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The Introduction of Proof in Secondary Geometry Textbooks

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Abstract

Explicit reasoning-and-proving opportunities in the United States are often relegated to a single secondary geometry course. This study analyzed the reasoning-and-proving opportunities in six U.S. geometry textbooks, giving particular attention to the chapter that introduced proof. Analysis focused on the types of reasoning-and-proving activities expected of students and the type of mathematical statement around which the reasoning-and-proving took place, be it general or particular. Results include the fact that reasoning-and-proving opportunities in student exercises were predominantly of the particular type, whereas textbook exposition most commonly had general statements. Within the chapters introducing proof, opportunities for students to develop proofs were less common than exercises involving conjectures and statements or exercises about the reasoning-and-proving process. Opportunities to reflect on the reasoning-and-proving process were prevalent in the introduction chapters, though rare in the remainder of the books.

Keywords: Reasoning, Proof, Geometry, Secondary, Textbook analysis

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1. Introduction

Proof has a central role in the discipline of mathematics (Hanna, 2000; Hersh, 2009) and represents, for students of school mathematics, a formal culmination of the processes of sense-making and justification (National Council of Teachers of Mathematics [NCTM], 2009) that, ideally, began at an early age. Moreover, proof and justification processes in school mathematics can not only function to explain and verify but also as a means for fostering valuable skills and dispositions in students and providing formative assessment information for teachers (Staples, Bartlo, & Thanheiser, 2012). Because of these beneficial roles of proof, efforts have been undertaken in different parts of the world to integrate proving more fully into school curricula. In the United States, for example, NCTM has made recent efforts through policy publications and teacher professional development efforts (NCTM, 2009, 2011) to promote reasoning and sense-making for students of all ages. They point out that the exact form reasoning and sense-making takes depends upon classroom contexts and varies along a continuum from informal observations and explanations to formal conjectures and logical arguments, with mathematical proof constituting the formal endpoint of reasoning. The hyphenated term "reasoning-and-proving" (Stylianides, 2009) has arisen to capture the fact that the process of proving is much larger than proof itself, consisting of empirical explorations, conjecturing, justifying, refining, explaining, and so forth. With this broad notion of reasoning-and-proving, researchers have been able to examine reasoning-and-proving practices in various grade levels (e.g., Bieda, 2010; Stylianides, 2007; Stylianou, Blanton, & Knuth, 2009).

Because reasoning-and-proving is such a broad practice, however, it is possible that students do not recognize their informal reasoning experiences as being related to proof. For example, students who explain how they know their solution is correct may not realize that this explanation could essentially prove their result. Evidence of such disconnects has been found in studies where students had limited conceptions of the role of proof in mathematics (Herbst & Brach, 2006; Schoenfeld, 1988; Soucy McCrone & Martin, 2009). This potential disconnect between broad reasoning-and-proving processes and those activities under the specific label of proof points to the importance of the way in which mathematical proof is explicitly introduced to students. Is proof introduced in ways that connect to students' past experiences, allowing them to reflect on how their previous processes of sense-making, justification, and explanation relate to the more formal process of proof? Does the introduction of proof highlight how all of these processes can be powerful sources of understanding (de Villiers, 1995; Hanna, 1990) and lay the foundation for students' future experiences with reasoning-and-proving?

One of the most important aspects of this foundation for students' future experiences is that students are guided to see the intellectual necessity of transitioning from more informal ways of reasoning to formal mathematical proof, and in particular, deductive argumentation. Harel and Tall (1991) articulated the *necessity principle* as a way to think about this issue in general. The necessity principle states that subject matter should be presented in ways that encourage learners to see its intellectual necessity, "[f]or if students do not see the rationale for an idea, the idea would seem to them as being evoked arbitrarily; it does not become a concept *of the students*" (p. 41, emphasis in original). There is evidence, however, that students in many countries often fail to see the intellectual impetus behind proof in mathematics (Chazan, 1993; Fujita, Jones, & Kunimune, 2009; Harel & Sowder, 2007; Porteous, 1990; Soucy McCrone & Martin, 2009).

The explicit introduction of proof typically occurs in a secondary-level geometry course, both in the United States (Herbst, 2002) and elsewhere (Hanna & de Bruyn, 1999; Jones, Fujita, Clarke, & Lu, 2008). This study examines the chapters in geometry textbooks that introduce proof and gives particular attention to the ways in which the reasoning-and-proving opportunities might support students in seeing the necessity of deductive forms of reasoning. Details about how we have operationalized this dimension of analysis are contained in Section 2.2. In a past study (Otten, Gilbertson, Males, & Clark, 2011), we characterized in aggregate the reasoning-and-proving opportunities in secondary-level geometry textbooks. Here, we focus on the following research questions:

1. What is the nature of reasoning-and-proving opportunities in the textbook chapters that introduce proof?
2. How do the reasoning-and-proving opportunities in these chapters compare to the reasoning-and-proving opportunities in the remainder of the textbooks?

3. What opportunities do the chapters that introduce proof provide for students to make reasoning-and-proving an object of attention or reflection?

The site for this study is student editions of geometry textbooks published for use in the United States. Although we analyzed United States textbooks, the issues raised are international in scope. In any nation, it is worthwhile to carefully consider the ways in which a process such as reasoning-and-proving, which is integral to what it means to do mathematics, is treated during the explicit shift toward deductive reasoning. Even if other textbooks do not have all the same characteristics as those in the present study, much may still be learned by comparing and contrasting other textbooks with those here and clarifying where the differences lie and why they might be important. With that being said, there are also known similarities with respect to reasoning-and-proving between textbooks from different countries. For example, similar to arguments made about textbooks in the United States, Fujita et al. (2009) found that Japanese textbooks treat proof in a way that does not "illustrate convincingly for students the difference between formal proof and experimental verification" (p. 176).

