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FROBENIUS BETTI NUMBERS AND MODULES OF FINITE PROJECTIVE DIMENSION

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ABSTRACT. Let $(R, \mathfrak{m}, K)$ be a local ring, and let M be an R-module of finite length. We study asymptotic invariants, $\beta_i^F(M, R)$, defined by twisting with Frobenius the free resolution of M. This family of invariants includes the Hilbert-Kunz multiplicity ($e_{HK}(\mathfrak{m}, R) = \beta_0^F(K, R)$). We discuss several properties of these numbers that resemble the behavior of the Hilbert-Kunz multiplicity. Furthermore, we study when the vanishing of $\beta_i^F(M, R)$ implies that M has finite projective dimension. In particular, we give a complete characterization of the vanishing of $\beta_i^F(M, R)$ for one-dimensional rings. As a consequence of our methods we give conditions for the non-existence of syzygies of finite length.

1. Introduction. Let $(R, \mathfrak{m}, K)$ denote an F-finite local ring of dimension d and characteristic $p > 0$, and let $\alpha = \log_p[K : K^p]$. Given an R-module M and an integer $e \geqslant 0$, ${}^eM$ denotes the R-module structure on M given by $r * m = r^{p^e} m$ for every $m \in {}^eM$ and $r \in R$. In addition, $\lambda_R(M)$, or simply $\lambda(M)$ when the ring is clear from the context, denotes the length of M as an R-module.

Let $q = p^e$ be a power of p. For an ideal $I \subseteq R$, let $I^{[q]} = (i^q \mid i \in I)$ be the ideal generated by the qth powers of elements in I. If I is $\mathfrak{m}$-primary, the *Hilbert-Kunz multiplicity of I in R* is defined by

$$e_{HK}(I, R) = \lim_{e \to \infty} \frac{\lambda(R/I^{[q]})}{q^d}.$$

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The existence of the previous limit was proven by Monsky [**25**]. Under mild conditions, $e_{HK}(\mathfrak{m}, R) = 1$ if and only if R is a regular ring [**34**]. The Hilbert-Kunz multiplicity can be interpreted as a measure of singularity: the smaller it is, the nicer is the ring. For instance, Aberbach and Enescu proved rings with small Hilbert-Kunz multiplicity are Gorenstein and F-regular [**1**] (see also [**7**]). We have that

$$\lambda(R/I^{[q]}) = q^{\alpha}\lambda(R/I \otimes_R {}^eR) = q^{\alpha}\lambda(\operatorname{Tor}_0^R(R/I, {}^eR)).$$

This gives rise to the following extension of the Hilbert-Kunz multiplicity for higher Tor functors. Let N be a finitely generated R-module, and let M be an R-module of finite length. For an integer $i \geqslant 0$, define

$$\beta_i^F(M, N) = \lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}_i^R(M, {}^eN))}{q^{(d+\alpha)}}.$$

We denote $\beta_i^F(K, R)$ by $\beta_i^F(R)$ and call it the *ith Frobenius Betti number of R*.

These higher invariants also detect regularity, namely, Aberbach and Li [**3**] showed that R is a regular ring if and only if $\beta_i^F(R) = 0$ for some $i \geqslant 1$. Note that R is regular if and only if K has finite projective dimension as R-module.

In this manuscript, we seek an answer to the following question.

Question 1.1. *Let M be an R-module of finite length. What vanishing conditions on $\beta_i^F(M, R)$ imply that M has finite projective dimension?*

Miller [**23**] showed that, if R is a complete intersection and M is an R-module of finite length, then the vanishing of $\beta_i^F(M, R)$ for some $i \geqslant 1$ implies that M has finite projective dimension. We refer to [**13**] for related results for Gorenstein rings. In Section 4, we answer this question for rings that have small regular algebras, and for rings that have F-contributors. Later, we focus on one-dimensional rings and give the following characterization for the vanishing of $\beta_i^F(M, R)$.

Theorem (see Theorem 4.7). *Let $(R, \mathfrak{m}, K)$ be a one-dimensional local ring of positive characteristic p, and let M be an R-module of finite length. Let $(G_j, \varphi_j)_{j\geqslant 0}$ be a minimal free resolution of M. Then the following are equivalent*:

(i) $\operatorname{Im}(\varphi_{i+1}) \subseteq H^0_{\mathfrak{m}}(G_i)$.
(ii) $\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$ *for all* $e \geqslant 0$, *for all* $\mathfrak{p} \in \operatorname{Min}(R)$.
(iii) $\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$ *for all* $e \gg 0$, *for all* $\mathfrak{p} \in \operatorname{Min}(R)$.
(iv) $\beta_i^F(M, R) = 0$.

Assume, in addition, that R *is complete and* K *is algebraically closed. If* V *denotes the integral closure of* R *in its ring of fractions, then the conditions above are equivalent to*

(v) $\operatorname{Tor}_i^R(M, V) = 0$.

As a consequence of this theorem, we show that, if R is a one dimensional Cohen-Macaulay local ring and $\lambda(M) < \infty$, then $\beta_i^F(M, R) = 0$ for any $i \geqslant 1$ implies that M has finite projective dimension (see Corollary 4.8). Furthermore, we prove that the vanishing of two consecutive $\beta_i^F(M, R)$ implies that M has finite projective dimension in every one-dimensional local ring (see Corollary 4.9).

From the above theorem we have that $\beta_i^F(M, R) = 0$ if and only the $(i+1)$-syzygy has finite length. On the other hand, there are modules of infinite projective dimension over one-dimensional rings which have second syzygies of finite length (see Example 5.1). Motivated by Iyengar's question about the eventual stability of dimensions of syzygies and by our results regarding $\beta_i^F(M, R)$, we ask the following question.

Question 1.2. *Let* R *be a* d*-dimensional local ring, and let* M *be a finitely generated* R*-module such that* $\operatorname{pd}_R(M) = \infty$ *and* $\lambda(M) < \infty$. *If* $i > d+1$, *then must the length of the* i*th syzygy be infinite?*

In Section 5, we study this question, mainly for one-dimensional rings. In particular, we show that the answer to Question 1.2 is positive for one-dimensional Buchsbaum rings (see Proposition 5.3). We also obtain a partial answer for modules whose Betti numbers are eventually non-decreasing (see Proposition 5.7). Furthermore, we show that the first and third syzygies of M are either zero or have infinite length for every finite length module M over a one-dimensional ring (see Corollary 5.10). The assumption of M having finite length is necessary, as shown in Example 5.11. Aside from the study of projective dimension, we study basic properties of the higher invariants that resemble the Hilbert-Kunz multiplicity in other aspects.

2. Notation and terminology. Throughout this article, $(R, \mathfrak{m}, K)$ will denote a local ring of Krull dimension $\dim(R) = d$. For a finitely generated R-module M, we define $\dim(M) = \dim(R/(0 :_R M))$, where $0 :_R M = \{x \in R \mid xM = 0\}$. When $M = 0$, we set $\dim(M) = -1$. An element $x \in R$ such that $\dim(R/(x)) = d-1$ will be called a *parameter of* R. Given a finitely generated R-module M, a *minimal free resolution* $(G_\bullet, \varphi_\bullet)$ of M is an exact sequence

$$\cdots \to G_{i+1} \xrightarrow{\varphi_{i+1}} G_i \xrightarrow{\varphi_i} \cdots \to G_1 \xrightarrow{\varphi_1} G_0 \to M \to 0$$

such that $G_i \cong R^{\beta_i(M)}$ are free R-modules and $\operatorname{Im}(\varphi_{i+1}) \subseteq \mathfrak{m} G_i$. The integers $\beta_i(M) = \operatorname{rk}(G_i) = \lambda(\operatorname{Tor}_j^R(M, K))$ are called the *Betti numbers of* M. If $\beta_i(M) = 0$ for some i, we say that M has *finite projective dimension*, and that it is equal to $\operatorname{pd}_R(M) = \max\{i \in \mathbb{N} \mid \beta_i(M) \neq 0\}$. We adopt the convention that $\operatorname{pd}_R(M) = -\infty$, when $M = 0$. For all $i \geqslant 0$, we set $\Omega_i(M) = \operatorname{Coker}(\varphi_i)$, and we call it the *ith syzygy of the module* M. Note that $\Omega_0(M) = M$. When no confusion may arise, we will denote $\Omega_i(M)$ simply by Ω_i.

Herein, we often use local cohomology tools. For every $k \in \mathbb{N}$, the quotient map $R/\mathfrak{m}^{k+1} \to R/\mathfrak{m}^k$ induces maps of functors

$$\operatorname{Ext}^i_R(R/\mathfrak{m}^k, -) \longrightarrow \operatorname{Ext}^i_R(R/\mathfrak{m}^{k+1}, -).$$

For an R-module M, we define the *ith local cohomology of* M *with support on* $\mathfrak{m}$ by

$$H^i_{\mathfrak{m}}(M) = \lim_{k \to \infty} \operatorname{Ext}^i_R(R/\mathfrak{m}^k, M).$$

In particular,

$$H^0_{\mathfrak{m}}(M) = \bigcup_{k \in \mathbb{N}} 0 :_M \mathfrak{m}^k = \{v \in M \mid \mathfrak{m}^k v = 0 \text{ for some } k \in \mathbb{N}\}.$$

For a non-zero finitely generated R-module M, $\operatorname{depth}(M)$ denotes the smallest integer j such that $H^j_{\mathfrak{m}}(R) \neq 0$. When $\operatorname{depth}(M) = \dim(M)$, the module is called *Cohen-Macaulay*, and M is called *maximal Cohen-Macaulay* if $\operatorname{depth}(M) = \dim(R)$.

We now review some basic facts regarding integral closures. For an ideal $I \subseteq R$ and an element $x \in R$, we say that x is integral over I if it satisfies an equation of the form $x^n + r_1 x^{n-1} + \cdots + r_n = 0$, where $r_j \in I^j$ for all $j = 1, \ldots, n$. The set of elements integral over I forms an ideal, which is called the *integral closure of* I, and denoted $\overline{I}$. For an ideal $J \subseteq I$, we say that J is a *reduction of* I if $\overline{J} = \overline{I}$. We say that J is a *minimal reduction of* I if it is a reduction of I which is minimal with respect to containment. We refer the reader to [**32,** Chapter 8] for more details about reductions. For a domain R, let V be the integral closure of R in its field of fractions L. We define the *conductor of* R as the set of all elements $z \in L$ such that $zV \subseteq R$, and we denote it by C. When V is finite over R, it can be shown that C is the largest ideal which is common to R and V, and that C contains a non-zero divisor for R [**32,** Exercise 2.11]. In particular, if $(R, \mathfrak{m}, K)$ is an excellent one-dimensional local domain, the conductor is $\mathfrak{m}$-primary. See [**32,** Chapter 12] for more results about conductors.

We also need the notion of *dualizing complex*. We refer to [**27,** page 51] or to [**15,** Chapter V] for more details.

Definition 2.1. Let $(S, \mathfrak{n}, L)$ be a local ring of dimension d. We say that a complex $D^\bullet$ is a dualizing complex of S, if

(i) $D^i = \bigoplus_{\dim S/\mathfrak{p} = d-i} E_S(S/\mathfrak{p})$.
(ii) The cohomology $H^i(D^\bullet)$ is finitely generated.

Remark 2.2. If $(S, \mathfrak{n}, L)$ is a complete ring, then S has a dualizing complex, $D_S^\bullet$ [**15,** page 299]. If $\mathfrak{p}$ is a prime ideal such that $\dim S/\mathfrak{p} = \dim S$, we have that $S_\mathfrak{p}$ is Artinian, hence complete. In addition, $D_{S_\mathfrak{p}}^\bullet := D_S^\bullet \otimes S_\mathfrak{p}$ is a dualizing complex for $S_\mathfrak{p}$. Furthermore, $H^j(D_{S_\mathfrak{p}}^\bullet) = H^j(D_S^\bullet) \otimes S_\mathfrak{p} = 0$ for $j > 0$ and $\omega_{S_\mathfrak{p}} \cong H^0(D_{S_\mathfrak{p}}^\bullet) = E_{S_\mathfrak{p}}(S_\mathfrak{p}/\mathfrak{p}S_\mathfrak{p})$, since $S_\mathfrak{p}$ is Artinian, and thus it is Cohen-Macaulay.

