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The S(q,t) Summability Transform

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MATHEMATICS

THE S(q,t) SUMMABILITY TRANSFORM [1]

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1. Introduction. Robert E. Powell [7] has introduced a matrix of Summability which is a generalization of the Taylor or T(r) method. He studies this special class of Summability matrices, related to the Laguerre polynomials in the following way. For complex r and t, denote by L(r,t) the matrix whose elements are $a_{n,k} = 0$ when $k < n$,

and

$$a_{n,k} = (1 - r)^{n+1} \exp(tr/1-r)\, L^{n}_{k-n}(t)\, r^{k-n} \quad \text{when } k \geq n$$

$L^{n}_{j}(t)$ being the Laguerre polynomial of degree j, given by

$$L^{n}_{j}(t) = \sum_{i=0}^{j} \binom{j+n}{j-i} (-t)^{i}/i!$$

In this paper we consider the associated matrix S(q,t) defined as:

$$b_{n,k} = (1 - q)^{n+1} \exp(tq/1-q)\, L^{n}_{k}(t)\, q^{k} \quad k,n = 0,1,2,\ldots$$

The matrix S(q,t) can be obtained by shifting each row of the matrix L(q,t) to the left until the diagonal elements appear in the first column, thus making a matrix which contains no zeros (except in the trivial case of q = 0 or q = 1). The special case S(q,0) is the well-known associated Taylor matrix S(q) [5]. Thus S(q,t) is a generalization of S(q).

In § 2 we examine the regularity of S(q,t). In § 3 we examine the summability of the geometric series by means of the S(q,t) transform. In § 4 we study the summability of the geometric series by means of the iterated product method A·S(q,t) where A denotes the Abel method of summability. Finally in § 5 we determine sufficient conditions on p and q which ensure that each sequence that is summable by the Euler or E(p) method is summable by the S(q,t) method for all t.

2. Regularity. A matrix $C = (c_{n,k})$ is regular if and only if the well-known Silverman-Toeplitz conditions:

[1] This paper is a portion of the author's doctoral dissertation written at the University of Pittsburgh in 1967-1968 under the direction of Professor George Laush.

$$(1) \qquad \lim_{n\to\infty} \sum_{k=0}^{\infty} c_{n,k} = 1,$$

$$(2) \qquad \lim_{n\to\infty} c_{n,k} = 0 \quad k = 0,1,\ldots,$$

$$(3) \qquad \sup_{n} \{ \sum_{k=0}^{\infty} |c_{n,k}| \} < \infty$$

are satisfied.

Theorem 1. (i) If $S(z,t)$ is regular for some real or complex t, then $|z| \leqslant 1$.

(ii) If $S(z,t)$ is regular for some $t \leqslant 0$, then $\mathrm{Im}(z) = o$, and $0 \leqslant \mathrm{Re}(z) < 1$.

(iii) For a given value z, $S(z,t)$ is regular for each t if and only if $\mathrm{Im}(z) = 0$, and $0 \leqslant \mathrm{Re}(z) < 1$.

(iv) If $t \leqslant 0$, $S(z,t)$ is regular if and only if $\mathrm{Im}(z) = 0$ and $0 \leqslant \mathrm{Re}(z) < 1$.

Proof: (i) The generating formula for the Laguerre polynomials is given by

$$(4) \qquad \sum_{k=0}^{\infty} L_k^n (t) z^k = (1 - z)^{-n-1} \exp(-tz/1-z) \qquad ([10], \text{ p. } 100).$$

For t real or complex $\sum_{k=0}^{\infty} L_k^n (t) z^k$ is a power series in z with radius of convergence equal to one. Hence $\sum_{k=0}^{\infty} b_{n,k}$ and $\sum_{k=0}^{\infty} |b_{n,k}|$ can converge for $|z| \leqslant 1$ only.

Thus, we must have $|z| \leqslant 1$, if $S(z,t)$ is to be regular.