As Stylianides argued in the editorial of this issue, curriculum materials are an important factor in students' educational experiences (Grouws, Smith, & Sztajn, 2004; McCrory, Francis, & Young, 2008; Ni & Cai, 2011), and although teachers report substantial reliance on mathematics textbooks (Banilower et al., 2013), we recognize that teachers mediate the influence of those materials in important ways (Cohen, Raudenbush, & Ball, 2003; Tarr, Chavez, Reys, & Reys, 2006). With regard to reasoning-and-proving in textbooks, several past studies have focused on students' opportunities to engage in reasoning-and-proving activities, such as conjecturing, finding a counterexample, or proving (e.g., Davis, 2010; Stylianides, 2009; Thompson, Senk, & Johnson, 2012). These studies have documented that reasoning-and-proving opportunities outside of geometry are limited and insufficiently robust in guiding students to develop deep understanding of reasoning-and-proving, unless teachers fill in substantial gaps.

2. Methods

2.1. Textbook sample

This study involved six U.S. textbooks (see Table 1) designed for stand-alone geometry courses at the secondary level (i.e., texts meant to be used for a course covering geometry content only, as opposed to an integrated mathematics course), for students 13–16 years old. The textbook series that include these geometry texts constitute the mathematics textbooks used by approximately 90% of the U.S. secondary student population (Dossey, Halvorsen, & Soucy McCrone, 2008; Banilower et al., 2013).

Within each chapter, the textbooks contained numbered sections, which we call *lessons*, and each lesson contained expository text as well as student exercises. These lessons were the primary focus of analysis, although we also included in the analysis a section from the end of each chapter that revisited the content from each of the lessons in the chapter. We refer to these latter sections as chapter reviews. Supplementary activities outside of the canonical lessons, such as

Table 1. Geometry textbooks analyzed in this study.

*Title**	*Publisher*	*Authors*	*Year*
Geometry (CME)	Pearson Prentice Hall	CME Project	2009
Geometry (Glencoe)	Glencoe McGraw Hill	Carter, Cuevas, Day, Malloy, & Cummins	2010
Geometry (Holt)	Holt	McDougal Burger, Chard, Kennedy, Leinwand, Renfro, Roby, Seymour, & Waits	2011
Discovering Geometry (Key) 2008		Key Curriculum Press	Serra
Geometry (Prentice)	Pearson Prentice Hall	Bass, Charles, Hall, Johnson, & Kennedy	2009
Geometry (UCSMP)	Wright Group McGraw Hill	Benson, Klein, Miller, Capuzzi-Feuerstein, Fletcher, Marino, Powell, Jakucyn, & Usiskin	2009

* The term in parenthesis will be used to refer to each textbook.

explorations, hands-on activities, extensions, or connections to technology, were excluded from analysis.

To determine the chapters that explicitly introduced mathematical proof, we noted that across all six textbooks there was a difference between the content and focus of Chapter 2 and the other chapters in the textbook. The titles for Chapter 2 explicitly included the word "proof" or "reasoning" (or both), with the exception of UCSMP, whose chapter was entitled, *The Language and Logic of Geometry*. All six textbooks formally introduced students to proof in Chapter 2 and, with the exception of UCSMP, all explicitly included a lesson on deductive reasoning. These chapters included multiple lessons related to reasoning and logic on topics such as conditional statements, inductive reasoning, the use of definitions, and conjecturing. This is not to say that the introduction of proof was entirely contained within Chapter 2 for each textbook, but it was the first predominant site for the notion of proof. Although most textbooks in the sample predominantly focused on developing important ideas related to proof in Chapter 2, there was a notable exception of UCSMP that waited until Chapter 5 to develop additional concepts related to proving. (See Fujita & Jones, this volume, for an analysis of reasoning-and-proving in the triangle congruence chapters of geometry textbooks.) Thus, we divided each textbook into two components—the *introduction to proof chapter* that explicitly introduces proof (i.e., Chapter 2) and the remainder of the book, including Chapter 1.

Within the introduction to proof chapters, all canonical lessons and the selected chapter reviews were coded using the framework and procedures described below. Within the remainder of the textbooks, we used a stratified random sample to select lessons for analysis, with the textbook chapters serving as strata. The random samples included a minimum of 30% of the lessons from each chapter. Our sampling procedure treated all chapters and lessons equally, even though they may not all have the same likelihood of being enacted (Tarr et al., 2006). This choice is reflective of the fact that we are characterizing the opportunities to reason-and-prove as present in the textbooks themselves, not in classroom enactment. Table 2 contains information about the sampled lessons and chapter reviews.

Table 2. Number of lessons and chapter reviews analyzed in each textbook.

	Introduction to proof chapter		*Remaining chapters*	
Textbook	*Lessons*	*Chapter reviews*	*Lessons*	*Chapter reviews*
CME	18	1	32	7
Glencoe	8	1	30	12
Holt	7	1	28	11
Key	6	1	36	13
Prentice	5	1	29	11
UCSMP	7	1	38	13

Note: CME had more lessons per chapter, but roughly the same number of pages per chapter, when compared with the other textbooks.

2.2. Analytic framework

Following the recommendations of other textbook analysts (Li, 2000; Thompson, Senk, & Johnson, 2012), we attended to both textbook exposition and student exercises because both aspects of a textbook contribute to the potential opportunities students have to engage with reasoning-and-proving. By textbook exposition, we mean the paragraphs of narrative text in the body of a lesson including worked examples, as well as the text-boxes that contain formulas, theorems, or key ideas. By student exercise, we mean a numbered item in which students are expected to take an active role by answering a question, solving a problem, or completing a particular task. The majority of student exercises were located in a separate section of a lesson (i.e., an exercise set) after the textbook exposition, but there were also instances of student exercises interspersed with expository text.