We now introduce Buchsbaum rings. We study Question 1.2 in Section 5.

Definition 2.3. Let $(R, \mathfrak{m}, K)$ be a local ring of dimension d. We say that R is a *Buchsbaum ring* if, for any system of parameters $x_1, \ldots, x_d$, we have

$$(x_1, \ldots, x_{i-1}) : x_i = (x_1, \ldots, x_{i-1}) : \mathfrak{m}$$

for every $i = 1, \ldots, d$. When $i = 1$, the ideal $(x_1, \ldots, x_{i-1})$ is simply the zero ideal.

There are several equivalent ways for defining Buchsbaum rings, but that above is the most convenient for our purposes.

Remark 2.4. Let $(R, \mathfrak{m}, K)$ be a one-dimensional local ring. Suppose that R is not Cohen-Macaulay, so that $H^0_{\mathfrak{m}}(R) \neq 0$. Then there exists a parameter x of R such that $H^0_{\mathfrak{m}}(R) = 0 :_R x$. In fact, fix an integer $n \in \mathbb{N}$ such that $\mathfrak{m}^n H^0_{\mathfrak{m}}(R) = 0$, using that $H^0_{\mathfrak{m}}(R) \subseteq R$ is an ideal; hence, it is finitely generated. Take any parameter $y \in \mathfrak{m}$, and set $x = y^n$. With this choice, we have $xH^0_{\mathfrak{m}}(R) \subseteq \mathfrak{m}^n H^0_{\mathfrak{m}}(R) = 0$, so that $H^0_{\mathfrak{m}}(R) \subseteq 0 :_R x$. On the other hand, there exists a $k \in \mathbb{N}$ such that $\mathfrak{m}^k \subseteq (x)$. Therefore, if $r \in 0 :_R x$, we get $r\mathfrak{m}^k \subseteq r(x) = 0$, so that $r \in H^0_{\mathfrak{m}}(R)$. We conclude that $H^0_{\mathfrak{m}}(R) = 0 :_R x$.

Remark 2.5. Let $(R, \mathfrak{m}, K)$ be a one-dimensional Buchsbaum ring. By Remark 2.4, there exists a parameter $x \in R$ such that $0 :_R x = H^0_{\mathfrak{m}}(R)$. By the definition of the Buchsbaum ring, we have that

$$H^0_{\mathfrak{m}}(R) = 0 :_R x = 0 :_R \mathfrak{m}.$$

In particular, $\mathfrak{m}H^0_{\mathfrak{m}}(R) = 0$, that is, $H^0_{\mathfrak{m}}(R) \cong \bigoplus_{j=1}^t K$ is a finite-dimensional K-vector space.

For the rest of the section, assume that $(R, \mathfrak{m}, K)$ is a local ring of characteristic $p > 0$. For an integer $e \geqslant 1$, we consider the eth iteration of the Frobenius endomorphism $F^e : R \to R$, $F^e(r) = r^{p^e}$ for all $r \in R$. For an R-module M, we can consider M with the action induced by restriction of scalars, via F^e. We denote this module by ${}^eM$. More explicitly, for $r \in R$ and $m \in {}^eM$, we have $r * m = r^{p^e} m$.

Definition 2.6. We say that R is *F-finite* if ${}^1R$ is a finitely generated R-module.

Note that R is F-finite if and only if ${}^eR$ is a finitely generated R-module for any $e \geqslant 1$ or, equivalently, for all $e \geqslant 1$. Furthermore, F-finite rings are excellent [**20,** Theorem 2.5]. When R is F-finite, we have that $[K : K^p] < \infty$. In this case, we set $\alpha = \log_p[K : K^p]$.

3. Definition and properties of $\beta_i^F(M, N)$ and $\mu_i^F(M, N)$. We begin by defining the Frobenius Betti numbers and showing basic properties that resemble the Hilbert-Kunz multiplicity.

Definition 3.1 (see also [**22**])**.** Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, let M be an R-module of finite length, and let N be a finitely generated R-module. Define

$$\beta^F_{i,R}(M,N) = \lim_{e\to\infty} \frac{\lambda(\mathrm{Tor}^R_i(M, {}^eN))}{q^{(d+\alpha)}}.$$

We denote $\beta^F_{i,R}(K,R)$ by $\beta^F_{i,R}(R)$ and call it the *ith Frobenius Betti number of* R. If the ring is clear from the context, we only write $\beta^F_i(M,N)$. The above limit exists by the main result in [**29**].

We point out that Li [**22**] focused on $\beta^F_i(R/I,R)$, which he denoted by $t_i(I,R)$.

Example 3.2. Suppose that $R = S/fS$, where S is an F-finite regular local ring of characteristic $p > 0$, and $f \in S$. We write ${}^eR \cong R^{a_e} \oplus M_e$, where M_e has no free summands. The limit $s(R) := \lim_{e\to\infty}(a_e/q^{(d+\alpha)})$ exists [**33,** Theorem 4.9], and it is called the *F-signature of* R, which is an important invariant related to strong F-regularity [**2,** Theorem 0.2]. We consider the minimal free resolution of ${}^eR$:

$$\cdots \to R^{\beta_i({}^eR)} \to R^{\beta_{i-1}({}^eR)} \to \cdots \to R^{\beta_0({}^eR)} \to {}^eR \to 0.$$

We note that $\beta_0({}^eR) = a_e + \beta_0(M_e)$ and $\beta_i({}^eR) = \beta_i(M_e)$ for $i > 0$. Since M_e is a maximal Cohen-Macaulay module with no free summands, we have that $\beta_i(M_e) = \beta_0(M_e)$ for $i > 0$ [**14,** Proposition 5.3 and Theorem 6.1]. Then,

$$\begin{aligned}\beta^F_0(R) = e_{HK}(\mathfrak{m},R) &= \lim_{e\to\infty}\frac{\beta_0({}^eR)}{q^{(d+\alpha)}}\\ &= \lim_{e\to\infty}\frac{a_e}{q^{(d+\alpha)}} + \lim_{e\to\infty}\frac{\beta_0(M_e)}{q^{(d+\alpha)}}\\ &= s(R) + \lim_{e\to\infty}\frac{\beta_0(M_e)}{q^{(d+\alpha)}}.\end{aligned}$$

Hence,

$$\begin{aligned}\beta^F_i(R) &= \lim_{e\to\infty}\frac{\beta_i({}^eR)}{q^{(d+\alpha)}} = \lim_{e\to\infty}\frac{\beta_i(M_e)}{q^{(d+\alpha)}}\\ &= \lim_{e\to\infty}\frac{\beta_0(M_e)}{q^{(d+\alpha)}} = e_{HK}(\mathfrak{m},R) - s(R)\end{aligned}$$

for $i > 0$.

As for the Hilbert-Kunz multiplicity, the Frobenius Betti numbers also increase after taking the quotient by a nonzero divisor.

Proposition 3.3. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, M an R-module of finite length, and $x \in \operatorname{ann}(M)$ a nonzero divisor on R. Then,*

$$\beta^F_{i,R}(M,R) = \lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}_i^R(M, {}^eR))}{q^{(d+\alpha)}} \leqslant \beta^F_{i,R/(x)}(N, R/(x)) = \lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}_i^{R/(x)}(M, {}^e(R/(x))))}{q^{(d-1+\alpha)}},$$

where the subscripts indicate over which ring we are computing the Frobenius Betti numbers. In particular, $\beta^F_{i,R}(R) \leqslant \beta^F_{i,R/(x)}(R/(x))$.

Proof. Let $G_\bullet \to {}^eR$ be a minimal free resolution of ${}^eR$. Let $\overline{R}$ denote R/xR. We have that $\overline{G}_\bullet = G_\bullet \otimes_R \overline{R}$ is a free resolution for ${}^eR \otimes_R \overline{R}$ as an $\overline{R}$-module. Furthermore, we have that $H_0(\overline{G}_\bullet) = {}^eR \otimes_R \overline{R}$. This is a consequence of the fact that $H_i(\overline{G}_\bullet) = \operatorname{Tor}_i^R({}^eR, \overline{R}) = 0$ for $i > 0$ since x is a nonzero divisor on R and ${}^eR$.

Due to the fact that $x \in \operatorname{ann}(M)$, we have

$$\begin{aligned}\operatorname{Tor}_i^R(M, {}^eR) &= H_i(M \otimes_R G_\bullet) = H_i(M \otimes_{\overline{R}} \overline{R} \otimes_R G_\bullet)\\ &= H_i(M \otimes_{\overline{R}} \overline{G}_\bullet)\\ &= \operatorname{Tor}_i^{\overline{R}}(M, {}^eR \otimes_R \overline{R}).\end{aligned}$$

Since x is a nonzero divisor on R, there is a filtration

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_q = {}^eR \otimes_R \overline{R}$$

such that $L_{r+1}/L_r = {}^e(\overline{R})$. As a consequence, $\lambda(\operatorname{Tor}_i^{\overline{R}}(M, {}^eR \otimes_R \overline{R})) \leqslant q \cdot \lambda(\operatorname{Tor}_i^{\overline{R}}(M, {}^e\overline{R}))$. Then,

$$\lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}_i^R(M, {}^e R))}{q^{(d+\alpha)}} \leqslant \lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}_i^{\overline{R}}(M, {}^e\overline{R}))}{q^{(d-1+\alpha)}} \qquad \square$$

We now introduce $\mu_i^F(M,N)$, a dual version of $\beta_i^F(M,N)$, which is defined in terms of Ext. In Proposition 3.11, we establish a relation between these asymptotic invariants.

Definition 3.4. Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, let M be an R-module of finite length, and let N be a finitely generated R-module. We define

$$\mu_i^F(M, N) = \lim_{e \to \infty} \frac{\lambda(\operatorname{Ext}_R^i(M, {}^eN))}{q^{(d+\alpha)}},$$

Next, we prove that the numbers $\mu_i^F(M, N)$ are well defined. The proof is essentially the same as that for $\beta_i^F(M, N)$, as it uses the main result in [**29**]. Nonetheless, we include it here for completeness.

Proposition 3.5. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, let M be an R-module of finite length, and let N be a finely generated R-module. Then, $\lim_{e\to\infty}[\lambda(\operatorname{Ext}_R^i(N,{}^e M))]/q^{(d+\alpha)}$ exists. Moreover, if*

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0$$

is a short exact sequence, then

$$\lim_{e \to \infty} \frac{\lambda(\operatorname{Ext}_R^i(M, {}^eN_2))}{q^{(d+\alpha)}} = \lim_{e \to \infty} \frac{\lambda(\operatorname{Ext}_R^i(M, {}^eN_1))}{q^{(d+\alpha)}} + \lim_{e \to \infty} \frac{\lambda(\operatorname{Ext}_R^i(M, {}^eN_3))}{q^{(d+\alpha)}}.$$

Proof. Let $G_\bullet \to M$ be a minimal free resolution of M, and define

$$g_e(N) = \lambda(H^i(\operatorname{Hom}_R(G_\bullet), {}^eN)).$$

Let $0 \to N_1 \to N_2 \to N_3 \to 0$ be a short exact sequence of finitely generated R-modules. We have that $g_e(N_2) \leqslant g_e(N_1) + g_e(N_3)$, and equality holds if the sequence splits. Then,

$$\lim_{e \to \infty} \frac{g_e(N)}{q^{(d+\alpha)}} = \lim_{e \to \infty} \frac{\lambda(\operatorname{Ext}_R^i(M, {}^eN))}{q^{(d+\alpha)}}$$

exists, and it is additive in short exact sequences [**29**]. □

Proposition 3.6. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, M an R-module of finite length, and N a finitely generated R-module. Let Λ be the set of all prime ideals $\mathfrak{p}$ such that $\dim R/\mathfrak{p} = \dim R$. We have that*

$$\beta_i^F(M, N) = \sum_{\mathfrak{p} \in \Lambda} \beta_i^F(M, R/\mathfrak{p}) \lambda_{R_\mathfrak{p}}(N_\mathfrak{p})$$

and

$$\mu_i^F(M,N) = \sum_{\mathfrak{p}\in\Lambda} \mu_i^F(M,R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(N_\mathfrak{p}).$$

