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EXTENSIONS OF LOCAL DOMAINS WITH TRIVIAL GENERIC FIBER

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EXTENSIONS OF LOCAL DOMAINS WITH TRIVIAL GENERIC FIBER

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Dedicated to Phil Griffith, in honor of his contributions to commutative algebra

ABSTRACT. We consider injective local maps from a local domain R to a local domain S such that the generic fiber of the inclusion map $R \hookrightarrow S$ is trivial, that is, $P \cap R \neq (0)$ for every nonzero prime ideal P of S. We present several examples of injective local maps involving power series that have or fail to have this property. For an extension $R \hookrightarrow S$ having this property, we give some results on the dimension of S; in some cases we show $\dim S = 2$ and in some cases $\dim S = 1$.

1. Introduction and background

Our work in this paper originates with the following question raised by Melvin Hochster and Yongwei Yao.

QUESTION 1.1. Let R be a complete local domain. Can one describe or somehow classify the injective local maps of R to a complete local domain S such that $U^{-1}S$ is a field, where $U = R \setminus (0)$, i.e., such that the generic fiber of $R \hookrightarrow S$ is trivial?

By Cohen's structure theorems [4], [15, (31.6)], a complete local domain R is a finite integral extension of a complete regular local domain R_0. If R has the same characteristic as its residue field, then R_0 is a formal power series ring over a field. The generic fiber of $R \hookrightarrow S$ is trivial if and only if the generic fiber of $R_0 \hookrightarrow S$ is trivial. Thus as Hochster and Yao remark: if R is equal characteristic zero one obtains extensions as in Question 1.1 by starting with

$$R_0 := K[[x_1, \ldots, x_n]] \hookrightarrow T := L[[x_1, \ldots, x_n, y_1, \ldots, y_m]],$$

where K is a subfield of L and the x_i, y_j are formal indeterminates. Let P be a prime ideal of T maximal with respect to being disjoint from the image

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of $R_0 \setminus \{0\}$. Then the composite map $R_0 \hookrightarrow T \to T/P =: S$ is an extension of this type. Of course, such prime ideals P are maximal in the generic fiber $(R_0 \setminus \{0\})^{-1}T$ of the embedding $R_0 \hookrightarrow T$.

In [11], we study the generic fiber of extensions of power series rings over the same base field. With $R = K[[x_1, \dots, x_n]]$ as above and $T = R[[y_1, \dots, y_m]]$, we show in [11, Theorem 7.2] that, if P is maximal in the generic fiber of $R \hookrightarrow T$ and $S = T/P$, then $\dim S$ is either 2 or n. This answers Question 1.1 in the case where $R = K[[x_1, \dots, x_n]]$ is a complete regular local domain with coefficient field K and S is a complete local domain that also has coefficient field K.

Definition 1.2. If $R \hookrightarrow S$ is an injective map of integral domains, we say that S is a *trivial generic fiber extension*, *TGF extension*, of R if each nonzero ideal of S has a nonzero intersection with R, or equivalently, if each nonzero element of S has a nonzero multiple in R. Since ideals of S maximal with respect to not meeting the multiplicative system of nonzero elements of R are prime ideals, S is a TGF extension of R if and only if $P \cap R \neq (0)$ for each nonzero prime ideal P of S. Another condition equivalent to S is a TGF extension of R is that $U^{-1}S$ is a field, where $U = R \setminus (0)$.

Let $(R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ be an injective local homomorphism of complete local domains, so that $\mathbf{n} \cap R = \mathbf{m}$. We say that S is a *TGF-complete extension* of R if S is a TGF extension of R.

In [12] we consider the TGF property for extensions of mixed polynomial/power series rings over the same base field and we partially characterize the prime ideal spectra of such rings. For example, we consider the nested mixed polynomial/power series rings

$$A := k[x,y] \hookrightarrow B := k[[y]]\,[x] \hookrightarrow C := k[x]\,[[y]] \hookrightarrow E := k[x, 1/x]\,[[y]], \tag{1.1}$$

$$C \hookrightarrow D_1 := k[x]\,[[y/x]] \hookrightarrow \cdots \hookrightarrow D_n := k[x]\,[[y/x^n]] \hookrightarrow \cdots \hookrightarrow E, \tag{1.2}$$

where k is a field and x and y are indeterminates over k. In Sequence (1.1) the maps are all flat. In Sequence (1.2), for n a positive integer, the map $C \hookrightarrow D_n$ is not flat, but $D_n \hookrightarrow E$ is a localization followed by an adic completion of a Noetherian ring and therefore is flat. All of the extensions in (1.1) and (1.2) except those that begin with A are TGF. The extensions that begin with A are not TGF. In dimension 3 we consider in [12] embeddings such as

$$k[x,y,z] \overset{\alpha}{\hookrightarrow} k[[z]]\,[x,y] \overset{\beta}{\hookrightarrow} k[x]\,[[z]]\,[y] \overset{\gamma}{\hookrightarrow} k[x,y]\,[[z]] \overset{\delta}{\hookrightarrow} k[x]\,[[y,z]],$$

$$k[[z]]\,[x,y] \overset{\epsilon}{\hookrightarrow} k[[y,z]]\,[x] \overset{\zeta}{\hookrightarrow} k[x]\,[[y,z]] \overset{\eta}{\hookrightarrow} k[[x,y,z]],$$

where k is a field and x, y and z are indeterminates over k. Here all of the proper inclusions fail to be TGF. Takehiko Yasuda in [18] gives additional information on the TGF property. In particular, he shows in [18, Theorem

2.7] that

$$\mathbb{C}[x,y][[z]] \hookrightarrow \mathbb{C}[x,x^{-1},y][[z]]$$

is not TGF, where $\mathbb{C}$ is the field of complex numbers.

In this article we discuss several additional topics and questions related to Question 1.1 and the TGF property. In Section 2 we record several basic facts about TGF extensions. We prove in Proposition 2.6 that if $A \hookrightarrow B$ is a TGF extension, where B is a Noetherian integral domain, then $\dim A \geq \dim B$.