Reid and Knipping (2010) have pointed out that it is beneficial for researchers to be explicit about how they use terms related to reasoning-and-proving. In an attempt to heed this call, we clarify that we use the term *claim* to mean a statement with a truth-value and *conjecture* to mean a claim whose truth-value is unknown in a given context, though there may be some evidence for or against it. The term *argument* refers to a series of reasons intended to support the truth or falsehood of a claim and the term *proof* refers to an argument that proceeds deductively and is valid. In this way, *proof* is being used as an object whereas *reasoning-and-proving* refers to a broad

	Exposition	Student Exercises	
	Properties, Theorems, or Claims	Related to Mathematical Claims	Related to Mathematical Arguments
Mathematical Statement or Situation	• General • Particular	• General • Particular • General with particular instantiation provided	
Expected Student Activity		• Make a conjecture, refine a statement, or draw a conclusion • Fill in the blanks of a conjecture • Investigate a conjecture or statement	• Construct a proof • Develop a rationale or other non-proof argument • Outline a proof or construct a proof given an outline • Fill in the blanks of an argument or proof • Evaluate or correct an argument or proof • Find a counterexample
Attending to or Reflecting on Reasoning-and-Proving	• Statements *about* reasoning-and-proving	• Exercises *about* reasoning-and-proving	

Fig. 1. An analytic framework for reasoning-and-proving in geometry textbooks.

set of processes. Other terms, such as *rationale* or *general*, are defined through their operationalization in the framework.

The analytic framework for this study (see Fig. 1) was developed from that of Thompson et al. (2012) with influences from Stylianides (2009), regarding non-proof arguments, and Hanna and de Bruyn (1999), regarding reflection on reasoning-and-proving. The full framework is described in detail elsewhere (Otten, Gilbertson, Males, & Clark, in press) and includes an additional dimension of justification type that is omitted in the present study.

First, based on the necessity principle (Harel & Tall, 1991), the framework includes a dimension for mathematical statement or situation, with codes *general*, *particular*, and *general with particular instantiation provided*. A general statement is one made about an entire class of mathematical objects or situations. For example, claims about all parallel lines or about all isosceles triangles are general in nature. A particular statement, in contrast, concerns a specific mathematical object or situation (or a finite subset of an infinite set of objects). Two examples of particular statements are depicted in Fig. 2. In student exercises, it is also possible for an item to concern a general statement but for a particular instantiation to be provided, thus allowing the student to treat the item as particular in nature. Fig. 3 contains an example of this type of situation. The *general with particular instantiation provided* category was treated separately, rather than as a subset, of the *general* category. Furthermore, it was not considered to be

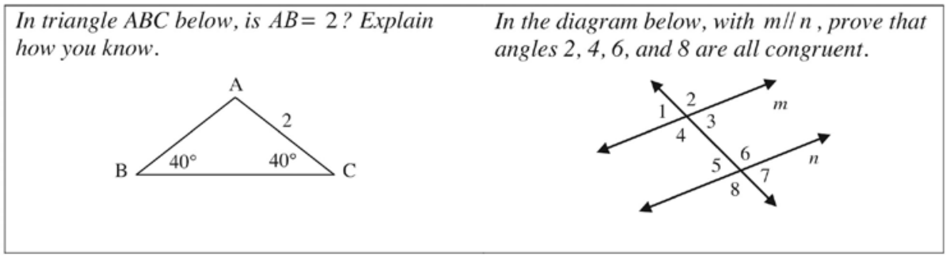

Fig. 2. Reasoning-and-proving opportunities with particular mathematical statements.

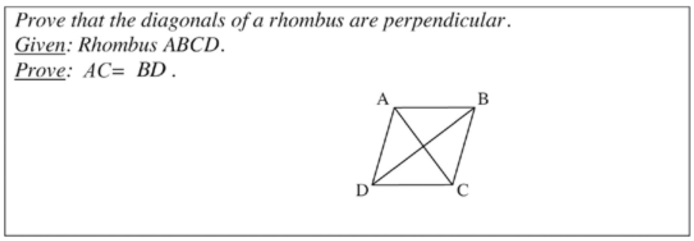

Fig. 3. A reasoning-and-proving opportunity with a general statement and particular instantiation provided.

a code within textbook exposition because, when reasoning around a general claim, providing a particular instantiation for oneself is a normal part of such reasoning, in contrast with the case of student exercises where the textbook authors are essentially providing the particular instantiation for the student. Because general statements cannot be proven empirically, they, as a category of reasoning-and-proving opportunities, have a greater likelihood of adhering to the necessity principle. Moreover, general statements require the logical step of universal generalization (Hurley, 2006) in order to proceed deductively, and in this way necessitate deductive reasoning to a greater extent than the other types of statements.

Second, as with past work (e.g., Thompson et al., 2012), the framework distinguishes between several types of expected student activity. The *construct a proof* code is reserved for instances that explicitly call for a proof and so are distinct from the *develop a rationale or other non-proof argument* code, which applies to exercises where students are asked to justify or explain why a claim is true or false but does not explicitly call for a proof. Other codes related to mathematical arguments involve something short of the students constructing a full proof, either by working from a given proof outline or only being asked to provide a proof outline, by filling in certain claims or justifications missing from a given proof, or evaluating a given argument. We used a separate code for exercises that asked students to supply a counterexample because, although counterexamples can be used to prove that something is not the case, they are pedagogically different than standard proof exercises and most counterexample items in the textbooks did not explicitly require students to explain how their counterexample proved or disproved the claim. With regard to opportunities related to mathematical claims, we coded for exercises that asked students to generate a conjecture or refine a given statement into something the student believed to be true, for exercises that supplied a portion of a conjecture and asked students to fill it in, and for exercises that asked students to investigate a given conjecture, making a judgment regarding the truth-value of a claim. When asked to investigate, students may also be required to supply a rationale or proof to support that judgment. If such a justification is required, then a second code (e.g., construct a proof) was used together with the *investigate* code.

