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## ANALYTIC CONTINUATION BY MEANS OF THE $L(r,t)$ – SUMMABILITY TRANSFORM

Stanley D. Luke

1. Let  $A = (a_{nm})$  and  $x = [S_m]$  ( $n, m = 0, 1, 2, \dots$ ) be a matrix and a sequence of complex numbers, respectively. We write
  - (1)  $t_n \equiv A_n(x) = \sum_{m=0}^{\infty} a_{nm} S_m$ , and say that the sequence  $x$  (and the corresponding series  $\sum_{m=0}^{\infty} (S_m - S_{m-1})$ , with  $S_{-1} = 0$ ) is summable  $A$  to the sum  $t$  if each of the series in (1) converges and  $\lim_n t_n$  exists and equals  $t$ . We say that the method  $A$  is regular provided it sums every convergent sequence to its limit. The method  $A$  is regular if and only if
    - (2)  $\sum_{m=0}^{\infty} |a_{nm}| \leq k$  ( $n = 0, 1, 2, \dots$ ),
    - (3)  $\lim_{n \rightarrow \infty} a_{nm} = 0$  ( $m = 0, 1, \dots$ ),
    - (4)  $\lim_{n \rightarrow \infty} \sum_{m=0}^{\infty} a_{nm} = 1$

where  $k$  is a constant independent of  $n$ . These are so called Silverman Toeplitz conditions.

2. The Method  $L(r,t)$ . For complex  $r$  and  $t$  denote by  $L(r,t)$  the matrix whose elements are  $A_{nk} = 0$  when  $k < n$ , and  $A_{nk} = (1-r)^{n+1} \exp(tr/1-r) L_{k-n}^n(t) r^{k-n}$ ,  $k \geq n$ .  $L_j^n(t)$  being the Laguerre polynomial of degree  $j$ , given by

$$L_j^n(t) = \sum_{i=0}^j \binom{j+n}{j-i} \frac{(-t)^i}{i!}$$

*Theorem 1.* For a given value of  $z$ ,  $L(z,t)$  is regular if and only if  $\text{Im}(z) = 0$  and  $0 \leq \text{Re}(z) < 1$ . (R.E. Powell [1])

*Theorem 2.* Let  $|r| < 1$ . For each  $t$ , the  $L(r,t)$  transform continues the geometric series analytically into the region

$$\{z: \left| \frac{(1-r)z}{1-rz} \right| < 1\} \cap \{z: |rz| < 1\}$$

(R.E. Powell, *Canad. J. Math.* 18(1966), 1251-1260).

3. We shall now investigate summability of Power Series by the  $L(r,t)$  method. Let  $\sum_{n=0}^{\infty} a_n z^n$  be a power series with finite radius of

convergence  $R$ , and let  $F(z)$  denote the function obtained by analytic continuation along radial lines from the origin. The region of analyticity of  $F(z)$  is the Mittag – Leffler Star  $S$  associated with  $F(z)$ . For each index  $r$  and vertex  $\xi$  of  $S$ , let  $C(r, \xi)$  denote the set of points

$$(5) \quad |rz| < |\xi|, |z - rz| < |\xi - rz|.$$

It should be noted that  $C(r, \xi)$  is a neighborhood of the origin bounded by circular arcs and having  $\xi$  on its boundary.

Now let  $C(r)$  denote the set of inner points of the intersection of the sets  $C(r, \xi)$  determined by  $\xi$  of  $S$ .  $C(r) \subset S$ .

*Theorem 3.* For each fixed  $r$  satisfying  $0 < r < 1$ , the series  $\sum_{n=0}^{\infty} a_n z^n$  is summable  $L(r, t)$  to  $F(z)$  at every interior point  $z$  of the region  $C(r)$ .

*Proof:* If  $z = 0$ , the theorem is trivially true. Suppose, then, the  $z \neq 0$  and  $z$  is a fixed point of  $C(r)$ . Let  $C_1$  denote the circle consisting of points  $U$  for which

$$(6) \quad |rz| = |U| \text{ holds, and let } C_2 \text{ denote the circle consisting of points } U \text{ for which}$$

$$(7) \quad |z - rz| = |U - rz| \text{ holds.}$$

Inequalities (5) tell us that both  $C_1$  and  $C_2$  are wholly contained inside  $S$ . Therefore, a simple closed curve  $\gamma$  can be constructed inside  $S$  which contains  $C_1$  and  $C_2$  in its interior.

