The ultimate goal of this section is a Russo-Dye type theorem showing that the convex hull of the isometries in any free semigroup algebra contains the whole unit ball. A number of structural results of independent interest are needed along the way.

The first step is to show that in an algebra $\mathfrak{S}$ of type L, every isometry in $\mathfrak{L}_n$ is sent to an isometry in $\mathfrak{S}$. The argument is easy given what we already know, but we give it the status of a theorem because it is not at all clear just from a completely isometric isomorphism.

Theorem 4.1. *Let $\mathfrak{S}$ be a free semigroup algebra of type* L, *and let φ be the canonical isomorphism of $\mathfrak{L}_n$ onto $\mathfrak{S}$. Then for each isometry $V \in \mathfrak{M}_k(\mathfrak{L}_n)$, the element $A = \varphi^{(k)}(V)$ is an isometry in $\mathfrak{M}_k(\mathfrak{S})$.*

Proof. It suffices to establish this for some ampliation, as A is an isometry if and only if $A^{(p)}$ is isometric. We choose p using Corollary 1.9. Thus we may suppose that $p = 1$ and that $\mathscr{H}$ is spanned by wandering vectors. If ξ is any wandering vector in $\mathscr{H}$, the restriction of $\mathfrak{M}_k(\mathfrak{S})$ to direct sum of k copies of $\mathfrak{S}[\xi]$ is unitarily equivalent to $\mathfrak{M}_k(\mathfrak{L}_n)$. Hence A is isometric on this subspace. As φ is completely isometric, we know that $\|A\| = 1$. A contraction which is isometric on a spanning set is an isometry because of the elementary fact that the set of vectors on which any operator achieves its norm is a closed subspace. □

In order to construct sufficiently many isometries in $\mathfrak{S}$, we need a factorization result for $I - A^*A$ when A is a strict contraction. Let us say that $\mathfrak{S}$ satisfies the *factorization property* if whenever A is an element of $\mathfrak{S}$ with $\|A\| < 1$, then there is some $B \in \mathfrak{S}$ such that $A^*A + B^*B = I$. It is a special case of Popescu [20], Theorem 4.1 that $\mathfrak{L}_n$ has this property. Moreover, B may be chosen to be invertible. We shall use the previous theorem to extend this to all type L algebras. Then by combining our Structure Theorem with Cholesky's algorithm, we shall extend this to all free semigroup algebras.

For the reader's convenience, we provide a proof of Popescu's Lemma for $\mathfrak{L}_n$. Let R_i generate the right regular representation on Fock space by $R_i \xi_w = \xi_{wi}$ for $1 \leqq i \leqq n$ and

$w \in \mathbb{F}_n^+$. Recall that the WOT-closed algebra $\mathfrak{R}_n$ generated by the R_is is unitarily equivalent to $\mathfrak{L}_n$, and that $\mathfrak{R}_n' = \mathfrak{L}_n$.

Lemma 4.2 (Popescu). *Suppose that $A \in \mathfrak{L}_n$ and $\|A\| < 1$. Then there is an invertible element $B \in \mathfrak{L}_n$ such that $A^*A + B^*B = I$.*

Proof. Let $T = I - A^*A$, which is positive and invertible. Define $S_i = T^{1/2} R_i T^{-1/2}$. An easy computation shows that for $1 \leqq i, j \leqq n$,

$$\begin{aligned} S_j^* S_i &= T^{-1/2} R_j^* T R_i T^{-1/2} \\ &= T^{-1/2}(R_j^* R_i - A^* R_j^* R_i A) T^{-1/2} = \delta_{ij} I. \end{aligned}$$

Thus the S_i are isometries with orthogonal ranges.

We claim that the S_i are simultaneously unitarily equivalent to the R_i. To see this, notice that

$$\sum_{|w|=k} \operatorname{Ran} S_w = T^{1/2} \sum_{|w|=k} \operatorname{Ran} R_w = T^{1/2} \operatorname{span}\{\xi_v \colon |v| \geqq k\}.$$

Hence these ranges intersect to $\{0\}$, so there is no Cuntz part in the Wold decomposition. Moreover $\sum_{i=1}^{n} \operatorname{Ran} S_i$ has codimension 1, so that the multiplicity of the pure part is 1. Therefore there is a unitary operator W such that $S_i = W^* R_i W$ for $1 \leqq i \leqq n$.