Third, the framework includes a code for opportunities that students have to think *about reasoning-and-proving* in both the textbook exposition and the student exercises. These opportunities are characterized by statements or exercises that take the reasoning-and-proving process as an explicit object of attention or reflection. For example, expository text may describe the limitations of a particular type of logical argument or an exercise might ask students to write about the benefits and drawbacks of various formats of written proofs.

2.3. Analytic procedures

Within each sampled lesson, we identified any expository mathematical statements such as theorems, postulates, properties, formulas, or identities. Mathematical definitions were not coded, but worked examples that included a prompt for reasoning-and-proving were. The statements included in the analysis were then coded for their mathematical statement and their justification-type; however, because justification-type results are not presented in this article they are not discussed any further. A reliability check on a 20% sub-sample of lessons yielded 95% agreement on statement-type within exposition sections.

For student exercises in the sampled lessons and chapter reviews, the reasoning-and-proving framework was taken to be exhaustive of the possible types of reasoning-and-proving items. Exercises that involved at least one of the reasoning-and-proving activities were coded for that expected student activity as well as the type of mathematical statement or situation involved. The unit of analysis was items as partitioned by numbers or letters in the textbook. So, for example, an exercise with parts a–d was considered as four separate items. A single item, however, could receive multiple codes if it involved multiple reasoning-and-proving activities. The reliability check in the case of student exercises yielded 92% agreement on mathematical statement and 93% agreement on expected student activity. Within textbook exposition and student exercises, we coded statements or exercises *about* the practice of reasoning-and-proving. The unit of analysis in the exposition was the theme, not the word or sentence, and in the exercises it was the numbered items as before.

To compare introduction to proof chapters and the remaining chapters of each book, expository and exercise codes were tabulated separately and descriptive statistics were compared for trends as well as between- and within-textbook differences. Because our prior analysis revealed limitations in students' opportunities to actively reason with general claims, we began our comparative analysis by looking at statement-types. We also investigated differences in the expected student activities present in exercises and the opportunities to think *about* reasoning-and-proving.

3. Results

To give an overall sense of the reasoning-and-proving opportunities, Fig. 4 shows the percentage of reasoning-and-proving exercises by chapter for each textbook. One can see that relatively high percentages of reasoning-and-proving opportunities existed in Chapters 2–7, with percentages declining for Chapters 8 and beyond. Partial exceptions to this trend are CME, Key, and UCSMP, which had fairly high percentages of reasoning-and-proving in their final chapters—Chapters 8, 13, and 14, respectively. The other textbooks' final chapters were 13 for Glencoe, 12 for Holt, and 12 for Prentice.

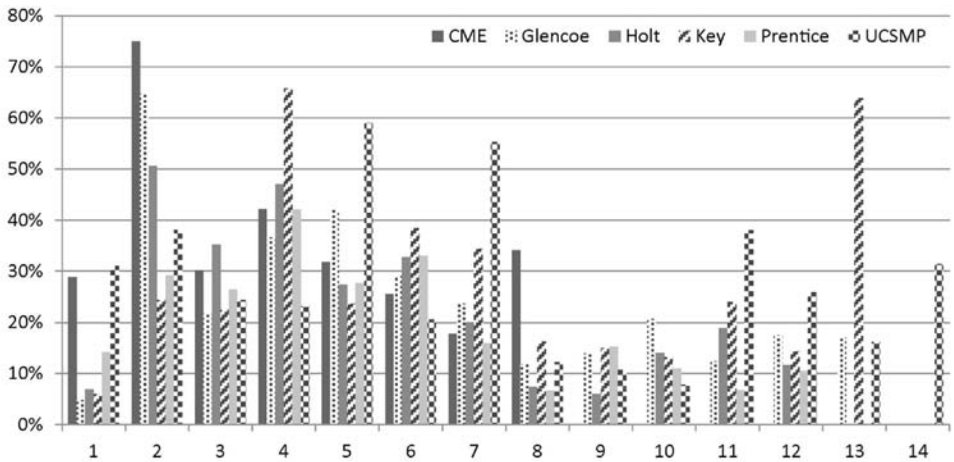

Fig. 4. Percent of reasoning-and-proving exercises by chapter. *Note*: Chapter 0 from Glencoe and Key were omitted because they did not contain reasoning-and-proving exercises in our sampled sections.

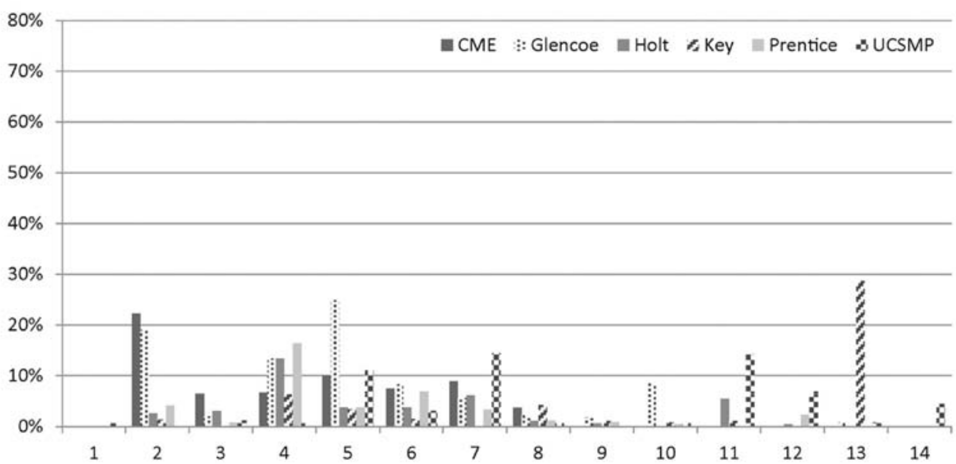

Fig. 5. Percent of exercises, by chapter, that involve constructing a proof.

