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MULTI-INTERVAL STURM-LIOUVILLE TRANSFORM AND ITS APPLICATIONS[1]

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I. INTRODUCTION

The method of finite integral transforms (Eringen, 1954; Churchill, 1955; Sneddon, 1955; Cinelli, 1965; Roettinger, 1947) has gradually replaced the classical method of separation of variables. This is undoubtedly due to its compact and almost drill-like formalism. Wherever applicable, it has also replaced the Laplace transform with the resulting avoidance of cumbersome contour integrals in the inversion.

In this paper, we are concerned with solving, by such a transformational or operational method, the following general boundary-value problem:

$$\begin{cases} L_i(f_i) + M(f_i) = g_i(x), \quad C_{i-1}<x<C & (1) \\ a_1 f_1'(C_o) - a_2 f_1(C_o) = D_1 & (2) \\ a_3 f_n'(C_n) + a_4 f_n(C_n) = D_2 & (3) \\ A_{1j} f_j'(C_j^-) = A_{1(j+1)} f_{j+1}'(C_j^+) & (4) \\ \alpha_j A_{1j} f_j'(C_j^-) = H_j[f_{j+1}(C_j^+) - f_j(C_j^-)] & (5) \end{cases}$$

where

$$i = 1, 2, 3, \ldots., n \tag{6}$$

$$j = 1, 2, 3, \ldots., (n-1) \tag{7}$$

$$L_i(f_i) = \frac{1}{A_{3i}p_3(x)}[A_{1i}p_1(x)f_i']' - \frac{A_{2i}p_2(x)}{A_{3i}p_3(x)}f_i \tag{8}$$

and M is certain linear differential operator involving independent variables other than x, with associated linear conditions not exhibited. $g_i(x)$ are given bounded functions of x, which may depend on the other independent variables. D_1 and D_2 are given functions of independent variables other than x. a_1, a_2, a_3, a_4, (a_1 and a_2 are not both zero, a_3 and a_4 are not both zero) A_{1i}, A_{2i}, A_{3i}, H_j and α_j are all given constants; a_1, a_3 and α_j being either 1 of 0. $p_1(x)$, $p_2(x)$ and $p_3(x)$ are given continuous functions. The prime represents here differentiation with respect to x. If $p_1(C_o) = 0$, condition (2) is to be replaced by

$$f_1(C_o),\ f'(C_o) \text{ finite} \tag{2'}$$

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With respect to x, the above system represents a boundary-value problem in an interval (C_o, C_n) which is divided into n sub-intervals (C_o, C_1), (C_1, C_2), ... (C_{n-1}, C_n); with boundary conditions (2) and (3), and matching conditions (4) and (5). These conditions are very general. If $a_1 = 0$, we have a boundary condition of the first kind at $x = C_o$. If $a_4 = 0$, we have that of the second kind at $x = C_n$, and so forth. So, the boundary conditions at $x = C_o$ and C_n can be any one of the three kinds independently of each other. Two constants on each end are used in writing out the boundary conditions to avoid having to take the limit of something going to infinity, which process is sometimes rather involved. For the same reason, we have also introduced the constant α_j. Then, if $\alpha_j = 0$, condition (5) simply becomes

$$f_j(C_j^+) = f_{j+1}(C_j^-) \tag{5'}$$

without having to let $H_i \to \infty$. Borrowing an expression from the theory of heat conduction, we say that (5') and (4) constitute the so called perfect contact at $x = C_j$. If $H_j = 0$, we simply have

$$f_j'(C_j^-) = 0 \tag{4'}$$

$$f_{j+1}'(C_j^+) = 0 \tag{5''}$$

The system is then decoupled into two boundary-value problems, one in the interval (C_o, C_j) and one in (C_j, C_n), independently of each other. We therefore assume that $H_j \neq 0$. When $\alpha_j = 1$, we say that a linear contact law with coefficient H_j is in effect at $x = C_j$.

Our general system contains many practical problems. For example, if $M = -\frac{\partial}{\partial t}$, A_{1i} = thermal conductivity, A_{3i} = density times specific heat capacity, H_j = contact heat transfer coefficient, $a_1 = a_3 = 0$, $p_1(x) = p_3(x) = 1$, and $p_2(x) = 0$, then, f_i becomes the transient temperature distribution in a n-layered slab. Other possible problems are heat conduction in multi-layered cylinders and spheres, oscillations of multi-sectional strings, transient multi-layered flow of liquids, forced convection in multi-layered duct flows, etc.

The problem of transient heat conduction in multi-layered bodies has induced recently several authors (Tittle, 1965; Bulavin and Kashcheev, 1965; Giere, 1968) to investigate, apparently independently of one another, the associated eigenvalues and eigenfunctions as a result of the classical separation of variables; none of them has approached it from the modern viewpoint of finite integral transforms. It is the purpose of this paper to establish a general, unifying, integral transform based on a Sturm-Liouville (S-L) equation whose coefficients jump in a number of sub-intervals. This is done in the same spirit as Eringen, 1954 and Churchill, 1955. The transform is then capable of handling our general system (1) - (8). After the general discussion, two special cases (the multi-interval Fourier and Hankel transforms) are worked out in detail because of their frequent use in connection with multi-layered slabs and cylinders. Finally, practical applications are illustrated by treating two transient flow problems whose solutions are unavailable in the literature.

It may be stated in short that this paper differs from the works quoted in the last paragraph not only in spirit, but also in scope and depth.

II. MULTI-INTERVAL S-L PROBLEM

For brevity, the system (1) - (8) can also be written symbolically as follows:

$$\begin{cases} L(f) + M(f) = g(x), \quad C_0<C_1<C_2<\cdots<C_{i-1}<x<C_i<\cdots<C_n & (9) \\ a_1 f'(C_0) - a_2 f(C_0) = D_1; \text{ or } f(C_0),\ f'(C_0) \text{ finite if } p_1(C_0) = 0 & (10) \\ a_3 f'(C_n) + a_4 f(C_n) = D_2 & (11) \\ f'(C_j^+) = \dfrac{A_1(C_j^-)}{A_1(C_j^+)} f'(C_j^-) & (12) \\ f(C_j^+) = \left(\dfrac{\alpha_j}{H_j}\right) A_1(C_j^-) f'(C_j^-) + f(C_j^-) & (13) \end{cases}$$

where

$$L(f) = \frac{1}{A_3(x)p_3(x)}[A_1(x)p_1(x)f']' - \frac{A_2(x)p_2(x)}{A_3(x)p_3(x)} f \qquad (14)$$

and

$$A_1(x) = A_{1i}, \quad C_{i-1}^+ \le x \le C_i^- \qquad (15)$$

$$A_2(x) = A_{2i}, \quad \text{Do.} \qquad (16)$$

$$A_3(x) = A_{3i}, \quad \text{Do.} \qquad (17)$$

Associated with this system, we have the following multi-interval S-L problem:

$$\begin{cases} L(\phi) + \lambda\phi = 0, \quad C_0<C_1<C_2<\cdots<C_{i-1}<x<C_i<\cdots<C_n & (18) \\ a_1\phi'(C_0) - a_2\phi(C_0) = 0; \text{ or } \phi(C_0),\ \phi'(C_0) \text{ finite if } p_1(C_0) = 0 & (19) \\ a_3\phi'(C_n) + a_4\phi(C_n) = 0 & (20) \\ \phi'(C_j^+) = \dfrac{A_1(C_j^-)}{A_1(C_j^+)}\phi'(C_j^-) & (21) \\ \phi(C_j^+) = \left(\dfrac{\alpha_j}{H_j}\right) A_1(C_j^-)\phi'(C_j^-) + \phi(C_j^-) & (22) \end{cases}$$

(This is again only symbolic. Whenever ambiguity is encountered, one should refer back to an extended form similar to (1) - (8).)

For simplicity, we will assume from now on that (i) $x \geqslant 0$; (ii) $p_1''(x)$, $p_2(x)$ and $p_3(x)$ are continuous in $(C_o\ C_n)$; (iii) $p_2(x) \geqslant 0$, $p_1(x)$ and $p_3(x) > 0$, but $p_1(C_o)$ may vanish; (iv) $A_1, A_2, A_3, > 0$; $H_j > 0$; (v) $a_2, a_4. \geqslant 0$. Some of these assumptions do not harm the generality of the discussion at all; some are always valid for practical problems of interest. The only important exception is $A_2 > 0$. So, in a way, this is the only simplifying assumption introduced in this paper.