Proof. We only prove the first statement, since the proof of the second is completely analogous. Let $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_h = N$ be a filtration for N such that $N_j/N_{j-1} \cong R/\mathfrak{p}_j$, where $\mathfrak{p}_j \subseteq R$ is a prime ideal; we have short exact sequences

$$0 \longrightarrow N_{j-1} \longrightarrow N_j \longrightarrow R/\mathfrak{p}_j \longrightarrow 0.$$

We deduce that

$$\beta_i^F(M,N) = \sum_{j=1}^{h} \beta_i^F(M,R/\mathfrak{p}_j)$$

[**29,** Proposition 1 (b)]. In addition, we have that $\beta_i^F(M,R/\mathfrak{p}_j) = 0$ whenever $\dim(R/\mathfrak{p}_j) < \dim(R)$ [**29,** Proposition 1 (a)]. In order to prove the result, we need to count the number of times that a prime $\mathfrak{p}$ such that $\dim R/\mathfrak{p} = \dim R$ appears in the prime filtration. This number is obtained by localizing the above filtration at $\mathfrak{p}$ and counting the length of the resulting chain. Since the localized chain is a composition series of the module $N_\mathfrak{p}$, we obtain that the number of times $\mathfrak{p}$ appears in the above prime filtration is given by $\lambda_{R_\mathfrak{p}}(N_\mathfrak{p})$. Then,

$$\beta_i^F(M,N) = \sum_{j=1 \, \mathfrak{p}_j\in\Lambda}^{h} \beta_i^F(M,R/\mathfrak{p}_j) = \sum_{\mathfrak{p}\in\Lambda} \beta_i^F(M,R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(N_\mathfrak{p}). \quad \square$$

Remark 3.7. It follows from Proposition 3.6 that, if $\beta_i^F(M,R) = 0$ for some $i \in \mathbb{N}$, we have that $\beta_i^F(M,R/\mathfrak{p}) = 0$ for every minimal prime of R such that $\dim(R/\mathfrak{p}) = d$. Therefore, if this is the case, $\beta_i^F(M,N) = 0$ for every finitely generated R-module N, again using Proposition 3.6. A similar statement holds for $\mu_i^F(M,R)$.

The following theorem is related to results of Chang [**10,** Lemma 1.20, Corollary 2.4], and in some cases it follows from them. We present a different proof that does not use spectral sequences.

Theorem 3.8. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, M an R-module of finite length, and N a finitely generated R-module. Then*

$$\lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^eN))}{q^{(i+1+\alpha)}} = 0$$

for $i < d$. In particular, $\mu_i^F(M, N) = 0$ for $i < d$.

Proof. Our proof follows by induction on $n = \dim(N)$.

If $n = 0$, we have that $h = \lambda(N)$ is finite. There is a filtration

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_h = N$$

such that $N_j/N_{j-1} \cong K$. From the short exact sequences

$$0 \longrightarrow N_{j-1} \longrightarrow N_j \longrightarrow K \longrightarrow 0,$$

we have that

$$\lambda(\operatorname{Ext}^i_R(M, {}^eN_j))) \leqslant \lambda(\operatorname{Ext}^i_R(K, {}^eN_{j-1})) + \lambda(\operatorname{Ext}^i_R(K, {}^eK)).$$

Since

$$\begin{aligned}\lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^eK))}{q^{(i+1+\alpha)}} &= \lim_{e\to\infty} \frac{q^\alpha\lambda(\operatorname{Ext}^i_R(M, K))}{q^{(i+1+\alpha)}} \\ &= \lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, K))}{q^{(i+1)}} = 0,\end{aligned}$$

we have that

$$\lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^eN))}{q^{(i+1+\alpha)}} = 0$$

by an inductive argument.

Suppose that our claim is true for modules of dimension less than or equal to $n - 1$. There is a filtration

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_h = N$$

such that $N_j/N_{j-1} \cong R/\mathfrak{p}_j$, where $\mathfrak{p}_j \subset R$ is a prime ideal. From the short exact sequences $0 \to N_{j-1} \to N_j \to R/\mathfrak{p}_j \to 0$, we have that

$$\lambda(\operatorname{Ext}^i_R(M, {}^eN_j)) \leqslant \lambda(\operatorname{Ext}^i_R(M, {}^eN_{j-1})) + \lambda(\operatorname{Ext}^i_R(M, {}^e(R/\mathfrak{p}_j))).$$

It suffices to show that

$$\lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^e(R/\mathfrak{p}_j)))}{q^{(i+1+\alpha)}} = 0 \tag{3.1}$$

for primes $\mathfrak{p}_j$ such that $\dim_R(R/\mathfrak{p}_j) = n = \dim_R N$. Let $T = R/\mathfrak{p}_j$. Let $x \in \operatorname{Ann}_R M \setminus \mathfrak{p}_j$, which exists because $\dim_R T = \dim_R N > 0 = \dim_R M$. We have a short exact sequence

$$0 \longrightarrow {}^eT \xrightarrow{x} {}^eT \longrightarrow {}^eT/x({}^eT) \longrightarrow 0,$$

which induces a long exact sequence
(3.2)

$$\cdots \longrightarrow \operatorname{Ext}^i_R(M, {}^eT) \xrightarrow{0} \operatorname{Ext}^i_R(M, {}^eT) \longrightarrow \operatorname{Ext}^i_R(M, {}^eT/x({}^eT)) \longrightarrow \cdots.$$

Then, for every i,

$$\lambda(\operatorname{Ext}^i_R(M, {}^eT)) \leqslant \lambda(\operatorname{Ext}^{i-1}_R(M, {}^eT/x({}^eT))).$$

We have a filtration

$$0 = L_0 \subseteq L_1 \subseteq \cdots \subseteq L_q = {}^eT/x({}^eT)$$

such that $L_{r+1}/L_r = {}^e(T/xT)$ since x is not a zero divisor of T. From the induced long exact sequence by $\operatorname{Ext}^i_R(M, -)$, we have that

$$\lambda(\operatorname{Ext}^i_R(M, {}^eT/x({}^eT))) \leqslant q \cdot \lambda(\operatorname{Ext}^i_R(M, {}^e(T/xT)))).$$

Therefore,

$$\begin{aligned}
\lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^i_R(M, {}^eT))}{q^{(i+1+\alpha)}} &\leqslant \lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^{i-1}_R(M, {}^eT/x({}^eT)))}{q^{(i+1+\alpha)}} \\
&\leqslant \lim_{e\to\infty} \frac{q \cdot \lambda(\operatorname{Ext}^{i-1}_R(M, {}^e(T/xT)))}{q^{(i+1+\alpha)}} \\
&= \lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}^{i-1}_R(K, {}^e(T/xT))))}{q^{(i+\alpha)}} \\
&= 0 \quad \text{since } \dim T/xT = n-1.
\end{aligned}$$

□

Corollary 3.9. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$. Let N be a finitely generated R-module, and let C be an R-module such that, for all $e \gg 0$, C^{θ_e} is a direct summand of ${}^eN$ for some $\theta_e \in \mathbb{N}$. Assume that $\theta = \limsup_{e\to\infty}(\theta_e/q^{(d+\alpha)}) > 0$. Then, for all R-modules*

M of finite length, and all integers i, we have

$$\mu_i^F(M,N) \geqslant \theta \cdot \lambda(\operatorname{Ext}_R^i(M,C)).$$

In particular, C is a maximal Cohen-Macaulay module.

Proof. We have

$$\mu_i^F(M,N) = \lim_{e\to\infty} \frac{\lambda(\operatorname{Ext}_R^i(M,{}^eN))}{q^{(d+\alpha)}} \geqslant \limsup_{e\to\infty} \frac{\theta_e \cdot \lambda(\operatorname{Ext}_R^i(M,C))}{q^{(d+\alpha)}}$$
$$= \theta \cdot \lambda(\operatorname{Ext}_R^i(M,C)).$$

Using $M = K$ in Theorem 3.8, we obtain that $\mu_i^F(K,N) = 0$ for all $i < d$. It follows from the inequality that $\operatorname{Ext}_R^i(K,C) = 0$ for all $i < d$, and then C is a maximal Cohen-Macaulay module. □

Remark 3.10. Let $(R,\mathfrak{m},K)$ be a local ring of characteristic $p > 0$, and let N be a finitely generated R-module. We say that an R-module C is an F-contributor of N if C^{θ_e} is a direct summand of ${}^eN$ for $e \gg 0$, and $\limsup_{e\to\infty}(\theta_e/q^{(d+\alpha)}) > 0$ [**35**]. Corollary 3.9 shows that every F-contributor is a maximal Cohen-Macaulay module. This was already noted by Yao [**35,** Lemma 2.2] when N has finite F-representation type.

The next proposition shows that taking limits with respect to Tor or Ext give the same invariants up to a shift in the homological degrees.

Proposition 3.11. *Let $(R,\mathfrak{m},K)$ be a local ring of characteristic $p > 0$ and M an R-module of finite length. Then,*

$$\beta_i^F(M,R) = \mu_{d+i}^F(M,R)$$

for every $i \in \mathbb{N}$.

Proof. Since $\beta_i^F(M,R)$ and $\mu_{d+i}^F(M,R)$ are not affected by completion at $\mathfrak{m}$, we may assume that R is a complete local ring. In this case, R has a dualizing complex $D_R^\bullet$ by Remark 2.2. We have that

$$\beta_i^F(M,R) = \mu_{d+i}^F(M,H^0(D_R^\bullet))$$

by [**10,** Proposition 2.3(2)]. Let Λ be the set of all prime ideals of R such that $\dim R/\mathfrak{p} = \dim R$. Let $\mathfrak{p} \in \Lambda$. We have that

$(H^0(D^\bullet_R))_\mathfrak{p} = H^0(D^\bullet_{R_\mathfrak{p}}) = \omega_{R_\mathfrak{p}}$ by Remark 2.2. We have that $\omega_{R_\mathfrak{p}} = \operatorname{Hom}_{R_\mathfrak{p}}(R_\mathfrak{p}, E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}))$ and $\lambda_{R_\mathfrak{p}}(\omega_{R_\mathfrak{p}}) = \lambda_{R_\mathfrak{p}}(R_\mathfrak{p})$. Finally, by Proposition 3.6,

$$\begin{aligned}\mu^F_{d+i}(M, H^0(D^\bullet_R)) &= \sum_{\mathfrak{p}\in\Lambda} \mu^F_{d+i}(M, R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(H^0(D^\bullet_{R_\mathfrak{p}}))\\ &= \sum_{\mathfrak{p}\in\Lambda} \mu^F_{d+i}(M, R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(\omega_{R_\mathfrak{p}})\\ &= \sum_{\mathfrak{p}\in\Lambda} \mu^F_{d+i}(M, R/\mathfrak{p})\lambda_{R_\mathfrak{p}}(R_\mathfrak{p})\\ &= \mu^F_{d+i}(M, R).\end{aligned}$$

□

Remark 3.12. If R itself has an F-contributor C, then we get a relation involving the β^F_i's. In fact, by Proposition 3.11, we have $\beta^F_i(M, R) = \mu^F_{d+i}(M, R)$ for all $i \in \mathbb{N}$. Thus, in the notation of Corollary 3.9, we have $\beta^F_i(M, R) \geqslant \theta \cdot \lambda(\operatorname{Ext}^{d+i}_R(M, C))$.

We end this section with a proposition which shows how $\beta^F_i(M, N)$ behaves under some flat ring extensions. First, we need a different way of computing $\beta^F_i(M, N)$.