Define $B = WT^{1/2}$. Clearly $B^*B = T = I - A^*A$ and B is invertible. Finally, for $1 \leqq i \leqq n$,

$$R_i B = R_i W T^{1/2} = W S_i T^{1/2} = W T^{1/2} R_i = B R_i.$$

Thus B belongs to $\mathfrak{R}_n' = \mathfrak{L}_n$. □

Theorem 4.3. *Every free semigroup algebra $\mathfrak{S}$ satisfies the factorization property.*

Proof. First suppose that $\mathfrak{S}$ is of type L, and let A be a strict contraction in $\mathfrak{S}$. Let φ be the canonical isomorphism of $\mathfrak{S}$ onto $\mathfrak{L}_n$. We apply Popescu's Lemma to the element $\varphi(A)$ to obtain an invertible element which we may call $\varphi(B)$ so that $\varphi(A)^*\varphi(A) + \varphi(B)^*\varphi(B) = I$. This is equivalent to the statement that $L_1\varphi(A) + L_2\varphi(B)$ is an isometry in $\mathfrak{L}_n$. So by Theorem 4.1, $S_1 A + S_2 B$ is an isometry in $\mathfrak{S}$. Again this is equivalent to $A^*A + B^*B = I$. Therefore $\mathfrak{S}$ has the factorization property, and moreover B is invertible in $\mathfrak{S}$.

Now consider a general free semigroup algebra $\mathfrak{S}$. By Theorem 2.6, $\mathfrak{S}$ decomposes via a projection P in $\mathfrak{S}$ as $\mathfrak{S} = \mathfrak{W}P + P^\perp \mathfrak{S} P^\perp$, where $\mathfrak{W}$ is the von Neumann algebra generated by S and $P^\perp \mathfrak{S}|_{P^\perp \mathscr{H}}$ is of type L.

Let $A \in \mathfrak{S}$ with $\|A\| < 1$. Decompose A into a matrix with respect to the decomposition $P\mathscr{H} \oplus P^\perp \mathscr{H}$ as $A = \begin{bmatrix} X & 0 \\ Y & Z \end{bmatrix}$. Note that since $P \in \mathfrak{S}$, each entry 'belongs' to $\mathfrak{S}$ in the sense that $X := PAP$, $Y := P^\perp AP$ and $Z := P^\perp AP^\perp$ belong to $\mathfrak{S}$. Now $\|Z\| < 1$ and Z belongs to an algebra of type L. Hence by the first paragraph, there is an invertible element T in $P^\perp\mathfrak{S}|_{P^\perp\mathscr{H}}$ such that $T^*T = P^\perp - Z^*Z$.

By Cholesky's algorithm in $\mathfrak{W}$, there is a lower triangular operator of the form $B = \begin{bmatrix} Q & 0 \\ R & T \end{bmatrix}$ such that $B^*B = I - A^*A$. Indeed, since

$$I - A^*A = \begin{bmatrix} I - X^*X - Y^*Y & -Y^*Z \\ -Z^*Y & I - Z^*Z \end{bmatrix},$$

we may define $R = -T^{*-1}Z^*Y$. The operator $P^\perp RP$ belongs to $\mathfrak{W}P$, and thus is in $\mathfrak{S}$. The Cholesky argument shows that

$$\begin{bmatrix} R^* \\ T^* \end{bmatrix}\begin{bmatrix} R & T \end{bmatrix} = \begin{bmatrix} R^*R & -Y^*Z \\ -Z^*Y & I - Z^*Z \end{bmatrix}$$

is the smallest positive operator with these coefficients in the 12, 21 and 22 entries. That is, $R^*R \leqq I - X^*X - Y^*Y$. Let

$$Q = (I - X^*X - Y^*Y - R^*R)^{1/2}.$$

So PQP lies in $P\mathfrak{W}P$ and hence in $\mathfrak{S}$. It is now easy to see that B lies in $\mathfrak{S}$ and $B^*B = I - A^*A$. $\square$

We now turn to the convex hull of the set $\mathrm{Isom}(\mathfrak{S})$ of all isometries in $\mathfrak{S}$. An integral argument yields the norm-closed convex hull. Then a finite approximation yields the algebraic result. We note that the necessary facts are factorization and the existence of two isometries with orthogonal ranges. So this argument works for some other operator algebras as well.