In each sub-interval (C_{i-1}, C_i), there are two linearly independent solutions of the S-L equation (18), say $E_i(x;\lambda)$ and $F_i(x;\lambda)$. Then the general solution of (18) must be

$$\phi(x;\lambda) = \gamma_i E_i(x;\lambda) + \delta_i F_i(x;\lambda),\ C_{i-1}<x<C_i \tag{23}$$

To determine γ_i and δ_i, we have to use the conditions (19) - (22). The matching conditions (21) and (22) yield,

$$\begin{cases} \gamma_{j+1}E'_{j+1}(C_j;\lambda) + \delta_{j+1}F'_{j+1}(C_j;\lambda) = \dfrac{A_{1j}}{A_{1(j+1)}}\sigma_j & (24) \\ \gamma_{j+1}E_{j+1}(C_j;\lambda) + \delta_{j+1}F_{j+1}(C_j;\lambda) = \left(\dfrac{\alpha_j}{H_j}\right)A_{1j}\sigma_j + \tau_j & (25) \end{cases}$$

where

$$\sigma_j = \gamma_j E'_j(C_j;\lambda) + \delta_j F'_j(C_j;\lambda) \tag{26}$$

$$\tau_j = \gamma_j E_j(C_j;\lambda) + \delta_j F_j(C_j;\lambda) \tag{27}$$

Thus,

$$\gamma_{j+1} = \frac{1}{\Delta_{j+1}} \begin{vmatrix} A_{1j}\sigma_j/A_{1(j+1)} & F'_{j+1}(C_j;\lambda) \\ \left(\dfrac{\alpha_j}{H_j}\right)A_{1j}\sigma_j + \tau_j & F_{j+1}(C_j;\lambda) \end{vmatrix} \tag{28}$$

$$\delta_{j+1} = \frac{1}{\Delta_{j+1}} \begin{vmatrix} E'_{j+1}(C_j;\lambda) & A_{1j}\sigma_j/A_{1(j+1)} \\ E_{j+1}(C_j;\lambda) & \left(\dfrac{\alpha_j}{H_j}\right)A_{1j}\sigma_j + \tau_j \end{vmatrix} \tag{29}$$

where

$$\Delta_{j+1} = \begin{vmatrix} E'_{j+1}(C_j;\lambda) & F'_{j+1}(C_j;\lambda) \\ E_{j+1}(C_j;\lambda) & F_{j+1}(C_j;\lambda) \end{vmatrix} \tag{30}$$

The recursive formulae (24) and (25) determine $\partial_2, \partial'_3, \ldots, \partial_n, \delta_2, \delta_3, \ldots$, and δ_n. They must be supplemented by the boundary condition (19), i.e.,

$$\gamma_1[a_1E'_1(C_0;\lambda) - a_2E_1(C_0;\lambda)] + \delta_1[a_1F'_1(C_0;\lambda) - a_2F_1(C_0;\lambda)] = 0 \tag{31}$$

$$\text{or, if } p_1(C_0) = 0,\ \gamma_1 \text{ or } \delta_1 = 0 \tag{32}$$

This latter is due to the fact that one of the linearly independent solutions of a singular S-L equation (i.e., with $p_1(C_0) = 0$) must be unbounded at $x = C_0$. Equation (31) can be conveniently written as

$$\gamma_1 = -\delta_1 \cdot \frac{a_1F'_1(C_0;\lambda) - a_2F_1(C_0;\lambda)}{a_1E'_1(C_0;\lambda) - a_2E_1(C_0;\lambda)} \tag{33}$$

where δ_1 can be set equal to any convenient constant. If $p_1(C_0) = 0$, we will agree to have $\partial_1 = 0$; δ_1 is then again a conveniently chosen constant. With ∂_1, δ_1 thus chosen or calculated, we can calculate $\partial'_2, \ldots, \partial'_n, \delta_2, \ldots, \delta_n$. Then, the boundary condition (20) yields

$$\begin{aligned} a_3[\gamma_n(\lambda)E'_n(C_n;\lambda) + \delta_n(\lambda)F'_n(C_n;\lambda)] \\ + a_4[\gamma_n(\lambda)E_n(C_n;\lambda) + \delta_n(\lambda)F_n(C_n;\lambda)] = 0 \end{aligned} \tag{34}$$

where $\partial_n(\lambda)$ and $\delta_n(\lambda)$ are displayed to show that ∂'_n and δ_n involve λ.

The parameter λ in (23) must now satisfy (34) in order that $\phi(x;\lambda)$ is a solution of the S-L problem, except for the trivial case where ϕ vanishes identically.

The roots of the transcendental equation (34) are the eigenvalues of the problem. For each root $\lambda^{(m)}$, there corresponds a nontrivial function $\phi^{(m)}(x;\lambda^{(m)})$, called an eigenfunction. Some fundamental properties of $\lambda^{(m)}$ and $\phi^{(m)}$ will be established in the next section.

III. EIGENVALUES AND EIGENFUNCTIONS

Suppose that $\phi^{(m)}$ and $\phi^{(\mu)}$ are two eigenfunctions associated with eigenvalues $\lambda^{(m)}$ and $\lambda^{(\mu)}$, respectively. That is ,

$$\left.\begin{aligned} L(\phi^{(m)}) + \lambda^{(m)}\phi^{(m)} &= 0 \\ L(\phi^{(\mu)}) + \lambda^{(\mu)}\phi^{(\mu)} &= 0 \end{aligned}\right\} \quad , \quad C_0<x<C_n$$

or, more precisely,

$$A_{1i}[p_1\phi_i^{(m)\prime}]^{\prime} - A_{2i}p_2\phi_i^{(m)} + \lambda^{(m)}A_{3i}p_3\phi_i^{(m)} = 0, \quad C_{i-1}<x<C_i \tag{35}$$

$$A_{1i}[p_1\phi_i^{(\mu)\prime}]^{\prime} - A_{2i}p_2\phi_i^{(\mu)} + \lambda^{(\mu)}A_{3i}p_3\phi_i^{(\mu)} = 0, \quad C_{i-1}<x<C_i \tag{36}$$

Multiplying (35) by $\phi^{(\mu)}$, (36) by $\phi^{(m)}$, subtracting and rearranging, we have

$$A_{1i}[p_1(\phi_i^{(\mu)}\phi_i^{(m)\prime} - \phi_i^{(m)}\phi_i^{(\mu)\prime})]^{\prime} = (\lambda^{(\mu)} - \lambda^{(m)})A_{3i}p_3\phi_i^{(m)}\phi_i^{(\mu)} \tag{37}$$

For the time being, we will assume that $\lambda^{(m)} \neq \lambda^{(\mu)}$. Next, integrating (37) from $x = C_{i-1}$ to $x = C_i$ and summing over $i = 1, 2, \ldots, n$, we have

$$\begin{aligned} &(\lambda^{(\mu)} - \lambda^{(m)}) \int_{C_0}^{C_n} A_3p_3\phi^{(m)}\phi^{(\mu)}dx \\ &= (\lambda^{(\mu)} - \lambda^{(m)}) \sum_{i=1}^{n} A_{3i} \int_{C_{i-1}}^{C_i} p_3\phi_i^{(m)}\phi_i^{(\mu)}dx \\ &= \sum_{i=1}^{n} A_{1i}[p_1(\phi_i^{(\mu)}\phi_i^{(m)\prime} - \phi_i^{(m)}\phi_i^{(\mu)\prime})]\Big|_{x = C_{i-1}^{+}}^{x = C_i^{-}} \\ &= A_{1n}p_1(C_n)(\phi_n^{(\mu)}\phi_n^{(m)\prime} - \phi_n^{(m)}\phi_n^{(\mu)\prime})|_{x = C_n} \\ &\quad - A_{11}p_1(C_o)(\phi_1^{(\mu)}\phi_1^{(m)\prime} - \phi_1^{(m)}\phi_1^{(\mu)\prime})|_{x = C_o} \end{aligned} \tag{38}$$

