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From only a brief study of the history of science, one realizes that science is a dynamic enterprise. The Ptolemaic theory of celestial motion was replaced by the Copernician theory of the solar system, the phlogiston theory was replaced by the theory of oxidation brought forth by the discoveries of Priestley and Lavoisier, the mechanics of Galileo was replaced by the mechanics of Newton, Newton's was replaced by the mechanics of Einstein. But do all such replacements basically have the same conceptual structure? It seems that they do not. For when the phlogiston theory was replaced by the theory of oxidation, the phlogiston theory was rejected as scientifically unacceptable and was no longer taught in the universities except as part of the history of science. But when Galilean mechanics was replaced by Newtonian mechanics, Galilean mechanics was not completely rejected but was considered as incomplete and shown its proper domain of application. This is because a case can be made that Galilean mechanics is reducible to Newtonian mechanics.

Just what it means to say that one theory is reducible to another theory is a different question to answer; but it has been informally characterized as one theory explaining another or as one theory being derivable or following from another theory. Such informal characterizations might be intuitively helpful, but they cannot be accepted as giving substantial insight into the nature of scientific reduction. Therefore, a need is developed for what might be called a metatheory of reduction. Now reduction has been discussed by such philosophers as Nagel, Adams, Schaffner, Kemeny, and Oppenheim; but since Adams' approach seems more applicable than the rest in constructing a metatheory of reduction, I will center this paper around his approach by first giving a brief exposition of one of his papers and then making some general comments concerning the significance of his method as a general approach to reduction.

In his paper, Adams (1959:250) has three purposes: "(1) to give a system of axioms for classical rigid body mechanics (henceforth abbreviated 'RBM'); (2) to show how these axioms can be derived from those of particle mechanics (abbr. 'PM'); and (3), using the foregoing derivation as an example, to give a general characterization of the notion of 'reduction' of theories in the natural sciences." Adams begins his paper by presenting the following two axiomatizations and their intended interpretations. The first axiomatization was developed by McKinsey, Sugar, and Suppes (McKinsey, 1953) and has been extensively discussed by Leinfellner (1966); the second axiomatization was developed by Adams (1956) in his doctoral dissertation.

I. An ordered quintuple $\langle P,T,m,S,F \rangle$ is a system of classical r-dimensional particle mechanics if and only if it satisfies axioms P1 to P6.

P1. P is a non-empty finite set.

P2. T is an interval of real numbers.

P3. S is an r-vector valued function with domain PxT such that for all p in P and t in T, $d^2/dt^2(S(p,t))$ exists.

P4. m is a positive real-valued function with domain P.

P5. F is an r-vector valued function with domain PxTxN, where N is the set of positive integers, and for all p in P and t in T the series $\sum_{i=1}^{\infty} F(p,t,i)$ is absolutely convergent.

P6. For all p in P and t in T,

$$m(p)\frac{d^2}{dt^2}S(p,t) = \sum_{i=1}^{\infty} F(p,t,i).$$

*Intended Interpretation:* P is a set of particles; T is an interval of clock readings; m(p) is the mass of particle p; S(p,t) is the position vector of p at time t; F(p,t,i) is the vector representing the $i^{th}$ force acting on p at time t.

II. An ordered septuple $\langle K,T,g,R,H,u,W \rangle$ is a system of r-dimensional rigid body mechanics if and only if it satisfies axioms R1 to R5.

R1. H is a function with domain KxTxN taking as values ordered pairs of r-vectors, and if $H^1$ and $H^2$ are r-vector valued functions with domain KxTxN such that for all k in K, t in T, and i in N,

$$H(k,t,i) = \langle H^1(k,t,i), H^2(k,t,i) \rangle,$$

then $\langle K,T,g,R,H^1 \rangle$ is a system of classical r-dimensional particle mechanics.

R2. W is a function with domain KxT taking as values r by r orthogonal matrices, such that for all k in K and t in T, $d^2/dt^2(W(k,t))$ exists.

R3. u is a function with domain KxT taking as values r by r symmetric positive semi-definite matrices of rank r or r − 1.

R4. For all k in K and t in T, the series $\sum_{i=1}^{\infty} H^2(k,t,i) \times H^1(k,t,i)$ is absolutely convergent.

