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The frontiers of mathematics lie in two directions: higher towards more and more sophisticated and complicated constructions based on accepted principlės and methods; and lower towards deeper and deeper foundations of the superstructure of accepted mathematics. Frege was a great pioneer on the latter frontier and single-handedly opened up a deep and rich area of research from which mathematicians and philosophers still profit. In this paper I will describe Frege's contributions to the foundations of number theory and analysis and then discuss the tenability of his program.

In a nutshell, Frege's principle goal was to reduce mathematics to logic. His greatest obstacle lay in the weakness of the logical systems prevalent in his day, consisting as they did mainly of syllogistics and smatterings of propositional and modal logics. Clearly, logic as conceived in his day was inadequate for treating mathematics, as it did not even include an adequate theory of relations and quantification. So Frege built up a modern and precise logic which to this day serves as a standard of thoroughness and precision. How this logic may be described relative to mathematics is a problem I will discuss *after* I describe how Frege reduced natural numbers and real numbers to the concepts of his logical system.

Frege's approach to understanding the concept of number differed from that of Cantor and Weierstrass mainly because he defined numbers *explicitly in terms of their roles in measurement.* A natural number was defined explicitly as that which all things of the same cardinal measure have in common; a real number was defined explicitly as that which all magnitudes have in common which have the same measure relative to their respective units of measurement.

What is a system of measurement in general? It is a (relational) structure held in common by all sets of objects which can be given a similar assignment of numbers. The *basis* A/for a measurement system is a relational system describing a set of objects A/in terms of certain relations $R_1, \ldots, R_n$. Therefore we say

$A = \langle A, R_1, \ldots, R_n \rangle$ is a relational system
iff A is a domain of objects over which the m_i-place
relations R_i are defined.

Now the essential step in setting up the system of measurement is to find another relational system where A consists of *numbers* with the same structure as A. An example would be $N = \langle N, \leqslant, + \rangle$, the relational system whose domain N is the natural numbers and whose relations are 'not larger than' and 'plus'. Then we may say that A is measured by N iff there is an assignment of numbers from N to each object a/in A designated by 'f(a)' such that if a, a', a'' ϵ A, R(a,a') iff f(a) $\leqslant$ f(a'), and R'(a,a',a'') iff f(a) + f(a') = f(a''), where R and R' are two of the relations among $R_1,\ldots, R_n$ and all the other relations among them are defineable in terms of R and R'.

In the case of natural numbers, the measurement system is absolute, because any given set of objects can only have one cardinal number. In the case of real numbers, however, the measurement is relative to the unit of measurement applied. Hence Frege defined real numbers as sets of relations between units of measurement and magnitudes of objects measured, and he would say for instance that the relation between the standard meter and Wilt Chamberlain's height is the same as, e.g. the relation between the standard unit for electrical current and the amount of current the generator of my car puts out (both approximately 2).

Frege's method therefore went exactly according to modern measurement theory: he defined the relational structures which natural and real numbers have, he introduced sets of objects whose existence is logically certain and then showed that they satisfy those structures and can hence be used as numbers. First let us see how Frege characterized the relational structure of natural numbers in his monographs *Begriffschrift* (1879) and *Grundlagen der Arithmetik* (1884), which came n.b. prior to the publications of Dedekind and Peano. There are five axioms for this structure, given in modern notation as follows:

1. $O \epsilon N$ (the progression is non-empty)
2. $x \epsilon N \rightarrow S(x) \epsilon N$ (each member has a successor)
3. $x \epsilon N \rightarrow S(x) \neq O$ (one member (zero) has no predecessor)
4. $[x,y \epsilon N \wedge S(x) = S(y)] \rightarrow x = y$ (no branches among predecessors)
5. $[P(O) \wedge (y)((y \epsilon N \wedge P(y)) \rightarrow P(S(y)))] \rightarrow (x)(x \epsilon N \rightarrow P(x))$
   (principle of induction for progressions)

