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Relationships Among the Parameters of Milk Production, Constituent Production, and Constituent Percentage

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Abstract

A method of finding the parameters pertaining to one of the variables of milk production, constituent production, and constituent percentage when the parameters pertaining to the two other variables are known is presented. The method is based on the expectation of products of random variables. This method and the delta approximation method are compared to sampling estimates from simulated data.

Knowledge of the parameters of any two of milk production, constituent production, and constituent percentage can be used to find the parameters pertaining to the third. Eisen (1) presented formulae giving these relationships, the formulae being based on the delta approximation presented in Kendall and Stuart (3). Alternative formulae are presented here, these being based on known expectations of the products of two, three, or four normally distributed variables. The case of milk production, milk fat production, and milk fat percentage (test) is used as an illustration.

The mean of milk fat production can be found as:

$$E(f) = E(mt), \qquad = \sigma_{mt} + \mu_m\mu_t, \qquad [1]$$

in which

f is milk fat production,
m is milk production,
t is test,
σ_{mt} is the covariance (phenotypic or genetic) between milk and test,
μ_m and μ_t are the means of milk and test, respectively, and
E denotes expectation.

The covariance between milk and milk fat, σ_{mf}, is

$$\sigma_{mf} = E(fm) - E(f)\ E(m), \qquad = E(mtm) - E(mt)\ E(m).$$

$E(mtm)$ can be determined from the known expectation of the product of three normally distributed variables. The expectation of three variables, say x, y, and z, can be found as:

$$\begin{aligned} E(xyz) &= E[xE(yz|x)], \\ &= E[x\{E(y|x)E(z|x) + \sigma_{zy|x}\}], \\ &= E\left[x\{[\mu_y + \frac{\sigma_{yx}}{\sigma_x^2}(x-\mu_x)]\cdot[\mu_z + \frac{\sigma_{zx}}{\sigma_x^2}(x-\mu_x)] + [\sigma_{yz} - \frac{\sigma_{yx}\sigma_{zx}}{\sigma_x^2}]\}\right], \\ &= \mu_x\mu_y\mu_z + \mu_y\sigma_{xz} + \mu_z\sigma_{xy} + \frac{\sigma_{xy}}{\sigma_x^2}\frac{\sigma_{xz}}{\sigma_x^2}[E(x^3) - 2\mu_x E(x^2) + \mu_x^3] + \mu_x\sigma_{yz} - \mu_x\frac{\sigma_{xy}\sigma_{xz}}{\sigma_x^2}, \\ &= \mu_x\mu_y\mu_z + \mu_y\sigma_{xz} + \mu_z\sigma_{xy} + \frac{\sigma_{xy}\sigma_{xz}}{\sigma_x^2\sigma_x^2}[\mu_x^3 + 3\mu_x\sigma_x^2 - 2\mu_x^3 - 2\mu_x\sigma_x^2 + \mu_x^3] + \mu_x\sigma_{yz} - \mu_x\frac{\sigma_{xy}\sigma_{xz}}{\sigma_x^2}, \\ &= \mu_x\mu_y\mu_z + \mu_x\sigma_{yz} + \mu_y\sigma_{xz} + \mu_z\sigma_{xy}. \end{aligned}$$

Therefore, the covariance between milk and milk fat is:

$$\begin{aligned} \sigma_{mf} &= \mu_m^2\mu_t + 2\mu_m\sigma_{mt} + \mu_t\sigma_m^2 - (\sigma_{mt} + \mu_m\mu_t)(\mu_m), \\ &= \mu_m\sigma_{mt} + \mu_t\sigma_m^2. \end{aligned} \qquad [2]$$

The variance of milk fat production, σ_f^2, is

$$\begin{aligned} \sigma_f^2 &= E(f^2) - [E(f)]^2, \\ &= E(mtmt) - [E(mt)]^2. \end{aligned}$$

The expectation of the product of four normally distributed variables, say w, x, y, and z, is found as:

$$\begin{aligned} E(wxyz) &= E\{wE(xyz|w)\}, \\ &= E\{w(\mu_{x|w}\mu_{y|w}\mu_{z|w} + \mu_{x|w}\sigma_{yz|w} + \mu_{y|w}\sigma_{xz|w} + \mu_{z|w}\sigma_{xy|w})\} \\ &= \mu_w\mu_x\mu_y\mu_z + \mu_w\mu_x\sigma_{yz} + \mu_w\mu_y\sigma_{xz} + \mu_w\mu_z\sigma_{xy} + \mu_x\mu_y\sigma_{wz} + \mu_x\mu_z\sigma_{wy} \\ &\quad + \mu_y\mu_z\sigma_{wx} + \sigma_{wx}\sigma_{yz} + \sigma_{wy}\sigma_{xz} + \sigma_{wz}\sigma_{xy}. \end{aligned}$$