The other terms all cancel out in pairs:

$$A_{1j}(\phi_j^{(\mu)}\phi_j^{(m)\prime} - \phi_j^{(m)}\phi_j^{(\mu)\prime})|_{x = C_j^-}$$

$$- A_{1(j+1)}(\phi_{j+1}^{(\mu)}\phi_{j+1}^{(m)\prime} - \phi_{j+1}^{(m)}\phi_{j+1}^{(\mu)\prime})|_{x = C_j^+}$$

$$= A_{1j}\phi_j^{(m)\prime}(C_j^-)[\phi_j^{(\mu)}(C_j^-) - \phi_{j+1}^{(\mu)}(C_j^+)]$$

$$- A_{1j}\phi_j^{(\mu)\prime}(C_j^-)[\phi_j^{(m)}(C_j^-) - \phi_{j+1}^{(m)}(C_j^+)]$$

$$= -A_{1j}^2(\frac{\alpha_j}{H_j})[\phi_j^{(m)\prime}(C_j^-)\phi_j^{(\mu)\prime}(C_j^-) - \phi_j^{(\mu)\prime}(C_j^-)\phi_j^{(m)\prime}(C_j^-)] = 0 \quad (39)$$

where matching conditions (21) and (22) are used.

It is also easy to verify that the remaining two terms on the right hand side of (38) vanish separately because of boundary conditions (19) and (20).

$$\therefore \quad \int_{C_o}^{C_n} A_3 p_3 \phi^{(m)} \phi^{(\mu)} dx = 0 \quad (40)$$

since it is assumed that $\lambda^{(m)} \neq \lambda^{(\mu)}$.

If $\lambda^{(m)} = \lambda^{(\mu)}$, there are two posibilities: (i) $m = \mu$ and (ii) $m \neq \mu$. For case (i), we have simply

$$N^{(m)} = \int_{C_o}^{C_n} A_3 p_3 [\phi^{(m)}]^2 dx = \sum_{i=1}^{n} A_{3i} \int_{C_{i-1}}^{C_i} p_3 [\phi_i^{(m)}]^2 dx > 0 \quad (41)$$

$N^{(m)}$, the norm of the eigenfunctions, is to be calculated by direct integration.

When $\lambda^{(m)} = \lambda^{(\mu)}$ while $m \neq \mu$, we have the degenerate case where two eigenfunctions $\phi^{(m)}$ and $\phi^{(\mu)}$ are associated with one single eigenvalue. In such a case, we have to go back to equation (37) and conclude that

$$A_{1i} p_1 (\phi_i^{(\mu)}\phi_i^{(m)\prime} - \phi_i^{(m)}\phi_i^{(\mu)\prime}) = D_i^*, \text{ a constant} \quad (42)$$

From (39),

$$A_{1j}(\phi_j^{(\mu)}\phi_j^{(m)\prime} - \phi_j^{(m)}\phi_j^{(\mu)\prime})|_{x = C_j^-}$$

$$= A_{1(j+1)}(\phi_{j+1}^{(\mu)}\phi_{j+1}^{(m)\prime} - \phi_{j+1}^{(m)}\phi_{j+1}^{(\mu)\prime})|_{x = C_j^+}$$

$$\therefore \quad D_1^* = D_2^* = \ldots = D_n^*$$

Equation (42) can therefore be written symbolically as

$$A_1 p_1 (\phi^{(\mu)}\phi^{(m)\prime} - \phi^{(m)}\phi^{(\mu)\prime}) = \text{constant}$$

But this expression vanishes at $x = C_o$ and C_n because of the boundary conditions. Therefore,

$$A_1 p_1 (\phi^{(\mu)}\phi^{(m)\prime} - \phi^{(m)}\phi^{(\mu)\prime}) = 0$$

identically in (C_o,C_n). Now $A_1 p_1$ is not identically zero in (C_o,C_n). It follows then that

$$\frac{\phi^{(m)\prime}}{\phi^{(m)}} = \frac{\phi^{(\mu)\prime}}{\phi^{(\mu)}}$$

i.e.,

$$\phi^{(m)} \propto \phi^{(\mu)}$$

Thus, $\phi^{(m)}$ and $\phi^{(\mu)}$ are not two distinctive eigenfunctions, after all.

We next prove that $\lambda^{(m)}$ must be real. We can do this symbolically for the entire interval (C_o,C_n) without referring to the sub-intervals.

If $\lambda^{(m)}$ is complex, the corresponding $\phi^{(m)}$ may also be complex. In our S-L problem, we can take the conjugate ( ) of all terms and obtain

$$L\overline{\phi^{(m)}} + \overline{\lambda^{(m)}}\;\overline{\phi^{(m)}} = 0$$

where $\overline{\phi^{(m)}}$ satisfies the same boundary and matching conditions as $\overline{\phi^{(m)}}.$

Therefore, $\overline{\phi^{(m)}}$ is an eigenfunction associated with the eigenvalue $\overline{\lambda^{(m)}}$. We must then have

$$(\lambda^{(m)} - \overline{\lambda^{(m)}}) \int_{C_o}^{C_n} A_3 p_3 \phi^{(m)} \overline{\phi^{(m)}} dx = 0$$

Now, $\phi^{(m)}\overline{\phi^{(m)}} = |\phi^{(m)}|^2 . > 0$. We have also assumed that $A_3, p_3 > 0$. So, the integrand does not change sign in (C_o, C_n), and the integral must be some (positive) non-zero number. Therefore, we must have

$$\lambda^{(m)} - \overline{\lambda^{(m)}} = 0$$

i.e., $\lambda^{(m)}$ is real. The corresponding $\phi^{(m)}$ is then real also, making the norm real and positive.

Actually, the eigenvalues are not only real but also non-negative as we have assumed that $p_2 \geqslant 0$ and $A_2 > 0$. To see this, we multiply (35) by $\phi^{(m)}$ and integrate from $x = C_o$ to $x = C_n$; thus,

$$\lambda^{(m)} N^{(m)} = - \sum_{i=1}^{n} A_{1i} \int_{C_{i-1}}^{C_i} \phi_i^{(m)} [p_1 \phi_i^{(m)\prime}]' dx + \sum_{i=1}^{n} A_{2i} \int_{C_{i-1}}^{C_i} p_2 [\phi_i^{(m)}]^2 dx$$

$$= - \sum_{i=1}^{n} A_{1i} [p_1 \phi_i^{(m)} \phi_i^{(m)\prime}] \Big|_{x = C_{i-1}^+}^{x = C_i^-} + \sum_{i=1}^{n} \left\{ A_{1i} \int_{C_{i-1}}^{C_i} p_1 [\phi_i^{(m)\prime}]^2 dx + A_{2i} \int_{C_{i-1}}^{C_i} p_2 [\phi_i^{(m)}]^2 dx \right\} \quad (43)$$

In the first sum, we see that

$$A_{1j} p_1 (C_j) \phi_j^{(m)} (C_j^-) \phi_j^{(m)\prime} (C_j^-)$$

$$- A_{1(j+1)} p_1(C_j) \phi_{j+1}^{(m)}(C_j^+) \phi_{j+1}^{(m)\prime}(C_j^+)$$

$$= - A_{1j}^2 p_1(C_1) [\phi_j^{(m)\prime}(C_j^-)]^2 \left(\frac{\alpha_1}{H_1}\right) \leq 0$$

because of matching conditions (21) and (22). From the boundary conditions (19) and (20), it is also easy to see that

$$A_{1n} p_1(C_n) \phi_n^{(m)}(C_n) \phi_n^{(m)\prime}(C_n)$$

$$- A_{11} p_1(C_o) \phi_1^{(m)}(C_o) \phi_1^{(m)\prime}(C_o) \leq 0$$

So, the right-hand side of (43) is non-negative, or,

$$\lambda^{(m)} \geq 0 \tag{44}$$

Incidentally, we have also shown, at the same time, that

$$\int_{C_o}^{C_n} \phi^{(m)} L(\phi^{(m)}) A_3 p_3 dx \geq 0 \tag{45}$$

Because $\lambda^{(m)}$ is real and non-negative, we can write

$$\lambda^{(m)} = [\xi^{(m)}]^2$$

and consider $\xi^{(m)} \geqslant 0$ as the eigenvalues. Equation (34) with λ substituted by ξ^2 will be referred to as the ξ-equation. The non-negative roots of the ξ-equation constitute the set of eigenvalues which are from now on ordered such that $\xi^{(m+1)} > \xi^{(m)}$, m = 1, 2, 3, ...