In all six textbooks, as explained above, Chapter 2 contained the formal introduction of the notion of *proof*. Fig. 5 traces the *construct a proof* code throughout the chapters of the textbooks. Even though Chapter 2 contained the introduction of proof, it was not until Chapters 4–7 that many of the books reached their highest levels of proof-construction opportunities. Only CME and Glencoe had more than 5% of the exercises in Chapter 2 involve constructing a proof. Additionally, in correspondence with their overall reasoning-and-proving percentages mentioned above, UCSMP and Key also had relatively high percentages of proof opportunities in later chapters, whereas most textbooks had few proof opportunities after Chapter 8.

In the next sections, we go into more detail on the opportunities depicted in Figs. 4 and 5. The structure follows the three research questions. First, we present detailed results for the introduction to proof chapters of each textbook and then compare these results to those found in the remaining chapters. To address the third research question, we share examples of statements or exercises *about* reasoning-and-proving.

3.1. The nature of reasoning-and-proving in the chapters that introduce proof

3.1.1. Exposition

The frequencies of each type of reasoning-and-proving statement appearing in the textbook exposition of the proof chapters are presented in Table 3. The textbooks varied noticeably in their total numbers of reasoning-and-proving statements, yet, in the proof chapters of all six textbooks, reasoning-and-proving statements tended to more often involve general situations than particular ones, except in UCSMP where the difference between general and particular was negligible. There was wide variability in the ratio of general to particular statements from nearly equal amounts (UCMSP) to nearly four times as many (Prentice) and there were substantial numbers of statements *about* reasoning-and-proving, which are addressed in detail in Section 3.3.

Table 3. Types of statements in textbook exposition of the introduction to proof chapters.

Textbook	*No. of general statements (%)*	*No. of particular statements (%)*	*No. of statements about reasoning-and-proving (%)*	*Total*
CME	31 (36)	14 (16)	40 (47)	85
Glencoe	45 (44)	31 (30)	26 (25)	102
Holt	40 (49)	14 (17)	27 (33)	81
Key	10 (32)	5 (16)	16 (52)	31
Prentice	22 (54)	6 (15)	13 (32)	41
UCSMP	6 (29)	5 (24)	10 (48)	21

Table 4. Frequency of reasoning-and-proving exercises in the introduction to proof chapters.

Textbook	*No. of lessons analyzed*	*No. of exercises analyzed*	*No. of reasoning-and-proving exercises (%)*	*Average no. of reasoning-and-proving exercises per lesson*
CME	19	468	298 (64)	15.7
Glencoe	9	593	383 (65)	42.6
Holt	8	482	242 (50)	30.3
Key	7	205	77 (27)	11.0
Prentice	6	357	136 (38)	22.7
UCSMP	8	334	103 (31)	12.9

Note: Lessons in this table include chapter reviews.

3.1.2. Student exercises

Because the focus of many lessons in the introduction to introduction to proof chapters was on developing students' understanding of reasoning-and-proving, one might expect to find a majority of homework exercises related to this topic. This expectation was confirmed in some, but not all, cases (see Table 4). In Glencoe and CME, for example, nearly two-thirds of student exercises in the introduction to proof chapters involved reasoning-and-proving. In contrast, Key, UCSMP, and Prentice, had less than half of their exercises involve reasoning-and-proving. The number of reasoning-and-proving exercises per lesson varied, largely due to the fact that the textbooks varied in the total number of exercises per lesson.

None of this is to say that one textbook is more or less deficient than another in their handling of reasoning-and-proving in student exercises. Rather, the variation simply highlights that the textbook authors seemed to take different approaches in their introduction to proof chapters. To uncover more about these approaches, we turn to the reasoning-and- proving activities expected of students. Fig. 6 indicates the frequency of various types of reasoning-and-proving activities within the introduction to proof chapter of each textbook, with percentages taken out of only those exercises involving reasoning-and-proving. One can see that many exercises involve students working with conjectures, either by investigating conjectures, making conjectures, or occasionally filling in the blanks of given conjectures. Overall, the introduction to proof chapters were more focused on having students make or explore conjectures than writing proofs. Only

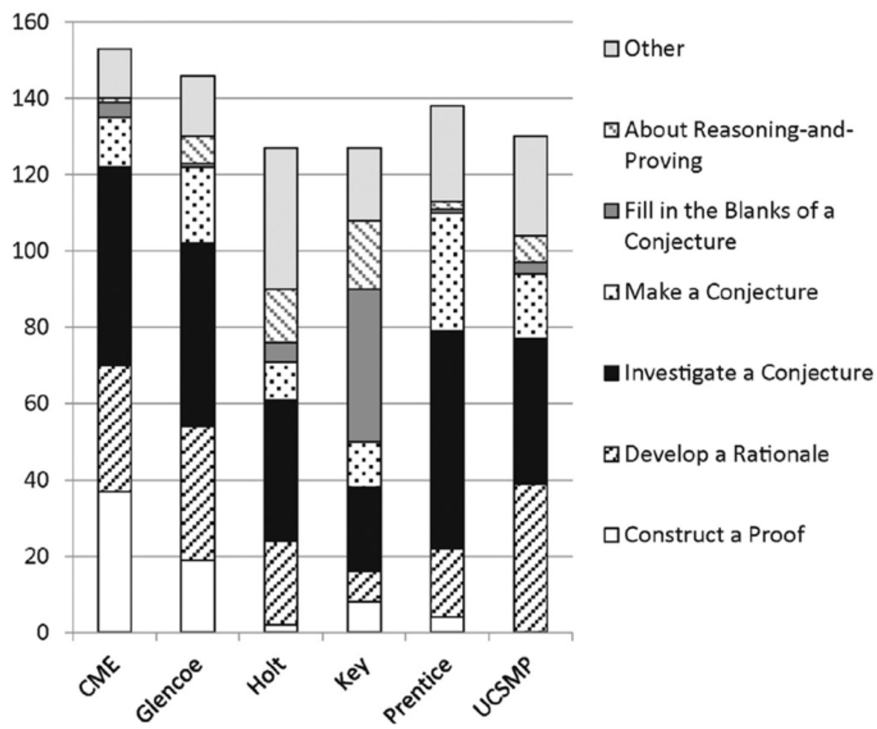

Fig. 6. Percent of reasoning-and-proving exercises of various activity-types in the introduction to proof chapters. *Note*: Sums are greater than 100% because a single exercises can involve multiple reasoning-and-proving activities.