We prove in Corollary 3.3 that if $(A,\mathbf{m}) \hookrightarrow (S,\mathbf{n})$ is a TGF-complete extension, where A is equicharacteristic with $\dim A = n \geq 2$ and $S/\mathbf{n}$ finite algebraic over $A/\mathbf{m}$, then either $\dim S = n$ and S is a finite integral extension of A or $\dim S = 2$. We also include in Section 3 other remarks concerning TGF-complete extensions having finite residue field extension. For each $n \geq 2$ and $R = k[[X]]$ a formal power series ring in n variables over a field k, we describe in (3.5) a TGF-complete extension $R \hookrightarrow S$, where S is a power series ring in 2 variables over k.

In Section 4 we consider a TGF-complete extension $(R,\mathbf{m}) \hookrightarrow (S,\mathbf{n})$, where $S/\mathbf{n}$ is transcendental over $R/\mathbf{m}$. We address, but do not resolve, the question of whether in this situation $\dim S \leq 1$. We prove in Theorem 4.8 that if $(A,\mathbf{m}) \hookrightarrow (B,\mathbf{n})$ is an injective local homomorphism of 2-dimensional regular local rings such that $B/\mathbf{n}$ as a field extension of $A/\mathbf{m}$ is not algebraic, then $A \hookrightarrow B$ is not TGF. We deduce that for indeterminates x,y,z,w,t over a field k, if $\varphi : R = k[[x,y]] \hookrightarrow S := k(t)[[z,w]]$ is an injective local k-algebra homomorphism, then $\varphi(R) \hookrightarrow S$ is not TGF.

There is much in the literature concerning homomorphisms of formal power series rings; see, for example, the articles of Abhyankar-Moh [2], Matsumura [13], Rotthaus [16].

2. Trivial generic fiber (TGF) extensions, general remarks

We record in Proposition 2.1 several basic facts about TGF extensions. We omit the proofs since they are straightforward.

Proposition 2.1. *Let $R \hookrightarrow S$ and $S \hookrightarrow T$ be injective maps, where R, S and T are integral domains.*

1. *If $R \hookrightarrow S$ and $S \hookrightarrow T$ are TGF extensions, then so is the composite map $R \hookrightarrow T$. Equivalently, if the composite map $R \hookrightarrow T$ is not TGF, then at least one of the extensions $R \hookrightarrow S$ or $S \hookrightarrow T$ is not TGF.*
2. *If $R \hookrightarrow T$ is TGF, then $S \hookrightarrow T$ is TGF.*
3. *If the map $\operatorname{Spec} T \to \operatorname{Spec} S$ is surjective and $R \hookrightarrow T$ is TGF, then $R \hookrightarrow S$ is TGF.*

We consider in Proposition 2.2 the relatively easy case where the base ring has dimension one.

Proposition 2.2. *Let* $(R, \mathbf{m})$ *be a complete one-dimensional local domain. Assume that* $(S, \mathbf{n})$ *is a TGF-complete extension of* R. *Then:*

(1) $\dim(S) = 1$ *and* $\mathbf{m}S$ *is* $\mathbf{n}$*-primary.*
(2) *If* $[S/\mathbf{n} : R/\mathbf{m}] < \infty$, *then* S *is a finite integral extension of* R.

Thus, if $R \hookrightarrow S$ *is a TGF-extension with finite residue extension and* $\dim S \geq 2$, *then* $\dim R \geq 2$.

Proof. By Krull's Principal Ideal Theorem [14, Theorem 13.5], $\mathbf{n}$ is the union of the height-one primes of S. If $\dim S > 1$, then S has infinitely many height-one primes. Each nonzero element of $\mathbf{n}$ is contained in only finitely many of these height-one primes. If $\dim S > 1$, then the intersection of the height-one primes of S is zero. Since $\dim R = 1$, every nonzero prime of S contains $\mathbf{m}$. Thus $\dim S = 1$ and $\mathbf{m}S$ is $\mathbf{n}$-primary. Moreover, if $[S/\mathbf{n} : R/\mathbf{m}] < \infty$, then S is finite over R by [14, Theorem 8.4]. □

Remarks 2.3. (1) Notice that there exist TGF-complete extensions of R that have an arbitrarily large extension of residue field. For example, if k is a subfield of a field F and x is an indeterminate over F, then $R := k[[x]] \subseteq S := F[[x]]$ is a TGF-complete extension.

(2) Let $(R, \mathbf{m}) \hookrightarrow (T, \mathbf{q})$ be an injective local homomorphism of complete local domains. For $P \in \operatorname{Spec} T$, $S := T/P$ is a TGF-complete extension of R if and only if P is an ideal of T maximal with respect to the property that $P \cap R = (0)$.

Remarks 2.4. Let $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_m\}$ and $Z = \{z_1, \ldots, z_r\}$ be algebraically independent finite sets of indeterminates over a field k, where $n \geq 2$, $m, r \geq 1$. Set $R := k[[X]]$ and let P be a prime ideal of $k[[X, Y, Z]]$ that is maximal with respect to $P \cap R = (0)$. Then we have the inclusions

$$R := k[[X]] \overset{\sigma}{\hookrightarrow} S := k[[X, Y]]/(P \cap k[[X, Y]]) \overset{\tau}{\hookrightarrow} T := k[[X, Y, Z]]/P.$$

By Remark 2.3(2), $\tau \cdot \sigma$ is a TGF extension. By Proposition 2.1(2), $S \hookrightarrow T$ is TGF.

(1) If the map $\operatorname{Spec} T \to \operatorname{Spec} S$ is surjective, then $\sigma : R \hookrightarrow S$ is TGF by Proposition 2.1(3).
(2) If $R \hookrightarrow T$ is finite, then $R \hookrightarrow S$ is also finite, and so $\sigma : R \hookrightarrow S$ is TGF.
(3) If $R \hookrightarrow T$ is not finite, then $\dim T = 2$ by [11, Theorem 7.2].
(4) If $P \cap k[[X, Y]] = 0$, then $S = R[[Y]]$ and $R \hookrightarrow S$ is not TGF. (We show in Example 3.10 that this can occur.)