Remark 3.13. Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, let M be an R-module of finite length, and let N be a finitely generated R-module. Let $G_\bullet = (G_j, \varphi_j)_{j\geqslant 0}$ denote a minimal free resolution of M. Let $G^{[q]}_\bullet$ be the complex $(G_j, \varphi^{[q]}_j)_{j\geqslant 0}$, where the matrix of $\varphi^{[q]}_j$ has as entries the qth powers of the entries in the matrix of φ_j. We have that

$$\lambda(\operatorname{Tor}^R_i(M, {}^eN)) = q^\alpha \lambda(H_i(G^{[q]}_\bullet \otimes_R N)).$$

Hence,

$$\beta^F_i(M, N) = \lim_{q\to\infty} \frac{\lambda(H_i(G^{[q]}_\bullet \otimes_R N))}{q^d}.$$

Proposition 3.14. *Let $(R, \mathfrak{m}, K) \to (S, \mathfrak{n}, L)$ be a flat extension of two F-finite local rings of characteristic $p > 0$. Let M be a finite length R-module. Let $\alpha = \log_p[K : K^p]$ and $\theta = \log_p[L : L^p]$. Suppose that*

$\mathfrak{m}S = \mathfrak{n}$. *Then,*

$$\begin{aligned}\beta^F_{i,R}(M,R) &= \lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}^R_i(M, {}^eR))}{p^{e(d+\alpha)}}\\ &= \lim_{e\to\infty} \frac{\lambda(\operatorname{Tor}^S_i(M\otimes_R S, {}^eS))}{p^{e(d+\theta)}}\\ &= \beta^F_{i,S}(M\otimes_R S, S).\end{aligned}$$

In particular, we have that $\beta^F_{i,R}(M,R) = \beta^F_{i,\widehat{R}}(\widehat{M},\widehat{R})$.

Proof. Let $q = p^e$. We have:

$$\begin{aligned}\frac{\lambda_R(\operatorname{Tor}^R_i(M, {}^eR))}{q^\alpha} &= \lambda_R(H_i(G_\bullet^{[q]})) \quad \text{by Remark 3.13}\\ &= \lambda_S(H_i(G_\bullet^{[q]}\otimes_R S))\\ &\quad \text{since } S \text{ is flat and } \mathfrak{m}S=\mathfrak{n}\\ &= \lambda_S(H_i((G_\bullet\otimes_R S)^{[q]}))\\ &\quad \text{since } G_\bullet \text{ is free}\\ &= \frac{\lambda_S(\operatorname{Tor}^S_i(M\otimes_R S, {}^eS)}{q^\theta}\\ &\quad \text{by Remark 3.13 and since } S \text{ is flat.}\end{aligned}$$

After dividing by q^d and taking limits, we have that

$$\beta^F_{i,R}(M,R) = \beta^F_{i,S}(M\otimes_R S, S). \qquad \square$$

4. Relations with projective dimension. Let $(R,\mathfrak{m},K)$ be a local F-finite ring of characteristic $p>0$, and let M be an R-module of finite length. In this section, we investigate when the vanishing of $\beta^F_i(M,R)$ detects whether M has finite projective dimension.

First we recall known results in this direction. We have that R is a regular ring if and only if $\beta^F_i(R) = \beta^F_i(K,R) = 0$ for some $i \geqslant 1$ [**3,** Corollary 3.2]. Let M be a finitely generated R-module. If M has finite projective dimension, then $\operatorname{Tor}^R_i(M, {}^eR) = 0$ for all $i>0$ and all $e\geqslant 0$ [**26,** Theorem 1.7]. Conversely, if $\operatorname{Tor}^R_i(M, {}^eR)=0$ for infinitely many e and all $i>0$, then M has finite projective dimension [**16,** Theorem 3.1]. In fact, even more is true: if $\operatorname{Tor}^R_i(M, {}^eR) = 0$ for $\operatorname{depth}(R)+1$

consecutive values of i and some $e \gg 0$, then M has finite projective dimension [**18,** Proposition 2.6] (see also [**24,** Theorem 2.2.8]). Now, suppose that R is a complete intersection. If $\beta_i^F(M,R) = 0$ for some $i > 0$, then M has finite projective dimension by [**23,** Corollary 2.5] (see also [**13,** Corollary 4.11]).

Proposition 4.1. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$, and let M be an R-module of finite length. Suppose that there is a regular local ring $(A, \mathfrak{n}, L)$ and a map of local rings $\phi : R \to A$ such that A is finitely generated as an R-module, and* $\dim A = d$*. If*

$$\beta_j^F(M,R) = \beta_{j+1}^F(M,R) = \cdots = \beta_{j+d}^F(M,R) = 0,$$

then M has finite projective dimension.

Proof. We note that $\log_p[L : L^p] = \log_p[K : K^p] = \alpha < \infty$, and thus, A is F-finite. Since A is regular and local, ${}^eA \cong \bigoplus^{q^{(d+\alpha)}} A$. Let $x_1, \ldots, x_d \in A$ be a set of generators for $\mathfrak{n}$, and let $I_r := (x_1, \ldots, x_r)A$. By induction on r, we will show that

$$\operatorname{Tor}_{j+r}^R(M, A/I_r) = \cdots = \operatorname{Tor}_{j+d}^R(M, A/I_r) = 0 \tag{4.1}$$

for every r. If $r = 0$, we have that $\operatorname{Tor}_i^R(M, {}^eA) = \oplus^{q^{(d+\alpha)}} \operatorname{Tor}_i^R(M, A)$ for every $i \in \mathbb{N}$. Then, $\lambda(\operatorname{Tor}_i^R(M, {}^eA)) = q^{(d+\alpha)}\lambda(\operatorname{Tor}_i^R(M, A))$, and thus,

$$\beta_i^F(M, A) = \lambda(\operatorname{Tor}_i^R(M, A)).$$

Since A is finitely generated, and since $\beta_i^F(M,R) = 0$ for $i = j, \ldots, j+d$ by assumption, we have that $\beta_j^F(M,A) = \cdots = \beta_{j+d}^F(M,A) = 0$ by Remark 3.7. Hence, $\operatorname{Tor}_j^R(M,A) = \cdots = \operatorname{Tor}_{j+d}^R(M,A) = 0$. We suppose that (4.1) holds for $r-1$ and prove it for r. We have a short exact sequence

$$0 \longrightarrow A/I_{r-1} \xrightarrow{x_r} A/I_{r-1} \longrightarrow A/I_r \longrightarrow 0.$$

This induces a long exact sequence

$$\cdots \longrightarrow \operatorname{Tor}_i^R(M, A/I_{r-1}) \xrightarrow{x_r} \operatorname{Tor}_i^R(M, A/I_{r-1}) \longrightarrow \operatorname{Tor}_i^R(M, A/I_r) \longrightarrow \operatorname{Tor}_{i-1}^R(M, A/I_{r-1}) \longrightarrow \cdots.$$

Since $\operatorname{Tor}_{j+r-1}^R(M, A/I_{r-1}) = \cdots = \operatorname{Tor}_{j+d}^R(M, A/I_{r-1}) = 0$, we have that $\operatorname{Tor}_{j+r}^R(M, A/I_r) = \cdots = \operatorname{Tor}_{j+d}^R(M, A/I_r) = 0$, proving the claim.

In particular, we obtain $\operatorname{Tor}^R_{j+d}(M, A/I_d) = 0$. Since $L = A/I_d$ is a finite field extension of K, we have

$$0 = \lambda(\operatorname{Tor}^R_{j+d}(M, A/I_d)) = [L : K] \cdot \lambda(\operatorname{Tor}^R_{j+d}(M, K)).$$

Therefore, $\operatorname{Tor}^R_{j+d}(M, K) = 0$ and M has finite projective dimension. $\square$

Lemma 4.2. *Let $(R, \mathfrak{m}, K)$ be a local ring of characteristic $p > 0$. Suppose that there is an R-module N of dimension d that has an F-contributor C. Let M be an R-module of finite length. If $\beta^F_i(M, N) = 0$, then $\operatorname{Tor}^R_i(M, {}^eC) = 0$ for every $e \geqslant 0$. In particular, if R is strongly F-regular of positive dimension d, and $\beta^F_i(M, R) = 0$ for d consecutive values of i, then M has finite projective dimension.*

Proof. For $e' \gg 0$ and $q' = p^{e'}$, we have that $C^{\theta_{e'}}$ is a direct summand of ${}^{e'}N$, for some $\theta_{e'} \in \mathbb{N}$ such that $\limsup \theta_{e'}/q'^{(d+\alpha)} > 0$. We note that $({}^eC)^{\theta_{e'}}$ is a direct summand of ${}^{e+e'}N$ for all $e \geqslant 0$. Then,

$$\begin{aligned}\left(\limsup_{e'\to\infty} \frac{\theta_{e'}}{q'^{(d+\alpha)}}\right) \frac{\lambda(\operatorname{Tor}^R_i(M, {}^eC))}{q^{(d+\alpha)}} &\leqslant \lim_{e'\to\infty} \frac{\lambda(\operatorname{Tor}^R_i(M, {}^{e+e'}N))}{qq'^{(d+\alpha)}} \\ &= \beta^F_i(M, N) = 0.\end{aligned}$$

It follows that $\operatorname{Tor}^R_i(M, {}^eC) = 0$. If R is strongly F-regular, then R is an F-contributor of itself. In addition, R is Cohen-Macaulay and, if $\operatorname{Tor}_i(M, {}^eR) = 0$ for d consecutive values of i and for $e \gg 0$, we have that M has finite projective dimension [**18,** Proposition 2.6] (see also [**24,** Theorem 2.2.11]). $\square$

Proposition 4.3. [**12,** Corollary 3.3] *Let $(R, \mathfrak{m}, K)$ be a local ring, let I be an integrally closed $\mathfrak{m}$-primary ideal, and let N be a finitely generated R-module. Then, $\operatorname{pd}_R(N) < i$ if and only if $\operatorname{Tor}^R_i(N, R/I) = 0$.*

In particular, Proposition 4.3 shows that, if $\operatorname{Tor}^R_i(R/I, {}^eR) = 0$ for some $e \geqslant 1$, then R is regular [**19,** Theorem 2.1].

We now present a similar result for Frobenius Betti numbers.

Proposition 4.4. *Let $(R, \mathfrak{m}, K)$ be a reduced local ring of characteristic $p > 0$. Suppose that there exists an R-module N of dimension d that* *has an F-contributor C. If I is an integrally closed $\mathfrak{m}$-primary ideal such that $\beta_i^F(R/I, N) = 0$ for some $i > 0$, then R is regular.*

Proof. By Lemma 4.2, we have that $\operatorname{Tor}_i(R/I, {}^eC) = 0$ for every $e \geqslant 0$, and thus, ${}^eC$ has finite projective dimension by [**12,** Corollary 3.3]. Since ${}^eC$ is a maximal Cohen-Macaulay module [**35,** Lemma 2.2], see Remark 3.10, we have that ${}^eC$ is a free module for every $e \geqslant 0$. In particular, ${}^1C \cong \bigoplus_n R$ and ${}^2C = {}^1(\bigoplus_n R) \cong \bigoplus_n {}^1R$ is free as well. Therefore, ${}^1R$ is free, and R is regular [**19,** Theorem 2.1]. $\square$

We now focus on one-dimensional rings. In this case, we can find a characterization of the vanishing of $\beta_i^F(M, R)$. We first prove two lemmas.

Lemma 4.5. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional complete local domain of characteristic $p > 0$, with K algebraically closed. Then, there exists a parameter $x \in R$ such that $(x^q) = \mathfrak{m}^{[q]}$ for all $q = p^e \gg 0$. Furthermore, if V denotes the integral closure of R in its field of fractions, then ${}^eR \cong \bigoplus V$ for all $e \gg 0$ (as R-modules).*

Proof. Since R is a complete domain, we have that $(V, \mathfrak{m}_V, K)$ is a one-dimensional, integrally closed, local domain. Hence, V is a DVR. Let $x \in R$ be a minimal reduction of $\mathfrak{m}$, and let v denote the order valuation on V. Let $x, y_1, \ldots, y_n$ be a minimal generating set of the maximal ideal. We claim that we can choose the elements y_i's such that $v(x) < v(y_i)$ for all $i = 1, \ldots, n$. We have $v(x) \leqslant v(y_i)$ for all i since x is a minimal reduction of $\mathfrak{m}$ [**32,** Proposition 6.8.1]. If equality holds, say for $i = 1$, we have that $y_1/x = \alpha \in K_V = K$ since K is algebraically closed. Fix a lifting $u \in R$ of α. If we replace y_1 for $y_1' := y_1 - ux$, we have that $x, y_1', \ldots, y_n$ is still a minimal generating set of $\mathfrak{m}$. Now $v(x) < v(y_1')$, since $y_1'/x \in \mathfrak{m}_V$. Similarly, if necessary, we may replace each y_i to obtain our claim. Since the conductor C is $\mathfrak{m}_V$-primary, for all $e \gg 0$ and all $i = 1, \ldots, n$, we have that

$$(y_i/x)^q = r_i \in \mathfrak{m}_V^{[q]} \subseteq C \subseteq R.$$

Thus, $y_i^q = r_i x^q \in (x)^q$. This shows the first part of the lemma.