IV. MULTI-INTERVAL S-L TRANSFORM

With reference to the multi-interval S-L problem we can now define the general multi-interval S-L transform of a function f(x):

$$\tilde{f}(m) = \int_{C_o}^{C_n} f(x) \phi^{(m)}(x; \xi^{(m)}) A_3 p_3(x) dx$$

$$= \sum_{i=1}^{n} A_{3i} \int_{C_{i-1}}^{C_i} f(x)\phi_i^{(m)}(x;\xi^{(m)})p_3(x)dx \tag{46}$$

If we assume that f and f′ are continuous in the sub-intervals (C_{i-1}, C_i) as is common in physical applications, we see easily that

$$\widetilde{L(f)} = -[\xi^{(m)}]^2\tilde{f} + \sum_{i=1}^{n} A_{1i}[p_1(f'\phi_i^{(m)} - \phi_i^{(m)\prime}f)]_{C_{i-1}}^{C_i} \tag{47}$$

To go further from here, we will assume that f also satisfies the same matching conditions as $\phi^{(m)}$ at $x = C_j$. Then, (47) becomes

$$\widetilde{L(f)} = -[\xi^{(m)}]^2\tilde{f} + U^{(m)}(a_1f' - a_2f)|_{x=C_o}$$

$$+ V^{(m)}(a_3f' + a_4f)|_{x=C_n} \tag{48}$$

where

$$U^{(m)} = -\frac{A_{11}p_1(C_o)}{a_1}\phi_1^{(m)}(C_o),\quad a_1 \neq 0$$

$$= -\frac{A_{11}p_1(C_o)}{a_2}\phi_1^{(m)\prime}(C_o),\quad a_1 = 0 \tag{49}$$

$$V^{(m)} = \frac{A_{1n}p_1(C_n)}{a_3}\phi_n^{(m)}(C_n),\quad a_3 \neq 0$$

$$= -\frac{A_{1n}p_1(C_n)}{a_4}\phi_n^{(m)\prime}(C_n),\quad a_3 = 0 \tag{50}$$

the rest of the summation being cancelled out in pairs just as in (39).

Another important property of the transform is its inversion formula: If f(x) has only a finite number of finite jumps in (C_o,C_n),

$$(1/2)[f(x^+) + f(x^-)] = \sum_{m=1}^{\infty} \tilde{f}(m)\phi^{(m)}(x)/N^{(m)}$$

$$= \sum_{m=1}^{\infty} \tilde{f}(m)\phi_i^{(m)}(x)/N^{(m)}, \quad C_{i-1} < x < C_i \tag{51}$$

(51) implies a convergence in the mean of the S-L series on the right-hand side just as in the single-interval case from which we also expect to inherit such characteristics as Gibbs phenomenon where f jumps, deterioration of convergence through term-wise differentiation, and improvement of convergence through term-wise integration.

The inversion formula is stated here in a language sufficiently general for physical applications. For the proof of a much more general statement, see, for example, Naimark, 1968.

The S-L transform can now be employed to solve the boundry-value problem (9) - (13). The transformed system is

$$M(\tilde{f}) - [\xi^{(m)}]^2\tilde{f} + U^{(m)}\mathcal{D}_1 + V^{(m)}\mathcal{D}_2 = \tilde{g} \tag{52}$$

together with conditions associated with M. Once f is solved from the transformed system, (51) yields the solution f of the original problem. The transform of g(x), i.e.,

$$\tilde{g} = \int_{C_o}^{C_n} g(x)\phi^{(m)}A_3p_3dx$$

is to be computed from the given g(x). The result, just as g itself, may involve the other independent variables.

V. MULTI-INTERVAL FOURIER AND HANKEL TRANSFORMS

Details about two special cases which are most often used will be given in this section to facilitate applications. These are the multi-interval counterparts of the various Fourier and Hankel transforms.

Case 1 Multi-Interval Fourier Transform.

$$E_i(x;\xi) = \cos\left(\frac{\xi x}{\beta_i}\right) \tag{53}$$

$$F_i(x;\xi) = \sin\left(\frac{\xi x}{\beta_i}\right) \tag{54}$$

$$\phi_i^{(m)}(x) = \gamma_i \cos\left(\frac{\xi^{(m)}x}{\beta_i}\right) + \delta_i \sin\left(\frac{\xi^{(m)}x}{\beta_i}\right)$$

where

$$\beta_i = \sqrt{A_{1i}/A_{3i}} > 0 \tag{55}$$

Without loss of generality, we will always take $C_0 = 0$ and $C_n = X$. The transform is then

$$\tilde{f}(m) = \sum_{i=1}^{n} A_{3i} \int_{C_{i-1}}^{C_i} f(x)\left[\gamma_i \cos\left(\frac{\xi^{(m)}x}{\beta_i}\right) + \delta_i \sin\left(\frac{\xi^{(m)}x}{\beta_i}\right)\right]dx \tag{56}$$

where

$$\gamma_{j+1} = \frac{\xi^{(m)}}{\beta_{j+1}}\{\left[\left(\frac{\alpha_j}{H_j}\right)A_{1j}\sigma_j + \tau_j\right]\left(\frac{\xi^{(m)}}{\beta_{j+1}}\right)\cos\left(\frac{\xi^{(m)}C_j}{\beta_{j+1}}\right)$$

$$- \frac{A_{1j}\sigma_j}{A_{1(j+1)}} \sin\left(\frac{\xi^{(m)}C_j}{\beta_{j+1}}\right)\} \tag{57}$$

$$\delta_{j+1} = \frac{\xi^{(m)}}{\beta_{j+1}}\{\left[\left(\frac{\alpha_j}{H_j}\right)A_{1j}\sigma_j + \tau_j\right]\left(\frac{\xi^{(m)}}{\beta_{j+1}}\right)\sin\left(\frac{\xi^{(m)}C_j}{\beta_{j+1}}\right)$$

$$+ \frac{A_{1j}\sigma_j}{A_{1(j+1)}} \cos\left(\frac{\xi^{(m)}C_j}{\beta_{j+1}}\right)\} \tag{58}$$

$$\sigma_j = -\gamma_j\left(\frac{\xi^{(m)}}{\beta_j}\right)\sin\left(\frac{\xi^{(m)}C_j}{\beta_j}\right) + \delta_j\left(\frac{\xi^{(m)}}{\beta_j}\right)\cos\left(\frac{\xi^{(m)}C_j}{\beta_j}\right) \tag{59}$$

$$\tau_j = \gamma_j\cos\left(\frac{\xi^{(m)}C_j}{\beta_j}\right) + \delta_j \sin\left(\frac{\xi^{(m)}C_j}{\beta_j}\right) \tag{60}$$

and it is convenient to set

$$\gamma_1 = a_1\left(\frac{\xi^{(m)}}{\beta_1}\right),\ a_2 \neq 0; = 1,\ a_2 = 0 \tag{61}$$

$$\delta_1 = a_2 \tag{62}$$

The operator L is

$$L(\ ) = \frac{A_{1i}}{A_{3i}} (\ )'', \quad C_{i-1}<x<C_i \tag{63}$$

and

$$\widetilde{L(f)} = -[\xi^{(m)}]^2\tilde{f} + U^{(m)}(a_1 f' - a_2 f)|_{x = 0}$$

$$+ V^{(m)}(a_3 f' + a_4 f)|_{x = \chi} \tag{64}$$

where

$$U^{(m)} = -(\frac{A_{11}}{a_1})\gamma_1, \quad a_1 \neq 0; = -(\frac{A_{11}}{a_2})\delta_1(\frac{\xi^{(m)}}{\beta_1}), \quad a_1 = 0 \tag{65}$$

$$V^{(m)} = \frac{A_{1n}}{a_3}[\gamma_n \cos(\frac{\xi^{(m)}\chi}{\beta_n}) + \delta_n \sin(\frac{\xi^{(m)}\chi}{\beta_n})], \quad a_3 \neq 0;$$

$$= -(\frac{A_{1n}}{a_4})(\frac{\xi^{(m)}}{\beta_n})[-\gamma_n \sin(\frac{\xi^{(m)}\chi}{\beta_n}) + \delta_n \cos(\frac{\xi^{(m)}\chi}{\beta_n})], \quad a_3 = 0 \tag{66}$$