Remarks and Question 2.5. (1) With notation as in Remarks 2.4 and with $Y = \{y\}$, a singleton set, it is always true that $\operatorname{ht}(P \cap R[[y]]) \leq n - 1$, [11, Theorem 7.1]. Moreover, if $\operatorname{ht}(P \cap R[[y]]) = n-1$, then $R \hookrightarrow S$ is TGF. Thus if $n = 2$ and $P \cap R[[y]] \neq 0$, then $R \hookrightarrow S$ is TGF.

(2) With notation as in (1) and $n = 3$, it can happen that $P \cap k[[X, y]] \neq (0)$ and $R \hookrightarrow R[[y]]/(P \cap R[[y]])$ is not a TGF extension. To construct an example of such a prime ideal P, we proceed as follows: Since $\dim k[[X, y]] = 4$, there exists a prime ideal Q of $k[[X, y]]$ with $\operatorname{ht} Q = 2$ and $Q \cap k[[X]] = (0)$, [11, Theorem 7.1]. Let $\mathbf{p} \subset Q$ be a prime ideal with $\operatorname{ht} \mathbf{p} = 1$. Since $\mathbf{p} \subsetneq Q$ and $Q \cap k[[X]] = (0)$, the extension $k[[X]] \hookrightarrow k[[X, y]]/\mathbf{p}$ is not a TGF extension. In particular, it is not finite. Let $P \in \operatorname{Spec} k[[X, y, Z]]$ be maximal with respect to $P \cap k[[X, y]] = \mathbf{p}$. By Corollary 3.4 below, $\dim k[[X, y, Z]]/P = 2$. Hence P is maximal in the generic fiber over $k[[X]]$.

(3) If $(R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ is a TGF-complete extension with $S/\mathbf{n}$ finite algebraic over $R/\mathbf{m}$, can the transcendence degree of S over R be finite but nonzero?

(4) If $(R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ is a TGF-complete extension as in (3) with R equicharacteristic and $\dim R \geq 2$, then by Corollary 3.3 below it follows that either S is a finite integral extension of R or $\dim S = 2$.

PROPOSITION 2.6. *Let $A \hookrightarrow B$ be a TGF extension, where B is a Noetherian integral domain. For each $Q \in \operatorname{Spec} B$, we have $\operatorname{ht} Q \leq \operatorname{ht}(Q \cap A)$. In particular, $\dim A \geq \dim B$.*

Proof. If $\operatorname{ht} Q = 1$, it is clear that $\operatorname{ht} Q \leq \operatorname{ht}(Q \cap A)$ since $Q \cap A \neq (0)$. Let $\operatorname{ht} Q = n \geq 2$, and assume by induction that $\operatorname{ht} Q' \leq \operatorname{ht}(Q' \cap A)$ for each $Q' \in \operatorname{Spec} B$ with $\operatorname{ht} Q' \leq n-1$. Since B is Noetherian,

$$(0) = \bigcap \{Q' \mid Q' \subset Q \quad \text{and} \quad \operatorname{ht} Q' = n-1\}.$$

Hence there exists $Q' \subset Q$ with $\operatorname{ht} Q' = n-1$ and $Q' \cap A \subsetneq Q \cap A$. We have $n - 1 \leq \operatorname{ht}(Q' \cap A) < \operatorname{ht}(Q \cap A)$, so $\operatorname{ht}(Q \cap A) \geq n$. □

3. TGF-complete extensions with finite residue field extension

SETTING 3.1. Let $n \geq 2$ be an integer, let $X = \{x_1, \ldots, x_n\}$ be a set of independent variables over the field k and let $R = k[[X]]$ be the formal power series ring in n variables over the field k.

THEOREM 3.2. *Let $R = k[[X]]$ be as in Setting 3.1. Assume that $R \hookrightarrow S$ is a TGF-complete extension, where $(S, \mathbf{n})$ is a complete Noetherian local domain and $S/\mathbf{n}$ is finite algebraic over k. Then either $\dim S = n$ and S is a finite integral extension of R or $\dim S = 2$.*

Proof. It is clear that if S is a finite integral extension of R, then $\dim S = n$. Assume S is not a finite integral extension of R. Let $b_1, \ldots, b_m \in \mathbf{n}$ be such that $\mathbf{n} = (b_1, \ldots, b_m)S$, and let $Y = \{y_1, \ldots, y_m\}$ be a set of independent variables over R. Since S is complete the R-algebra homomorphism $\varphi : T :=$

$R[[Y]] \to S$ such that $\varphi(y_i) = b_i$ for each i with $1 \le i \le m$ is well defined. Let $Q = \ker \varphi$. We have

$$R \hookrightarrow T/Q \hookrightarrow S.$$

By [14, Theorem 8.4], S is a finite module over T/Q. Hence $\dim S = \dim(T/Q)$ and the map $\operatorname{Spec} S \to \operatorname{Spec} T/Q$ is surjective, so by Proposition 2.1(3), $R \hookrightarrow T/Q$ is TGF. By [11, Theorem 7.2], $\dim(T/Q) = 2$, so $\dim S = 2$. □

Corollary 3.3. *Let $(A, \mathbf{m})$ and $(S, \mathbf{n})$ be complete equicharacteristic local domains with $\dim A = n \ge 2$ and suppose that $A \hookrightarrow S$ is a local injective homomorphism and that the residue field $S/\mathbf{n}$ is finite algebraic over the residue field $A/\mathbf{m} := k$. If $A \hookrightarrow S$ is a TGF-complete extension, then either $\dim S = n$ and S is a finite integral extension of A or $\dim S = 2$.*

Proof. By [14, Theorem 29.4(3)], A is a finite integral extension of $R = k[[X]]$, where X is as in Setting 3.1. We have $R \hookrightarrow A \hookrightarrow S$. By Proposition 2.1(1), $R \hookrightarrow S$ is TGF. By Theorem 3.2, either $\dim S = n$ and S is a finite integral extension of A or $\dim S = 2$. □

For example, if $R = k[[x_1, \dots, x_4]]$ and $S = k[[y_1, y_2, y_3]]$, then every k-algebra embedding $R \hookrightarrow S$ fails to be TGF.