Since  $C_1$  includes the origin in its interior and  $z$  lies on  $C_2$ , then  $\gamma$  includes the origin and the point  $z$  in its interior. Inside and on  $\gamma$ ,  $F(z)$  is analytic. Consequently, for each point  $w$  on  $\gamma$  the following inequalities hold:

$$(8) \quad |F(w)| \leq M, \quad M = \text{Max } |F(w)| \text{ for } w \text{ on } \gamma;$$

$$(9) \quad |w - z| \geq N > 0, \quad N = \text{min } |w - z| \text{ for } w \text{ on } \gamma;$$

$$(10) \quad |rz| < |w|;$$

$$(11) \quad \left| \frac{z - rz}{w - rz} \right| < \theta < 1, \quad \theta = \max \left| \frac{z - rz}{w - rz} \right| \quad \text{for } w \text{ on } \gamma.$$

Returning now to the series  $\sum_{n=0}^{\infty} a_n z^n$ , the coefficients  $a_n$  are given by the Cauchy formula

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{F(w) dw}{w^{n+1}} \quad n = 0, 1, 2, \dots$$

For each  $k$ , we may write the partial sum

$$\begin{aligned}
 S_k(z) &= \sum_{n=0}^k a_n z^n \\
 &= \sum_{n=0}^k z^n \frac{1}{2\pi i} \int_{\gamma} \frac{F(w) dw}{w^{n+1}} \\
 &= \frac{1}{2\pi i} \int_{\gamma} \frac{F(w)}{w} \sum_{n=0}^k \left(\frac{z}{w}\right)^n dw \\
 &= \frac{1}{2\pi i} \int_{\gamma} \frac{F(w)}{(w-z)} - \frac{1}{2\pi i} \int_{\gamma} \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw \\
 &= F(z) - \frac{1}{2\pi i} \int_{\gamma} \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw
 \end{aligned}$$

The  $L(r,t)$ -transform of  $S_k(z)$  is given by

$$(1-r)^{n+1} \exp(tr/(1-r)) \sum_{k=n}^{\infty} L_{k-n}^n(t) r^{k-n} \left[ F(z) - \frac{1}{2\pi i} \int_{\gamma} \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw \right]$$

=  $A - B$ , where

$$A = (1-r)^{n+1} \exp(tr/(1-r)) \sum_{k=n}^{\infty} L_{k-n}^n(t) r^{k-n} F(z)$$

$$= (1-r)^{n+1} \exp(tr/(1-r)) \sum_{k=0}^{\infty} L_k^n(t) r^k F(z)$$

=  $F(z)$ , and

$$B = \frac{(1-r)^{n+1} \exp(tr/(1-r))}{2\pi i} \sum_{k=n}^{\infty} L_{k-n}^n(t) r^{k-n} \int_{\gamma} \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw$$

$$= \frac{(1-r)^{n+1} \exp(tr/(1-r))}{2\pi i} \sum_{k=0}^{\infty} L_k^n(t) r^k \int_{\gamma} \left(\frac{z}{w}\right)^{n+1} \frac{F(w)}{(w-z)} \left(\frac{z}{w}\right)^k dw.$$

Because of (10), the order of summation and integration may be changed, hence

$$\begin{aligned}
 B &= \frac{(1-r)^{n+1} \exp(tr/(1-r))}{2\pi i} \int_{\gamma} \left(\frac{z}{w}\right)^{n+1} \frac{F(w)}{(w-z)} \sum_{k=0}^{\infty} L_k^n(t) \left(\frac{rz}{w}\right)^k dw \\
 &= \frac{(1-r)^{n+1} \exp(tr/(1-r))}{2\pi i} \int_{\gamma} \left(\frac{z}{w}\right)^{n+1} \frac{F(w)}{(w-z)} \left(1 - \frac{rz}{w}\right)^{-n-1} \exp\left(\frac{trz/w}{1-rz/w}\right) dw \\
 &= \frac{\exp(tr/(1-r))}{2\pi i} \int_{\gamma} \frac{F(w)}{(w-z)} \left(\frac{z-rz}{w-rz}\right)^{n+1} \exp\left(\frac{-trz}{w-rz}\right) dw .
 \end{aligned}$$

With the help of (8), (9), (11), and the fact that  $\gamma$  has finite length, it becomes evident that  $\lim_{n \rightarrow \infty} B = 0$ ; and hence  $\lim_{n \rightarrow \infty} t_n = F(z), z \in C(r)$ .

4. We define the product of two methods of summability  $A_1, A_2$  as the  $A_1$ -transform of the  $A_2$ -transform of the sequence  $[S_k]$  and refer to it as the  $A_1 \cdot A_2$  method of summability. If  $A_1, A_2$  are regular methods of summability, then  $A_1 \cdot A_2$  is regular. ([2], p. 83).