R5. For all k in K and t in T,

$$W(k,t) \times \left[u(k)\frac{d^2}{dt^2} W(k,t)\right] = \sum_{i=1}^{\infty} H^2(k,t,i) \times H^1(k,t,i).$$

*Intended Interpretation:* K is a set of rigid bodies; T is an interval of clock readings; g(k) is the mass of rigid body k; R(k,t) is the vector representing the position of the center of mass of k as measured relative to a system of cartesian coordinate axes; $H^1(k,t,i)$ is the vector representing the $i^{th}$ applied force on k at time t; $H^2(k,t,i)$ is the vector representing the point of application of the $i^{th}$ force acting on k at time t relative to a set of coordinate axes parallel to the original reference frame and having their origin at the center of mass of k; u(k) is a matrix representing the moment of inertia tensor of k relative to a set of coordinate axes which rotate with k so that they always maintain a fixed relation to the parts of k; W(k,t) is a matrix which represents the orientation of the set of rotating coordinates of k at time t relative to the axes of space.

Next, Adams develops the concept of a theory in such a way that it will be possible to characterize what is involved when one theory is reduced to another theory. For Adams, "a theory T is to be construed as an ordered-pair of sets T=⟨C,I⟩ such that C is the set of all entities satisfying the axioms of T, and I is the set of all intended interpretations of T" (Adams, 1959: 259), where the intended interpretations are those "systems which, for one reason or another, it is demanded that the axioms of T conform to" (Adams, 1959: 258). With these ideas in mind, Adams explains that for a reduction to take place, every intended interpretation of the theory to be reduced must have a special relation R to an intended interpretation of the reducing theory. Adams places two necessary conditions on this relation R.

*Condition A:* Let $T' = \langle C', I' \rangle$ and $T'' = \langle C'', I'' \rangle$ be two theories such that $T''$ is reduced to $T'$ by relation R. Then for all $i''$ in $I''$ there exists $i'$ in $I'$ such that $i'' R i'$.

*Condition B.* Let $T' = \langle C', I' \rangle$ and $T'' = \langle C'', I'' \rangle$ be two theories such that $T''$ is reduced to $T'$ by relation R. Then for all $c'$ and $c''$, if $c'$ is in $C'$ and $c'' R c'$, then $c''$ is in $C''$.

Then, Adams gives a brief sketch of how the reduction R is to be formulated for the reduction of RBM to PM. For example, R holds between an ordered septuple $\langle K,T,g,R,H,u,W \rangle$ and an ordered quintuple $\langle P,T,m,S,F \rangle$ if and only if the following conditions hold.

1. K is a partition of P such that if p and q are elements of k then for all t in T, $|s(q,t) - s(p,t)|$ is a constant.
2. For all k in K, $g(k) = \sum_{p \in k} m(p)$.
3. For all k in K and t in T, $R(k,t) = \dfrac{\sum_{p \in k} m(p) S(p,t)}{\sum_{p \in k} m(p)}$

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Further conditions relating H, u, and W to S, m, and F.

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Finally, Adams concludes by explaining that the reduction of RBM to PM under the relation R can be shown to satisfy condition B by the usual derivations given in textbooks to show that the laws of RBM follow from the laws of PM, and that the reduction can be shown to satisfy condition A if rigid bodies can be construed as being composed of particles.

This completes my exposition of Adams' approach, and now I would like to make some general comments concerning the methodology which Adams used to analyze the concept of reduction. First of all, there seems to develop an odd consequence from Adams' approach to reduction. It has been shown by Adams that RBM is most likely reducible to PM; but it also appears that if

Adams' approach is correct, then PM is most likely reducible to RBM. For example, we can define a relation R′ as holding between an ordered quintuple $\gamma=\langle P,T',m,S,F\rangle$ and an ordered septuple $\Gamma=\langle K,T,g,R,H,u,W\rangle$ if and only if the following conditions hold.

1. K=P.
2. T=T′.
3. g=m.
4. $R_1$=S.
5. H =F.

Under this relation R′, we can make a strong case that PM is reducible to RBM. First, this reduction under R′ will satisfy condition B. To prove this we must show that for any septuple $\Gamma = \langle K,T,g,R,H,u,W\rangle$ and quintuple $\gamma = \langle P,T',m,S,F\rangle$ such that Γ satisfies the axioms of RBM and $\gamma R'\Gamma$, then γ satisfies the axioms of PM. Since Γ satisfies the axioms of RBM, we know by the first axiom of RBM that the ordered quintuple $\langle K,T,g,R,H^1\rangle$ satisfies the axioms of PM. From the definition of R′ it trivially follows that γ satisfies the axioms of PM if it is the case that $\gamma R'\Gamma$. Therefore, the reduction of PM to RBM under R′ satisfies condition B. Secondly, this reduction appears to satisfy condition A. To prove this we must show that for any intended interpretation γ of PM, there exists an intended interpretation of RBM such that $\gamma R'\Gamma$. Now let $\gamma = \langle P,T',m,S,F\rangle$ be an intended interpretation of PM. Since γ is an intended interpretation of PM, P will be a set of particles, T′ an interval of clock readings, m a function attributing a mass to each particle, S a function relating each particle to its position vector, and F a function relating each particle to the forces acting on it. It is possible to construct a septuple $\Gamma = \langle K,T,g,R,H,u,W\rangle$ such that $\gamma R'\Gamma$ if we let