(ii) By (i) we have $|z| \leqslant 1$. For $t \leqslant 0$,

$$L_k^n (t) = \sum_{\mu=0}^{k} \binom{k+n}{k-\mu} (-t)^{\mu}/\mu! \geq 0, \text{ for all } k \text{ and } n.$$

Hence,

$$\sum_{k=0}^{\infty} |b_{n,k}| = |1 - z|^{n+1}| \exp(tz/1-z)| \sum_{k=0}^{\infty} L_k^n (t)|z|^k.$$

Suppose $|z| < 1$. Then by (4),

$$\sum_{k=0}^{\infty} |b_{n,k}| = (|1-z|/1-|z|)^{n+1} |\exp(tz/1-z)| \exp(-t|z|/1-|z|),$$

which is uniformly bounded for $n \geqslant 0$ if and only if

$$|1 - z|/1-|z| \leq 1.$$

However, $|1 - z| \geqslant 1 - |z|$; thus we must have $\operatorname{Im}(z) = 0$ and $0 \leqslant \operatorname{Re}(z) < 1$.

Now, suppose $|z| = 1$. Then

$$\sum_{k=0}^{\infty} |b_{n,k}| = |1 - z|^{n+1} |\exp(tz/1-z)| \sum_{k=0}^{\infty} L_k^n(t).$$

For $t \leqslant 0$, $L_k^n(t) \geqslant 0$ and

$$\sum_{k=0}^{\infty} L_k^n(t) = \lim_{x \to 1-0} \sum_{k=0}^{\infty} L_k^n(t)\, x^k = \lim_{x \to 1-0} (1-x)^{-n-1} \exp(-xt/1-x) = +\infty$$

by Abel's Theorem. So $\sum_{k=0}^{\infty} L_k^n(t)$ diverges, and we cannot have $|z| = 1$.

(iii) Let z be given. If $S(z,t)$ is regular for each t, it is regular for any $t \leqslant 0$. Hence, by (ii), $\operatorname{Im}(z) = 0$, and $0 \leqslant \operatorname{Re}(z) < 1$.

Now, let $\operatorname{Im}(z) = 0$, and $0 \leqslant \operatorname{Re}(z) < 1$ and fix t complex. The condition (1) of regularity is satisfied if $|z| < 1$; that is,

$$\lim_{n \to \infty} \sum_{k=0}^{\infty} b_{n,k} = 1, \text{ because}$$

$$\sum_{k=0}^{\infty} b_{n,k} = (1 - z)^{n+1} \exp(tz/1-z) \sum_{k=0}^{\infty} L_k^n(t)\, z^k$$

$$= (1 - z)^{n+1} \exp(tz/1-z)\, (1 - z)^{-n-1} \exp(-tz/1-z) =$$

Proceeding to condition (2), let us fix k, and show that

$$\lim_{n \to \infty} b_{n,k} = \lim_{n \to \infty} (1 - z)^{n+1} \exp(tz/1-z)\, L_k^n(t) z^k = 0.$$

This is equivalent to

$$\lim_{n \to \infty} (1 - z)^{n+1} L_k^n(t) = 0,$$

which follows by showing that $\sum_{n=0}^{\infty} (1 - z)^{n+1} L_k^n(t)$ converges.

This, in turn, is a consequence of the following calculation:

$$\sum_{n=0}^{\infty} (1-z)^{n+1} L_k^n(t) = \sum_{n=0}^{\infty} (1-z)^{n+1} \sum_{j=0}^{k} \binom{n+k}{k-j} (-t)^j/j!$$

$$= \sum_{j=0}^{k} (-t)^j/j! \sum_{n=0}^{\infty} (1-z)^{n+1} \binom{n+k}{n+j}$$

$$= \sum_{j=0}^{k} (-t)^j/j! \; (1-z)^{1-j} \sum_{m=j}^{\infty} (1-z)^m \binom{m-j+k}{m},$$

since $\sum_{m=0}^{\infty} (1-z)^m \binom{m-j+k}{m}$, and hence $\sum_{m=j}^{\infty} (1-z)^m \binom{m-j+k}{m}$,

converges under the hypothesis, then $\sum_{k=0}^{\infty} (1-z)^{n+1} L_k^n(t)$ converges and

$$\lim_{n\to\infty} (1-z)^{n+1} L_k^n(t) = 0.$$

Hence, condition (2) holds.