*BOREL METHOD.* Consider the sequence  $[S_k]$ . The B-transform of  $[S_k]$  is the function

$$F(x) = e^{-x} \sum_{k=0}^{\infty} S_k \frac{x^k}{k!} .$$

If the power series  $\sum_{k=0}^{\infty} S_k \frac{x^k}{k!}$  converges everywhere and if, for real values of  $x, \lim_{x \rightarrow \infty} F(x) = S$ , then the sequence  $[S_k]$  is B-summable to the value  $S$ . The Borel Method is regular.

*Theorem 4.* If  $0 < r < 1$ , the series  $\sum_{n=0}^{\infty} z^n$  is summable to  $(1-z)^{-1}$  by the iterated product method B.  $L(r,t)$  for all  $z$  in the region

$$(12) \quad |z| < r^{-1} \cap \left| z - \left(\frac{r^{-1}+1}{2}\right) \right| > \left(\frac{r^{-1}-1}{2}\right)$$

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Proof: Consider the geometric series  $\sum_{n=0}^{\infty} z^n$  and the sequence of

partial sums  $[S_k]$  with  $S_k(z) = \frac{1-z^{k+1}}{1-z}$ ,  $z \neq 1$ . The  $L(r,t)$ -transform of  $S_k(z)$ ,  $0 < r < 1$ , is the sequence

$$t_n(z) = (1-r)^{n+1} \exp(tr/(1-r)) \sum_{k=n}^{\infty} L_{k-n}^n(t) r^{k-n} S_k(z)$$

and it is clear that  $L(r,t)$  is applicable to  $S_k(z)$  if and only if  $|z| < r^{-1}$ .

The B-transform of  $[t_n(z)]$  is the function

$$\begin{aligned} F(x) &= e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} (1-r)^{n+1} \exp(tr/(1-r)) \sum_{k=n}^{\infty} L_{k-n}^n(t) r^{k-n} \frac{1-z^{k+1}}{1-z} \\ &= e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} (1-r)^{n+1} \exp(tr/(1-r)) \sum_{k=0}^{\infty} L_k^n(t) r^k \frac{1-z^{k+n+1}}{1-z} \\ &= \frac{1}{1-z} - e^{-x} \exp(tr/(1-r)) \exp(-trz/(1-rz)) \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{x(1-r)z}{1-rz} \right)^n \\ &= \frac{1}{1-z} - e^{-x} \exp(tr/(1-r)) \exp(-trz/(1-rz)) \exp(x(1-r)z/(1-rz)) \frac{(1-r)z}{(1-z)(1-rz)} \\ (13) &= \frac{1}{1-z} - \exp(tr/(1-r)) \exp(-trz/(1-rz)) \exp(x(z-1)/(1-rz)) \frac{(1-r)z}{(1-z)(1-rz)}. \end{aligned}$$

From (13) we see that in order for  $\lim_{x \rightarrow \infty} F(x) = \frac{1}{1-z}$ ,  $z$  must be such that  $\operatorname{Re}((z-1)/(1-rz)) < 0$ .

Put  $z = u + iv$ .

$$\operatorname{Re}((z-1)/(1-rz)) = \frac{(u-1) - r(u^2 + v^2) + ru}{(1-ru)^2 + r^2v^2}.$$

This is negative provided,  $(u-1) - r(u^2 + v^2) + ru < 0$ ,

which is equivalent to  $(u - (1+r)/2r)^2 + v^2 > ((1-r)/2r)^2$ ;

which in terms of  $z$  is given by  $|z - (1+r)/2r| > |(1-r)/2r|$ .

Therefore, the geometric series is summable to  $(1-z)^{-1}$  by the product B- $L(r,t)$ -method for all  $z$  in the region (12).

Now, as before consider the power series  $\sum_{n=0}^{\infty} a_n z^n$  with finite positive radius of convergence  $R$ , and let  $F(z)$  denote the function obtained by analytic continuation along radial lines from the origin.

Corresponding to each singularity  $\xi$  of  $F(z)$  and with  $r$  fixed, let  $C(r, \xi)$  denote the set of points  $z$  for which

$$(14) \quad |z| < r^{-1} |\xi| \cap \left| \frac{z}{\lambda} - \frac{(r^{-1}\xi + \xi)}{2} \right| > \left| \frac{r^{-1}\xi - \xi}{2} \right|$$

whenever  $\lambda \geq 1$ , and let  $C(r)$  denote the intersection of the set  $C(r, \xi)$ .

*Theorem 5.* For each fixed  $r$  satisfying  $0 < r < 1$ , the series  $\sum_{n=0}^{\infty} a_n z^n$  is summable B-L( $r, t$ ) to  $F(z)$  at every interior point  $z$  of the region  $C(r)$ .