Corollary 3.4. *Let $R = k[[X]]$ be as in Setting 3.1. Let $Y = \{y_1, \dots, y_m\}$ be a set of m independent variables over R and let $S = R[[Y]]$. If $P \in \operatorname{Spec} R$ is such that $\dim R/P \ge 2$ and $Q \in \operatorname{Spec} S$ is maximal with respect to $Q \cap R = P$, then either*

(i) $\dim S/Q = 2$, *or*
(ii) $R/P \hookrightarrow S/Q$ *is a finite integral extension (and so* $\dim R/P = \dim S/Q$*).*

Proof. Let $A := R/P \hookrightarrow S/Q =: B$, and apply Corollary 3.3. □

General Example 3.5. It is known that, for each positive integer n, the power series ring $R = k[[x_1, \dots, x_n]]$ in n variables over a field k can be embedded into a power series ring in two variables over k. The construction is based on the fact that the power series ring $k[[z]]$ in the single variable z contains an infinite set of algebraically independent elements over k. Let $\{f_i\}_{i=1}^{\infty} \subset k[[z]]$ with $f_1 \ne 0$ and $\{f_i\}_{i=2}^{\infty}$ algebraically independent over $k(f_1)$. Let $(S := k[[z, w]], \mathbf{n} := (z, w))$ be the formal power series ring in the two variables z, w. Fix a positive integer n and consider the subring $R_n := k[[f_1 w, \dots, f_n w]]$ of S with maximal ideal $\mathbf{m}_n = (f_1 w, \dots, f_n w)$. Let $x_1, \dots, x_n$ be new indeterminates over k and define a k-algebra homomorphism $\varphi : k[[x_1, \dots, x_n]] \to R_n$ by setting $\varphi(x_i) = f_i w$ for $i = 1, \dots, n$.

Claim 3.6. *(cf. [19, pp. 219-220]) φ is an isomorphism.*

Proof. Suppose $g = \sum_{m=0}^{\infty} g_m$, where g_m is a form of degree m in $k[x_1, \dots, x_n]$. Then

$$\varphi(g) = \sum_{m=0}^{\infty} \varphi(g_m) \quad \text{and} \quad \varphi(g_m) = g_m(f_1 w, \dots, f_n w) = w^m g_m(f_1, \dots, f_n),$$

where $g_m(f_1, \dots, f_n) \in k[[z]]$. If $\varphi(g) = 0$, then $g_m(f_1, \dots, f_n) = 0$ for each m. Thus

$$0 = g_m(f_1, \dots, f_n) = \sum_{i_1+\cdots+i_n=m} a_{i_1,\dots,i_n} f_1^{i_1} \cdots f_n^{i_n},$$

where the $a_{i_1,\dots,i_n} \in k$ and the i_j are nonnegative integers. Our hypothesis on the f_j implies that each of the $a_{i_1,\dots,i_n} = 0$, and so $g_m = 0$ for each m. □

Proposition 3.7. *With notation as in Example 3.5, for each integer $n \geq 2$, the extension $(R_n, \mathbf{m}_n) \hookrightarrow (S, \mathbf{n})$ is nonfinite TGF-complete with trivial residue extension. Moreover* $\mathrm{ht}(P \cap R_n) \geq n - 1$*, for each nonzero prime* $P \in \operatorname{Spec} S$.

Proof. We have $k = R_n/\mathbf{m}_n = S/\mathbf{n}$, so the residue field of S is a trivial extension of that of R_n. Since $\mathbf{m}_n S$ is not $\mathbf{n}$-primary, S is not finite over R_n. If $P \cap R_n = \mathbf{m}_n$, then $\mathrm{ht}(P \cap R_n) = n \geq n - 1$. Since $\dim S = 2$, if $\mathbf{m}_n$ is not contained in P, then $\mathrm{ht}\, P = 1$, S/P is a one-dimensional local domain, and $\mathbf{m}_n(S/P)$ is primary for the maximal ideal $\mathbf{n}/P$ of S/P. It follows that $R_n/(P \cap R_n) \hookrightarrow S/P$ is a finite integral extension [14, Theorem 8.4]. Therefore $\dim R_n/(P \cap R_n) = 1$. Since R_n is catenary and $\dim R_n = n$, $\mathrm{ht}(P \cap R_n) = n - 1$. □

Corollary 3.8. *Let X and $R = k[[X]]$ be as in Setting 3.1. Then there exists an infinite properly ascending chain of two-dimensional TGF-complete extensions $R =: S_0 \hookrightarrow S_1 \hookrightarrow S_2 \hookrightarrow \cdots$ such that each S_i has the same residue field as R and S_{i+1} is a nonfinite TGF-complete extension of S_i for each i.*

Proof. Example 3.5 and Proposition 3.7 imply that R can be identified with a proper subring of the power series ring in two variables so that $k[[y_1, y_2]]$ is a TGF-complete extension of R and the extension is not finite. Now Example 3.5 and Proposition 3.7 can be applied again, to $k[[y_1, y_2]]$, and so on. □

Example 3.9. A particular case of Example 3.5.

For $R := k[[x, y]]$, the extension ring $S := k[[x, y/x]]$ has infinite transcendence over R [17]. The method used in [17] to prove that S has infinite transcendence degree over R is by constructing power series in y/x with 'special large gaps'. Since $k[[x]]$ is contained in R, it follows that S is a TGF-complete extension of R. To show this, it suffices to show $P \cap R \neq (0)$ for each $P \in \operatorname{Spec} S$ with $\mathrm{ht}\, P = 1$. This is clear if $x \in P$, while if $x \notin P$, then

$k[[x]] \cap P = (0)$, so $k[[x]] \hookrightarrow R/(P \cap R) \hookrightarrow S/P$ and S/P is finite over $k[[x]]$. Therefore $\dim R/(P \cap R) = 1$, so $P \cap R \neq (0)$.

Notice that the extension $k[[x,y]] \hookrightarrow k[[x,y/x]]$ is, up to isomorphism, the same as the extension $k[[x,xy]] \hookrightarrow k[[x,y]]$.

In Example 3.10 we show the situation of Remark 2.4(4) does occur.