1. K be the set of particles P,
2. T be the interval of clock readings T′,
3. g be the function m,
4. R be the function S,
5. H be a function with domain KxTxN such that for all k in K, t in T, i in N, $H(k,t,i) = \langle H^1(k,t,i),H^2(k,t,i)\rangle$ where $H^1$ is the function F and $H^2$ is a function with domain KxTxN such that $H^2(k,t,i)$ is the zero vector for all k in K, t in T, and i in N,
6. u be a function with domain K such that u(k) is a matrix with all zero elements for all k in K (this matrix will represent the moment of inertia tensor of k with respect to an appropriate rotating coordinate system),
7. W be a function with domain KxT such that W(k,t) is an orthogonal matrix such that $d^2/dt^2(W(k,t))$ exists for all k in K and t in T (this matrix will supposedly represent the orientation of a set of rotating coordinates).

From the above construction of Γ we can easily see that $\gamma R'\Gamma$. Now if we can show that Γ is an intended interpretation of RBM, we will have shown that the reduction of PM to RBM under R′ satisfies condition A. I can see no reason for not allowing Γ as an intended interpretation of RBM since it seems possible to construe a particle as a rigid body with zero spatial dimension ($H^2$(k,t,i) = ⟨0,0,0⟩ for all k in K, t in T, and i in N), since Γ with the above interpretation essentially satisfies the axioms of RBM if the set of particles P act according to Newton's laws of motion[1], and since RBM when applied to rigid bodies with zero spatial dimension will give the same results as PM.

So if Adams' approach to reduction is correct, we seemed to be forced into admitting that PM and RBM are most likely mutually reducible. On the surface, this appears to be an odd consequence since in cases of reduction.we usually think of a reduction between two theories as going only one way. For example, thermodynamics is reducible to statistical mechanics, but statistical mechanics is not reducible to thermodynamics. This odd consequence could easily cause one to doubt the soundness of Adams' approach to reduction.

[1]Actually Γ does not satisfy the third axiom of RBM since the rank of the r by r, zero matrix u(k) is not usually r or r-1; but it is not clear to me that this restriction on the rank of u(k) is a necessary part of an axiomatization of rigid body mechanics. And even if it was, Γ could possibly be reformulated to satisfy this restriction; or it could be argued that even though Γ does not satisfy the third axiom, Γ is still an intended interpretation of RBM.

But perhaps such hastily formed doubts are unwarrented. For the relation between PM and RBM, even though a case of reduction, could be a very special case of reduction (perhaps an internal reduction); and considering that both theories contain many of the same terms in their intended interpretations (such as mass, force, velocity, etc.), that both appear to have the same domain of reliable application, and that both are so-called branches of Newtonian mechanics, it would seem that if there was a reduction between PM and RBM such a reduction would be of a very special nature.

Now Adams' methodology reflects that a reduction between PM and RBM would most likely be of a special nature since two reduction relations R and R′ exist between PM and RBM. Therefore, I conclude that we have reasons to believe that Adams' approach or an approach similar to his would be useful in constructing a metatheory of reduction. Also and perhaps more importantly, it seems that if we based a metatheory of reduction around Adams' approach, we would gain insight into the different types of reduction by being able to formulate explicit definitions of when a reduction is of a certain type. This could be done by characterizing what conditions the reduction relations R must satisfy in order that a reduction will be of a certain type. For example, if it is decided that the reduction between PM and RBM is an internal reduction, we would naturally be led to define a reduction between two theories as internal if and only if there exists two reduction relations between the theories which meet certain conditions similar to the conditions characterizing R and R′. And finally considering that the reduction of Galilean mechanics to Newtonian mechanics has been called an internal reduction by some and a reduction by approximation by others, that the reduction of thermodynamics to statistical mechanics has been called the classical case of a micro-reduction, and that explicit definitions of when a reduction is of a certain type are lacking, such insight would definitely be welcomed.

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