Theorem 4.4. *The closed convex hull of* $\mathrm{Isom}(\mathfrak{S})$ *of the set of isometries in any free semigroup algebra* $\mathfrak{S}$ *is the whole closed unit ball.*

Proof. For $|a| < 1$, let $\omega_a(z) = \dfrac{\bar{a}(a - z)}{|a|(1 - \bar{a}z)}$ be the Möbius map which sends a to 0 and 0 to $|a|$. Notice that for all $z \in \overline{\mathbb{D}}$,

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} \omega_{re^{i\theta}}(z)\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi} (r - ze^{-i\theta})(1 - rze^{-i\theta})^{-1}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} (r - ze^{-i\theta}) \sum_{k \geqq 0} (rze^{-i\theta})^k\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} r - (1 - r^2) \sum_{k \geqq 1} r^{k-1}z^k e^{-ik\theta}\,d\theta = r. \end{aligned}$$

Let $A \in \mathfrak{S}$ with $\|A\| = s < 1$. Then choose $s < r < 1$, and let $A_\theta = \omega_{re^{i\theta}}(S_1)r^{-1}A$. By the previous integral calculation, we observe that

$$A = \frac{1}{2\pi}\int_0^{2\pi} A_\theta \, d\theta.$$

So it suffices to show that each A_θ belongs to $\operatorname{conv}\big(\operatorname{Isom}(\mathfrak{S})\big)$.

Let T_θ be an isometry in $\mathfrak{S}$ whose range is orthogonal to the range of $\omega_{re^{i\theta}}(S_1)$. One such isometry is

$$T_\theta = \sqrt{1-r^2}(I - re^{-i\theta}S_1)^{-1}S_2.$$

This may be seen by direct computation. (However it was really found by using an isometric automorphism Θ_X of $\mathfrak{S}$ [23], [11] associated to the matrix $X = \begin{bmatrix} x_0 & \eta^* \\ \eta & X_1 \end{bmatrix}$ where $x_0 = 1/(1-r^2)$, $\eta = \dfrac{re^{-i\theta}}{\sqrt{1-r^2}} e_1$ and $X_1 = (1-r^2)^{-1}E_{11} + E_{11}^\perp$. After noting that $\omega_{re^{i\theta}}(S_1) = \Theta_X(-S_1)$, set $T_\theta = \Theta_X(S_2)$, which is the formula given above.)

By Theorem 4.3, the invertible operator $I - A_\theta^* A_\theta$ factors as $D_\theta^* D_\theta$ for some $D_\theta \in \mathfrak{S}$. Then a simple calculation shows that $S_\theta^\pm = A_\theta \pm T_\theta D_\theta$ are isometries in $\mathfrak{J}$ with average A_θ. $\square$

If we work harder, we can show that all elements of the open ball are in fact a finite convex combination of isometries.

Theorem 4.5. *The convex hull of* $\operatorname{Isom}(\mathfrak{S})$ *contains the open unit ball of* $\mathfrak{S}$. *Moreover, if* $\|A\| < 1 - \dfrac{1}{k}$ *for* $k > 0$ *an even integer, then* A *is the average of* $6k$ *isometries.*