And to show how natural number theory is reducible to purely logical notions, Frege gave the following definitions for the two basic notions O and S in the *Grundlagen* as follows: first define [x] as the cardinal number of x in terms of one-to-one mappings f. Then let the cardinal number [x] be the *equivalence class* of x whereby f is the equivalence relation used; formally,

$$y \epsilon [x] \leftrightarrow (z)(z \epsilon y \leftrightarrow f(z) \epsilon x).$$

The justification for this procedure is that counting takes place by comparing members of a set to be counted one for one with members of a set whose number we already know, this comparison being carried out by using the one-to-one mapping f. Now the first number, zero, may be defined with reference to the null set, for we know this set has no members. Furthermore, we know that, if cardinal numbers up to n are defined, the n+1-tuple containing those numbers and also zero must have cardinality n + 1, which permits the definition of the successor function S. Formally, we have

$0 =_{df} [\Lambda]$; letting $g(n) = \langle 0, \ldots, n \rangle$, we say
$S(n) =_{df} [g(n)]$.

For example, if we take n = 1, we define 2 = S(1) by using the fact that $\langle 0, 1 \rangle$ contains two elements by logical necessity.

Frege characterized the relational structure of real numbers in the second volume of his *Grundgesetze der Arithmetik* as follows (again in modern notation):

1. $x > 0 \wedge x = 0 \vee -x > 0 \rightarrow x \in Rd$ (definition of the set of real magnitudes relative to the universal null magnitude)
2. $x - x = 0$ (definition of the null magnitude)
3. $x \not> x$ (irreflexivityof >)
4. $x,y > 0 \rightarrow x + y > y \vee x = y \vee y > x \rightarrow (E x) (y > z \wedge \sim P (z))) \rightarrow x \in Re$ (the least upper bound of any bounded property of magnitudes is a magnitude)
7. $x > 0 \rightarrow (Ey) (x > y > 0)$ (there are arbitrarily small magnitudes)

Again, Frege intended to define the fundamental notions of real magnitude theory in purely logical terms, find purely logical concepts, i.e. concepts whose existence is provable from logic alone, which have the structure of real magnitudes, and then define real numbers as relations between the magnitudes. This program was interrupted at the end of *Grundgesetze* II by Russell's discovery of a paradox in set theory, to which Frege's theory of concepts was subject. Nevertheless, we see that Frege intended to proceed in real number theory quite analogously to his method with natural numbers, viz. by a direct involvement of measurement theory. Even so, he was still able to derive many well-known properties of real magnitudes from his axioms, such as Archimedes' Law, the associativity and the commutativity of addition, etc.

Now the question arises in what sense Frege can be considered to have been successful in reducing mathematics to logic. Frege's thesis encountered two difficulties, the first concerning the adequacy of his proofs, the second concerning whether his premises were really entirely logical and not covertly mathematical after all. As it turns out, both difficulties are closely tied together.

As is well known, all set theories and theories of concepts and properties were shown to contain vexing contradictions by Bertrand Russell in 1902. In this year he wrote his famous letter to Frege describing the antinomy now called by his name and showing how Frege's sytem in the *Grundgesetze* was inconsistent. Unbeknownst to Russell, Cantor had earlier discovered for himself, revealed only to Dedekind, a related antinomy now called Cantor's paradox which generated contradictions as efficiently as Russell's paradox. Because Frege's system also contained all these paradoxes, it may be concluded that he did indeed *fail* to derive mathematics from logic in a strict sense. *But* it must be noted that in the same sense, *all* nineteenth century foundations of mathematics failed, including the famous reductions of Cantor, Dedekind and Weierstrass of analysis to arithmetic, as these reductions all relied essentially on an unsound set theory. Of course this fact was never used against the reduction of analysis to arithmetic even though it was used frequently against the attempted reduction by Frege of arithmetic to set theory. In this unfair manner Frege's career and reputation suffered, whereas those of the other reductionists did not.

