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Attempted Resolutions Of The "Allais" Paradox

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A long standing debate among decision and value theorists concerns the consistency of individual as well as collective preferences. Two traditional yet diametrically opposed positions have been staked out. A von Neumann-Morgenstern decision theorist (hereafter reference to the von Neumann-Morgenstern system will be abbreviated NM) maintains that individual preferences are exhibited behavioristically and that from empirical observations alone a consistent set of axioms can be (or have been) constructed which describe and predict individual decision making and, in addition, may be regarded as norms of future behavior (von Neumann and Morgenstern, 1953: 31-33). Others have denied that such a set of axioms fulfilling these descriptive, predictive and normative demands have been (or can be) constructed. Their counter-argument, in part, is based on the Allais paradox. Following Allais they contend that individual values are an intuitively given normative network which does not exhibit a sufficiently strict correlation with preferences so as to fulfill the NM demands. Consequently they claim that empirical behavioristic observations alone will not provide one with the information from which a consistent set of axioms can be constructed such that these axioms will be complete in the NM sense, i.e., at once descriptive, predictive and normative of individual decision making.

In this paper it will be demonstrated that any proposal which attempts a complete (1) descriptive, (2) predictive and (3) normative account of individual decision making within the NM system is futile. The outline of this proposed demonstration is as follows. First, the NM axioms are presented and the theorem of the maximization of expected utility is derived from them. Secondly, an account is given of the Allais paradox. This paradox resulted from Allais' questioning of the reliability of the NM system for the prediction of an individual's future choices. Thirdly, three proposed resolutions offered by Leonard J. Savage, Donald Morrison and Karl Borch, respectively, are presented. An examination of these proposals will reveal specific weaknesses of each one. Fourthly, it will be shown that all three solutions "fail" in a more general sense for they all exceed the boundaries of the NM system in a very fundamental way. Finally, a suggestion for a possible resolution of the Allais paradox will be put forth which will require the development of a new generalized decision theory.

The NM axioms (von Neumann and Morgenstern, 1953: 26,27):

We consider a system U of entities u,v,w, ... In U a relation is given, $u > v$, and for any number α, $(0 < \alpha < 1)$, an operation: $\alpha u + (1 - \alpha)v = w$.

These concepts satisfy the following axioms:

(3:A) $u > v$ is a complete ordering of U.

This means: Write $u < v$ when $v > u$. Then:

(3:A:a) For any two u, v one and only one of the following relations holds: $u = v$, $u > v$, $u < v$.

(3:A:b) $u > v$, $v > w$ imply $u > w$.

(3:B) Ordering and combining.

(3:B:a) $u < v$ implies that $u < \alpha u + (1 - \alpha)v$.

(3:B:b) $u > v$ implies that $u > \alpha u + (1 - \alpha)v$.

(3:B:c) $u < w < v$ implies the existence of an α with $\alpha u + (1 - \alpha)v < w$.

(3:B:d) $u > w > v$ implies the existence of an α with $\alpha u + (1 - \alpha)v > w$.

(3:C) Algebra of combining.

(3:C:a) $\alpha u + (1 - \alpha)v = (1 - \alpha)v + \alpha u$.

(3:C:b) $\alpha(\beta u + (1 - \beta)v = \Upsilon u + (1 - \Upsilon)v$ where $\Upsilon = \alpha\beta$.

The system U – i.e. in our present interpretation, the system of (abstract) utilities – is one of numbers up to a linear transformation.

The following is a condensation of a proof that the maximization of expected utility is a theorem of the NM axioms (von Neumann and Morgenstern, 1953: 618-632).

If we define $u(a_1) = 1$, $u(a_n) = 0$ then by (3:A:a) $a_1 \gtrsim x \gtrsim a_n$.

If there exists an α_1, then by (3:B:c) and (3:B:d) $x \sim (\alpha_1 a_1, \alpha_n a_n)$.

In a similar manner for y: $y \sim (\beta_1 a_1, \beta_n a_n)$.

By (3:C:a) and (3:B:d): $(x \gtrsim y) \rightarrow (\alpha_1 a_1, \alpha_n a_n) \gtrsim (\beta_1 a_1, \beta_n a_n)$ if and only if $\alpha_1 \geqslant \beta_1$.