We now focus on the second part of the lemma. Since K is algebraically closed, R and K have the same residue field. It then follows that $R \subseteq V = R + \mathfrak{m}_V$. Since R is a domain, we can identify ${}^eR$ with $R^{1/q}$, the ring of qth roots of R. For $w \in V$, we can write $w = u+v$, for some $u \in R$ and $v \in \mathfrak{m}_V$. Therefore, we have that $\mathfrak{m}_V^{[q]} \subseteq C \subseteq R$ for $e \gg 0$, since C is $\mathfrak{m}_V$ primary. This shows that $w^q = u^q + v^q \in R$, that is, $w \in R^{1/q}$. Thus, for $e \gg 0$, we have $R \subseteq V \subseteq {}^eR$. Hence, ${}^eR$ is a V-module. Since V is a DVR, ${}^eR$ decomposes into a V-free part and a V-torsion part. However, ${}^eR$ is torsion free as a V-module because R is a domain. Thus,

$$ {}^eR \cong \bigoplus_q V. $$

Finally, the V-module structure on ${}^eR$ is compatible with the inclusion $R \subseteq V$; therefore, ${}^eR \cong \bigoplus V$ is also an isomorphism of R-modules. $\square$

Lemma 4.6. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional local ring of characteristic $p > 0$. Let $(G_j, \varphi_j)_{j \geqslant 0}$ be a minimal free resolution of a finite length R-module M. Suppose that there exists an $i \geqslant 0$ such that $\operatorname{Im}(\varphi_{i+1}) \nsubseteq \mathfrak{p}G_i$ for some $\mathfrak{p} \in \operatorname{Min}(R)$. Then*

$$ \beta_i^F(M, R) = \lim_{e \to \infty} \frac{\lambda(\operatorname{Tor}_i^R(M, {}^eR))}{q^\alpha} > 0. $$

Proof. We can write $\widehat{R} = K[[x_1, \dots, x_n]]/I$ for some $n \in \mathbb{N}$ and some ideal $I \subseteq K[\![x_1, \dots, x_n]\!]$ by the Cohen structure theorem. Let $S = L[\![x_1, \dots, x_n]\!]/I'$, where L is the algebraic closure of K and $I' = I\ L[\![x_1, \dots, x_n]\!]$. Every inclusion $K \to L$ gives a flat extension $R \to S$ such that $\mathfrak{m}S$ is the maximal ideal of S. If $\operatorname{Im}(\varphi_{i+1} \otimes_R 1_S) = \operatorname{Im}(\varphi_{i+1}) \otimes_R S$ is contained in a minimal prime of S, then $\operatorname{Im}(\varphi_{i+1})$ must be contained in the contraction of such a minimal prime to R. Then, we can assume that R is complete and that K is algebraically closed by Proposition 3.14.

Let $\overline{R}$ denote $R/\mathfrak{p}$, $\overline{x}$ the class of the element x modulo $\mathfrak{p}$ and V the integral closure of $\overline{R}$. Since $R/\mathfrak{p}$ is a one-dimensional complete local domain, by Lemma 4.5, we can choose $0 \neq \overline{x} \in \overline{R}$ a minimal reduction of $\overline{\mathfrak{m}} := \mathfrak{m}/\mathfrak{p}$ and $q_0 = p^{e_0}$ such that $\overline{\mathfrak{m}}^{[q]} = (\overline{x}^q)$ for $q \geqslant q_0$. We may also choose q_0 large enough such that $\overline{x}^q V \cap \overline{R} \subseteq \overline{x}\overline{R}$ by using the Artin-Rees lemma and the fact that the conductor from $\overline{R}$ to V is primary to the maximal ideal. In particular, $(\overline{x}^q V :_V \overline{r}) \subseteq \mathfrak{m}_V$ for every $\overline{r} \in \overline{R}$ such that $\overline{r} \notin \overline{x}\overline{R}$, where $\mathfrak{m}_V$ is the maximal ideal of V, which is a DVR.

Fix $q \geqslant q_0$, and consider the matrix associated to $\overline{\varphi}_{i+1}^{[q]} := \varphi_{i+1}^{[q]} \otimes 1_{\overline{R}}$. Since $q \geqslant q_0$, $\operatorname{Im}(\varphi_{i+1}^{[q]} \otimes 1_{\overline{R}}) \subseteq \overline{\mathfrak{m}}^{[q]} G_i = (\overline{x}^q) G_i$. Due to the fact that $\operatorname{Im}(\varphi_{i+1}) \nsubseteq \mathfrak{p} G_i$, by changing the basis for G_{i+1} if needed, we can assume that the matrix

$$\overline{\varphi}_{i+1}^{[q]} = \overline{x}^{q+j} \left[\begin{array}{c|ccc} \overline{r}_1 & * & \cdots & * \\ \hline \overline{r}_2 & * & \cdots & * \\ \hline \vdots & \vdots & & \vdots \\ \hline \overline{r}_n & * & \cdots & * \end{array}\right],$$

where we have factored out the biggest possible power of $\overline{x}$, so that $\overline{r}_1 \notin (\overline{x})$. Here, $n = \operatorname{rk}(G_i)$.

Let $q' = p^{e'}$, and consider the matrix associated to $\overline{\varphi}_{i+1}^{[qq']}$:

$$\overline{\varphi}_{i+1}^{[qq']} = \overline{x}^{(q+j)q'} \left[\begin{array}{c|ccc} \overline{r}_1^{q'} & * & \cdots & * \\ \hline \overline{r}_2^{q'} & * & \cdots & * \\ \hline \vdots & \vdots & & \vdots \\ \hline \overline{r}_n^{q'} & * & \cdots & * \end{array}\right]. \tag{4.2}$$

We claim that $[\overline{r}_1^{q'}, \overline{r}_2^{q'}, \ldots, \overline{r}_n^{q'}]^T \in \operatorname{Ker}(\overline{\varphi}_i^{[qq']})$. In fact, we have that

$$\overline{x}^{qq'+jq'} \begin{bmatrix} \overline{r}_1{}^{q'} \\ \overline{r}_2{}^{q'} \\ \vdots \\ \overline{r}_n{}^{q'} \end{bmatrix} \in \operatorname{Im}\left(\overline{\varphi}_{i+1}^{[qq']}\right) \subseteq \operatorname{Ker}\left(\overline{\varphi}_i^{[qq']}\right);$$

therefore,

$$\overline{\varphi}_i^{[qq']} \left(\overline{x}^{qq'+jq'} \cdot \begin{bmatrix} \overline{r}_1{}^{q'} \\ \overline{r}_2{}^{q'} \\ \vdots \\ \overline{r}_n{}^{q'} \end{bmatrix} \right) = \overline{x}^{qq'+jq'} \cdot \overline{\varphi}_i^{[qq']} \left(\begin{bmatrix} \overline{r}_1{}^{q'} \\ \overline{r}_2{}^{q'} \\ \vdots \\ \overline{r}_n{}^{q'} \end{bmatrix} \right) = 0.$$

Since $\overline{x}^{qq'+jq'}$ is a nonzero divisor in $\overline{R}$, we have

$$\overline{\varphi}_i^{[qq']}\left(\begin{bmatrix} \overline{r}_1{}^{q'} \\ \overline{r}_2{}^{q'} \\ \vdots \\ \overline{r}_n{}^{q'} \end{bmatrix}\right) = 0,$$

which proves the claim. Thus,

$$\begin{aligned}\lambda(\operatorname{Tor}_i^R(M, {}^{e+e'}(R/\mathfrak{p}))) &\geqslant \lambda\left(\frac{\overline{R}[\overline{r}_1^{q'}, \ldots, \overline{r}_n^{q'}]^T + \operatorname{Im}\left(\overline{\varphi}_{i+1}^{[qq']}\right)}{\operatorname{Im}\left(\overline{\varphi}_{i+1}^{[qq']}\right)}\right) \\ &\geqslant \lambda\left(\frac{(\overline{r}_1^{q'}) + (\overline{x}^{qq'})}{(\overline{x}^{qq'})}\right),\end{aligned}$$

since $\operatorname{Im}(\overline{\varphi}_{i+1}^{[qq']}) \subseteq (\overline{x}^{qq'})G_i$. This comes from the expression of $\overline{\varphi}_{i+1}^{[qq']}$ in (4.2). We also have projected onto the first component of G_i. This yields a cyclic module which is isomorphic to the quotient of $\overline{R}$ by the ideal $(\overline{x}^{qq'} : \overline{r}_1^{q'})$.

We claim that there exists an integer $q_1 = p^{e_1}$ such that, for all q',

$$(\overline{x}^{qq'} : \overline{r}_1^{q'}) \subseteq (\overline{x}^{q'/q_1}).$$

Assuming the claim, and lifting back to R, we obtain:

$$\begin{aligned}\lambda(\operatorname{Tor}_i^R(M, {}^{e+e'}\overline{R})) &\geqslant \lambda\left(\frac{(\overline{r}_1^{q'}) + (\overline{x}^{qq'})}{(\overline{x}^{qq'})}\right) \\ &\geqslant \lambda\left(\frac{\overline{R}}{(\overline{x}^{q'/q_1})}\right).\end{aligned}$$

Dividing by qq' and taking the limit as $e' \to \infty$, we get

$$\begin{aligned}\beta_i^F(M, \overline{R}) &= \lim_{e' \to \infty} \frac{\lambda(\operatorname{Tor}_i^R(M, {}^{e+e'}(\overline{R})))}{qq'} \\ &\geqslant \lim_{e' \to \infty} \frac{\lambda(\overline{R}/(\overline{x}^{q'/q_1}))}{qq'} \\ &= \frac{1}{qq_1} \operatorname{e}_{HK}(x, \overline{R}) > 0.\end{aligned}$$

Since $\dim(R/\mathfrak{q}) = \dim(R)$ for $\mathfrak{q} \in \operatorname{Spec}(R)$ if and only if $\mathfrak{q} \in \operatorname{Min}(R)$, we have

$$\beta_i^F(M,R) = \sum_{\mathfrak{q}\in\operatorname{Min}(R)} \left(\beta_i^F(M,R/\mathfrak{q})\lambda(R_{\mathfrak{q}})\right) \geqslant \beta_i^F(M,R/\mathfrak{p}) > 0,$$

by Proposition 3.6.

It remains to prove the claim. Suppose that $u \in (\overline{x}^{qq'} : \overline{r}_1^{q'})$. Then,

$$u \in (\overline{x}^{qq'} : \overline{r}_1^{q'})V \cap \overline{R} = (\overline{x}^q V :_V \overline{r}_1)^{[q']} \cap \overline{R} \subseteq \mathfrak{m}_V^{q'} \cap R,$$

by the choice of q. Since the conductor of $\overline{R}$ is primary to the maximal ideal, it follows that there exists a $q_1 = p^{e_1}$ such that $\mathfrak{m}_V^{q'} \cap \overline{R} \subseteq (\overline{x}^{q'/q_1})$, as claimed. $\square$

Theorem 4.7. *Let $(R,\mathfrak{m},K)$ be a one-dimensional local ring of characteristic $p > 0$ and M an R-module of finite length. Let $(G_j,\varphi_j)_{j\geqslant 0}$ denote a minimal free resolution of M. Then the following are equivalent*:

(i) $\operatorname{Im}(\varphi_{i+1}) \subseteq H^0_{\mathfrak{m}}(G_i)$.
(ii) $\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$ *for all* $e \geqslant 0$, *for all* $\mathfrak{p} \in \operatorname{Min}(R)$.
(iii) $\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$ *for all* $e \gg 0$, *for all* $\mathfrak{p} \in \operatorname{Min}(R)$.
(iv) $\beta_i^F(M,R) = 0$.

Assume, in addition, that R is complete and K is algebraically closed. If V denotes the integral closure of R in its ring of fractions, then the conditions above are equivalent to:

(v) $\operatorname{Tor}_i^R(M,V) = 0$.