Proof. The simple observation is that the integral of the previous theorem may be replaced with a finite sum. Indeed, fix $0 < r < 1$ and an integer $p \geqq 2$. Let $\alpha = e^{2\pi i/p}$. Then

$$\begin{aligned} f_{r,p}(z) := \frac{1}{p}\sum_{j=0}^{p-1} \frac{r - z\alpha^j}{1 - rz\alpha^j} &= \frac{1}{p}\sum_{j=0}^{p-1}\left(r - (1-r^2)\sum_{k\geqq 1} r^{k-1}z^k\alpha^{kj}\right) \\ &= r - (1-r^2)\sum_{k\geqq 1} r^{k-1}z^k \frac{1}{p}\sum_{j=0}^{p-1}\alpha^{kj} \\ &= r - (1-r^2)\sum_{k\geqq 1} r^{kp-1}z^{kp} \\ &= r - \frac{(1-r^2)r^{p-1}z^p}{1 - r^p z^p}. \end{aligned}$$

Thus if we set $r = 1 - \dfrac{1}{3k}$, $p = 6k$ and $f_k = f_{r,p}$ for $k \geqq 1$, then observing that $\left(1 - \dfrac{1}{3k}\right)^{3k} < \dfrac{1}{e}$, we obtain

$$\|r - f_k\|_\infty = \frac{(1-r^2)r^p}{r(1-r^p)} < \left(\frac{2}{3k-1}\right)\frac{e^{-2}}{1-e^{-2}} < \frac{1}{3(3k-1)} < \frac{1}{6k}.$$

In particular, $\|f_k^{-1}\|_\infty < \left(1 - \frac{1}{2k}\right)^{-1}$.

Now if $A \in \mathfrak{S}$ satisfies $\|A\| < 1 - \frac{1}{2k}$, then $\|f_k^{-1}(S_1)A\| < 1$. Thus the same is true for $A_j = \omega_{r\bar{\lambda}^j}(S_1)f_k^{-1}(S_1)A$. Now it follows from the calculation above that

$$\frac{1}{6k}\sum_{j=0}^{6k-1} A_j = f_k(S_1)f_k^{-1}(S_1)A = A.$$

As in the previous theorem, each A_j is the average of two isometries in $\mathfrak{S}$. Thus A is the average of $12k$ isometries. □

When $n = 1$, it is known that the convex hull of the inner functions contains the whole open ball of H^∞ [17]. An easier result shows that the convex hull of the continuous inner functions contains the open ball of the disk algebra [13]. We wonder if the same holds for the non-commutative disk algebra $\mathfrak{A}_n$.

5. Invariant subspaces

In this section, we consider the question of reflexivity and hyper-reflexivity. First we make an elementary observation which highlights yet again the importance of deciding the question of wandering vectors.

Remark 5.1. Suppose that there were a type L algebra $\mathfrak{S}$ without wandering vectors. Such an algebra is reductive by [9], Lemma 3.3. Hence $\operatorname{Alg}\operatorname{Lat}\mathfrak{S} = \mathfrak{W}$ is the von Neumann algebra it generates. As $\mathfrak{S}$ is isomorphic to $\mathfrak{L}_n$, it is not self-adjoint, and thus is a proper subalgebra of $\mathfrak{W}$. Consequently it would not be reflexive.

The existence of such an algebra would provide a concrete counter-example to the reductive algebra problem, which is essentially equivalent to the transitive algebra problem. The transitive algebra problem is in turn closely related to the invariant subspace problem. Thus, it seems unlikely that such examples exist.

The structure theorem reduces the question of reflexivity for arbitrary free semigroup algebras to those of type L, as we now show.

Lemma 5.2. *Let $\mathfrak{S}$ be a free semigroup algebra, and let P be the projection in Theorem 2.6. Then $\operatorname{Lat}\mathfrak{S}$ consists of all subspaces of the form $\mathscr{M} = \mathscr{M}_1 \vee \mathscr{M}_2$ where $\mathscr{M}_1$ belongs to $\operatorname{Lat}\mathfrak{W}$ and $\mathscr{M}_2 = 0 \oplus N$ for N in $\operatorname{Lat}\mathfrak{S}|_{P^\perp\mathscr{H}}$.*