By definition of u this is equivalent to $u(x) \geqslant u(y)$.

Thus the property of additivity of utility is satisfied under the NM axioms.

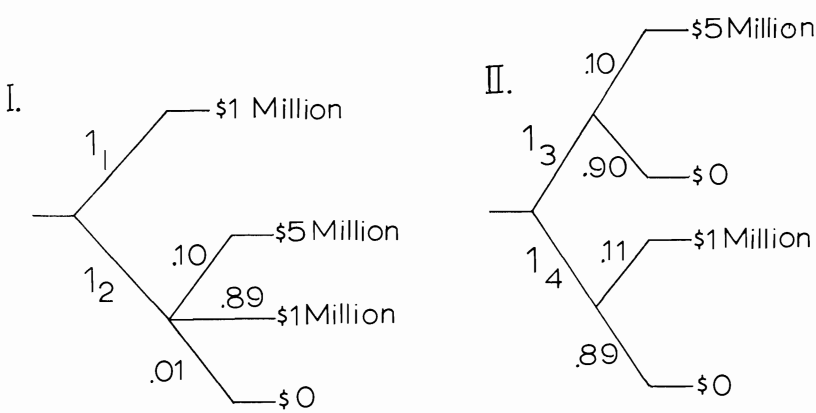
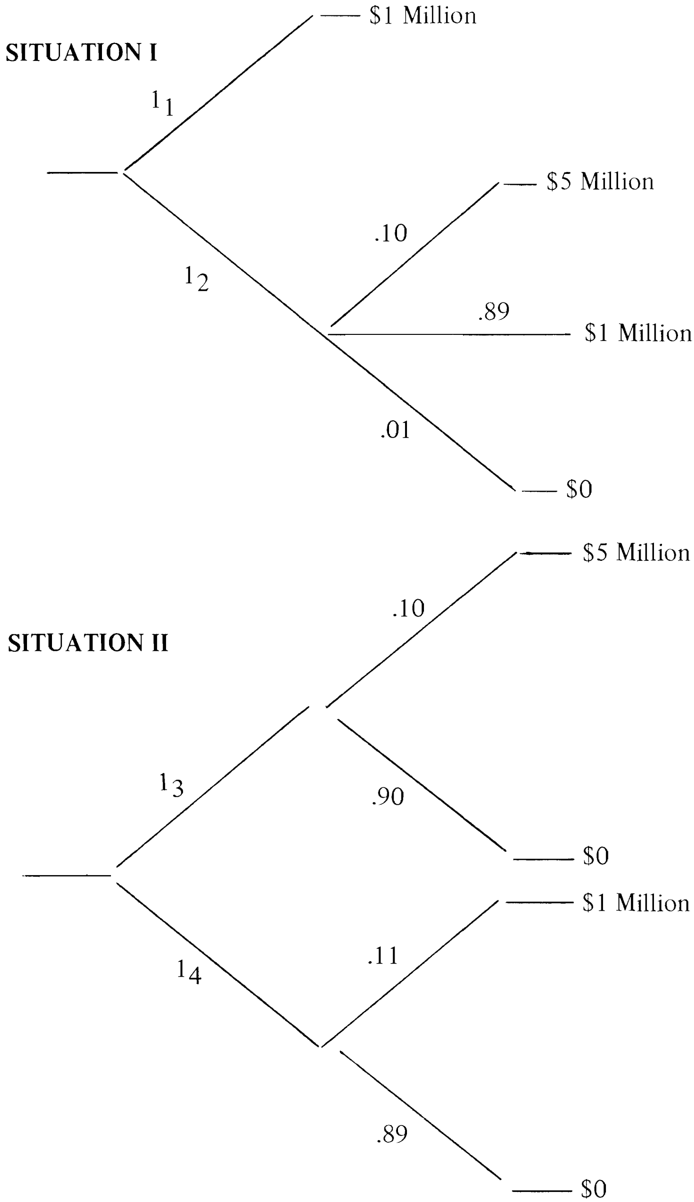