Therefore, the variance of milk fat production is:

$$\begin{aligned} \sigma_f^2 &= \mu_m^2\mu_t^2 + 4\mu_m\mu_t\sigma_{mt} + \mu_m^2\sigma_t^2 + \mu_t^2\sigma_m^2 + \sigma_m^2\sigma_t^2 + 2\sigma_{mt}^2 - \sigma_{mt}^2 - 2\mu_m\mu_t\sigma_{mt} - \mu_m^2\mu_t^2, \\ &= 2\mu_m\mu_t\sigma_{mt} + \mu_m^2\sigma_t^2 + \mu_t^2\sigma_m^2 + \sigma_m^2\sigma_t^2 + \sigma_{mt}^2. \end{aligned} \qquad [3]$$

This formula is the same as that presented by Goodman (2) for the variance of the product of two variables in the particular case that the two variables are normally distributed. This assumption of normality is examined later.

If the parameters for milk and milk fat production are known, the parameters concerning test can be found by algebraic rearrangement of Equations [1] to [3]. The mean for test (μ_t) can be found by noting that from [2]:

$$\sigma_{mt} = \frac{\sigma_{mf} - \mu_t\sigma_m^2}{\mu_m}, \qquad [2a]$$

so that substituting [2a] into [1] and rearranging leads to:

$$\mu_t = \frac{\mu_m\mu_f - \sigma_{mf}}{\mu_m^2 - \sigma_m^2}. \qquad [4]$$

Substitution of Equation [4] into [2a] gives:

$$\sigma_{mt} = \frac{\sigma_{mf} - \left[\dfrac{\mu_m\mu_f - \sigma_{mf}}{\mu_m^2 - \sigma_m^2}\right]\sigma_m^2}{\mu_m},$$

$$= \frac{\mu_m\sigma_{mf} - \mu_f\sigma_m^2}{\mu_m^2 - \sigma_m^2}. \quad [5]$$

Rearrangement of Equation [3] gives:

$$\sigma_t^2 = \frac{\sigma_f^2 - 2\mu_m\mu_t\sigma_{mt} - \mu_t^2\sigma_m^2 - \sigma_{mt}^2}{\mu_m^2 + \sigma_m^2}, \quad [6]$$

which can be obtained once the values of Equations [4] and [5] have been determined.

The validity of these formulae is dependent on the assumption of a normal distribution of milk and test or in general of the variables whose product is the third characteristic. The importance of the assumption of normality is examined by comparisons with simulated data. One of the variables in the product is considered to be the ratio of two normally distributed variables, which is approximately the case for test. This variable thus does not have a normal distribution.

Estimates of the phenotypic and genetic variances of test and correlations between milk and test from simulated data are compared (Tables 1 and 2) to the parameters determined by the delta approximation and to the parameters determined by the formulae described above. For each combination of phenotypic milk and milk fat standard deviations, milk and milk fat means and phenotypic correlation between milk and milk fat, 10,000 values for milk production and milk fat production were simulated. From these sample values estimates of the phenotypic mean and variance of test and the phenotypic correlation between milk and test were determined. Corresponding values for the mean and variance of test and the correlation between milk and test were determined by the delta approximation method and by the formulae just presented by substitution of the parameters concerning milk and milk fat production for that particular combination. In the expectation method computations the covariance between milk and test was found according to Equation [5] and then the correlation was determined. The parameters concerning milk and milk fat production used represent the present situation with some allowance for variation. The procedure was then repeated for combinations of genotypic parameters concerning milk and milk fat, to generate data from which the geno-

TABLE 1. Comparisons of phenotypic parameters obtained by expectation (E) and delta (D) methods with those obtained as sampling estimates from simulated data (S).