Proof: If  $z = 0$ , the theorem is trivially true. Suppose, then that  $z \neq 0$  and  $z$  is a fixed point of  $C(r)$ . Then  $|rz| < |\xi|$  for all singularities  $\xi$  of  $F(z)$ , and it follows that the circle  $C_1$  with center at the origin and radius  $|rz|$  is contained in the Mittag-Leffler star  $S$  of  $F$ . Also, when  $\lambda \geq 1$ ,

$$\left| \frac{z}{\lambda\xi} - \frac{1+r}{2r} \right| > \left| \frac{1-r}{2r} \right|; \quad \text{and, as we see by}$$

making inversion in which complex numbers are replaced by their reciprocals,

$$\left| \lambda\xi - \frac{z+rZ}{2r} \right| > \left| \frac{z-rZ}{2r} \right|.$$

This means that, whenever  $\xi$  is a singular point of  $F$  and  $\lambda \geq 1$ , the point  $\lambda\xi$  lies outside the circle  $C_2$  of points  $U$  for which

$$\left| U - \frac{z+rZ}{2} \right| = \left| \frac{z-rZ}{2} \right| \quad \text{lies inside } S.$$

This implies that we can choose a simple closed curve  $\gamma$  which has finite length, which lies in  $S$  and which contains both  $C_1, C_2$  in its interior.

Since the origin is in  $C_1$  and  $z$  is on  $C_2$ , both of these points are inside  $\gamma$ . Inside and on  $\gamma$   $F(z)$  is analytic. Consequently, for each  $w$  on  $\gamma$  the following inequalities hold:

$$(15) \quad |F(w)| \leq M, \quad M = \max |F(w)| \quad \text{for } w \text{ on } \gamma;$$

$$(16) \quad |w - z| \geq N > 0, \quad N = \min |w - z| \quad \text{for } w \text{ on } \gamma;$$

$$(17) \quad |rz| < |w|;$$

$$(18) \quad |w - rz| \geq P > 0, \quad P = \min |w - rz| \quad \text{for } w \text{ on } \gamma.$$

Returning now to the series  $\sum_{n=0}^{\infty} a_n z^n$ , the coefficients  $a_n$  are given by the Cauchy formula:

$$a_n = \frac{1}{2\pi i} \int_{\gamma} \frac{F(w) dw}{w^{n+1}} \quad n = 0, 1, 2, \dots$$

For each  $k$ , we may write the partial sum

$$\begin{aligned} S_k(z) &= \sum_{n=0}^k a_n z^n \\ &= \sum_{n=0}^k z^n \frac{1}{2\pi i} \int_{\gamma} \frac{F(w) dw}{w^{n+1}} \\ &= F(z) - \frac{1}{2\pi i} \int_{\gamma} \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw. \end{aligned}$$

The B-L(r,t)-transform of  $S_k(z)$  is therefore, given by

$$\begin{aligned} &= e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) \sum_{k=n}^{\infty} L_{k-n}^n(t) r^{k-n} \left[ F(z) - \frac{1}{2\pi i} \int_{\gamma} \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw \right] \\ (19) \quad &= F(z) - R_z(x), \text{ where } R_z(x) \text{ is given by} \end{aligned}$$

$$\begin{aligned} &= e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) \sum_{k=n}^{\infty} L_{k-n}^n(t) r^{k-n} \frac{1}{2\pi i} \int_{\gamma} \frac{zF(w)}{w(w-z)} \left(\frac{z}{w}\right)^k dw \\ &= e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) \sum_{k=0}^{\infty} L_k^n(t) r^k \frac{1}{2\pi i} \int_{\gamma} \frac{F(w)}{(w-z)} \left(\frac{z}{w}\right)^{n+1} \left(\frac{z}{w}\right)^k dw \end{aligned}$$



Because of (17), the order of summation and integration may be changed, hence

$$R_z(x) = \frac{e^{-x}}{2\pi i} \sum_{n=0}^{\infty} \frac{x^n}{n!} (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) \int_{\gamma} \left(\frac{z}{w}\right)^{n+1} \frac{F(w)}{(w-z)} \sum_{k=0}^{\infty} L_k^n(t) \left(\frac{rz}{w}\right)^k dw.$$

From (19), it is apparent that  $F(x) \rightarrow F(z)$  provided  $R_z(x) \rightarrow 0$  as  $x \rightarrow \infty$ . With the help of (15), (16), (18) and the fact that  $\gamma$  has finite length it becomes evident that  $\lim_{x \rightarrow \infty} R_z(x) = 0$ , provided

$$\operatorname{Re} \left( \frac{z-w}{w-zr} \right) < 0, \text{ which is equivalent to}$$

$$\left| w - \frac{1+r}{2} z \right| > \frac{1-r}{2} |z|.$$

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