Diagram for the explication of the Allais Paradox (Morrison, 1967: 374):



The paradox (Fig. 1) is named for the French economist who first proposed it, Maurice Allais (Allais, 1955). In this reconstruction of the paradox the subject is asked to pick one of the two alternatives depicted in the diagram. In situation I the subject must choose between lottery 1_1 (getting one million dollars for sure) and the lottery 1_2 (getting five million dollars with a probability of .10, one million dollars with a probability of .89, or – with a probability of .01 – ending up with nothing.) In situation II, 1_3 gives him a .10 probability of obtaining five million and a .90 probability of getting nothing; 1_4 gives a slightly better chance (.11) of obtaining only one million dollars and a .89 probability of getting nothing.

Most subjects (including myself) would prefer 1_1 to 1_2 if they were placed in situation I. If placed in the less favorable situation II, they would prefer 1_3 to 1_4. (This is my preference as well.)

The preference for 1_1 over 1_2 and, at the same time, 1_3 over 1_4 can be shown to be inconsistent with respect to the NM axioms and, in particular, inconsistent with the expected utility theorem derived from the NM axioms.[1] This demonstration reveals that the subject in Situation I, by selecting 1_1 over 1_2, is not maximizing his expected utility. But by virtue of his selection of 1_3 over 1_4 in Situation II the subject has made a choice which does maximize his expected utility. This inconsistency between two pairs of preferences made at the same time, a violation of the theorem of maximizing expected utility, is the Allais paradox.

First L. J. Savage's attempted resolution of this paradox will be examined. It is Savage's contention that the NM axioms are at once empirical and normative (Savage, 1954: 97) therefore the Allais paradox is for him a very real one. He admits that the selections of 1_1 over 1_2 and 1_3 over 1_4 at the same time have great intuitive appeal and indeed they are the choices which he himself would make. Although admitting that he has been lured into Allais' clever "trap" Savage is determined to escape. He writes (Savage, 1954: 102):

"In general, a person who has tentatively accepted a normative theory must conscientiously study situations in which the theory seems to lead him astray; he must decide for each by reflection – deduction will typically be of little relevance – whether to retain his initial impression of the situation or to accept the implications of the theory for it."

[1] Calculations – 1_1 = (1.0) U (\$1,000,000) $>$ 1_2 = (.10) U (\$5,000,000) + (.89) U (\$1,000,000) + (.01) U (\$0) which computes as 1_1 = U (\$1,000,000) $>$ 1_2 = U (\$1,390,000) which is false mathematically.

1_3 = (.10) U (\$5,000,000) + (.90) U (\$0) $>$ 1_4 = (.11) U (\$1,000,000) + (.89) U (\$0) which computes as 1_3 = U (\$500,000) $>$ 1_4 = U (\$110,000) which is true mathematically.

Because he believes his intuitive choices have victimized him into a violation of the NM axioms Savage decides that this situation requires "conscientious study." His reflection bears fruit in an ingenious scheme which for him has intuitive appeal and additionally allows him to alter his selections so that they are consistent with the NM axioms. Savage's result permits him to preserve his "rationality" in the NM sense.

Donald Morrison similarly admits he fell victim to the Allais paradox (Morrison, 1967: 373). He concedes that his choices are inconsistent with respect to the NM axioms. To begin his attempted resolution of the paradox Morrison first examines Savage's proposal. Morrison's analysis and criticism of this approach may be summarized as follows.

Savage maintains that since the NM axioms are both empirical and normative they completely characterize rational decision making, past, present and future. Since Savage wishes to behave in a rational, consistent manner he wishes to make his decisions in such a way that these choices are not inconsistent with the choices dictated by the NM axioms. What troubles Savage is that his original intuitive choices, though inconsistent with the NM system, did not seem to be irrational choices. Faced with this dilemma Savage's contention is that if a compelling argument can be put forth which induces one to alter his choices in favor of the NM axioms he will have succeeded in demonstrating that the original choices in violation of those axioms were, if not irrational, at least unthinking and that on reflection one would prefer the choices guided by the NM system.