EXAMPLE 3.10. Let k, $X = \{x_1, x_2\}$, $Y = \{y\}$, $Z = \{z\}$ and $R = k[[x_1, x_2]]$ be as in Remarks 2.4. Let $f_1, f_2 \in k[[z]]$ be algebraically independent over k. Let P denote the ideal of $k[[x_1, x_2, y, z]]$ generated by $(x_1 - f_1 y, x_2 - f_2 y)$, Then P is the kernel of the k-algebra homomorphism $\theta : k[[x_1, x_2, y, z]] \to k[[y, z]]$ obtained by defining $\theta(x_1) = f_1 y$, $\theta(x_2) = f_2 y$, $\theta(y) = y$ and $\theta(z) = z$. In the notation of Remark 2.4,

$$T = k[[x_1, x_2, y, z]]/P \cong k[[y, z]].$$

Let $\varphi := \theta|_R$ and $\tau := \theta|_{R[[y]]}$. The proof of Claim 3.6 shows that φ and τ are embeddings. Hence $P \cap R[[y]] = (0)$. By Proposition 3.7, φ and τ are TGF. We have

$$R \overset{\sigma}{\hookrightarrow} S = \frac{R[[y]]}{P \cap R[[y]]} = R[[y]] \overset{\tau}{\hookrightarrow} \frac{R[[y,z]]}{P} \cong k[[y,z]],$$

where $\sigma : R \hookrightarrow S$ is the inclusion map. Since $yS \cap R = (0)$, $\sigma : R \hookrightarrow S$ is not TGF.

QUESTIONS 3.11. (1) If $\varphi : R \hookrightarrow S$ is a TGF-complete nonfinite extension with finite residue field extension, is it always true that φ can be extended to a TGF-complete nonfinite extension $R[[y]] \hookrightarrow S$?

(2) Suppose that $R \hookrightarrow S$ is a TGF-complete extension and y is an indeterminate over S. It is natural to ask: Does $R[[y]] \hookrightarrow S[[y]]$ have the TGF property? Computing with elements, one may ask: For $s \in S \setminus R$, does $y + s$ have a multiple in $R[[y]]$? There is a $t \in S$ with $ts \in R$, but is there a $t' \in S$ with both $t't$ and $t'ts \in R$?

(3) A related question is whether the given $R \hookrightarrow S$ is extendable to an injective local homomorphism $\varphi : R[[y]] \hookrightarrow S$. For example, with k a field, $k[[x_1]][y]_{(x_1,y)} \hookrightarrow k[y][[x_1]]_{(x_1,y)}$ is TGF. Can we extend to $k[[x_1]][y][[x_2]]_{(x_1,x_2,y)} \hookrightarrow k[y][[x_1]]_{(x_1,y)}$, say by $x_2 \to \sum_{n=0}^{\infty} (yx)^n$, which is still local injective?

We show in Proposition 3.12 that the answer to Question 3.11(2) is 'no' if the answer to Question 3.11(3) is "yes", that is, the given $R \hookrightarrow S$ is extendable to an injective local homomorphism $R[[y]] \hookrightarrow S$. In Example 3.13 we present an example where this occurs.

PROPOSITION 3.12. *Let $R \hookrightarrow S$ be a TGF-complete extension and let y be an indeterminate over S. If $R \hookrightarrow S$ is extendable to an injective local homomorphism $\varphi : R[[y]] \hookrightarrow S$, then $R[[y]] \hookrightarrow S[[y]]$ is not TGF.*

Proof. Let $a := \varphi(y)$ and consider the ideal $Q = (y-a)S[[y]]$. The canonical map $S[[y]] \to S[[y]]/Q = S$ extends φ. Thus $Q \cap R[[y]] = (0)$ and $R[[y]] \hookrightarrow S[[y]]$ is not TGF. □

Example 3.13. Let $R := R_n = k[[f_1 w, \ldots, f_n w]] \hookrightarrow S := k[[z, w]]$ be as in Example 3.5 with $n \geq 2$. Define the extension $\varphi : R[[y]] \hookrightarrow S$ by setting $\varphi(y) = f_{n+1}w \in S$. By Proposition 3.7, $\varphi : R[[y]] \hookrightarrow S$ is TGF-complete. Thus by Proposition 3.12, $R[[y]] \hookrightarrow S[[y]]$ is not TGF.

Remark and Questions 3.14. Let $(R, \mathbf{m}) \hookrightarrow (S, \mathbf{n})$ be a TGF-complete extension. Assume that $[S/\mathbf{n} : R/\mathbf{m}] < \infty$ and that S is not finite over R. By [14, Theorem 8.4], $\mathbf{m}S$ is not $\mathbf{n}$-primary. Thus $\dim S > \mathrm{ht}(\mathbf{m}S)$. Therefore $\dim S > 1$, so by Proposition 2.2, $\dim R > 1$.

(1) If $(R, \mathbf{m})$ is equicharacteristic, then by Corollary 3.3, $\dim S = 2$. Is it true in general that $\dim S = 2$?
(2) Is it possible to have $\dim S - \mathrm{ht}(\mathbf{m}S) > 1$?

Examples 3.15. (1) Let $R := k[[x, xy, z]] \hookrightarrow S := k[[x, y, z]]$. We show this is not a TGF extension. By (3.9), $\varphi : k[[x, xy]] \hookrightarrow k[[x, y]]$ is TGF-complete. By Proposition 3.12, it suffices to extend φ to an injective local homomorphism of $k[[x, xy, z]]$ to $k[[x, y]]$. Let $f \in k[[x]]$ be such that x and f are algebraically independent over k, so $(1, x, f)$ is not a solution to any nonzero homogeneous form over k. As in (3.2) and (3.5), the extension of φ obtained by mapping $z \to fy$ is an injective local homomorphism.

(2) The extension $R = k[[x, xy, xz]] \hookrightarrow S = k[[x, y, z]]$ is also not a TGF-complete extension, since $R = k[[x, xy, xz]] \hookrightarrow k[[x, xy, z]] \hookrightarrow S = k[[x, y, z]]$ is a composition of two extensions that are not TGF by part (1). Now apply Proposition 2.1.

4. The case of transcendental residue extensions

In this section we address, but do not fully resolve, the following question.