The inversion formula is

$$f(x) = \sum_{m=1}^{\infty} \tilde{f}(m)[\gamma_i \cos(\frac{\xi^{(m)}x}{\beta_i}) + \delta_i \sin(\frac{\xi^{(m)}x}{\beta_i})]/N^{(m)} \tag{67}$$

where the summation is taken over all the non-negative roots of

$$a_3(\frac{\xi^{(m)}}{\beta_n})[-\gamma_n \sin(\frac{\xi^{(m)}\chi}{\beta_n}) + \delta_n \cos(\frac{\xi^{(m)}\chi}{\beta_n})]$$

$$+ a_4[\gamma_n \cos(\frac{\xi^{(m)}\chi}{\beta_n}) + \delta_n \sin(\frac{\xi^{(m)}\chi}{\beta_n})] = 0 \tag{68}$$

and

$$N^{(m)} = (1/2)\{\sum_{i=1}^{n} A_{3i}(\gamma_i^2 + \delta_i^2)(C_i - C_{i-1})$$

$$+ A_{11}a_1a_2(\gamma_1^2 + \delta_1^2)(\frac{1}{\beta_1})^2/[(\frac{a_1\xi^{(m)}}{\beta_1})^2 + a_2^2]$$

$$+ A_{1n}a_3a_4(\gamma_n^2 + \delta_n^2)(\frac{1}{\beta_n})^2/[(\frac{a_3\xi^{(m)}}{\beta_n})^2 + a_4^2]$$

$$+ \sum_{j=1}^{n-1} \frac{A_{1j}^2}{\beta_j^2}(\frac{\alpha_j}{H_j})[-\gamma_j \sin(\frac{\xi^{(m)}C_j}{\beta_j}) + \delta_j \cos(\frac{\xi^{(m)}C_j}{\beta_j})]\}$$

$$(\text{with } C_0 = 0,\ C_n = \chi) \qquad (69)$$

There are many possible (equivalent) forms for $N^{(m)}$. (69) is thought to be most convenient because of its symmetry.

Case 2 Multi-Interval Hankel Transforms

$$E_i(x;\xi) = J_\nu(\frac{\xi x}{\beta_i}) \qquad (70)$$

$$F_i(x;\xi) = Y_\nu(\frac{\xi x}{\beta_i}) \qquad (71)$$

$$\phi_i^{(m)} = \gamma_i J_\nu(\frac{\xi^{(m)} x}{\beta_i}) + \delta_i Y_\nu(\frac{\xi^{(m)} x}{\beta_i})$$

where J_ν is the Bessel function of the first kind, of order ν; Y_ν, of the second kind, if ν is an integer. If ν is not an integer, $Y_\nu = J_{-\nu}$. The order ν is non-negative. The transform is now

$$\tilde{f}(m) = \sum_{i=1}^{n} A_{3i} \int_{C_{i-1}}^{C_i} f(x)[\gamma_i J_\nu(\frac{\xi x}{\beta_i}) + \delta_i Y_\nu(\frac{\xi x}{\beta_i})]x\,dx$$

We will set for convenience[2]

$$\gamma_1 = a_1\left(\frac{\xi^{(m)}}{\beta_1}\right)\dot{Y}_\nu\left(\frac{\xi C_o}{\beta_1}\right) - a_2 Y_\nu\left(\frac{\xi^{(m)} C_o}{\beta_1}\right),\ a_2 \text{ and } C_o \neq 0;$$

$$= a_1\dot{Y}_\nu\left(\frac{\xi^{(m)} C_o}{\beta_1}\right),\ a_2 = 0 \text{ and } C_o \neq 0; = 1,\ C_o = 0 \tag{72}$$

$$\delta_1 = -[a_1\left(\frac{\xi^{(m)}}{\beta_1}\right)\dot{J}_\nu\left(\frac{\xi^{(m)} C_o}{\beta_1}\right) - a_2 J_\nu\left(\frac{\xi^{(m)} C_o}{\beta_1}\right)],\ a_2 \text{ and } C_o \neq 0;$$

$$= -a_1\dot{J}_\nu\left(\frac{\xi C_o}{\beta_1}\right),\ a_2 = 0 \text{ and } C_o \neq 0; = 0,\ C_o = 0 \tag{73}$$

$$\gamma_{j+1} = \frac{\pi C_j}{2}\{[\left(\frac{\alpha_j}{H_j}\right)A_{1j}\sigma_j + \tau_j]\left(\frac{\xi^{(m)}}{\beta_{j+1}}\right)\dot{Y}_\nu\left(\frac{\xi^{(m)} C_j}{\beta_{j+1}}\right)$$

$$- \frac{A_{1j}\sigma_j}{A_{1(j+1)}} Y_\nu\left(\frac{\xi^{(m)} C_j}{\beta_{j+1}}\right)\} \tag{74}$$

$$\delta_{j+1} = -\frac{\pi C_j}{2}\{[\left(\frac{\alpha_j}{H_j}\right)A_{1j}\sigma_j + \tau_j]\left(\frac{\xi^{(m)}}{\beta_{j+1}}\right)\dot{J}_\nu\left(\frac{\xi^{(m)} C_j}{\beta_{j+1}}\right)$$

$$- \frac{A_{1j}\sigma_j}{A_{1(j+1)}} J_\nu\left(\frac{\xi^{(m)} C_j}{\beta_{j+1}}\right)\} \tag{75}$$

$$\sigma_j = \gamma_j\left(\frac{\xi^{(m)}}{\beta_j}\right)\dot{J}_\nu\left(\frac{\xi^{(m)} C_j}{\beta_j}\right) + \delta_j\left(\frac{\xi^{(m)}}{\beta_j}\right)\dot{Y}_\nu\left(\frac{\xi^{(m)} C_j}{\beta_j}\right) \tag{76}$$

$$\tau_j = \gamma_j J_\nu\left(\frac{\xi^{(m)} C_j}{\beta_j}\right) + \delta_j Y_\nu\left(\frac{\xi^{(m)} C_j}{\beta_j}\right) \tag{77}$$

where $\dot{Y}_\nu(\frac{\xi x}{\beta})$, etc. represents $d\dot{Y}_\nu(\frac{\xi x}{\beta})/d(\frac{\xi x}{\beta})$, etc.

[2]This is quite different from the single-interval case.

The operator L(f) is

$$L(f) = \frac{A_{1i}}{A_{3i}}[(xf')' - \frac{\nu^2}{x}]/x, \quad C_{i-1}<x<C_i \tag{78}$$

and

$$\widetilde{L(f)} = -[\xi^{(m)}]^2\tilde{f} + U^{(m)}(a_1 f' - a_2 f)|_{x = C_o} + V^{(m)}(a_3 f' + a_4 f)|_{x = C_n} \tag{79}$$

where

$$U^{(m)} = -\frac{A_{11}C_o}{a_1}[\gamma_1 J_\nu(\frac{\xi^{(m)}C_o}{\beta_1}) + \delta_1 Y_\nu(\frac{\xi^{(m)}C_o}{\beta_1})], \quad a_1 \neq 0;$$

$$= -\frac{A_{11}C_o}{a_2}[\gamma_1(\frac{\xi^{(m)}}{\beta_1})\dot{J}_\nu(\frac{\xi^{(m)}C_o}{\beta_1}) + \delta_1(\frac{\xi^{(m)}}{\beta_1})\dot{Y}_\nu(\frac{\xi^{(m)}C_o}{\beta_1})], \quad a_1 = 0 \tag{80}$$

$$V^{(m)} = \frac{A_{1n}C_n}{a_3}[\gamma_1 J_\nu(\frac{\xi^{(m)}C_n}{\beta_n}) + \delta_1 Y_\nu(\frac{\xi^{(m)}C_n}{\beta_n})], \quad a_3 \neq 0;$$

$$= -\frac{A_{1n}C_n}{a_4}[\gamma_1(\frac{\xi^{(m)}}{\beta_n})\dot{J}_\nu(\frac{\xi^{(m)}C_n}{\beta_n}) + \delta_1\dot{Y}_\nu(\frac{\xi^{(m)}C_n}{\beta_n})], \quad a_3 = 0 \tag{81}$$

Note that in the above we have chosen $A_2 = A_1$. Furthermore, $\nu \neq 0$ is carried for generality. In applications, the case $\nu = 0$ is the most important.