Phenotypic parameters[a]				Ratios of milk fat percentage (test) means		Ratios of test variances		Correlations between milk and test		
μ_M	μ_F	σ_F	r_{MF}	E/S	D/S	E/S	D/S	S	E	D
(kg)										
5,443	196	36	.85	.997	.996	.756	.781	—.481	—.492	—.462
			.95	.999	.999	.773	.798	—.502	—.519	—.489
		46	.85	1.002	1.002	.836	.871	—.082	—.101	—.094
			.95	1.000	1.001	.817	.851	.165	.169	.158
	207	36	.85	.997	.996	.730	.751	—.572	—.572	—.540
			.95	.998	.998	.747	.767	—.614	—.641	—.605
		46	.85	1.000	1.000	.774	.806	—.209	—.199	—.186
			.95	1.000	1.000	.832	.868	.001	.000	.000
6,350	229	36	.85	1.001	1.001	.833	.854	—.480	—.484	—.463
			.95	1.001	1.000	.825	.845	—.502	—.511	—.489
		46	.85	.999	.999	.889	.917	—.093	—.099	—.094
			.95	.999	1.000	.894	.922	.158	.166	.158
	241	36	.85	.998	.997	.813	.831	—.559	—.563	—.540
			.95	1.000	.999	.803	.819	—.624	—.631	—.605
		46	.85	1.000	.999	.865	.891	—.180	—.196	—.186
			.95	1.001	1.001	.861	.888	.000	.000	.000
7,258	261	36	.85	.999	.999	.893	.910	—.473	—.479	—.462
			.95	1.001	1.000	.847	.863	—.504	—.506	—.489
		46	.85	1.000	1.000	.870	.891	—.084	—.098	—.094
			.95	1.001	1.001	.905	.927	.152	.164	.158
	276	36	.85	1.000	.999	.897	.912	—.542	—.558	—.540
			.95	.999	.999	.834	.847	—.620	—.625	—.605
		46	.85	.999	.999	.891	.912	—.192	—.193	—.186
			.95	.999	.999	.916	.939	—.020	.000	.000

[a] Mean for milk (μ_M), mean for milk fat (μ_F), standard deviation of milk fat (σ_F), and correlation between milk and milk fat (r_{MF}). The means for milk fat are 3.6 and 3.8% of each mean for milk. Standard deviation of milk is held constant at 1,134 kg.

TABLE 2. Comparisons of genetic parameters obtained by expectation (E) and delta (D) methods with those obtained as sampling estimates from simulated data (S).

Genetic parameters[a]				Ratios of milk fat percentage (test) means		Ratios of test variances		Correlations between milk and test		
μ_M	μ_F	σ_F	r_{MF}	E/S	D/S	E/S	D/S	S	E	D
(kg)										
5,443	196	18	.70	.999	.999	.963	.971	−.510	−.519	−.511
			.80	.999	.999	.937	.945	−.488	−.484	−.476
		23	.70	1.001	1.001	.966	.976	−.260	−.274	−.270
			.80	1.000	1.000	.922	.932	−.173	−.167	−.164
	207	18	.70	.999	.999	.929	.936	−.577	−.572	−.564
			.80	1.000	1.001	.944	.952	−.543	−.551	−.542
		23	.70	1.001	1.000	.945	.954	−.335	−.336	−.330
			.80	1.000	.999	.945	.954	−.250	−.246	−.242
6,350	229	18	.70	1.000	1.000	.955	.961	−.525	−.517	−.511
			.80	1.000	1.000	.943	.948	−.482	−.482	−.476
		23	.70	1.000	1.000	.966	.973	−.270	−.273	−.270
			.80	1.000	1.000	.957	.965	−.167	−.166	−.164
	241	18	.70	1.001	1.001	.978	.983	−.561	−.570	−.564
			.80	1.000	1.000	.977	.982	−.534	−.548	−.542
		23	.70	1.000	1.000	.954	.961	−.327	−.334	−.330
			.80	1.001	1.001	.957	.964	−.247	−.245	−.242
7,258	261	18	.70	.999	.999	.976	.981	−.513	−.516	−.511
			.80	1.000	1.000	.970	.974	−.475	−.481	−.476
		23	.70	1.000	1.000	.957	.963	−.275	−.272	−.270
			.80	1.001	1.001	.975	.981	−.177	−.166	−.164
	276	18	.70	1.001	1.001	.959	.963	−.567	−.568	−.564
			.80	1.000	1.000	.952	.956	−.550	−.547	−.542
		23	.70	1.000	1.000	.951	.957	−.336	−.333	−.330
			.80	1.000	1.000	.973	.978	−.242	−.245	−.242

[a] Mean for milk (μ_M), mean for milk fat (μ_F), standard deviation of milk fat (σ_F), and correlation between milk and milk fat (r_{MF}). The means for milk fat are 3.6 and 3.8% of each mean for milk. Standard deviation of milk is constant at 567 kg. Genetic variances for milk and milk fat are one-fourth the phenotypic variances.

typic mean and variance of test and genotypic correlation between milk and test were estimated.

The means determined by the formulae appear to be quite close to the sampling estimates of the means. The calculated variances, particularly the phenotypic variances at low mean levels of milk production, are lower than the sampling estimates. The ratio of expectation method variances to sampling estimates appears to be lower than the ratio of delta method variances to sampling estimates, although only slightly so. The ratios of calculated variances to sampling estimates change with changes in the known parameters. The phenotypic and genetic correlations calculated by both the expectation and delta method are quite close to the sampling estimates, the closeness depending on the magnitude of the known parameters.

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