Question 4.1. If $(S, \mathbf{n})$ is a TGF-complete extension of $(R, \mathbf{m})$ and if $S/\mathbf{n}$ is transcendental over $R/\mathbf{m}$ does it follow that $\dim S \leq 1$?

In Proposition 4.2 we prove every complete local domain of positive dimension has a one-dimensional TGF-complete extension.

Proposition 4.2. *Let $(R, \mathbf{m})$ be a local domain of positive dimension.*

(1) *There exists a one-dimensional complete local domain $(S, \mathbf{n})$ that is a TGF extension of R.*
(2) *If R is complete, there exists a one-dimensional TGF-complete extension of R.*

Proof. It is well known that there exists a discrete rank-one valuation domain $(S, \mathbf{n})$ that dominates R (see, for example, [3]). The $\mathbf{n}$-adic completion $\widehat{S}$ of S is a one-dimensional local ring that dominates R and each minimal prime $\mathbf{p}_i$ of $\widehat{S}$ intersects S in zero, so $\widehat{S}/\mathbf{p}_i$ is a one-dimensional complete local domain that dominates R. Moreover, if $(S, \mathbf{n})$ is a one-dimensional local domain that dominates a local domain $(R, \mathbf{m})$ of positive dimension, then it is obvious that S is a TGF extension of R, so if R and S are also complete, then S is a TGF-complete extension of R. □

Setting 4.3. Let $n \geq 2$ be an integer, let $X = \{x_1, \dots, x_n\}$ be a set of independent variables over the field k and let $R = k[[X]]$ be the formal power series ring in n variables over the field k. Let z, w, t, v be independent variables over R.

Proposition 4.4. *Let notation be as in Setting 4.3.*

(1) *There exists a TGF embedding $\theta : k[[z, w]] \to k(t)[[v]]$ defined by $\theta(z) = tv$ and $\theta(w) = v$.*
(2) *Moreover, the composition $\psi = \theta \circ \varphi$ of θ with $\varphi : R \to k[[z, w]]$ given in General Example 3.5 is also TGF.*

Proof. Suppose $f \in \ker \theta$. Write $f = \sum_{n=0}^{\infty} f_n(z, w)$, where f_n is a homogeneous form of degree n with coefficients in k. We have

$$0 = \theta(f) = \sum_{n=0}^{\infty} f_n(tv, v) = \sum_{n=0}^{\infty} v^n f_n(t, 1)\,.$$

This implies $f_n(t, 1) = 0$ for each n. Since t is algebraically independent over k, we have $f_n(z, w) = 0$ for each n. Thus $f = 0$ and θ is an embedding. Since θ is a local homomorphism and $\dim k(t)[[v]] = 1$, it is clear that θ is TGF.

For the second part, we use Proposition 2.1 together with the observation in General Example 3.5 that φ is a TGF embedding. □

As a consequence of Proposition 4.4, we prove:

Corollary 4.5. *Let $R = k[[X]]$ be as above and let $A = k(t)[[X]]$. There exists a prime ideal $P \in \operatorname{Spec} A$ in the generic fiber over R with $\operatorname{ht} P = n - 1$. In particular, the inclusion map $R = k[[X]] \hookrightarrow A = k(t)[[X]]$ is not TGF.*

Proof. Define $\varphi : R \to k[[z, w]] := S$, by

$$\varphi(x_1) = z, \quad \varphi(x_2) = h_2(w)z, \quad \dots, \quad \varphi(x_n) = h_n(w)z,$$

where $h_2(w), \dots, h_n(w) \in k[[w]]$ are algebraically independent over k. Also define $\theta : S \to k(t)[[v]] := B$ by $\theta(z) = tv$ and $\theta(w) = v$. Consider the following diagram

$$\begin{array}{ccc} R = k[[X]] & \xrightarrow{\subset} & A = k(t)[[X]] \\ \varphi\big\downarrow & & \Psi\big\downarrow \\ S = k[[z,w]] & \xrightarrow{\theta} & B = k(t)[[v]], \end{array}$$

where $\Psi : A \to B$ is the identity map on $k(t)$ and is defined by

$$\Psi(x_1) = tv, \quad \Psi(x_2) = h_2(v)tv, \quad \dots, \quad \Psi(x_n) = h_n(v)tv.$$

Notice that $\Psi|_R = \psi = \theta \circ \varphi$. Therefore the diagram is commutative. Let $P = \ker \Psi$. Since Ψ is surjective, $\operatorname{ht} P = n-1$. Commutativity of the diagram implies that $P \cap R = (0)$. □

Discussion 4.6. Let us describe generators for the prime ideal $P = \ker \Psi$ given in Corollary 4.5. Under the map Ψ, $x_1 \mapsto tv$, and so $\frac{x_1}{t} \mapsto v$. Since also $x_2 \mapsto h_2(v)tv, \dots, x_n \mapsto h_n(v)tv$, we see that

$$(x_2 - h_2(\frac{x_1}{t})x_1,\, x_3 - h_3(\frac{x_1}{t})x_1, \dots, x_n - h_n(\frac{x_1}{t})x_1)A \subseteq P$$

(that is, $\Psi(x_2 - h_2(x_1/t)x_1) = h_2(v)tv - h_2(v)tv = 0$, etc.) Since the ideal on the left-hand-side is a prime ideal of height $n-1$, the inclusion is an equality. Thus we have generators for the prime ideal $P = \ker \Psi$ resulting from the definitions of φ and θ given in the corollary.

On the other hand, in Corollary 4.5 if we change the definition of θ and we define $\theta' : k[[z,w]] \to k(t)[[v]]$ by $\theta'(z) = v$ and $\theta'(w) = tv$ (but we keep φ as above), then ψ' defined by $\psi'|_R = \theta' \cdot \varphi$ maps $x_1 \to v$, $x_2 \to h_2(tv)v$, $\dots$, $x_n \to h_n(tv)v$. In this case

$$(x_2 - h_2(tx_1)x_1,\, x_3 - h_3(tx_1)x_1, \dots, x_n - h_n(tx_1)x_1)A \subseteq \ker \Psi' = P'.$$

Again the ideal on the left-hand-side is a prime ideal of height $n-1$, so we have equality. This yields a different prime ideal P'.