As for the inversion formula, we have

$$f(x) = \sum_{m=1}^{\infty} \tilde{f}(m)[\gamma_i J_\nu(\frac{\xi^{(m)}x}{\beta_i}) + \delta_i Y_\nu(\frac{\xi^{(m)}x}{\beta_i})]/N^{(m)} \tag{82}$$

summed over all the non-negative roots of

$$a_3(\frac{\xi^{(m)}}{\beta_n})[\gamma_n \dot{J}_\nu(\frac{\xi^{(m)}C_n}{\beta_n}) + \delta_n \dot{Y}_\nu(\frac{\xi^{(m)}C_n}{\beta_n})]$$

$$+ a_4[\gamma_n J_\nu(\frac{\xi^{(m)} C_n}{\beta_n}) + \delta_n Y_\nu(\frac{\xi^{(m)} C_n}{\beta_n})] = 0 \tag{83}$$

where

$$N^{(m)} = \frac{C_n^{\,2} - C_o^{\,2}}{2} + \frac{1}{2[\xi^{(m)}]^2} \sum_{i=1}^{n} \beta_i^2 [(x\phi_i^{(m)\prime})^2 - \nu^2 \phi_i^2]_{C_{i-1}}^{C_i} \tag{84}$$

Again (84) is only one of many possible forms.

VI. TRANSIENT TWO-LAYER LIQUID FLOW

The general multi-interval transform has applications in many fields of engineering sciences. For applications to transient heat conduction in multi-layered bodies, see Tittle, 1965; Bulavin and Kashcheev, 1965; Giere, 1968. (Although the method of multi-interval transform was not employed in these investigations, it should be extremely easy to regain the results in these references.) For illustrations, we will work out in this section two unsolved problems in transient, incompressible, fluid flow: (i) the stoppage of a two-layer Couette flow; (ii) the turning-off of the pumping device in a two-layer Poisseuille flow. (For classical Couette and Poisseuille flows, see, for example, Schlichting, 1968.) Both problems have counterparts in the theory of heat conduction; but we prefer to treat them in the hydrodynamic framework.

(i) The system to be solved here is as follows:

$$\frac{\partial w_1}{\partial t} = \frac{\eta_1}{\rho_1} \frac{\partial^2 w_1}{\partial x^2}, \quad 0<x<C \tag{85}$$

$$\frac{\partial w_2}{\partial t} = \frac{\eta_2}{\rho_2} \frac{\partial^2 w_2}{\partial x^2}, \quad C<x<\mathcal{X} \tag{86}$$

$$x = 0 : w_1 = 0 \tag{87}$$

$$x = \mathcal{X} : w_2 = 0 \tag{88}$$

$$x = C : w_1 = w_2 \tag{89}$$

$$\eta_1 \frac{\partial w_1}{\partial x} = \eta_2 \frac{\partial w_2}{\partial x} \tag{90}$$

$$t = 0 : w_i = w_i^o, \ i = 1, 2 \tag{91}$$

where

$$\left\{\begin{array}{ll} 0 = \frac{\eta_1}{\rho_1}\frac{d^2 w_1^o}{dx^2}, \quad 0<x<C & (92) \\ 0 = \frac{\eta_2}{\rho_2}\frac{d^2 w_2^o}{dx^2}, \quad C<x<X & (93) \\ x = 0 : w_1^o = 0 & (94) \\ x = X : w_2^o = W & (95) \\ x = C : w_1^o = w_2^o & (96) \\ \qquad \eta_1 \frac{dw_1^o}{dx} = \eta_2 \frac{dw_2^o}{dx} & (97) \end{array}\right.$$

In the above, subscripts 1 and 2 refer to liquid layers 1 and 2. w_i is the transient velocity for $t > 0$ parallel to the fixed plates $x = 0$ and $x = X$; while w_i^o is the steady velocity for $t < 0$, with the plate $x = X$ moving at a constant rate W. ρ_i and η_i are, respectively, the density and viscosity.

To solve the problem it is obvious that we shou ld use the Fourier transform with $n = 2$, $C_1 = C$, $A_{1i} = \mu_i$, $A_{3i} = \rho_i$, $a_1 = 0$, $a_3 = 0$, $a_2 = 1$, $a_4 = 1$, $\alpha_1 = 0$; i.e.

$$\bar{f}(m) = \int_0^C f\rho_1 \sin\frac{\xi^{(m)}x}{\beta_1}dx + \int_C^X f\rho_2[\gamma_2\cos\frac{\xi^{(m)}x}{\beta_2} + \delta_2\sin\frac{\xi^{(m)}x}{\beta_2}]dx \qquad (98)$$

where

$$\beta_i = \sqrt{\eta_i/\rho_i}, \qquad (99)$$

$$\gamma_2 = \sin\frac{\xi^{(m)}C}{\beta_1}\cos\frac{\xi^{(m)}C}{\beta_2} - \frac{\beta_2}{\beta_1}\left(\frac{\eta_1}{\eta_2}\right)\cos\frac{\xi^{(m)}C}{\beta_1}\sin\frac{\xi^{(m)}C}{\beta_2} \qquad (100)$$

$$\delta_2 = \sin\left(\frac{\xi^{(m)}C}{\beta_1}\right)\sin\left(\frac{\xi^{(m)}C}{\beta_2}\right) + \frac{\beta_2}{\beta_1}\left(\frac{\eta_1}{\eta_2}\right)\cos\frac{\xi^{(m)}C}{\beta_1}\cos\frac{\xi^{(m)}C}{\beta_2} \qquad (101)$$

Applying this transform to the system, we have

$$\begin{cases} \dfrac{d\tilde{w}}{dt} = -[\xi^{(m)}]^2\tilde{w} & (102) \\ t = 0 : \tilde{w} = \tilde{w}^0 & (103) \end{cases}$$

where

$$-[\xi^{(m)}]^2\tilde{w}^0 - \eta_2 w(\frac{\xi^{(m)}}{\beta_2})[\delta_2\cos\frac{\xi^{(m)}X}{\beta_2} - \gamma_2\sin\frac{\xi^{(m)}X}{\beta_2}] = 0 \qquad (104)$$

$$\therefore \quad \tilde{w} = \tilde{w}_0 \exp\{-[\xi^{(m)}]^2 t\} \qquad (105)$$

where

$$\tilde{w}_0 = -\frac{\eta_2 W}{\beta_2\xi^{(m)}}[\delta_2\cos\frac{\xi^{(m)}X}{\beta_2} - \gamma_2\sin\frac{\xi^{(m)}X}{\beta_2}] \qquad (106)$$

Finally, the inversion formula yields

$$w = \sum_{m=1}^{\infty} \tilde{w} \sin\frac{\xi^{(m)}x}{\beta_1}/N^{(m)} \quad , \; 0<x<C$$

$$= \sum_{m=1}^{\infty} \tilde{w}[\gamma_2\cos\frac{\xi^{(m)}x}{\beta_2} + \delta_2\sin\frac{\xi^{(m)}x}{\beta_2}]/N^{(m)} \quad , \; C<x<X \qquad (107)$$

where

$$N^{(m)} = (1/2)\,\rho_1 C + \rho_2(\gamma_2^2 + \delta_2^2)(X - C)\} \qquad (108)$$

and the summation is over all the non-negative roots of

$$\gamma_2\cos\frac{\xi^{(m)}X}{\beta_2} + \delta_2\sin\frac{\xi^{(m)}X}{\beta_2} = 0 \qquad (109)$$