While Morrison does not specifically reject Savage's resolution it is clear that he is not extremely impressed with it for two reasons. First, Savage's argument used to persuade one to alter his original choices, if indeed one has made choices which are Allais paradoxical, does not have the intuitive appeal that the original choices themselves have. Damaging though this criticism may be Savage himself agrees with it (Savage, 1954: 103). Secondly, Morrison questions Savage's implied assumption that a person put in Situation II "should" have the same preferences when he is put in Situation I.

Morrison's response to Savage's proposed resolution of the Allais paradox is to suggest an alternative approach; essentially Morrison turns Savage's solution upside-down. Instead of changing his choices so that they are consistent with the NM axioms Morrison suggest adding assumptions such that the original, seemingly paradoxical, choices are absorbed consistently within a newly expanded axiomatic system.

Before proceeding with this approach Morrison suggests another possible resolution of the paradox. Morrison contends that it might not be meaningful to compare the two lottery situations because they are so different. If such is the case this incomparability would not allow one to assert that any possible pairs of preferences were inconsistent. Morrison dismisses this tack as simply ad hoc.

Morrison's serious proposal consists in formalizing three assumptions which when appended to the NM axioms offer one the opportunity to make seemingly Allain paradoxical choices but which, by virtue of his additional assumptions, are consistent within this expanded axiomatic system (Fig. 2)

Morrison's assumptions are (Morrison, 1967: 378,379):

1. The desirability of ending up with five million dollars is the same whether the subject's initial asset position was zero or one million dollars. Similarly, the desirability of ending up with one million dollars is the same regardless of whether the initial asset position was zero or one million dollars. Also, five million dollars is preferable to one million dollars.

2. It is less desirable to have an initial asset position of one million dollars and lose it than to start with nothing and lose nothing.

3. Letting the gain of five million dollars equal C^*, and the loss of going from one million dollars to zero equal C_* there exist equilibrating π's for the two intermediate prizes of gaining one million dollars and remaining at zero dollars.

It can be shown that for certain pairs of π_1's and π_0's perferring 1_1 and 1_3 need not be inconsistent with respect to Morrison's expansion of the NM system (Morrison, 1967: 381). Morrison's resolution consists in formalizing the concept of a subject's asset position into additional assumptions which when appended to the NM axioms has the result of confirming, under certain conditions, the consistency of typical Allain choices.

Karl Borch, yet another victim of the Allais paradox, proposes yet another resolution (Borch, 1968: 488,489). His proposal is fashioned after Savage in that he intends to provide an intuitively appealing argument which would persuade one to realign his choices in conformity with the NM system if they do not already so conform. But Borch goes a bit further than Savage because unlike Savage Borch does not consider the Allais paradox a genuine one; rather he calls it a "trap." Presumably it is a "trap" for the "unsophisticated," as Borch labels them, because the question, the Allais choices, are not presented in a "neutral" manner. For this reason Borch concludes that even Savage and Morrison are trapped into believing that normal people would prefer a sure million to a lottery among chances at five million, a million and nothing. Borch writes (Borch, 1968: 489),

> "Both authors and the subjects may have been dazzled at the prospect of receiving a million with certainty, and forget to ask themselves what they actually will do with the money once they get it. It is not likely that they can or will spend it all, nor that they will put all the money into a savings bank. We would not be surprised if they spent only a small part of the money, and invested the rest in securities with a good growth potential and small risk. This means, however, that 1_1 is exchanged for something very similar to 1_2."

Concerning Borch's conclusion Ole Hagen has this rebuttal. He writes, in

1. $((0 \to 5) \sim (1 \to 5)) \succ ((0 \to 1) \sim (1 \to 1))$

2. $(0 \to 0) \succ (1 \to 0)$

3.