In this case one can also see directly for

$$P' = (x_2 - h_2(tx_1)x_1,\, x_3 - h_3(tx_1)x_1, \dots, x_n - h_n(tx_1)x_1)A$$

that $P' \cap R = (0)$. We have $\Psi : A \to A/P' = k(t)[[v]]$. Suppose $f \in R \cap P'$. We write $f = \sum_{\ell=0}^{\infty} f_\ell(x_1, \dots, x_n)$, where $f_\ell \in k[x_1, \dots, x_n]$ is a homogeneous form of degree ℓ. We have

$$0 = \Psi'(f) = \sum_{\ell=0}^{\infty} f_\ell(v, h_2(tv)v, \dots, h_n(tv)v) = \sum_{\ell=0}^{\infty} v^\ell f_\ell(1, h_2(tv), \dots, h_n(tv)).$$

This implies $f_\ell(1, h_2(tv), \dots, h_n(tv)) = 0$ for each ℓ. Since $h_2, \dots, h_n$ are algebraically independent over k, each of the homogeneous forms $f_\ell(x_1, \dots, x_n) = 0$. Hence $f = 0$.

QUESTION 4.7. With notations as in Corollary 4.5, does every prime ideal of A maximal in the generic fiber over R have height $n-1$?

THEOREM 4.8. *Let $(A, \mathbf{m}) \hookrightarrow (B, \mathbf{n})$ be an extension of two-dimensional regular local domains. Assume that B dominates A and that $B/\mathbf{n}$ as a field extension of $A/\mathbf{m}$ is not algebraic. Then $A \hookrightarrow B$ is not TGF.*

Proof. Since $\dim A = \dim B$, the assumption that $B/\mathbf{n}$ is transcendental over $A/\mathbf{m}$ implies that B is not algebraic over A [14, Theorem 15.5]. If $\mathbf{m}B$ is $\mathbf{n}$-primary, then B is faithfully flat over A [14, Theorem 23.1], and [6, Theorem 1.12] implies that $A \hookrightarrow B$ is not TGF in this case.

If $\mathbf{m}B$ is principal, then $\mathbf{m}B = xB$ for some $x \in \mathbf{m}$ since B is local. It follows that $\mathbf{m}/x \subset B$. Localizing $A[\mathbf{m}/x]$ at the prime ideal $\mathbf{n} \cap A[\mathbf{m}/x]$ gives a local quadratic transform $(A_1, \mathbf{m}_1)$ of A. If $\dim A_1 = 1$, then $A_1 \hookrightarrow B$ is not TGF because only finitely many prime ideals of B can contract to the maximal ideal of A_1. Hence $A \hookrightarrow B$ is not TGF if $\dim A_1 = 1$. If $\dim A_1 = 2$, then $(A_1, \mathbf{m}_1)$ is a 2-dimensional regular local domain dominated by $(B, \mathbf{n})$ and the field $A_1/\mathbf{m}_1$ is finite algebraic over $A/\mathbf{m}$, and so $B/\mathbf{n}$ is transcendental over $A_1/\mathbf{m}_1$. Thus we can repeat the above analysis: If $\mathbf{m}_1 B$ is $\mathbf{n}$-primary, then as above $A \hookrightarrow B$ is not TGF. If $\mathbf{m}_1 B$ is principal, we obtain a local quadratic transform $(A_2, \mathbf{m}_2)$ of A_1. If this process does not end after finitely many steps, we have a union $V = \bigcup_{n=1}^{\infty} A_n$ of an infinite sequence A_n of quadratic transforms of a 2-dimensional regular local domains. Then V is a valuation domain of rank at most 2 contained in B, and so at most finitely many of the height-one primes of B have a nonzero intersection with V. Therefore $V \hookrightarrow B$ is not TGF and hence also $A \hookrightarrow B$ is not TGF.

Thus by possibly replacing A by an iterated local quadratic transform A_n of A, we may assume that $\mathbf{m}B$ is neither $\mathbf{n}$-primary nor principal. Let $\mathbf{m} = (x, y)A$. There exist $f, g, h \in B$ such that $x = gf, y = hf$ and g, h is a regular sequence in B. Hence $(g, h)B$ is $\mathbf{n}$-primary. Let $f = f_1^{e_1} \cdots f_r^{e_r}$, where $f_1 B, \dots f_r B$ are distinct height-one prime ideals and the e_i are positive integers. Then $f_1 B, \dots, f_r B$ are precisely the height-one primes of B that contain $\mathbf{m}$.

Let $t \in B$ be such that the image to t in $B/\mathbf{n}$ is transcendental over $A/\mathbf{m}$. Modifying t if necessary by an element of $\mathbf{n}$ we may assume that t is transcendental over A. We have $\mathbf{n} \cap A[t] = \mathbf{m}[t]$. Let $A(t) = A[t]_{\mathbf{m}[t]}$. Notice that $A(t)$ is a 2-dimensional regular local domain with maximal ideal $\mathbf{m}A(t)$ that is dominated by $(B, \mathbf{n})$. We have

$$A \hookrightarrow A[t] \hookrightarrow A(t) \hookrightarrow B.$$

Let $P = (xt - y)A(t)$. Then $P \cap A = (0)$. We have $PB = (gft - hf)B = f(gt - h)B$. Also $gt - h$ is a nonunit of B. Let Q be a minimal prime of $(gt - h)B$. Then $Q \notin \{f_1 B, \dots, f_r B\}$. Hence $\mathbf{m}A(t) \not\subseteq Q$. Therefore $Q \cap A(t)$

has height one. Since $P \subseteq (gt-h)B \subseteq Q$, we have $Q \cap A(t) = P$. Thus $Q \cap A = (0)$. This completes the proof. □

We have the following immediate corollary to Theorem 4.8.

COROLLARY 4.9. *Let x, y, z, w, t be indeterminates over the field k and let*

$$\varphi : R = k[[x,y]] \hookrightarrow S := k(t)[[z,w]]$$

be an injective local k-algebra homomorphism. Then $\varphi(R) \hookrightarrow S$ is not TGF.