Finally,

$$\sum_{k=0}^{\infty} |b_{n,k}| = |1-z|^{n+1} |\exp(tz/1-z)| \sum_{k=0}^{\infty} |L_k^n(t)| \; |z|^k$$

$$\leq |1-z|^{n+1} |\exp(tz/1-z)| \sum_{k=0}^{\infty} \sum_{j=0}^{k} \binom{n+k}{k-j} |t|^j/j! \; |z|^k$$

$$= |1-z|^{n+1} |\exp(tz/1-z)| \sum_{j=0}^{\infty} \sum_{k=j}^{\infty} \binom{n+k}{k-j} (|t|^j/j!) |z|^k$$

$$= |1-z|^{n+1} |\exp(tz/1-z)| \sum_{j=0}^{\infty} (|tz|^j/j!)(1/1-|z|)^{n+j+1}$$

$$= (|1-z|/1-|z|)^{n+1} |\exp(tz/1-z)| \; \exp(|tz|/1-|z|),$$

which is uniformly bounded for $n \geqslant 0$. Thus, condition (3) of regularity holds. So $S(z,t)$ is regular for each t.

(iv) Let $t \leqslant 0$. If $\mathrm{Im}(z) = 0$ and $0 \leqslant \mathrm{Re}(z) < 1$, then, by

(iii) above $S(z,t)$ is regular.

If $S(z,t)$ is regular then by (ii), $\mathrm{Im}(z) = 0$, and $0 \leqslant \mathrm{Re}(z) < 1$.

3. Summability of the geometric series. Letting DW denote the domain of values of z for which the geometric series is summable to $(1 - z)^{-1}$ by the method W, we have

Therom 2.Let $0 < |q| < 1$. For each t, the $S(q,t)$ - transform continues the geometric series analytically into the region

$$DS(q,t):\ \{z:\ |z| < |q^{-1}|\} \cap \{z:\ |z-q^{-1}| > |1-q^{-1}|\}$$

Proof. We transform the sequence

$$S_k(z) = \sum_{n=0}^{k} z^n = (1-z^{k+1})/(1-z)$$

by the matrix $S_{n,k}(q,t)$. Using the condition $0 < |q| < 1$, and since $\sum_{k=0}^{\infty} S_{n,k}(q,t) = 1$, we obtain

$$t_n(z) = 1/(1-z) - \frac{z}{1-z}(1-q)^{n+1}\exp(tq/1-q)\sum_{k=0}^{\infty} L_k^n(t)\,(qz)^k$$

$$= \frac{1}{1-z} - \frac{z}{1-z}\left(\frac{1-q}{1-qz}\right)^{n+1}\exp\left(\frac{tq(1-z)}{(1-q)(1-qz)}\right),$$

if $|z| < |q^{-1}|$; and $\lim_{n\to\infty} t_n(z) = \frac{1}{1-z}$ if $\left|\frac{1-q}{1-qz}\right| < 1$.

We define the product of two methods of summability A_1 and A_2 as the A_1-transform of the A_2-transform of the sequence $\{s_k\}$ and refer to it as the $A_1 \cdot A_2$ method of summability.

4. $A \cdot S(q,t)$ summability of the geometric series. Theorem 3. For $0 < |q| < 1$, the geometric series $\sum_{n=0}^{\infty} z^n$ is $A \cdot S(q,t)$ summable for each z, satisfying $|z| < |q^{-1}|$, and $|1-q| \leqslant |1-qz|$.

Proof. The kth partial sum of $\sum_{n=0}^{\infty} z^n$ is

$$s_k(z) = \frac{1-z^{k+1}}{1-z}.$$

The $S(q,t)$-transform of the sequence $\{s_k(z)\}$ $0 < |q| < 1$, is the sequence

$$t_n(z) = (1-q)^{n+1}\exp(tq/1-q)\sum_{k=0}^{\infty} L_k^n(t)\,q^k\,\frac{1-z^{k+1}}{1-z}.$$

$\{t_n(z)\}$ exists for each n if and only if $|z| < |q-1|$. The Abel transform of

$\{t_n(z)\}$ is the function

$$F(x) = (1-x) \sum_{n=0}^{\infty} x^n t_n(z);$$

hence the $A \cdot S(q,t)$-transform of $S_k(z)$ becomes

$$F(x) = (1-x) \sum_{n=0}^{\infty} x^n (1-q)^{n+1} \exp\left(\frac{tq}{1-q}\right) \sum_{k=0}^{\infty} L_k^n(t) q^k \frac{1-z^{k+1}}{1-z}$$

$$= (1-x) \sum_{n=0}^{\infty} \frac{x^n(1-q)^{n+1}}{1-z} \left[\exp\left(\frac{tq}{1-q}\right) \sum_{k=0}^{\infty} L_k^n(t) q^k - \exp\left(\frac{tq}{1-q}\right) z \sum_{k=0}^{\infty} L_k^n(t)(qz)^k\right]$$