CME had exercises involving the writing of proofs as one of their top two activity-types. Although Glencoe had nearly one in five reasoning-and-proving exercises focused on constructing proofs, the other four textbooks had less than ten percent and as little as zero.

As Table 5 shows, all textbooks except UCSMP had more particular statements in the reasoning-and-proving exercises in the introduction to proof chapters than general statements, even when considering general statements with particular instantiation provided. In this way, the student exercises contrast with the textbook exposition in the introduction to proof chapters where general statements were more common than particular statements (see Table 3). With respect to exercises *about* reasoning-and-proving in the introduction to proof chapters, they accounted for between 10% and 20% of reasoning-and-proving exercises in some textbooks (i.e., Holt and Key) but only 1% or 2% of such exercises in other textbooks (i.e., CME and UCSMP).

Table 5. Percentages of statement-types in reasoning-and-proving exercises in the introduction to proof chapters.

Textbook	*No. of particular statements (%)*	*No. of general statements (%)*	*No. of general statements with instantiation (%)*	*No. of exercises about reasoning-and-proving (%)*
CME	135 (45)	157 (53)	4 (1)	2 (1)
Glencoe	147 (38)	201 (52)	10 (3)	25 (7)
Holt	81 (33)	118 (49)	8 (3)	35 (14)
Key	23 (30)	39 (51)	1 (1)	14 (18)
Prentice	32 (24)	96 (71)	5 (4)	3 (2)
UCSMP	48 (47)	46 (45)	2 (2)	7 (7)

3.2. Comparing the introduction to proof chapters with the remaining chapters

3.2.1. Exposition

Table 6 reports for each textbook the percentage of each type of statement in the exposition section for the introduction to proof chapters and the remaining chapters. The introductory proof chapter percentages are reprinted from Table 3 for ease of comparison.

In the remaining chapters of each textbook, general statements were always more prevalent than particular statements and those *about* reasoning-and-proving. In introduction to proof chapters, however, the distribution was somewhat different. Although there were more general than particular statements, in some textbooks (i.e., CME, Key, UCSMP) the percentage of statements about

Table 6. Percentages of statement-types in textbook exposition.

Textbook	*General statements*	*Particular statements*	*Statements about reasoning-and-proving*
CME	36 | 79	16 | 18	47 | 3
Glencoe	44 | 72	30 | 23	25 | 5
Holt	49 | 77	17 | 19	33 | 4
Key	32 | 85	16 | 6	52 | 9
Prentice	54 | 75	15 | 19	32 | 3
UCSMP	29 | 75	24 | 10	48 | 15

Note: The introductory proof chapter percentage is located to the left of the "|" and the remaining chapter percentage is to the right.

reasoning-and-proving was higher than that of general statements. Overall, the percentage of statements *about* reasoning-and-proving was markedly higher in the introduction to proof chapters than the remaining chapters. Although these high percentages could be due to a reduction in the number of general and particular statements, it is our impression that the high percentages stemmed from an actual prevalence of statements about reasoning-and-proving.

3.2.2. Student exercises

Table 7 shows the percentages of the types of statements in the exercises for the introduction to proof chapters in comparison to the remaining chapters. Statement types in textbook exercises were fairly uniform across introduction to proof chapters and the remaining chapters. Particular statements were always more prevalent than other statement types, except in UCSMP's proof chapter, where the number of general statements was slightly higher than the number of particular statements.

Table 8 contains the percentages of activity-types in the student exercises. Generally, the introduction to proof chapters included greater emphasis on investigating conjectures and finding counter-examples, whereas the remaining chapters had more instances relative to other activities of constructing proofs and developing rationales. For all textbooks, except for CME, the most striking difference in the activities is the decreased opportunities in the introduction to proof chapters to construct proofs and develop rationales. Although in the introduction to proof chapters most textbooks devoted a large number of

Table 7. Percentage of statement-types in student exercises.

Textbook	*General statements*	*Particular statements*	*General statements with instantiation*
CME	45 | 40	53 | 56	1 | 3
Glencoe	38 | 28	52 | 59	3 | 11
Holt	33 | 18	49 | 76	3 | 5
Key	30 | 44	51 | 49	1 | 5
Prentice	24 | 29	71 | 64	4 | 7
UCSMP	47 | 35	45 | 60	2 | 3

Note: The introductory proof chapter percentage is located to the left of the "|" and the remaining chapter percentage is to the right.

Table 8. Percentage of activity-types in student exercises.

Textbook	*Construct a proof*	*Develop a rationale*	*Investigate a statement or find a counterexample*	*Make or fill-in a conjecture*	*About reasoning-and-proving*	*Other*
CME	37 | 22	33 | 46	59 | 46	17 | 18	1 | 1	6 | 1
Glencoe	19 | 27	35 | 56	60 | 33	21 | 12	7 | 1	4 | 5
Holt	2 | 15	22 | 48	57 | 40	15 | 6	14 | 1	17 | 14
Key	8 | 13	8 | 53	40 | 69	52 | 20	18 | 2	1 | 6
Prentice	4 | 16	18 | 58	72 | 50			10 | 8
UCSMP	0 | 16	39 | 40	60 | 34	20 | 11	7 | 2	4 | 0

Note: The introductory proof chapter percentage is located to the left of the "|" and the remaining chapter percentage is to the right.

instances to developing rationale, ranging from 8% to 39% of activities, this was often lower than the percent in their remaining chapters. There was an even greater difference in the percent devoted to constructing a proof across the proof and remaining chapters. Holt and Prentice devoted less than 5% of activities in their introduction to proof chapters to constructing a proof and UCSMP did not have any opportunities for students to construct a proof in their introduction to proof chapter. UCSMP, however, devoted an almost equal percentage of opportunities in their introduction to proof chapter and remaining chapters to developing a rationale. CME was the only textbook that included more opportunities for constructing proofs in their introduction to proof chapter (37%) than their remaining chapters (22%). At first glance this may be surprising, however, because these introduction to proof chapters served to introduce students to mathematical proof, it makes sense that students do not engage heavily in constructing proofs until later chapters after building up the necessary tools and skills for constructing these proofs (e.g., postulates, theorems). We return to this point in the discussion.