In relation to Question 4.1, Example 4.10 is a TGF extension $A \hookrightarrow B$ that is not complete for which the residue field of B is transcendental over that of A and $\dim B = 2$.

EXAMPLE 4.10. Let $A = k[x,y,z,w]_{(x,y,z,w)}$, where k is a field and $xw = yz$. Thus A is a 3-dimensional normal local domain with maximal ideal $\mathbf{m} := (x,y,z,w)A$ and residue field $A/\mathbf{m} = k$. Notice that $C := A[y/x] = k[y/x = w/z, x, z]$ is a polynomial ring in 3 variables over k. Thus $B := C_{(x,z)}$ is a 2-dimensional regular local domain with maximal ideal $\mathbf{n} = (x,z)B$. Notice that $(B,\mathbf{n})$ birationally dominates $(A,\mathbf{m})$. Hence $(A,\mathbf{m}) \hookrightarrow (B,\mathbf{n})$ is a TGF extension. Also $B = k(y/x)[x,z]_{(x,z)}$, so $k(y/x)$ is a coefficient field for B. The image t of y/x in $B/\mathbf{n}$ is transcendental over k and $B/\mathbf{n} = k(t)$. The completion of A is the normal local domain $\widehat{A} = k[[x,y,z,w]]$, where $xw = yz$. By a form of Zariski's subspace theorem [1, (10.6)], $\widehat{A}$ is dominated by $\widehat{B} = k(t)[[x,z]]$. Thus we have $\varphi : \widehat{A} \hookrightarrow \widehat{B}$, where $\varphi(x) = x, \varphi(z) = z, \varphi(y/x) = t = \varphi(w/z)$ and so also $\varphi(y) = tx, \varphi(w) = tz$, $\varphi(xw) = xtz = \varphi(yz)$.

In Example 4.10, $\widehat{A} \hookrightarrow \widehat{B}$ is not a TGF-complete extension. Equivalently, the inclusion map

$$R := k[[x,z,tx,tz]] \hookrightarrow k(t)[[x,z]] := S$$

is not a TGF-extension. We hope to expand on this in a future publication.

References

[1] S. S. Abhyankar, *Resolution of singularities of embedded algebraic surfaces*, Pure and Applied Mathematics, Vol. 24, Academic Press, New York, 1966. MR 0217069 (36 #164)

[2] S. S. Abhyankar and T. T. Moh, *On analytic independence*, Trans. Amer. Math. Soc. **219** (1976), 77–87. MR 0414546 (54 #2647)

[3] C. Chevalley, *La notion d'anneau de décomposition*, Nagoya Math. J. **7** (1954), 21–33. MR 0067866 (16,788g)

[4] I. S. Cohen, *On the structure and ideal theory of complete local rings*, Trans. Amer. Math. Soc. **59** (1946), 54–106. MR 0016094 (7,509h)

[5] A. Grothendieck, *Élements de Géométrie Algébrique. IV*, Inst. Hautes Études Sci. Publ. Math. **24** (1965).

[6] W. Heinzer and C. Rotthaus, *Formal fibers and complete homomorphic images*, Proc. Amer. Math. Soc. **120** (1994), 359–369. MR 1189544 (94d:13020)

[7] W. Heinzer, C. Rotthaus, and S. Wiegand, *Noetherian domains inside a homomorphic image of a completion*, J. Algebra **215** (1999), 666–681. MR 1686210 (2000e:13031)
[8] ______, *Building Noetherian and non-Noetherian integral domains using power series*, Ideal theoretic methods in commutative algebra (Columbia, MO, 1999), Lecture Notes in Pure and Appl. Math., vol. 220, Dekker, New York, 2001, pp. 251–264. MR 1836605 (2002b:13031)
[9] ______, *Examples of integral domains inside power series rings*, Commutative ring theory and applications (Fez, 2001), Lecture Notes in Pure and Appl. Math., vol. 231, Dekker, New York, 2003, pp. 233–254. MR 2029829 (2004k:13037)
[10] ______, *Building Noetherian domains inside an ideal-adic completion*, Abelian groups, module theory, and topology (Padua, 1997), Lecture Notes in Pure and Appl. Math., vol. 201, Dekker, New York, 1998, pp. 279–287. MR 1651173 (99m:13035)
[11] ______, *Generic fiber rings of mixed power series/polynomial rings*, J. Algebra **298** (2006), 248–272. MR 2215127 (2006m:13013)
[12] ______, *Mixed polynomial/power series rings and relations among their spectra*, Multiplicative ideal theory in commutative algebra: a tribute to the work of Robert Gilmer, (J. Brewer, S. Glaz, W. Heinzer, B. Olberding, eds.), Springer, New York, 2006, pp. 227–242. MR 2265811
Volume in honor of Robert Gilmer, to appear.
[13] H. Matsumura, *On the dimension of formal fibres of a local ring*, Algebraic geometry and commutative algebra, Vol. I, Kinokuniya, Tokyo, 1988, pp. 261–266. MR 977763 (90a:13027)
[14] ______, *Commutative ring theory*, second ed., Cambridge Studies in Advanced Mathematics, vol. 8, Cambridge University Press, Cambridge, 1989. MR 1011461 (90i:13001)
[15] M. Nagata, *Local rings*, Interscience Tracts in Pure and Applied Mathematics, No. 13, Interscience Publishers, John Wiley & Sons, New York-London, 1962. MR 0155856 (27 #5790)
[16] C. Rotthaus, *On rings with low-dimensional formal fibres*, J. Pure Appl. Algebra **71** (1991), 287–296. MR 1117639 (92j:13006)
[17] P. B. Sheldon, *How changing $D[[x]]$ changes its quotient field*, Trans. Amer. Math. Soc. **159** (1971), 223–244. MR 0279092 (43 #4818)
[18] T. Yasuda, *On subschemes of formal schemes*, preprint, `arXiv:math.AG/0602542`.
[19] O. Zariski and P. Samuel, *Commutative algebra. Vol. II*, The University Series in Higher Mathematics, D. Van Nostrand Co., Inc., Princeton, N. J.-Toronto-London-New York, 1960. MR 0120249 (22 #11006)

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