(ii) The system to be solved here is, similarly to (i):

$$\frac{\partial w_1}{\partial t} = \frac{\eta_1}{\rho_1}\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w_1}{\partial r}) + D_1 e^{-\delta t} \tag{110}$$

$$\frac{\partial w_2}{\partial t} = \frac{\eta_2}{\rho_2}\frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial w_2}{\partial r}) + D_2 e^{-\delta t} \tag{111}$$

$$r = 0 : w_1, \text{ etc., finite} \tag{112}$$

$$r = R : w_2 = 0 \tag{113}$$

$$r = C : w_1 = w_2 \tag{114}$$

$$\eta_1\frac{\partial w_1}{\partial r} = \eta_2\frac{\partial w_2}{\partial r} \tag{115}$$

$$t = 0 : w_i = w_i^o \tag{116}$$

where

$$0 = \frac{\eta_1}{\rho_1}\frac{1}{r}\frac{d}{dr}(r\frac{dw_1^o}{dr}) + D_1 \tag{117}$$

$$0 = \frac{\eta_2}{\rho_2}\frac{1}{r}\frac{d}{dr}(r\frac{dw_2^o}{dr}) + D_2 \tag{118}$$

$$r = 0 : w_1^o, \text{ etc., finite} \tag{119}$$

$$r = R : w_2^o = 0 \tag{120}$$

$$r = C : w_1^o = w_2^o \tag{121}$$

$$\eta_1\frac{dw_1^o}{dr} = \eta_2\frac{dw_2^o}{dr} \tag{122}$$

The duct here is of circular sections. The pressure-gradient parameters associated with the pumping device are D_1, D_2. The diminishing factor of the turning off is δ.

We use the special double-interval Hankel transform, with $\nu = 0$ (the other parameters are the same as in (i)):

$$\tilde{f}(m) = \int_0^C f\rho_1 J_o(\frac{\xi^{(m)}r}{\beta_1})rdr + \int_C^R f\rho_2[\gamma_2 J_o(\frac{\xi^{(m)}r}{\beta_2}) + \delta_2 Y_o(\frac{\xi^{(m)}r}{\beta_2})]rdr \tag{123}$$

where

$$\gamma_2 = -\frac{\pi}{2}(\xi^{(m)}C)\{(\frac{\eta_1}{\eta_2})(\frac{1}{\beta_1})\dot{J}_o(\frac{\xi^{(m)}C}{\beta_1})Y_o(\frac{\xi^{(m)}C}{\beta_2})$$

$$-(\frac{1}{\beta_2})\dot{Y}_o(\frac{\xi^{(m)}C}{\beta_2})J_o(\frac{\xi^{(m)}C}{\beta_1})\} \quad (124)$$

$$\delta_2 = -\frac{\pi}{2}(\xi^{(m)}C)\{(\frac{1}{\beta_2})\dot{J}_o(\frac{\xi^{(m)}C}{\beta_2})J_o(\frac{\xi^{(m)}C}{\beta_1})$$

$$-\frac{\eta_1}{\eta_2}(\frac{1}{\beta_1})\dot{J}_o(\frac{\xi^{(m)}C}{\beta_1})J_o(\frac{\xi^{(m)}C}{\beta_2})\} \quad (125)$$

$$\frac{d\tilde{w}}{dt} = -[\xi^{(m)}]^2\tilde{w} + \tilde{\mathcal{D}}e^{-\delta t} \quad (126)$$

$$t = 0 : \tilde{w} = \tilde{w}^o \quad (127)$$

where

$$0 = -[\xi^{(m)}]^2\tilde{w}^o + \tilde{\mathcal{D}} \quad (128)$$

and

$$\tilde{\mathcal{D}} = \frac{\rho_1\beta_1\mathcal{D}_1C}{\xi^{(m)}}J_o(\frac{\xi^{(m)}C}{\beta_1})$$

$$+\frac{\rho_2\beta_2\mathcal{D}_2}{\xi^{(m)}}\{\gamma_2[RJ_o(\frac{\xi^{(m)}R}{\beta_2}) - CJ_o(\frac{\xi^{(m)}C}{\beta_2})]$$

$$+\delta_2[RY_o(\frac{\xi^{(m)}R}{\beta_2}) - CY_o(\frac{\xi^{(m)}C}{\beta_2})]\} \quad (129)$$

Thus

$$\tilde{w} = \{\tilde{w}_o - \frac{[\xi^{(m)}]^2 - \tilde{\mathcal{D}}}{\delta}\}\exp\{-[\xi^{(m)}]^2t\} + \frac{[\xi^{(m)}]^2\tilde{\mathcal{D}}}{\delta}e^{-\delta t} \quad (130)$$

where

$$\tilde{w}_o = \tilde{\mathcal{D}}/[\xi^{(m)}]^2 \quad (131)$$

The inversion formula yields

$$w = \sum_{m=1}^{\infty} \tilde{w} J_0(\frac{\xi^{(m)} r}{\beta_1})/N^{(m)}, \quad 0<x<C$$

$$= \sum_{m=1}^{\infty} \tilde{w}[\gamma_2 J_0(\frac{\xi^{(m)} r}{\beta_2}) + \delta_2 Y_0(\frac{\xi^{(m)} r}{\beta_2})]/N^{(m)}, \quad C<x<R \tag{132}$$

where

$$N^{(m)} = \frac{R^2}{2} + \frac{1}{2}[C\dot{J}_0(\frac{\xi^{(m)} C}{\beta})]^2$$

$$+ \{R[\gamma_2 \dot{J}_0(\frac{\xi^{(m)} R}{\beta_2}) + \delta_2 \dot{Y}_0(\frac{\xi^{(m)} R}{\beta_2})]\}^2$$

$$- \{C[\gamma_2 \dot{J}_0(\frac{\xi^{(m)} C}{\beta_2}) + \delta_2 \dot{Y}_0(\frac{\xi^{(m)} C}{\beta_2})]\}^2 \tag{133}$$

and the summation is over all the non-negative roots of

$$\gamma_2 J_0(\frac{\xi^{(m)} R}{\beta_2}) + \delta_2 Y_0(\frac{\xi^{(m)} R}{\beta_2}) = 0 \tag{134}$$

VII. A FINAL COMMENT

The previous special cases and applications illustrate only a portion of the general scope. As a final comment, we wish to point out especially one possibility which has no single-interval counterpart. This is best described with respect to the following example.

Consider the double-interval system:

$$A_{11}\phi_1'' - A_{21}\phi_1 + \lambda A_{31}\phi_1 = 0, \quad 0<x<C_1 \tag{135}$$

$$A_{12}\phi_2'' - A_{22}\phi_2 + \lambda A_{32}\phi_2 = 0, \quad C_1<x<X \tag{136}$$

$$\phi_1(C_1^-) = \phi_2(C_1^+) \tag{137}$$

$$A_{11}\phi_1'(C_1^-) = A_{12}\phi_2'(C_1^+) \tag{138}$$

$$\phi_1(0) = 0 \tag{139}$$

$$\phi_2(X) = 0 \tag{140}$$

The eigenfunctions here are

$$\phi_1^{(m)} = \begin{Bmatrix}\sin \\ \sinh\end{Bmatrix}\left(\frac{\lambda^{(m)}A_{31} - A_{21}}{A_{11}}\right)^{1/2}x, \quad \lambda^{(m)}\begin{Bmatrix}\geq \\ <\end{Bmatrix}A_{21}/A_{31} \tag{141}$$

$$\phi_2^{(m)} = \alpha_2\begin{Bmatrix}\sin \\ \sinh\end{Bmatrix}\left(\frac{\lambda^{(m)}A_{32} - A_{22}}{A_{12}}\right)^{1/2}x$$

$$+ \beta_2\begin{Bmatrix}\cos \\ \cosh\end{Bmatrix}\left(\frac{\lambda^{(m)}A_{32} - A_{22}}{A_{12}}\right)^{1/2}x, \quad \lambda^{(m)}\begin{Bmatrix}\geq \\ <\end{Bmatrix}A_{22}/A_{32} \tag{142}$$

where

$$\alpha_2 = - \left| \begin{matrix} \begin{Bmatrix}\sin \\ \sinh\end{Bmatrix}\left(\frac{\lambda^{(m)}A_{31} - A_{21}}{A_{11}}\right)^{1/2}C_1 \\ \left(\frac{A_{11}}{A_{12}}\right)^{3/2}\left(\frac{\lambda^{(m)}A_{31} - A_{21}}{\lambda^{(m)}A_{32} - A_{22}}\right)^{1/2}\begin{Bmatrix}\cos \\ \cosh\end{Bmatrix}\left(\frac{\lambda^{(m)}A_{31} - A_{21}}{A_{11}}\right)^{1/2}C_1 \end{matrix} \right.$$