Proof. It is evident that subspaces of the given form are invariant for $\mathfrak{S}$. So we show that an arbitrary invariant subspace $\mathscr{M}$ has this form. Since $P \in \mathfrak{S}$, we have $P\mathscr{M}$ and $\mathscr{M}_2 = P^\perp\mathscr{M}$ are contained in $\mathscr{M}$. Moreover $\mathscr{M}_2$ is invariant for $\mathfrak{S}$ and thus has the form $\mathscr{M}_2 = 0 \oplus N$ for N in $\operatorname{Lat}\mathfrak{S}|_{P^\perp\mathscr{H}}$. Hence $\mathscr{M}$ contains $\mathscr{M}_1 = \mathfrak{S}P\mathscr{M} = \mathfrak{W}P\mathscr{M}$, which lies in $\operatorname{Lat}\mathfrak{W}$ and contains $P\mathscr{M}$. So $\mathscr{M} = P\mathscr{M} \oplus \mathscr{M}_2 = \mathscr{M}_1 \vee \mathscr{M}_2$. □

Proposition 5.3. *Let $\mathfrak{S}$ be a free semigroup algebra, and let P be the projection in Theorem 2.6. Then $\mathfrak{S}$ is reflexive if and only if the type L algebra $\mathfrak{S}|_{P^\perp\mathscr{H}}$ is reflexive.*

Proof. Every von Neumann algebra is reflexive. So it is clear that $\operatorname{Alg}\operatorname{Lat}\mathfrak{S}$ is contained in $\mathfrak{W}P + (0 \oplus \operatorname{Alg}\operatorname{Lat}\mathfrak{S}|_{P^\perp\mathscr{H}})$. However, every such operator lies in $\mathfrak{W}$ and leaves the subspaces $0 \oplus N$ for N in $\operatorname{Lat}\mathfrak{S}|_{P^\perp\mathscr{H}}$ invariant. Thus such an operator belongs to $\operatorname{Alg}\operatorname{Lat}\mathfrak{S}$. It follows that $\mathfrak{S}$ is reflexive if and only if $\mathfrak{S}|_{P^\perp\mathscr{H}}$ is. □

Recall that an operator algebra $\mathfrak{A}$ is hyper-reflexive if there is a constant C so that

$$\operatorname{dist}(T, \mathfrak{A}) \leqq C \sup_{P \in \operatorname{Lat}\mathfrak{A}} \|P^\perp T P\| \quad \text{for all } T \in \mathscr{B}(\mathscr{H}).$$

Many free semigroup algebras have been shown to have this property, namely the atomic representations [10] and the finitely correlated ones [9], both with distance constant at most 5.

It is plausible that all free semigroup algebras are hyper-reflexive but we are not close to resolving that at this time. It is certainly helpful in proving such an estimate if the enveloping von Neumann algebra is hyper-reflexive. Christensen [8] showed that a von Neumann algebra $\mathfrak{W}$ is hyper-reflexive if and only if the cohomology group $H^1\big(\mathfrak{W}', \mathscr{B}(\mathscr{H})\big) = 0$. He verified this in the case that $\mathfrak{W}'$ is either infinite or injective or cyclic or of type II_1 isomorphic to the spatial tensor product of itself with the hyperfinite II_1 factor. Thus the hyper-reflexivity of von Neumann algebras is unresolved only when the commutant is type II_1 without various special properties.

A WOT-closed algebra $\mathfrak{A}$ has property $X_{0,1}$ if for every weak$*$ continuous functional f on $\mathfrak{A}$, there is a sequence of vectors ζ_k and η_k in $\mathscr{H}$ such that

(i) $f = [\zeta_k \eta_k^*]$, where the square brackets denote restriction to $\mathfrak{A}$,

(ii) $\lim \|\zeta_k\|\,\|\eta_k\| = \|f\|$, and

(iii) $\lim \|[\xi \eta_k^*]\| = \lim \|[\zeta_k \xi^*]\| = 0$ for all $\xi \in \mathscr{H}$.

In [3], Bercovici showed that any algebra $\mathfrak{A}$ commuting with two isometries with orthogonal ranges has property $X_{0,1}$. This includes $\mathfrak{L}_n$ and all type III von Neumann algebras. Moreover, he shows that any algebra with property $X_{0,1}$ is hyper-reflexive with constant at most 3.