On the other hand, as various apparently consistent set theories had been developed later on and applied to the foundation of mathematics, it turned out that the methods of introducing numbers required little modification compared to their original application in the contradictory set theories. Hence the reductions of Frege must also be regarded as successful insofar as they are easily adaptable to modern, apparently consistent set theories. It is unfortunate that Frege suffered the brunt of the hysteria at the time concerning the paradoxes mainly because he made himself more prominent through his rigor, whereas those such as Cantor and Dedekind who had not axiomatized their set theory as precisely as Frege managed to elude attack through vagueness.

In the course of the development of set theories such as Russell's type theory, the classic systems of Zermelo, Fraenkel, von Neumann, Gödel and Quine, the combinatory systems of Curry and Fitch, etc., etc., the view became more predominant that these set theories were not really logical at all, due to the extent to which apparently non-logical and sometimes vary ad hoc steps were taken to avoid paradoxes and yet still have a set theory strong enough for mathematical application. This was despite the fact that set theory is clearly an extensional theory of concepts and is intimately connected with traditionally logical and ontological problems reaching back into antiquity. Theories of concepts and terms, whether intensional or extensional, have always been reckoned to logic. One argument in favor of the new view was that a multiplicity of conflicting set theories had evolved, on which no general agreement could be made and where there seemed to be no consensus of intuitive conviction. This allegedly contrasts with the situation in logic, which was allegedly monolithic and relied on widespread agreement about axioms. But merely to state the latter is to evoke rebuttal, for in *all branches* of logic, conflicting systems with uncertain intuitive bases vie with each other.

Another popular argument is that set theory is not logical because it deals with concepts of infinity and hence must belong to mathematics. However, it turns out that the notion of infinity is definable and can be discussed even in terms of the weakest notions in logic, viz. that of the subsumption of one property under another. Even in terms of purely nominalistic logic, which is weaker still, having only the relation of part to whole, many statements about infinity may be made. Consequently, this argument fails.

One of the more usual recent arguments against placing set theory in logic is that the two fields differ very much in terms of methodology: quantification theory of predicate logic is complete, whereas even weak set theories are not. But then second order logic is already not complete and hence should be reckoned to mathematics (misnomer or no). (Strictly speaking, second order logic is one of the weakest set theories.) Are second order and higher order logics not parts of logic? Where can a line be drawn? Is the operation of abstraction or hypostasis, that classically logical notion on whose basis the higher order logics are generated, no longer to be considered logical?

The most sensible solution and the one to which I subscribe is to declare mathematics and logic to be identical. There is much to recommend this view besides the accidental fact that by far the most work being done in logic nowadays is coming out of mathematics departments. On the one hand, mathematics has long since ceased to be identified exclusively with number theories and geometry, having long ago branched out into abstract algebra and topology, which strictly speaking belong to the logic of relations. On the other hand, logic has also ceased to be identified with the trivial, tautological disciplines of syllogistics and propositional logics and has long since taken the clearly intuitive connection between the notions of predicate, concept and set and made them the major research areas.

Hence my conclusion is that, of the three schools concerning the foundations of mathematics, that of Frege is the least problematic and stands on the firmest basis. Hilbert's program of formalism has undergone such drastic changes since the publication of the proofs of Gödel and Church as to render it unrecognizable and hardly distinguishable from "logicism." Intuitionism is still a minority movement of a highly sectarian nature which, although it elicits the interest and sympathy of many mathematicians is still considered too restrictive. Of course, it must be admitted that "logicism," which is what Frege's approach is called today, can no longer be maintained with respect to any monolithic and universally accepted doctrine of logical truth in the present-day situation of pluralism and tolerance. But such an idea was really no more tenable in Frege's day than it is now; and yet there is still more to say in favor of the unity of logic and mathematics today than heretofore.