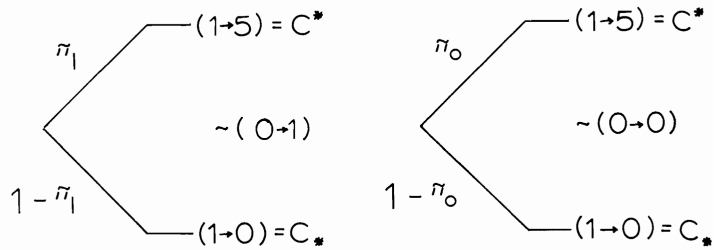
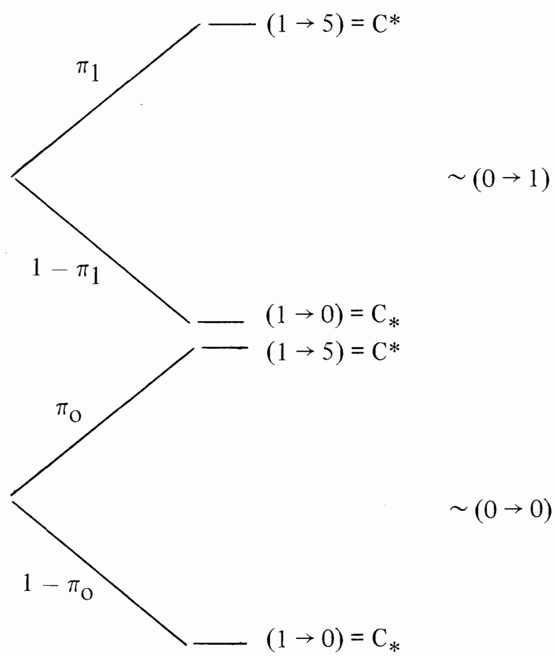


FIGURE 2

Diagram for the explication of the Morrison Assumptions (Morrison, 1967: 379):

1. $((0 \to 5)) \sim (1 \to 5)) > ((0 \to 1) \sim (1 \to 1))$
2. $(0 \to 0) > (1 \to 0)$
3.



a letter communicating with the editor of the journal *Theory and Decision* W. Leinfellner,

> "The suggestion that the person who would reject the 1% risk of losing a fortune in return for a 10% chance of increasing that fortune five times is one who is acting in an ill-considered fashion because should he get his sure million he would proceed to invest "most" of his fortune in growth stocks is beside the point. The point is that even if the possessor of the million dollars did invest most of his money he would not risk losing it all. Therefore the comparison Borch strikes is fraudulent. Borch's comparison, if accepted, is the real "trap."

Hagen apparently is suggesting that Borch is illegitmately converting the Allais choices, which involve a sure-thing alternative, into choices none of which is a sure-thing and all of which involve some element of risk. Hagen's criticism is well-taken. Further Borch is advising that one should treat all one's decisions as risky ones. It is this "should" which gives rise to concern.

What is to be made of these proposals? Savage's solution would appear to make only the rigidly rule-bound individuals rational. But is this always so?

Borch's resolution appears to make only the stock-broker types rational. But is this always so?

Both Savage and Borch conclude that Allain paradoxical choices are irrational, or at least ill-considered. Their premise is that the NM axioms provide a complete descriptive, predictive and normative framework for individual and collective decision making. Is this premise acceptable?

Morrison's reformulation of the consistency conditions embodies an appeal to a more broadly based reconstrual of "rational" behavior. Critics of Morrison's proposal may contend that his resolution makes a mockery of a general decision procedure. This criticism is made plausible by the specific character of Morrison's additional assumptions which clearly reveals the ad hoc nature of his proposal.

Morrison's proposed "resolution" is, however, open to a more damaging criticism. Rather than the resolution of a paradox Morrison has, instead, produced an antinomy; i.e., the mutual contradiction of two inferences deducible from one set of premises.

Allais apparently was looking for a resolution of his paradox within the NM system, if such a resolution were to be found. Morrison has not supplied this. It is true that Morrison's additions to the NM axioms have made it possible to deduce the preferences which Allais found intuitively attractive. Unfortunately the preferences which are NM consistent are also derivable from this expanded system. But, of course, these pairs of preferences contradict one another in the sense that the NM pair exhibit the expected utility theorem whereas the Allain pair do not. Thus an antinomy is revealed.