Although not immediately apparent from looking at Table 8 because of merged categories, some activities were much more prevalent in introduction to proof chapters than the remaining chapters for particular textbooks. Key devoted 40% of their introductory proof chapter reasoning-and-proving exercises to filling in the blanks of conjectures, an activity for which they devoted only 10% in the remaining chapters. Key also had 27% of the reasoning-and-proving exercises in their remaining chapters involve finding a counterexample, up from 18% in the proof chapter. With respect to exercises *about* reasoning-and-proving, several textbooks had markedly higher percentages in their introduction to proof chapters than elsewhere. For example, Holt and Key devoted many more instances to thinking *about* reasoning-and-proving in their introduction to proof chapters (14% and 18%, resp.) than in their remaining chapters (1% and 2%, resp.). Similarly, though to a lesser extent, Glencoe, Prentice, and UCSMP also had higher percentages in their introduction to proof chapters. In light of our third guiding question, we use the next section to look in more detail at these opportunities to explicitly reflect on or attend to issues related to reasoning-and-proving.

3.3. Attending to or reflecting on reasoning-and-proving

One of the most striking differences between the introduction to proof chapters and remaining chapters was in the frequency of opportunities for students to think *about* the process of reasoning-and-proving. Fig. 7 shows the percentage of statements *about* reasoning-and-proving in the introduction to proof chapters and the remaining chapters. It is important to consider the types of opportunities provided to think *about* reasoning-and-proving in introduction to proof chapters

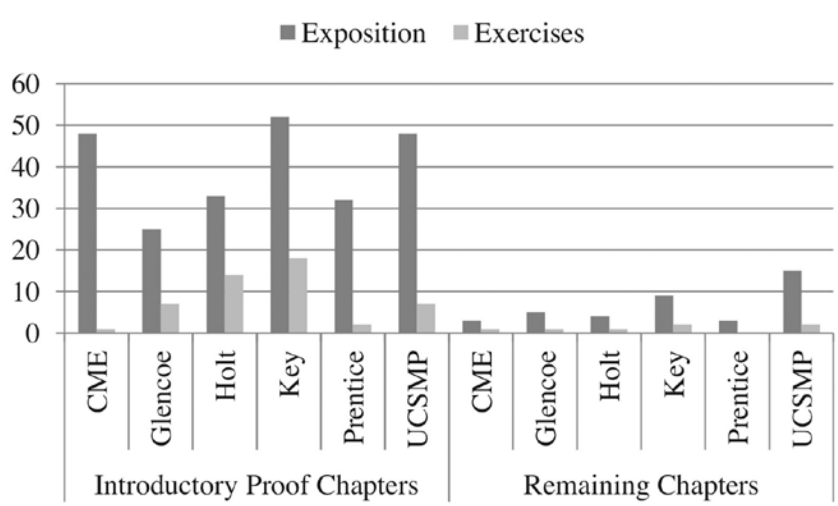

Fig. 7. Percent of statements and exercises *about* reasoning-and-proving.

because this is likely the first place where students are introduced to the formal process of proving. We turn now to a more detailed look at these opportunities.

Common foci of expository statements *about* reasoning-and-proving were explanations of the deductive reasoning process and descriptions of the role of conditional statements. For example, Prentice described deductive reasoning as “the process of reasoning logically from given statements to a conclusion. If the given statements are true, deductive reasoning produces a true conclusion” (p. 94). Many textbooks then discussed conditional statements as a means for linking givens and conclusions, with the resulting argument constituting a proof. Another example of exposition along these lines is found in UCSMP, where they explained that “a conditional statement is *true* if, for every possible case in which the antecedent is true, the consequent is also true. . . To prove that a conditional is *false*, all you need is one instance in which the antecedent is true, but the consequent is false” (p. 68, emphasis in original).

Most textbooks address the role of inductive reasoning in the reasoning-and-proving process either before or in conjunction with the introduction of deductive reasoning. For example, before formally introducing deductive reasoning in the introductory proof chapter, Holt spent a few lessons on inductive reasoning. They described how inductive reasoning, based on “specific cases” (p. 74), leads to conjectures. After providing students with several opportunities to generate conjectures from specific cases, Holt enumerated the steps of inductive reasoning (see Fig. 8). This representation of inductive reasoning, however, does not align with their definition because inductive reasoning was not defined as that particular set of steps but as the thought process that leads from step 1 to step 2. Furthermore, step 3 of inductive reasoning is later described as entailing deductive reasoning, which might lead students to believe that deductive reasoning

Inductive Reasoning
1. Look for a pattern
2. Make a conjecture
3. Prove the conjecture or find a counterexample

Fig. 8. The steps of inductive reasoning as displayed in Holt (p. 75).

is situated within inductive reasoning rather than being distinct from it. Because the steps in Fig. 8 explicitly include "prove" as a part of inductive reasoning, the relationship between proof and deduction (and also between deduction and induction) is somewhat muddled.