Proof. We will show that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i). We assume (i). Let $\mathfrak{p} \in \operatorname{Min}(R)$. Since M has finite length we have $M_{\mathfrak{p}} = 0$, and thus,

$$\operatorname{Tor}_j^R(M, {}^e(R/\mathfrak{p}))_{\mathfrak{p}} = \operatorname{Tor}_j^{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, {}^e(R/\mathfrak{p})_{\mathfrak{p}}) = 0$$

for all $j \geqslant 0$. In particular, the complex

$$(G_\bullet \otimes {}^e(R))_{\mathfrak{p}} : \cdots \longrightarrow (G_{i+1})_{\mathfrak{p}} \xrightarrow{(\varphi_{i+1}^{[q]})_{\mathfrak{p}}} (G_i)_{\mathfrak{p}} \xrightarrow{(\varphi_i^{[q]})_{\mathfrak{p}}} (G_{i-1})_{\mathfrak{p}} \longrightarrow \cdots \longrightarrow (G_0)_{\mathfrak{p}} \longrightarrow 0$$

is split exact. All the entries in a matrix associated to φ_{i+1} are in $H^0_{\mathfrak{m}}(R)$, and in particular, they are nilpotent. We choose $q_0 = p^{e_0}$ such that $\operatorname{Im}(\varphi_{i+1}^{[q]}) = 0$ for all $q \geqslant q_0$. For such a q, we have $(\varphi_{i+1}^{[q]})_{\mathfrak{p}} \equiv 0$; therefore, $(G_i)_{\mathfrak{p}}$ splits inside $(G_{i-1})_{\mathfrak{p}}$ via $(\varphi_i^{[q]})_{\mathfrak{p}}$. This means that

$$b_i := \operatorname{rk}((G_i)_{\mathfrak{p}}) = \operatorname{rk}(G_i) = \operatorname{rk}((\varphi_i^{[q]})_{\mathfrak{p}}) \quad \text{and} \quad I_{b_i}(\varphi_i^{[q]}) \not\subseteq \mathfrak{p}, \tag{4.3}$$

where $I_r(\psi)$ denotes the Fitting ideal of a homomorphism $\psi : G \to H$ of rank r between two free modules, G and H. Note that localizing and taking powers only decreases the rank of φ_i, and b_i is already the maximal possible rank. Thus, $b_i = \operatorname{rk}(\varphi_i^{[q]})$ for all $q \geqslant 1$. Furthermore, if $I_{b_i}(\varphi_i)$ were contained in $\mathfrak{p}$, then so would be $I_{b_i}(\varphi_i^{[q]})$. Hence, (4.3) holds in fact for all $q = p^e$.

Consider the complex

$$0 \longrightarrow G_i \otimes R/\mathfrak{p} \xrightarrow{\varphi_i^{[q]} \otimes 1_{R/\mathfrak{p}}} G_{i-1} \otimes R/\mathfrak{p} \longrightarrow C_q \longrightarrow 0,$$

where C_q is the cokernel. By the Buchsbaum-Eisenbud theorem **[8]**, the two conditions (4.3) ensure that it is acyclic for all q. Then,

$$\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = \operatorname{Tor}_1^R(C_q, R/\mathfrak{p}) = 0,$$

for all $e \geqslant 0$. This holds for all $\mathfrak{p} \in \operatorname{Min}(R)$, proving (ii).

Clearly, (ii) implies (iii). We now show (iii) $\Rightarrow$ (iv). Since, for all $\mathfrak{p} \in \operatorname{Min}(R)$, we have $\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$ for $e \gg 0$, in particular, $\beta_i^F(M, R/\mathfrak{p}) = 0$. Hence,

$$\beta_i^F(M,R) = \sum_{\mathfrak{p} \in \operatorname{Min}(R)} \left[\beta_i^F(M, R/\mathfrak{p}) \lambda_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})\right] = 0.$$

We now prove (iv) $\Rightarrow$ (i). Suppose that $\beta_i^F(M,R) = 0$. By Lemma 4.6, we have

$$\operatorname{Im}(\varphi_{i+1}) \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Min}(R)} \mathfrak{p} G_i = \sqrt{0} G_i.$$

Since the image is nilpotent, as noticed above in (4.3) while taking $q = 1$, we have

$$b_i = \operatorname{rk}(G_i) = \operatorname{rk}(\varphi_i) \quad \text{and} \quad I_{b_i}(\varphi_i) \not\subseteq \mathfrak{p}$$

for all $\mathfrak{p} \in \operatorname{Min}(R)$. Localizing the resolution at any $\mathfrak{p} \in \operatorname{Min}(R)$ gives a split exact complex

$$(G_\bullet)_\mathfrak{p} : 0 \longrightarrow (G_i)_\mathfrak{p} \xrightarrow{(\varphi_i)_\mathfrak{p}} \cdots \longrightarrow (G_0)_\mathfrak{p} \longrightarrow 0.$$

In particular, $\operatorname{Im}((\varphi_{i+1})_\mathfrak{p}) = (\operatorname{Im}(\varphi_{i+1}))_\mathfrak{p} = 0$. This holds for all minimal primes $\mathfrak{p}$ of R, proving that $\operatorname{Im}(\varphi_{i+1}) \subseteq H^0_\mathfrak{m}(G_i)$.

Finally, assume that R is complete and K is algebraically closed, and let V be an integral closure of R in its ring of fractions. Let $\mathfrak{p} \in \operatorname{Min}(R)$, and let $V(\mathfrak{p})$ be the integral closure of $R/\mathfrak{p}$, which is a DVR. By Lemma 4.5, we have that ${}^e(R/\mathfrak{p}) \cong \bigoplus V(\mathfrak{p})$ for all $e \gg 0$. Condition (iii) implies that

$$\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) \cong \bigoplus \operatorname{Tor}_i^R(M, V(\mathfrak{p})) = 0;$$

therefore, $\operatorname{Tor}_i^R(M, V(\mathfrak{p})) = 0$ for all $\mathfrak{p} \in \operatorname{Min}(R)$. Since $V \cong \bigoplus_{\mathfrak{p} \in \operatorname{Min}(R)} V(\mathfrak{p})$, we see that (iii) implies (v).

Conversely, if $\operatorname{Tor}_i^R(M, V) = 0$, by the same argument, we get that $\operatorname{Tor}_i^R(M, V(\mathfrak{p})) = 0$ implies $\operatorname{Tor}_i^R(M, {}^e(R/\mathfrak{p})) = 0$ for all $e \gg 0$ and for all $\mathfrak{p} \in \operatorname{Min}(R)$. Then, (v) implies (iii). $\square$

Corollary 4.8. *Let $(R, \mathfrak{m}, K)$ be a one dimensional Cohen-Macaulay local ring of characteristic $p > 0$ and M an R-module of finite length. Then the following are equivalent*:

(i) $\beta_i^F(M, R) = 0$ *for all* $i \geqslant 1$.
(ii) $\beta_i^F(M, R) = 0$ *for some* $i \geqslant 1$.
(iii) $\operatorname{pd}_R(M) < \infty$.

Proof. Clearly (i) implies (ii). Now assume (ii). We want to show that (iii) holds. By assumption, there exists an integer $i \geqslant 1$ such that $\beta_i^F(M, R) = 0$. Then, Theorem 4.7 implies that $\operatorname{Im}(\varphi_{i+1}) \subseteq H^0_\mathfrak{m}(G_i)$, where $(G_j, \varphi_j)_{j \geqslant 0}$ is a minimal free resolution of M. However, R has positive depth, and hence,

$$\operatorname{Im}(\varphi_{i+1}) = H^0_\mathfrak{m}(\operatorname{Im}(\varphi_{i+1})) \subseteq H^0_\mathfrak{m}(G_i) = 0,$$

since G_i is a free module. Thus, $\operatorname{Im}(\varphi_{i+1}) = 0$ and $\operatorname{pd}_R(M) < \infty$. Finally, if (iii) holds, we have $\operatorname{Tor}_i^R(M, {}^eR) = 0$ for all $i \geqslant 1$ and $e \geqslant 0$ [**26,** Theorem 1.7]. In particular, $\beta_i^F(M, {}^eR) = 0$ for all $i \geqslant 1$. $\square$

Corollary 4.9. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional local ring of characteristic $p > 0$, and let M be a finite length R-module. If $\beta_i^F(M, R) = \beta_{i+1}^F(M, R) = 0$ for some $i \geqslant 1$, then $\mathrm{pd}_R(M) < \infty$. In particular, for any parameter x, if $\beta_2^F(R/(x), R) = 0$, then R is Cohen-Macaulay.*

Proof. Let $(G_j, \varphi_j)_{j\geqslant 0}$ be a minimal free resolution of M. Since $\beta_i^F(M, R) = 0$, we have that $\mathrm{Im}(\varphi_{i+1})$ has finite length, and it is nilpotent. Take $q = p^e \gg 0$ such that $\mathrm{Im}(\varphi_{i+1}^{[q]}) = 0$. For such a q, we have $\mathrm{Ker}(\varphi_{i+1}) = G_{i+1}$. Since the resolution is minimal, we obtain

$$\lambda(\mathrm{Tor}_{i+1}^R(M, {}^eR)) = q^\alpha \lambda\left(\frac{G_{i+1}}{\mathrm{Im}(\varphi_{i+2}^{[q]})}\right) \geqslant q^\alpha \lambda\left(\frac{R}{\mathfrak{m}^{[q]}}\right),$$

where the last inequality comes from projecting onto one of the components of G_{i+1}. Dividing by q and taking limits, we get

$$\beta_{i+1}^F(M, R) = \lim_{e\to\infty} \frac{\lambda(\mathrm{Tor}_{i+1}^R(M, {}^eR))}{q^{(1+\alpha)}} \geqslant \lim_{e\to\infty} \frac{\lambda(R/\mathfrak{m}^{[q]})}{q} = \mathrm{e}_{HK}(\mathfrak{m}, R) > 0,$$

which is a contradiction.

The last claim follows from the fact that, for any parameter x, we have

$$\beta_1^F(R/(x), R) \leqslant \lim_{e\to\infty} \frac{\lambda(H_1(x^q; R))}{q} = 0,$$

where H_1 denotes the first Koszul homology, see **[28]** and **[17,** Theorem 6.2]). □

Lemma 4.10. *Let $(R, \mathfrak{m}, K)$ be a local ring of positive characteristic $p > 0$ and $\mathfrak{p} \in \mathrm{Spec}(R)$. If $\mathrm{pd}_R(\mathfrak{p}) < \infty$, then R is a domain.*

Proof. Since $\mathfrak{p}$ has finite projective dimension, given a minimal free resolution

$$0 \longrightarrow L_t \xrightarrow{\psi_t} \cdots \longrightarrow L_0 \longrightarrow R/\mathfrak{p} \longrightarrow 0$$

of $R/\mathfrak{p}$ over R, we have that

$$0 \longrightarrow L_t \xrightarrow{\psi_t^{[q]}} \cdots \longrightarrow L_0 \longrightarrow R/\mathfrak{p}^{[q]} \longrightarrow 0$$

is a minimal free resolution of $R/\mathfrak{p}^{[q]}$ over R **[26,** Exemples 1.3 d)]. Then, $\mathrm{Ass}_R(R/\mathfrak{p}^{[q]}) = \{\mathfrak{p}\}$, and thus, $\mathfrak{p}^{[q]}$ is $\mathfrak{p}$-primary for all $q = p^e$.