Lemma 5.4. *If $\mathfrak{S}$ is an algebra of type* L *generated by S, then the algebra $\mathfrak{A}$ generated by $S \oplus L$ has property $X_{0,1}$, and therefore is hyper-reflexive with constant* 3.

Proof. We only need to observe that if f is a functional on $\mathfrak{A}$, then it determines a functional of the same norm on $\mathfrak{L}_n$. Since $\mathfrak{L}_n$ is a direct summand, it is evident that choosing the sequence of vectors in this summand demonstrates property $X_{0,1}$. □

Theorem 5.5. *Let $\mathfrak{S}$ and $\mathfrak{T}$ be two free semigroup algebras of type* L, *generated by S and T respectively, which have wandering vectors. Then the algebra $\mathfrak{A}$ generated by $A = S \oplus T$ is hyper-reflexive with constant at most* 7.

Proof. Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be the spaces on which $\mathfrak{S}$ and $\mathfrak{T}$ act, let ξ_1 and ξ_2 be the wandering vectors which are given, let φ_i be the canonical identification of $\mathfrak{L}_n$ onto $\mathfrak{S}$ and $\mathfrak{T}$

respectively, and set $\varphi = \varphi_1 \oplus \varphi_2$. Notice that the restriction $\mathfrak{A}_1$ of $\mathfrak{A}$ to $\mathscr{K}_1 := \mathscr{H}_1 \oplus \mathfrak{T}[\xi_2]$ is unitarily equivalent to the algebra generated by $S \oplus L$. Likewise we define $\mathscr{K}_2$ and $\mathfrak{A}_2$.

For any operator $T \in \mathscr{B}(\mathscr{H})$ where $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$, let

$$\beta = \sup_{\mathscr{M} \in \operatorname{Lat} \mathfrak{A}} \|P_{\mathscr{M}}^{\perp} T P_{\mathscr{M}}\|.$$

Consider the compression T_1' of T to $\mathscr{K}_1$. It is evident that $\operatorname{Lat} \mathfrak{A}_1$ may be identified with a sublattice of $\operatorname{Lat} \mathfrak{A}$, and that $\beta_{\mathfrak{A}_1}(T_1') \leqq \beta$. Hence there is an element $A_1 \in \mathfrak{A}$ so that

$$\|T_1' - A_1|_{\mathscr{K}_1}\| \leqq 3\beta.$$

Consequently if T_1 is the compression of T to $\mathscr{H}_1$, we obtain

$$\|T_1 - A_1|_{\mathscr{H}_1}\| \leqq 3\beta.$$

Likewise, there is an element $A_2 \in \mathfrak{A}$ such that

$$\|T_2 - A_2|_{\mathscr{H}_2}\| \leqq 3\beta.$$

Moreover, we have

$$\begin{aligned} \|A_1 - A_2\| &= \|(A_1 - A_2)|_{\mathfrak{A}[\xi_1 \oplus 0]}\| \\ &\leqq \|A_1|_{\mathscr{H}_1} - T_1\| + \|T_2' - A_2|_{\mathscr{K}_2}\| \leqq 6\beta. \end{aligned}$$

Set $A = (A_1 + A_2)/2$, and note that $\|A - A_i\| \leqq 3\beta$ for $i = 1, 2$.

Since $\mathscr{H}_1$ and $\mathscr{H}_2$ lie in $\operatorname{Lat} \mathfrak{A}$, it is easy to show that

$$\|T - T_1 \oplus T_2\| \leqq \beta.$$

Hence

$$\begin{aligned} \|T - A\| &\leqq \beta + \max\{\|T_1 - A|_{\mathscr{H}_1}\|, \|T_2 - A|_{\mathscr{H}_2}\|\} \\ &\leqq \beta + 3\beta + \max\{\|A_1 - A\|, \|A_2 - A\|\} \\ &\leqq 4\beta + 3\beta = 7\beta. \quad \square \end{aligned}$$

Corollary 5.6. *If $\mathfrak{A}$ is of type* L, *then there is a finite ampliation $\mathfrak{A}^{(p)}$ which is hyperreflexive. If $\mathfrak{A}$ has a wandering vector, then $p = 2$ will suffice.*

For some representations of the Cuntz algebra obtained from states by the GNS construction, one can determine the type of the von Neumann algebra generated. Thus the following corollary is of interest because it applies in particular to all type III representations, since the commutant is then also type III and so is purely infinite.