$$= \frac{1-x}{1-z} \left[\sum_{n=0}^{\infty} x^n - \exp\left(\frac{tq}{1-q}\right) \exp\left(\frac{-tqz}{1-zq}\right) \frac{z(1-q)}{(1-qz)} \sum_{n=0}^{\infty} \left(\frac{x(1-q)}{1-qz}\right)^n\right]$$

$$= \frac{1}{1-z} - \frac{z(1-x)(1-q)}{(1-z)(1-qz)} \exp\left(\frac{tq}{1-q}\right) \exp\left(\frac{-tqz}{1-zq}\right) \frac{1}{1 - \frac{x(1-q)}{1-qz}},$$

provided $|x| < 1$, and $\left|\frac{x(1-q)}{1-qz}\right| < 1$. Therefore, if $0 < x < 1$, and

$\left|\frac{1-q}{1-qz}\right| \leq 1$, it follows that $F(x) \to \frac{1}{1-z}$ as $x \to 1-0$.

Comparison of theorems 2 and 3 indicates that the product method $A \cdot S(q,t)$ is more effective in that it extends the domain of summability of the geometric series to include certain parts of the boundary, subject to the values of q.

5. The relation $E(p) \subset S(q,t)$.

In this section we shall use the results of matrix products to suggest relationships between the orders p and q that may constitute necessary and sufficient conditions for inclusion. We shall compute the matrix product $S(q,t)E(p)$.

$$\sum_{k=0}^{\infty} s_{n,k}(q,t) E_{k,m}(p) = \sum_{k=0}^{\infty} (1-q)^{1+n} \exp\left(\frac{qt}{1-q}\right) \binom{k}{m} p^m (1-p)^{k-m} L_k^n(t) q^k$$

$$= (1-q)^{1+n} \exp\left(\frac{qt}{1-q}\right) \frac{p^m}{(1-p)^m} \sum_{k=m}^{\infty} \binom{k}{m} L_k^n(t)(q(1-p))^k;$$

since $n \geqslant 0$, $m \geqslant 0$ and $\binom{k}{m} = 0$ for $k < m$. On the right side let $k = i + m$. Then

$$\sum_{k=0}^{\infty} S_{n,k}(q,t)E_{k,m}(p) = (1-q)^{1+n} \exp\left(\frac{qt}{1-q}\right) p^m q^m \sum_{i=0}^{\infty} \binom{i+m}{m} L_{i+m}^n(t)(q(1-p))^i$$

$$= (1-q)^{1+n} \exp\left(\frac{qt}{1-q}\right) p^m q^m (1-q(1-p))^{-1-n-m} \exp\left(\frac{-tq(1-p)}{1-q(1-p)}\right) L_m^n \left(\frac{t}{1-q(1-p)}\right),$$

the last step being possible because of the following formula

$$(5) \quad \sum_{n=0}^{\infty} \frac{(n+k)!\, L_{n+k}^{\alpha}(x)}{n!\; k!} t^n = (1-t)^{-1-\alpha-k} \exp\left(\frac{-xt}{1-t}\right) L_k^{\alpha}\left(\frac{x}{1-t}\right), |t| < 1 \quad ([8], \text{p. } 211)$$

Then

$$\sum_{k=0}^{\infty} S_{n,k}(q,t)E_{k,m}(p) =$$

$$\left(\frac{1-q}{1-q(1-p)}\right)^{1+n} \left(\frac{pq}{1-q(1-p)}\right)^m \exp\left(\frac{pqt}{(1-q)(1-q+pq)}\right) L_m^n \left(\frac{t}{1-q+pq}\right),$$

provided $|q(1-p)| < 1$.

Let $u = \frac{pq}{1-q+pq}$. Then $1 - u = \frac{1-q}{1-q+pq}$, and

$$\sum_{k=0}^{\infty} S_{n,k}(q,t)E_{k,m}(p) = (1-u)^{1+n} \exp\left(\frac{ut}{1-q}\right) L_m^n\left(\frac{ut}{pq}\right) u^m$$

$$= S_{n,m}\left(u, \frac{ut}{pq}\right), \quad p,q \neq 0.$$

If p and q are real, we obtain Theorem 4. If $0<p<1$, $0<q<1$, and if $u = \frac{pq}{1-q+pq}$, then $0<u<1$ and $S(q,t)E(p) = S(u, \frac{ut}{pq})$.