Let $x \notin \mathfrak{p}$, and assume $xy = 0$ for $y \in R$. This implies that, for any q, we have $xy \in \mathfrak{p}^{[q]}$. We conclude that $y \in \mathfrak{p}^{[q]}$ since $x \notin \mathfrak{p}$. Thus,

$$y \in \bigcap_{q \geqslant 1} \mathfrak{p}^{[q]} = (0).$$

In particular, the localization map $R \to R_{\mathfrak{p}}$ is injective. We have that $\mathrm{pd}_R(R/\mathfrak{p}) < \infty$ implies $\mathrm{pd}_{R_{\mathfrak{p}}}(k(\mathfrak{p})) < \infty$. Then, $R_{\mathfrak{p}}$ is a regular local ring; in particular, it is a domain. Therefore, R is a domain. □

Proposition 4.11. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional local ring of characteristic $p > 0$, and let I be an $\mathfrak{m}$-primary integrally closed ideal. If $\beta_i^F(R/I, R) = 0$ for some $i > 0$, then R is regular.*

Proof. Let $\mathfrak{p}$ be a minimal prime of R. Since $\beta_i^F(R/I, R) = 0$, by Theorem 4.7, we have that $\mathrm{Tor}_i^R(R/I, R/\mathfrak{p}) = 0$. By Proposition 4.3, it follows that $\mathrm{pd}_R(R/\mathfrak{p}) < \infty$, and thus, R is a domain by Lemma 4.10. Since one-dimensional local domains are Cohen-Macaulay, by Corollary 4.8, we have that $\mathrm{pd}_R(R/I) < \infty$. In particular, $\mathrm{Tor}_j^R(R/I, K) = 0$ for $j \gg 0$. We conclude that $\mathrm{pd}_R(K) < \infty$ because R/I tests finite projective dimension **[9,** Theorem 5 (ii)]. Hence, R is regular. □

5. Syzygies of finite length. We now present several characteristic-free results. In particular, we do not always assume that the rings have positive characteristic. We focus on Question 1.2. Specifically, we give support to the claim that a finite length R-module M of infinite projective dimension cannot have a finite length syzygy Ω_i for $i > \dim(R)+1$. As a consequence of our methods, we describe, in some cases, the dimension of the syzygies.

It follows from Theorem 4.7 that, if $\dim(R) = 1$ and R has positive characteristic, then an affirmative answer to Question 1.2 is equivalent to the statement: for every M of finite length, $\beta_i^F(M, R) = 0$ for some $i > 1$ implies $\mathrm{pd}_R(M) < \infty$.

We now provide an example that shows that the requirement of $i > \dim(R) + 1$ in Question 1.2 is necessary for a positive answer.

Example 5.1. Let $R = \mathbb{F}_p[\![x,y]\!]/(x^2,xy)$ and $M = R/(x)$. Then $\dim(R) = 1$. In addition, $\mathrm{pd}_R(M) = \infty$ since R is not Cohen-Macaulay. We have that $\Omega_2 \cong H^0_{(x,y)}(R) = (x)$ has finite length.

Lemma 5.2. *Let $(R, \mathfrak{m}, K)$ be a local ring, and let M be a finite length R-module that has a finite length syzygy Ω_{i+1}, for some fixed $i > 0$. Then,*

$$\mathrm{Tor}_i^R\left(M, R/H^0_{\mathfrak{m}}(R)\right) = 0.$$

If R has positive characteristic p, then for all $e \geqslant 0$,

$$\mathrm{Tor}_i^R\left(M, {}^e\!\left(R/H^0_{\mathfrak{m}}(R)\right)\right) = 0.$$

Proof. Set $H := H^0_{\mathfrak{m}}(R)$. Let $(G_\bullet, \varphi_\bullet)$ be a minimal free resolution of M:

$$G_\bullet : \cdots \to G_{i+1} \xrightarrow{\varphi_{i+1}} G_i \xrightarrow{\varphi_i} G_{i-1} \xrightarrow{\varphi_{i-1}} G_{i-2} \to \cdots \to G_0 \to M \to 0.$$

Tensor $G_\bullet$ with R/H and denote by $\overline{G}_\bullet$ its residue class modulo H:

$$\overline{G}_{i+1} \xrightarrow{\overline{\varphi}_{i+1}} \overline{G}_i \xrightarrow{\overline{\varphi}_i} \overline{G}_{i-1}$$

Since $\mathrm{Im}(\varphi_{i+1}) = \Omega_{i+1}$ has finite length, by assumption, we have $\overline{\varphi}_{i+1} = 0$. We want to show that $\mathrm{Ker}(\overline{\varphi}_i) = 0$ as well. For any $\mathfrak{p} \in \mathrm{Spec}(R) \smallsetminus \{\mathfrak{m}\}$, the complex $(G_\bullet)_{\mathfrak{p}}$ is split exact:

$$0 \to (G_i)_{\mathfrak{p}} \xrightarrow{(\varphi_i)_{\mathfrak{p}}} (G_{i-1})_{\mathfrak{p}} \xrightarrow{(\varphi_{i-1})_{\mathfrak{p}}} (G_{i-2})_{\mathfrak{p}} \to \cdots \to (G_0)_{\mathfrak{p}} \to 0,$$

since M and Ω_{i+1} have finite length. We have that $\mathrm{rk}((\varphi_i)_{\mathfrak{p}})$ is maximal, due to the fact that $\mathrm{rk}(G_i) \leqslant \mathrm{rk}(G_{i-1})$ as the localized complex is split exact, and localizing only decreases the rank of a map. Thus, $r := \mathrm{rk}(G_i) = \mathrm{rk}((\varphi_i)_{\mathfrak{p}}) = \mathrm{rk}(\varphi_i)$. Furthermore, $I_r(\varphi_i) \not\subseteq \mathfrak{p}$, by split exactness. Since this holds for all $\mathfrak{p} \in \mathrm{Spec}(R) \smallsetminus \{\mathfrak{m}\}$, in particular, we have $\mathrm{depth}(I_r(\overline{\varphi}_i)) \geqslant 1$. By the Buchsbaum-Eisenbud criterion, we have that

$$0 \longrightarrow \overline{G}_i \xrightarrow{\overline{\varphi}_i} \overline{G}_{i-1} \longrightarrow \overline{\Omega}_{i-1} = \Omega_{i-1}/H\Omega_{i-1} \longrightarrow 0.$$

is an exact complex. Therefore $\mathrm{Ker}(\overline{\varphi}_i) = 0$, and hence, $\mathrm{Tor}_i^R(M, R/H) = 0$. For the second part of the Lemma, when R has positive characteristic, the argument is the same: just notice that the complex ${}^e(G_\bullet)_{\mathfrak{p}}$

is again split exact for all primes $\mathfrak{p} \neq \mathfrak{m}$ and apply the same argument as above to the map $\overline{\varphi}_i^{[q]}$. ☐

We now give results that support an affirmative answer to Question 1.2 for one-dimensional rings. Over Buchsbaum rings, the modules $H^i_{\mathfrak{m}}(R)$ are K-vector spaces for $i < \dim(R)$. Because of this fact, we can prove the following proposition using Lemma 5.2.

Proposition 5.3. *Let* $(R, \mathfrak{m}, K)$ *be a one-dimensional Buchsbaum ring. Then the answer to Question* 1.2 *is positive.*

Proof. Assume that there exists a finite length R-module M such that $\Omega_{i+1}(M)$ has finite length for some $i \geqslant 2$. By Lemma 5.2, we have

$$0 = \operatorname{Tor}_i^R \left(M, R/H^0_{\mathfrak{m}}(R)\right) \cong \operatorname{Tor}_{i-1}^R \left(M, H^0_{\mathfrak{m}}(R)\right),$$

where $i - 1 \geqslant 1$ for dimension shifting. By Remark 2.5, we have that $H^0_{\mathfrak{m}}(R) \cong \bigoplus_{j=1}^t K$. Therefore,

$$0 = \operatorname{Tor}_{i-1}^R \left(M, H^0_{\mathfrak{m}}(R)\right) = \bigoplus_{j=1}^t \operatorname{Tor}_{i-1}^R (M, K),$$

which implies $\operatorname{Tor}_{i-1}^R(M, K) = 0$. Hence, $\operatorname{pd}_R(M) \leqslant i - 2$. ☐

We now present two results about the dimension of syzygies of a finite-length module. These results will be used in Proposition 5.7 to give a case in which a finite-length module cannot have infinitely many syzygies of finite length.

Proposition 5.4. *Let* $(R, \mathfrak{m}, K)$ *be a local ring of dimension* d*, and let* M *be a finite length* R*-module. Let* $i \geqslant 1$*, and let* Ω_i *be the* i*th syzygy of* M*. Then, either* $\dim(\Omega_i) = d$ *or* Ω_i *has finite length.*

Proof. By way of contradiction, we suppose $\dim(\Omega_i) = k$ with $0 < k < d$. Let $G_\bullet \to M \to 0$ be a minimal free resolution of M. By our assumption on $\dim(\Omega_i)$, we can choose $\mathfrak{p} \in \operatorname{Min}(\operatorname{ann}(\Omega_i)) \smallsetminus (\{\mathfrak{m}\} \cup \operatorname{Min}(R))$ and localize $G_\bullet$ at $\mathfrak{p}$. The resulting complex is split exact, because $M_{\mathfrak{p}} = 0$. In particular, $(\Omega_i)_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module. By our choice of $\mathfrak{p}$, we have that $(\Omega_i)_{\mathfrak{p}}$ has finite length, and $\dim(R_{\mathfrak{p}}) > 0$, a contradiction. ☐

Proposition 5.5. *Let $(R, \mathfrak{m}, K)$ be a local ring of positive dimension. Suppose that there exists an R-module M of infinite projective dimension and finite length which has a finite length syzygy Ω_{i+1}, for some fixed $i > 0$. If $\beta_i(M) \geqslant \beta_{i-1}(M)$, then Ω_{i-1} has finite length as well and R is one-dimensional.*

Proof. Let $(G_\bullet, \varphi_\bullet)$ be a minimal free resolution of M:

$$\begin{array}{ccccccccccc} G_{i+1} & & \xrightarrow{\varphi_{i+1}} & & R^{\beta_i(M)} & & \xrightarrow{\varphi_i} & & R^{\beta_{i-1}(M)} & & \xrightarrow{\varphi_{i-1}} & & G_{i-2} \longrightarrow \cdots . \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \searrow & & \nearrow & \\ & & \Omega_{i+1} & & & & \Omega_i & & & & \Omega_{i-1} & & \end{array}$$

Let $\mathfrak{p} \in \operatorname{Spec}(R) \smallsetminus \{\mathfrak{m}\}$. We localize $G_\bullet$ at $\mathfrak{p}$. Since both M and Ω_{i+1} have finite length, we have a split exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow R_{\mathfrak{p}}^{\beta_i(M)} & & \longrightarrow & & R_{\mathfrak{p}}^{\beta_{i-1}(M)} & & \longrightarrow & & (G_{i-2})_{\mathfrak{p}} \longrightarrow \cdots \\ & \searrow^{\cong} & & \nearrow & & \searrow & & & \\ & & (\Omega_i)_{\mathfrak{p}} & & & & (\Omega_{i-1})_{\mathfrak{p}} . & & \end{array}$$

In particular, this implies $\beta_i(M) \leqslant \beta_{i-1}(M)$. Since the opposite inequality holds by our assumption, equality is obtained. Set $\beta = \beta_i(M) = \beta_{i-1}(M)$. From the above split exact sequence, we also get that $R_{\mathfrak{p}}^{\beta} \cong (\Omega_i)_{\mathfrak{p}}$; therefore, $(\Omega_{i-1})_{\mathfrak{p}} = 0$. Since $\mathfrak{p}$ is an arbitrary prime in $\operatorname{Spec}(R) \smallsetminus \{\mathfrak{m}\}$, we have that Ω_{i-1} has finite length. Thus, we have a free complex $0 \to F_1 = R^\beta \to F_0 = R^\beta \to 0$ with finite length homology. We conclude that R has dimension 1 by the New intersection theorem **[28]**. □

Remark 5.6. If, in Proposition 5.5, it is assumed that the sequence of Betti numbers $\{\beta_i(M)\}$ is non-decreasing, then the argument above may be repeated to show that i is necessarily odd, and $\beta_i(M) = \beta_{i-1}(M), \beta_{i-2}(M) = \beta_{i-3}(M), \ldots, \beta_1(M) = \beta_0(M)$. In addition, $\Omega_j(M)$ has finite length for all even j, $0 \leqslant j \leqslant i+1$. In particular, the typical situation to study would be $(R, \mathfrak{m}, K)$ a one-dimensional ring and a resolution

$$\begin{array}{ccccc} 0 \longrightarrow \Omega_4 \longrightarrow R^\beta \longrightarrow R^\beta & & \longrightarrow & & R^\alpha \longrightarrow R^\alpha \longrightarrow M \longrightarrow 0 \\ & \searrow & & \nearrow & \\ & & \Omega_2 & & \end{array}$$

with Ω_4 and Ω_2 of finite length.