Corollary 5.7. *Suppose that $\mathfrak{S}$ is a free semigroup algebra with enveloping von Neu-*

mann algebra $\mathfrak{W}$ *with purely infinite commutant* $\mathfrak{W}'$. *Let* P *be the projection provided by the Structure Theorem* 2.6. *Then*

(i) *the subspace* $P^{\perp}\mathscr{H}$ *is spanned by its wandering vectors, and*

(ii) $\mathfrak{S}$ *and* $\mathfrak{W}$ *have property* $X_{0,1}$ *and thus are hyper-reflexive with distance constant at most* 3.

Proof. When $\mathfrak{W}'$ is purely infinite, $\mathfrak{S}$ is unitarily equivalent to $\mathfrak{S}^{(\infty)}$. Therefore Theorem 1.8 applies to show that the wandering vectors span the whole type L subspace.

It is also easy to construct two isometries in $\mathfrak{W}'$ with orthogonal ranges. By [3], it follows that $\mathfrak{W}$ and $\mathfrak{S}$ have property $X_{0,1}$, and hence are hyper-reflexive with constant 3. □

6. Open questions

There remains a number of compelling questions about these algebras. The first two appear to be difficult, and clearly are of central interest.

Question 6.1. Can $\mathfrak{S}$ be a von Neumann algebra?

Question 6.2. Does every type L representation have wandering vectors?

These two problems are somewhat related. The existence of wandering vectors in arbitrary free semigroup algebras would imply that all are nonself-adjoint. However the $n = 1$ case includes any unitary operator U whose spectral measure does not dominate Lebesgue measure on the whole circle. Even if the spectrum of U is the whole circle, the algebra $W(U)$ is self-adjoint by Wermer's Theorem [24] as in Example 3.1. So there is a very real possibility of a self-adjoint free semigroup algebra. If so, then $\mathscr{B}(H)$ is generated by n isometries with orthogonal ranges as a WOT-closed algebra.

The example of a unitary with a singular measure opens up the question of whether there is something to replace the spectral measure for $n \geqq 2$. So far, we have nothing to replace the notions of singularity and absolute continuity in this context.

On the other hand, a type L algebra without wandering vectors is very unlikely indeed. Such an algebra would be a WOT-closed reductive algebra which is nonself-adjoint, providing a counterexample to a famous open problem closely related to the invariant subspace problem. Even if such examples exist, it is unlikely that they arise in such a concrete and rigid form.

As we saw in the last section, an answer to these questions is close to providing an answer to the question of (hyper)-reflexivity.

Question 6.3. Is every free semigroup algebra reflexive, or even hyper-reflexive?

The Structure Theorem focuses attention on representations of type L. We would like to understand these algebras in greater detail.

Question 6.4. Is every type L representation a direct summand of an inductive one?

Question 6.5. Is the restriction of a type L algebra to an invariant subspace also type L?

Clearly type L algebras of inductive type are inherently more easily understood than the others. If they play a central role in the theory, that would simplify various analyses. We don't even have an example which we can prove is not inductive, although Example 3.7 is a candidate.

Question 6.5 probably has a positive answer. Proposition 3.6 provides a positive answer if the subspace contains a wandering vector. It seems unlikely that there is an isometry S such that $S \oplus L$ is type L, but S generates a von Neumann algebra. This could be ruled out if there were a Kaplansky type density theorem for these algebras. Note that type L algebras do have the property that the unit ball of the norm-closed algebra generated by S is WOT-dense in the ball of $\mathfrak{S}$. This follows because in $\mathfrak{L}_n$, the Cesaro means of an element A converge even SOT-$*$ to A. The isometric weak$*$ homeomorphism between $\mathfrak{L}_n$ and $\mathfrak{S}$ transfers this convergence at least in the WOT topology.

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