Since $0<p<1$ and $0<q<1$ it readily follows that $0<u<1$ and also that $|q(1-p)|<1$.

The preceding computation also yields Theorem 5. If $|p|<1$, $|q|<1$, $|q(1-p)|<1$ and if $u = \frac{pq}{1-q+pq}$, then $S(q,t)E(p) = S(u, \frac{ut}{pq})$.

Because $E^{-1}(p) = E(1/p)$, ([1], p. 314), we are interested in a further result.

Theorem 6. $S(q,t)E(1/p) = S\left(\frac{q}{p-pq+q}, \frac{pt}{p-pq+q}\right)$, provided $|q(1-p^{-1})| < 1$.

If with suitable restrictions on p and q, it turns out that the S(q,t)-transform of the $E(p^{-1})$-transform of the sequence is the same as the transform of that sequence by the matrix product $S(q,t)E(p^{-1})$, then Theorem 6 will give us information about the possible inclusion of E(p) by S(q,t) because $S\left(\frac{q}{p-pq+q}, \frac{pt}{p-pq+q}\right)$ is regular if and only if $0< \frac{q}{p-pq+q} <1$.

Further clues about this inclusion, are provided by comparing DE(p) and DS(q,t) For p and q complex, the resulting domains are

(6) DE(p): $|z-(1-p^{-1})|<|p^{-1}|$, [1], p. 316

(7) DS(q,t): $|z|<|q^{-1}|$ $|z-q^{-1}|$. $<|q^{-1}-1|$.

We seek relations between p and q for which $0 < \frac{q}{p-pq+q} < 1$ or

$$1 = \left|\frac{q}{p+q-pq}\right| + \left|1 - \frac{q}{p+q-pq}\right|,$$

which is equivalent to

(8) $|p^{-1} + q^{-1} - 1| = |p^{-1}| + |q^{-1} - 1|$.

This implies the collinearity of the points q^{-1}, 1 and $1-p^{-1}$ with the point z = 1 lying between the other two points.

With p and q related by (8), upon comparing DS(q,t) and DE(p) we encounter another relationship between p and q which, in conjunction with (8), might guarantee inclusion of E(p) by S(q,t), namely

(9) $|q^{-1}| > |p^{-1}| + |1 - p^{-1}|$.

Conditions (8) and (9) turn out to be sufficient for the desired inclusion.

Theorem 7. If $|p|<1$, $|q|<1$ and if p and q satisfy conditions (8) and (9), then $S(q,t) \supset E(p)$.

Proof. Suppose that a given sequence $\{s_k\}$ is summable E(p) to S. The E(p)-transform of $\{s_k\}$ is given by

$$t_n = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} s_k$$

where $\lim_{n\to\infty} t_n = S$. Since any triangular matrix has an inverse if and only if all diagonal elements are non-zero, writing $\{s_n\}$ as the $E(p^{-1})$-transform of $\{t_k\}$ yields

$$s_n = \sum_{k=0}^{n} \binom{n}{k} (p^{-1})^k (1-p^{-1})^{n-k} t_k.$$

Then the S(q,t)-transform of $\{s_n\}$ is given by

$$(10) \quad \tau_m = (1-q)^{m+1} \exp\left(\frac{qt}{1-q}\right) \sum_{n=0}^{\infty} L_n^m(t)\, q^n \sum_{k=0}^{n} \binom{n}{k} (p^{-1})^k (1-p^{-1})^{n-k}\, t_k.$$

The convergence of $\{t_k\}$ implies that $\{t_k\} \leqslant M$ for $k = 0, 1, 2, \ldots$

Then

$$|\tau_m| \leq M|1-q|^{m+1} \left|\exp\left(\frac{qt}{1-q}\right)\right| \sum_{n=0}^{\infty} |L_n^m(t)|\, |q|^n \sum_{k=0}^{n} \binom{n}{k} |p^{-1}|^k |1-p^{-1}|^{n-k}$$

$$= M|1-q|^{m+1} \left|\exp\left(\frac{qt}{1-q}\right)\right| \sum_{n=0}^{\infty} |L_n^m(t)|\, |q|^n (|p^{-1}| + |1-p^{-1}|)^n$$

$$= M|1-q|^{m+1} \left|\exp\left(\frac{qt}{1-q}\right)\right| (1-|q|(|p^{-1}| + |1-p^{-1}|))^{-m-1} \exp\left(\frac{|t|\,|q|(|\frac{1}{p}|+|1-\frac{1}{p}|)}{|q|(|\frac{1}{p}|+|1-\frac{1}{p}|)-1}\right).$$