As a consequence of these results, we give a partial answer to Question 1.2 in the case where M has eventually non-decreasing Betti numbers. It is a conjecture of Avramov that every finitely generated module over a local ring has eventually non-decreasing Betti numbers [**4**]. The conjecture is known to be true in several cases [**5, 11, 14, 21, 30, 31**], in particular, for Golod rings [**21,** Corollaire 6.5].

Proposition 5.7. *Let $(R,\mathfrak{m},K)$ be a local ring, and let M be a finite length R-module of infinite projective dimension with eventually non-decreasing Betti numbers. Then, for all $i \gg 0$, there exists a $\mathfrak{p} \in \operatorname{Min}(R)$ such that $\dim(\Omega_i) = \dim(R/\mathfrak{p})$. In particular, M cannot have arbitrarily high syzygies of finite length.*

Proof. If $\operatorname{Supp}(\Omega_i) \cap \operatorname{Min}(R) \neq \emptyset$ for all $i \gg 0$, then we are done. By way of contradiction, assume that there exist infinitely many syzygies Ω_i of M such that $\operatorname{Supp}(\Omega_i) \cap \operatorname{Min}(R) = \emptyset$. Note that, by Proposition 5.4, such syzygies must have finite length. By replacing M with a high enough syzygy, we can then assume that M is a module of finite length with non-decreasing Betti numbers, and with infinitely many syzygies of finite length. We have that R is one-dimensional by Proposition 5.5. Furthermore, by Remark 5.6, we have $\beta_{2i} = \beta_{2i+1}$ for all $i \geqslant 0$. For $i \geqslant 0$, consider the short exact sequence

$$0 \longrightarrow \Omega_{2i+2} \longrightarrow R^{\beta} \xrightarrow{\varphi} R^{\beta} \longrightarrow \Omega_{2i} \longrightarrow 0,$$

where $\beta := \beta_{2i} = \beta_{2i+1}$. Let $S := R[\varphi]$. Then R^{β} becomes an S-module. The above exact sequence shows that $\Omega_{2i} \cong R^{\beta} \otimes_S S/(\varphi)$ and $\Omega_{2i+2} \cong (0 :_{R^{\beta}} \varphi)$. Then, by [**32,** Proposition 11.1.9 (2)],

$$\lambda(\Omega_{2i}) - \lambda(\Omega_{2i+2}) = e(\varphi; R^{\beta}),$$

where $e(\varphi; -)$ denotes the Hilbert-Samuel multiplicity with respect to the ideal (φ) in S. Since such a multiplicity is always positive, we have that $\lambda(\Omega_{2i+2}) < \lambda(\Omega_{2i})$, for all $i \geqslant 0$. Since there cannot be an infinite strictly decreasing sequence of such lengths, we obtain a contradiction. □

Remark 5.8. Proposition 5.7 also follows from [**6,** Theorem 8], and it gives another proof of the fact that, when M is a module of finite length with eventually non-decreasing Betti numbers and R is equidimensional, then the sequence of integers $\{\dim(\Omega_i)\}_{i=0}^{\infty}$ is constant for $i \gg 0$ (see [**6,** Corollary 2]).

Proposition 5.9. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional local ring. Suppose that there exists a finite length module M of infinite projective dimension that has a finite length syzygy Ω_{i+1}, for some fixed $i \geqslant 2$. Then,*

$$\lambda(\Omega_{i+1}) = \sum_{j=0}^{i} (-1)^{i-j+1} \lambda(\operatorname{Tor}_j^R(M, R/(x))),$$

where x is a suitable parameter.

Proof. Consider a minimal free resolution of M:

$$G_i \longrightarrow G_{i-1} \longrightarrow \cdots \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0.$$

$$G_i \searrow \Omega_i \hookrightarrow G_{i-1}, \qquad G_1 \searrow \Omega_1 \hookrightarrow G_0$$

For all $j = 1, \ldots, i+1$, this can be broken into short exact sequences:

$$0 \longrightarrow \Omega_j \longrightarrow G_{j-1} \longrightarrow \Omega_{j-1} \longrightarrow 0,$$

where $\Omega_0 := M$. These give two exact sequences:

$$0 \longrightarrow \Omega_{i+1} \longrightarrow H^0_{\mathfrak{m}}(G_i) \longrightarrow H^0_{\mathfrak{m}}(\Omega_i) \longrightarrow 0$$

and

$$0 \longrightarrow H^0_{\mathfrak{m}}(\Omega_j) \longrightarrow H^0_{\mathfrak{m}}(G_{j-1}) \longrightarrow H^0_{\mathfrak{m}}(\Omega_{j-1}).$$

The first short exact sequence comes from the fact that Ω_{i+1} has finite length, and thus, $H^1_{\mathfrak{m}}(\Omega_{i+1}) = 0$. Furthermore, the cokernel of the rightmost map in the second exact sequence, which may be proved to be the kernel of the leftmost map in

$$\Omega_j \otimes_R H^1_{\mathfrak{m}}(R) \longrightarrow G_{j-1} \otimes_R H^1_{\mathfrak{m}}(R) \longrightarrow \Omega_{j-1} \otimes_R H^1_{\mathfrak{m}}(R) \longrightarrow 0$$

is then $\operatorname{Tor}_1^R(\Omega_{j-1}, H^1_{\mathfrak{m}}(R))$. For simplicity, we denote $\omega_j := \lambda(H^0_{\mathfrak{m}}(\Omega_j))$, $g_j := \lambda(H^0_{\mathfrak{m}}(G_j))$ and $\alpha_j := \lambda(\operatorname{Tor}_1^R(\Omega_j, H^1(R)))$. Then, we have relations

$$\begin{aligned}
\omega_{i+1} &= g_i - \omega_i \\
\omega_i &= g_{i-1} - \omega_{i-1} + \alpha_{i-1} \\
&\vdots \\
\omega_2 &= g_1 - \omega_1 + \alpha_1 \\
\omega_1 &= g_0 - \lambda(M) + \lambda(\mathrm{Tor}_1(M, H^1_{\mathfrak{m}}(R))).
\end{aligned}$$

After localizing the resolution $G_\bullet$ at any minimal prime $\mathfrak{p}$, since $(\Omega_{i+1})_{\mathfrak{p}} = 0$, we obtain that $\sum_{j=0}^{i}(-1)^j\beta_j(M) = 0$. Then, $\sum_{j=0}^{i}(-1)^j g_j = 0$ because $g_j = \beta_j(M)\cdot\lambda(H^0_m(R))$. Therefore,

$$\begin{aligned}
\omega_{i+1} &= \lambda(\Omega_{i+1}) \\
&= \sum_{j=1}^{i-1}(-1)^{i-j}\alpha_j + (-1)^i\lambda(\mathrm{Tor}_1(M, H^1(R))) + (-1)^{i-1}\lambda(M).
\end{aligned}$$

Choose a parameter x such that $H^0_{\mathfrak{m}}(R) = 0 :_R x$, as in Remark 2.4. By similar considerations, we can also assume that $xM = 0$. From this choice, we have that $xH^0_{\mathfrak{m}}(\Omega_j) = 0$ for all $j = 0, \ldots, i+1$, since $\Omega_j \subseteq G_{j-1}$ is a free R-module. Since the Tor modules can be computed using flat resolutions, we have an exact sequence

$$0 \longrightarrow H^0_{\mathfrak{m}}(R) \longrightarrow R \longrightarrow R_x \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow 0.$$

Completion is produced on the left to obtain a flat resolution of $H^1_{\mathfrak{m}}(R)$:

$$\cdots \longrightarrow R^{\mu(H^0_{\mathfrak{m}}(R))} \longrightarrow R \longrightarrow R_x \longrightarrow H^1_{\mathfrak{m}}(R) \longrightarrow 0,$$

where $R \longrightarrow R_x$ factors as $R \longrightarrow R/H^0_{\mathfrak{m}}(R) \longrightarrow R_x$.

By our choice of x, we have that a free resolution of R/x begins as

$$\cdots \longrightarrow R^{\mu(H^0_{\mathfrak{m}}(R))} \longrightarrow R \longrightarrow R \longrightarrow R/(x) \longrightarrow 0.$$

For all $j = 1, \ldots, i-1$, we obtain

$$\mathrm{Tor}_1^R(\Omega_j, H^1_{\mathfrak{m}}(R)) \cong \mathrm{Tor}_1^R(\Omega_j, R/(x)) \cong \mathrm{Tor}_{j+1}^R(M, R/(x)),$$

where the last isomorphism comes from dimension shifting. In addition,

$$\operatorname{Tor}_1^R(M, H^1_{\mathfrak{m}}(R)) \cong \operatorname{Tor}_1^R(M, R/(x)).$$

Finally, since $xH^0_{\mathfrak{m}}(\Omega_0) = xM = 0$, we get

$$M \cong M/xM \cong \operatorname{Tor}_0^R(M, R/(x)),$$

and the proposition then follows. □

Corollary 5.10. *Let $(R, \mathfrak{m}, K)$ be a one-dimensional ring, and let M be a finite length module of infinite projective dimension. Then $\lambda(\Omega_1) = \lambda(\Omega_3) = \infty$.*

Proof. Note that $\lambda(\Omega_1) = \infty$; otherwise, we would have a short exact sequence

$$0 \longrightarrow \Omega_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

in which both Ω_1 and M have finite lengths. This cannot occur since $G_0 \neq 0$ is free and $\dim(R) = 1$.

Now, let us assume by way of contradiction that $\lambda(\Omega_3) < \infty$. Let $(G_\bullet, \varphi_\bullet)$ be a minimal free resolution of M:

$$0 \longrightarrow \Omega_3 \longrightarrow G_2 \xrightarrow{\varphi_2} G_1 \xrightarrow{\varphi_1} G_0 \longrightarrow M \longrightarrow 0.$$

Let $x \in R$ be a parameter such that $xM = xH^0_{\mathfrak{m}}(R) = 0$. Consider the short exact sequence

$$0 \longrightarrow (x) \longrightarrow R \longrightarrow R/(x) \longrightarrow 0.$$

By our choice of x we have $0 :_R x = H^0_{\mathfrak{m}}(R)$; hence, $(x) \cong R/H^0_{\mathfrak{m}}(R)$. After tensoring the sequence with M, we obtain that

$$0 \longrightarrow \operatorname{Tor}_1^R(M, R/(x)) \longrightarrow M/H^0_{\mathfrak{m}}(R)M \longrightarrow M \longrightarrow M/xM \longrightarrow 0.$$

Since $xM = 0$, we obtain

$$\lambda(\operatorname{Tor}_1^R(M, R/(x))) = \lambda(M/H^0_{\mathfrak{m}}(R)M).$$

Then, by Proposition 5.9, we have

$$\begin{aligned}\lambda(\Omega_3) &= -\lambda(\operatorname{Tor}_2^R(M, R/(x))) + \lambda(\operatorname{Tor}_1^R(M, R/(x))) - \lambda(M)\\ &\leqslant \lambda(\operatorname{Tor}_1^R(M, R/(x))) - \lambda(M) = \lambda(M/H^0_{\mathfrak{m}}(R)M) - \lambda(M) \leqslant 0,\end{aligned}$$

which gives a contradiction since $\Omega_3 \neq 0$, since M has infinite projective dimension. □

The next example is due to the second author and is taken from **[6]**. It shows the assumption that M has finite length is needed in Corollary 5.10.

Example 5.11. Let $S = \mathbb{Q}[x, y, z, u, v]$, and let $I \subseteq S$ be the ideal

$$I = (x^2, xz, z^2, xu, zv, u^2, v^2, zu + xv + uv, yu, yv, yx - zu, yz - xv).$$

Let $R = S/I$, which is a one-dimensional ring of depth 0. In this case, y is a parameter, $0 :_R y = (u, v, z^2)$ and $(y) = 0 :_R (0 :_R y)$. Let M be the cokernel of the rightmost map in the exact complex:

$$\cdots \longrightarrow R^3 \xrightarrow{\begin{bmatrix} u & v & z^2 \end{bmatrix}} R \xrightarrow{\ y\ } R \xrightarrow{\begin{bmatrix} u \\ v \\ z^2 \end{bmatrix}} R^3.$$

Then M is a one-dimensional module with first and third syzygies $\Omega_1 \cong R/(y)$ and $\Omega_3 \cong 0 :_R y$. Both are modules of finite length since y is a parameter.

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