Another common focus of expository text *about* reasoning-and-proving relates to the formats of proof arguments. Some textbooks wait until subsequent chapters to fully address proof formats, but several, by the end of the introductory proof chapter, introduce the two-column format as well as other formats such as paragraph proofs or flow proofs. Glencoe, for example, discusses two-column proofs as "*statements* and *reasons* organized into two columns" (p. 135, emphasis in original) and then provide several worked examples. They write *about* this proof format as an extension of work students have done in algebra where they solved an equation on the left and wrote the justifications for each step on the right. Glencoe also refers to two-column proofs as "formal proofs" and introduces paragraph proofs, which involve "writing a paragraph to explain why a conjecture for a given situation is true" (p. 127), as "informal proofs." This choice of language may be surprising since proofs in formal mathematical research are written in paragraph form, but Glencoe does clarify that the term "informal" is not meant to suggest that paragraph proofs are "any less valid than any other type of proof" (p. 127).

Other types of statements found in the textbook exposition related to the axiomatic system, such as how conjectures become theorems and what justifications are allowable in proofs, as well as some situating of proof in historical contexts. For example, CME wrote that "[t]he need for proof is a result of tradition, necessity, and culture" (p. 118).

Turning now to the exercises *about* reasoning-and-proving, the most common type of question was to ask students what form of reasoning was used to draw a given conclusion. For example, Holt included a multiple choice item in which students had to determine whether a claim was made based on "deductive reasoning, because the conclusion is based on logic," "deductive reasoning, because the conclusion is based on a pattern," "inductive reasoning, because the conclusion is based on logic," or "inductive reasoning, because the conclusion is based on a pattern" (p. 93). Other exercises addressed the differences between inductive and deductive reasoning in various ways. For instance, Key asked students to write about a situation in which they observed inductive reasoning being used incorrectly and why they "think it was an incorrect use of inductive reasoning" (p. 119).

Other exercises *about* reasoning-and-proving involved students considering the role of conjectures, postulates, and theorems in the axiomatic system or the rules of logic as they pertain to proof. For example, Glencoe asked students to "[c]ompare and contrast the Law of Syllogism and the Transitive Property" (p. 122). Whereas exercises that engage students in reasoning-and-proving may involve such properties within justifications, exercises *about* reasoning-and-proving shift the role of laws and properties to having students explicitly consider them as objects of study.

4. Discussion

In this study, we analyzed the reasoning-and-proving opportunities in the introduction to proof chapters of six U.S. geometry textbooks and compared aspects of those opportunities with the opportunities in other parts of the book. A finding that was true in both locations and for all six textbooks was that reasoning-and-proving statements in exposition were more likely to be general than particular, though in the introduction to proof chapters there was a much greater percentage of statements *about* reasoning-and-proving than in the remaining chapters. Within the student exercises, opportunities to explicitly attend to or reflect on reasoning-and-proving were also more common in the introduction to proof chapters than elsewhere, but, in contrast with the textbook exposition, reasoning-and-proving statements were more often particular than general. The only exception to these trends was the introductory proof chapter of UCSMP, which had nearly equal proportions of general and particular statements in the student exercises. These findings related to statement-types, when viewed through the lens of the necessity principle (Harel & Tall, 1991), have implications for students' opportunities to learn reasoning-and-proving as they may fail to see the need to employ deductive reasoning if they are reasoning predominantly around particular claims. They may, instead, view deduction as an arbitrary endeavor, as something demanded by a teacher or textbook rather than by the situation at hand and, indeed, such views have been observed by researchers (Soucy McCrone & Martin, 2009; Schoenfeld, 1988). Students may also misunderstand what is proven by deductive arguments (Chazan, 1993; Porteous, 1990) or view mathematical knowledge, such as proven general claims, as being produced by others rather than themselves (Schoenfeld, 1988). We recognize, however, that these potential problems can be substantially addressed by the teachers who enact the textbooks' reasoning-and-proving opportunities, whether they be general or particular (Gilbertson, Otten, Males, & Clark, 2013).

Within the chapters that introduced proof, we found that reasoning-and-proving activities tended to involve investigating conjectures, making conjectures, or developing rationales. Exercises that asked students to construct a proof were rare. In fact, UCSMP had no such exercises in its introductory proof chapter. In some sense, this is unsurprising because the introductory proof chapter may be introducing the notion of deductive reasoning and proof and laying the groundwork for students to eventually be able to produce proofs themselves in a later chapter (such as in the triangle congruence chapter analyzed by Fujita & Jones, this volume). It also emphasizes the importance of conjecturing and non-proof arguments in the broad reasoning-and-proving process. Nevertheless, this implies one is beginning the road to proof with something other than proof. The scope of this study does not allow us to say anything with respect to the effectiveness of one approach over another, but it does seem important for textbook authors and teachers (and students) to be aware of the fact that students may experience the building blocks of proof prior to engaging in proving itself, even in chapters that explicitly introduce proof. And the necessity principle (Harel & Tall, 1991) suggests that we should not simply move students on the road toward proof but should help them see *why* the road leads in that direction at all.

One of the ways that textbooks might embed opportunities to see the intellectual necessity of proof is through statements or exercises that make reasoning-and-proving an explicit object of attention or reflection. We coded these in exposition and student exercises as being *about* reasoning-and-proving, whereas Hanna and de Bruyn (1999) referred to similar items as "discussion of proof" and Thompson et al., 2012 as "principles of proof." Although such opportunities were quite common in the introduction to proof chapters of the geometry textbooks, they were uncommon in the remaining chapters. What role might these opportunities to step out and consider the reasoning-and-proving process play in students' developing understanding of what it means to prove and, more broadly, what it means to do mathematics? With regard to the textbooks in this study, students are provided early on with opportunities to consider what constitutes deductive reasoning, the differences between inductive and deductive reasoning, and the various formats of proof arguments. However, students are not yet given opportunities to construct proofs to a large degree. Given work such as Schoenfeld (1992) and Weber's (2001) with regard to meta-cognitive acts and their relation to students' reasoning, it would also seem to be prudent to provide students with opportunities to reflect on the nature of the reasoning-and-proving process beyond the introduction to proof chapter as they begin to produce proofs and after they have accumulated a broader array of proving experiences.

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