Finally,

$$|\tau_m| \leq M \left(\frac{|1-q|}{1-|q|(|p^{-1}|+|1-p^{-1}|)}\right)^{m+1} \left|\exp\left(\frac{qt}{1-q}\right)\right| \exp\left(\frac{|t|\,|q|(|p^{-1}|+|1-p^{-1}|)}{|q|(|p^{-1}|+|1-p^{-1}|)-1}\right)$$

where the last step is justified by (9). Since the series in (10) converge absolutely, it is permissible to invert the order of summation and write

$$\tau_m = (1-q)^{m+1}\exp\left(\frac{qt}{1-q}\right)\sum_{k=0}^{\infty}(p^{-1})^k(1-p^{-1})^{-k}\, t_k \sum_{n=k}^{\infty}\binom{n}{k}L_n^m(t)\,(q(1-p^{-1}))^n.$$

Let n = i + k. Then

$$\tau_m = (1-q)^{m+1}\exp\left(\frac{qt}{1-q}\right)\sum_{k=0}^{\infty}(q/p)^k\, t_k \sum_{i=0}^{\infty}\binom{i+k}{k}L_{i+k}^m(t)\,(q(1-p^{-1}))^i$$

$$= (1-q)^{m+1}\exp\left(\frac{qt}{1-q}\right)\sum_{k=0}^{\infty}\left(\frac{q}{p}\right)^k t_k(1-q(1-p^{-1}))^{-1-m-k}\exp\left(\frac{-tq(1-p^{-1})}{1-q(1-p^{-1})}\right)L_k^m\left(\frac{t}{1-q(1-p^{-1})}\right),$$

here the last step is justified by (5). Finally,

$$\tau_m = \left(\frac{1-q}{1-q(1-p^{-1})}\right)^{m+1}\exp\left(\frac{qt}{1-q}-\frac{tq(1-p^{-1})}{1-q(1-p^{-1})}\right)\sum_{k=0}^{\infty}L_k^m\left(\frac{t}{1-q(1-p^{-1})}\right)\left(\frac{q}{p+q-pq}\right)^k t_k,$$

provided $|q(1-p^{-1})| < 1$ or $|q^{-1}| > |1-p^{-1}|$. Condition (9) assures us of this.

Hence τ_m is the $S\left(\frac{q}{p-pq+q}, \frac{pt}{p-pq+q}\right)$-transform of t_k.

To show that lim τ_m = S, and

hence $\{s_k\}$ is summable S(q,t) to S, it is sufficient that the matrix

$S\left(\frac{q}{p-pq+q}, \frac{pt}{p-pq+q}\right)$, be regular which is the case when (8) holds.

Corollary. If $0 < p < 1$, $0 < q < 1$, and $1 + q^{-1} > 2p^{-1}$, then S(q,t)>E(p).

The necessity of the condition $|q^{-1}| > |p^{-1}| + |1-p^{-1}|$ for the inclusion of E(p) by S(q,t) is uncertain, but a slightly weaker result is contained in

Theorem 8. If $|p| < 1$, $|q| < 1$ and S(q,t)>E(p), then the following conditions hold:

(8) $|p^{-1} + q^{-1} - 1| = |p^{-1}| + |q^{-1} - 1$

(11) $|q^{-1}| \geqslant |p^{-1}| + |1 - p^{-1}|$.

Proof. Let c_1 denote the circle $|z-(1-p^{-1})| = |p^{-1}|$, and c_2 denote the circle $|z-q^{-1}| = |1 - q^{-1}|$. If (8) does not hold, the centers of c_1 and c_2 are not collinear with 1. Then it is readily seen that DE(p) and DS(q,t) overlap, and so (8) is necessary. Now suppose that (8) holds, but $|q^{-1}| < |p^{-1}| + |1-p^{-1}|$. In this case a point z can be chosen which lies inside c_1 but outside $|z| = |q^{-1}|$. This implies that $z \epsilon$ DE(p) but $z \epsilon$ DS(q,t), hence S(q,t)>E(p), and so (11) is necessary.

Corollary. If $0 < p < 1$, $0 < q < 1$, and S(q,t) E(p), then $1 + q^{-1} \geqslant 2p^{-1}$.

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