The source of the antinomy is apparent. In the expanded NM system we have Morrison's notion of utility which entails the maximizing of expected utility based on one's asset position juxtaposed with the NM concept of utility which is straight-forwardly a mathematical maximization of expected utility. Were these two concepts of utility formalized as axioms (von Neumann and Morgenstern have done this for their notion of utility; Morrison has not done this although as we have recounted he has sketched informally his intentions here) it would be clear that these axioms were contradictory thus rendering the putative NM-Morrison set of axioms inconsistent. Consequently Morrison's resolution reduces to the trivial for he has only deduced a contradiction from an inconsistent set of axioms.

Morrison's proposed resolution of the Allais paradox has failed. The first of his attempts was self-admittedly ad hoc. The second of his attempts is trivial.

In conclusion we claim that anyone who attempts a conterminous (1) descriptive, (2) predictive and (3) normative account of individual and collective decision making within the NM system (or in an expanded NM system in the intended sense of Morrison) makes extravagant demands which will remain unfulfilled. This conclusion is a consequence of the incompatibility of requirements 1, 2 and 3 within the NM framework.

The first demand, the descriptive, requires the collection of sufficient empirical preference data. Nothing, in principal, would frustrate the fulfillment of this request.

The second demand, the predictive, requires the assumption that human nature and, in particular, human preference behavior is essentially fixed. Theories in the physical sciences all make an implicit appeal to a similar principle of constancy. Perhaps in the physical sciences this assumption is innocuous. In the social sciences this assumption certainly requires investigation and, at our present state of knowledge, remains problematic at best.

The above aside we should like to focus briefly on the third demand, the normative.

One of Morrison's criticism of Savage's resolution was Savage's implicit assumption that a subject "should" respond in Situation I as he responded in Situation II for the sake of preserving his NM consistency. Borch's resolution was criticized for Borch demanded conformity with the NM axioms and further he demanded that one "should," when making decisions, always weigh uncertainty and risk. Morrison suggests that one "should" mind his asset position before he makes an economic plunge. This advice is offered within the context of the expanded NM system. All three of these gentlemen clearly demand that the subject oblige himself to follow certain norms. Thus the problem of the third, the normative, requirement.

Nowhere in the NM system or its modification as proposed by Morrison will one find the words, "should" or "ought" expressed in norms, obligations or deontic prescriptions. To satisfy the normative demand an assumption of this sort must be added to the system. But an addition of this kind of assumption takes us beyond the boundaries of the NM system as it is now conceived. Therefore within the context of the NM system no resolution of the Allais paradox is possible. Nevertheless even though the Allais paradox is insoluble in the NM framework it still remains a serious problem for decision theorists. It would be presumptuous to propose to eliminate this problem as casually as the above suggests. Accordingly the outline of a tentative solution of the general problem posed by the paradox is set forth.

Rules frequently govern individual and collective decision making as obligations or norms for a specific case. Sometimes these rules are made quite explicit and are codified as laws. More often these rules are informal, implicit, perhaps even unknown. Whatever the form these rules may take, because of their deontic aspect, they may allow us to give the NM system the same interpretation as regards individual choices for the future as we interpreted the system for individual choices in the past. What we desire then is a new generalized decision theory which will couple the NM system with obligations or norms of rational behavior.

For a start we may safely assume that norms of rational behavior, as all specific rules, will have an "if . . . then" character. For example, if you wish to win at chess then you must play expertly, of course, but first you must know and obey the rules of the game. If you do not wish to win at chess and further don't even wish to play the game you have, of course, no need to know or obey the rules. Similarly, if you are convinced of the efficacy of the NM axioms you may believe you should make choices in conformity with them. Perhaps you will select another system a la Morrison. Regardless, the advantage of viewing decisions as rule-bound is that it enables one to see clearly that a condemnation of particular choices as irrational makes no sense unless one understands the normative framework within which the subject believes himself to be operating.

Leinfellner has suggested an approach similar to this (Leinfellner, 1973). He has proposed that a union of the NM system with decision rules using deontic operators may permit us to give this supplemented NM system an interpretation concerning decision making for the future, depending upon the deontic formulation used. Whatever the form this final reconstruction may take we are sure that it will provide for a construal of the nature of values in the Allain sense along with the preservation of the NM system all within a new generalized decision theory. Only when this occurs can the Allais paradox truly be resolved.

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