$$\left. \begin{matrix} \begin{Bmatrix}\cos \\ \cosh\end{Bmatrix}\left(\frac{\lambda^{(m)}A_{32} - A_{22}}{A_{12}}\right)^{1/2}C_1 \\ \begin{Bmatrix}-\sin \\ \sinh\end{Bmatrix}\left(\frac{\lambda^{(m)}A_{32} - A_{22}}{A_{12}}\right)^{1/2}C_1 \end{matrix} \right| \tag{143}$$

$$\beta_2 = - \left| \begin{matrix} \begin{Bmatrix}\sin \\ \sinh\end{Bmatrix}\left(\frac{\lambda^{(m)}A_{32} - A_{22}}{A_{12}}\right)^{1/2}C_1 \\ \begin{Bmatrix}\cos \\ \cosh\end{Bmatrix}\left(\frac{\lambda^{(m)}A_{32} - A_{22}}{A_{12}}\right)^{1/2}C_1 \end{matrix} \right.$$

$$\left. \begin{matrix} \begin{Bmatrix}\sin \\ \sinh\end{Bmatrix}\left(\frac{\lambda^{(m)}A_{31} - A_{21}}{A_{11}}\right)^{1/2}C_1 \\ \left(\frac{A_{11}}{A_{12}}\right)^{3/2}\left(\frac{\lambda^{(m)}A_{31} - A_{21}}{\lambda^{(m)}A_{32} - A_{22}}\right)^{1/2}\begin{Bmatrix}\cos \\ \cosh\end{Bmatrix}\left(\frac{\lambda^{(m)}A_{31} - A_{21}}{A_{11}}\right)^{1/2}C_1 \end{matrix} \right| \tag{144}$$

The eigenvalues $\lambda^{(m)}$ (not $\xi^{(m)}$) are the non-negative real roots of the transcendental equation:

$$\alpha_2\left\{\begin{matrix}\sin\\ \sinh\end{matrix}\right\}\left(\frac{\lambda^{(m)}A_{32}-A_{22}}{A_{12}}\right)^{1/2}X$$

$$+\ \beta_2\left\{\begin{matrix}\cos\\ \cosh\end{matrix}\right\}\left(\frac{\lambda^{(m)}A_{32}-A_{22}}{A_{12}}\right)^{1/2}X = 0,\ \lambda^{(m)}\left\{\begin{matrix}\geq\\ <\end{matrix}\right\}A_{22}/A_{32} \tag{145}$$

The corresponding transform $\tilde{(\ )}$ may be called the double-interval, modified, Fourier transform. Its use lies in the following important property:

$$\widetilde{\left(\frac{A_1}{A_3}f'' - \frac{A_2}{A_3}f\right)} = -\lambda^{(m)}\tilde{f} + [A_{11}\phi_1^{(m)\prime}(0)]f(0) - [A_{12}\phi_2^{(m)\prime}(X)]f(X) \tag{146}$$

if f and $A_1 f$ are continuous throughout.

As an illustration, consider the commencement of the magnetohydrodynamic Couette flow (Sutton and Sherman, 1965) with two liquid layers, $0<x<C_1$ and $C_1<x<X$ of different electric conductivities (σ_1 and σ_2), viscosities (μ_1,μ_2) and densities (ρ_1,ρ_2). The lower plate x = 0 is fixed; the upper plate starts to move suddenly at time t = 0 with a uniform speed W in the z - direction, parallel to itself. For $t \leqslant 0$, the liquids are at rest and a magnetic field of strength H_0 is established in the x - direction. There is no current-return connections at y = ±∞. Since the magnetic Prandtl number is small ($\tilde{}10^{-6}$) even for liquid metals (Sears, 1961), we will take advantage of this fact and neglect the field change due to the fluid flow. On the other hand, we will assume that H_0 is so strong as to make itself felt by the flow field through a Lorentz force proportional to the local flow velocity w_1 or w_2. In short, we wish to solve the following problems:

$$\rho\frac{\partial w}{\partial t} = \mu\frac{\partial^2 w}{\partial x^2} - \sigma B_0^2 w \tag{147}$$

where

$$\left.\begin{matrix}w = w_1\\ \rho = \rho_1\\ \mu = \mu_1\\ \sigma = \sigma_1\end{matrix}\right\},\quad 0<x<C_1$$

$$\left.\begin{array}{l} w = w_2 \\ \rho = \rho_2 \\ \mu = \mu_2 \\ \sigma = \sigma_2 \end{array}\right\}, \quad C_1<x<X$$

and B_O is the magnetic induction due to H_O.

Initial condition:

$$w(x,0) = 0 \tag{148}$$

Boundary conditions:

$$w_1(0,t) = 0 \tag{149}$$

$$w_2(X,t) = W \tag{150}$$

Marching conditions:

$$w_1(C_1^-,t) = w_2(C_1^+,t) \tag{151}$$

$$\mu_1\frac{\partial w_1}{\partial x}\bigg|_{C_1^-} = \mu_2\frac{\partial w_2}{\partial x}\bigg|_{C_1^+} \tag{152}$$

To solve this problem we apply the double-interval modified Fourier transform, with $A_1 = \mu$, $A_2 = \sigma B_O^2 (A_{21} = \sigma_1 B_O^2, A_{22} = \sigma_2 B_O^2)$, $A_3 = \rho$:

$$\begin{cases} \dfrac{d\tilde{w}}{dt} = -\lambda^{(m)}\tilde{w} - [\mu_2\phi_2^{(m)\prime}(X)]W & (153) \\ \tilde{w}(m,0) = 0 & (154) \end{cases}$$

or,

$$\tilde{w} = \frac{[\mu_2\phi_2^{(m)\prime}(X)]W}{\lambda^{(m)}}(e^{-\lambda^{(m)}t}-1) \tag{155}$$

$$w_1 = \sum_{m=1}^{\infty} \frac{\tilde{w}}{N^{(m)}}\phi_1^{(m)}(x), \quad 0<x<C_1 \tag{156}$$

$$w_2 = \sum_{m=1}^{\infty} \frac{\tilde{w}}{N^{(m)}}\phi_2^{(m)}(x), \quad C_1<x<X \tag{157}$$

where

$$N^{(m)} = \int_0^{C_1} \rho_1[\phi_1^{(m)}]^2\, dx + \int_{C_1}^{X} \rho_2[\phi_2^{(m)}]^2 dx \tag{158}$$

can be calculated directly.

REFERENCES CITED

Bulavin, P. E., and Kashcheev, V. M., 1965. Solution of nonhomogeneous heat-conduction equation for multilayered bodies: International Chemical Engineering, 5: 112-115.

Churchill, R. V., 1955. Extensions of operational mathematics: Proceedings Conference on Differential Equations: 235-250: University of Maryland.

Cinelli, G., 1965. An extension of the finite Hankel transform and applications: International Journal of Engineering Science, 3:539-559.

Eringer, A. C., 1954. The finite Sturm-Liouville transform: Quarterly Journal of Mathematics, Oxford (2), 5:120-129.

Giere, A. C., 1968. An eigenfunction solution of a problem in heat conduction: American Journal of Physics, 36:994-1000.

Naimark, M. A., 1968. Linear differential operators, Part II, (English version): New York, F. Ungar Publishing Company, 352 p. and xv.

Roettinger, I., 1947. A generalization of the finite Fourier transformation and applications: Quarterly of Applied Mathematics, 5:298-319.

Schlichting, H., 1968. Boundary layer theory 6th edition: New York, McGraw-Hill Book Company, 747 p. and xix.

Sears, W. R., 1961. On a boundary-layer phenomenon in magneto-fluid dynamics: Astronautica Acta, 7: 223-236.

Sneddon, I. N., 1955. Functional analysis: Handbuch der Physik, II: 198-348: Berlin, Springer-Verlag.

Sutton, G. W., and Sherman, A., 1965. Engineering magnetohydrodynamics: New York, McGraw-Hill Book Company, 548 p. and xix.