1982

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1982. *Transactions of the Nebraska Academy of Sciences,* X:85–89.

FROM A MATHEMATICS OF NECESSITY TO A MATHEMATICS OF CONVENTION

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In Wittgenstein's speculation the transition from the philosophical point of view expressed in the *Tractatus logico-philosophicus* to the later development of his thought, as it is reflected in the *Philosophical investigations* and *On certainty,* is relevant also to his conception of mathematics.

In particular, while in the *Tractatus,* mathematics is not given an account of its own, independent of the account which is given to logic, the *Remarks on the foundations of mathematics* seems to offer sufficient evidence for the belief that, in the late stage of Wittgenstein's speculation, the analysis of the foundations of mathematics received an explicit treatment on its own.

This discussion is concerned with the attempt to illustrate the transformation which occurred in Wittgenstein's way of conceiving mathematics; that is to say, with the passage from the original idea of mathematics as a discipline, on a par with logic itself, reflecting the unmodifiable, and therefore necessary, features of the logical structure of the world, to the more "intuitionistic" conception of mathematics, as a constructive activity, leading to merely conventional truths.

The different connotations of the notion of "truth," in relation to these two different conceptions of mathematics, are also considered.

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The purpose of this paper is to sketch some features of the development and transformation of Wittgenstein's account of mathematics and mathematical truth from his first work (Wittgenstein, 1922) to the later development of his thought (Wittgenstein, 1956).

Let us start directly in outlining those main features of Wittgenstein's early conception of mathematics in such a way that, in the end, we will have the ground that we need for the comparison with the later view.

Three aspects will be addressed: (1) the form of mathematical propositions, (2) the relation between logic and mathematics, and (3) the character of necessity of mathematical truth.

The transition from the claim, "The propositions of logic are tautologies," to the remark, "Logic is not a body of doctrine, but a mirror-image of the world. Logic is transcendental," is significant.

In the following passage, both the link and the differentiation between logic and mathematics can be found. The assertion that "Mathematics is a logical method" defines the link, but already suggests a differentiation. "The propositions of mathematics are equations, and therefore pseudo-propositions" specifies the difference, without denying kinship.

Logical propositions cannot be regarded as standardly meaningful propositions, such as those of the natural sciences. The characterization of logical propositions as tautologies, *i.e.,* as propositions which say nothing because of their being "unconditionally true," and therefore, such as to admit "all possible situations," attributes to them "a unique status among all propositions" (Wittgenstein, 1922:69 and 121).

The truth-table method that allows the determination of the truth-value of complex propositions through knowledge of the truth-values of the elementary propositions with which they are constructed (Wittgenstein, 1922:73), and therefore theoretically succeeds in reducing the question of the truth-value of propositions to the empirically verifiable correspondence between states of affairs, actually existing in the world, and elementary propositions picturing them, brings into evidence the peculiarity of tautologies as propositions always true, regardless of the truth-values of their truth-functional components (Wittgenstein, 1922:69 and 121).

What about mathematical propositions? They are not tautologies, since they are not discovered to be "unconditionally true" independently of the truth of their truth-functional components. The peculiarity that makes them pseudo-propositions is, as in the case of logical propositions, a certain lacking of any "content." There is nothing comparable to what the pictured state of affairs represents for a proposition of the natural sciences.

The assertion that "A proposition of mathematics does not express a thought" (Wittgenstein, 1922:133) is a denial of any representational content in a mathematical proposition. Since "A thought is a proposition with a sense" (Wittgenstein, 1922:35) and "A proposition [it can be presupposed that here Wittgenstein was thinking of a 'proposition with a sense'] is a picture of reality" (Wittgenstein, 1922:39), and since "Instead of 'This proposition has such and such a sense' we can simply say 'This proposition represents such and such situation'" (Wittgenstein, 1922:43), it follows that a mathematical proposition does not represent any situation, *i.e.,* is not a picture of reality.

This lack of a *"sinn"* of mathematical propositions is shown in their form itself, since, because they are equations, they are pseudo-propositions (Wittgenstein, 1922:133).

Before advancing the analysis of the form of the equation, a too hastily formed conclusion that could be derived from the account of the senselessness of mathematical, and more generally, logical propositions, namely their absolute meaninglessness or uselessness is questioned.

It is essential to stress that for Wittgenstein to say that logical and mathematical propositions have no sense (they are *"sinnlos"*) is not the same as saying that they are nonsensical (*"unsinnig"*); as clearly stated in the propositions above mentioned, *"sinn"* is had only by propositions of natural sciences. The fact that logical and mathematical propositions lack it does not suffice to make them *"unsinnig"*—as for example philosophical propositions are (Wittgenstein, 1922:37).

This relevant difference cannot be justified unless the denial of sense of logical and mathematical propositions is not meant as equivalent to a denial of relevance or worth. Logical and mathematical propositions are not queer mixtures of signs; on the contrary, they perform the role of showing the logical structure of the world: "The logic of the world, which is shown in tautologies by the propositions of logic, is shown in equations by mathematics" (Wittgenstein, 1922:133).

Therefore, in order to explain both their emptiness and their function, it could be said that they have not a subject matter in the way in which empirical propositions do (Wittgenstein, 1922:129), or that the relation between a logical proposition and what is shown by it is different from the relation between empirical propositions and their subject matter.

Empirical propositions *picture* their subject matter; logical propositions *show* what only improperly can be called their subject matter, "the logic of the world."

Hence, the "unique status" of logical and mathematical propositions is not due merely to lack of sense or to a difference of subject matter in comparison with empirical propositions; rather, their status can be understood only by taking into consideration *how* these two types of propositions refer to the world, and this is primarily reflected in their *form.*

Mathematical propositions are all of the form a=a, or a=b. But, it is not the case that the supposed identity of reference of the two expressions on the two sides of the equality is asserted by the equation. Rather, the contrary is true: The equation can be written because of the already known identity of the two expressions (Wittgenstein, 1922:135 and 134).

What is stated, by asserting the identity of the two signs/symbols or expressions, is their capability of being substituted for one another. Therefore, for this reason, equations are in a certain way superfluous: ". . . the essential point about an equation is that it is not necessary in order to show that the two expressions connected by the sign of equality have the same meaning, since this can be seen from the two expressions themselves" (Wittgenstein, 1922:133).

The sign of identity cannot establish or produce the equivalence of the two expressions it unites, but on the contrary only because the two expressions have already been known to be equivalent, can the sign of identity be employed.

It really seems that there are only two alternatives with regard to the use of this sign: either it can be eliminated from our language (Wittgenstein, 1922:105), because the identity or the difference of objects is to be resolved by the mere identity or difference of signs denoting the objects, or its relevance has to be seen not in its power of equating two different expressions—since, were the equivalence of the two expressions not yet known, the sign of identity itself could not be used—but in being the instrument through which some, in a certain way useful, transitions from a set of expressions to another set of different expressions can be performed.

Being an equation is not something accidental for mathematical propositions: "It is the essential characteristic of mathematical method that it employs equations" (Wittgenstein, 1922:135).

Thus, mathematical propositions fall under the second of the possible alternatives. They have a function which requires their being equations. And since the legitimate use of the sign of identity is subsequent to the actual equivalence of the expressions, mathematical propositions, according to Wittgenstein, are not only true, but obviously true.

Similarly, as already mentioned, logical propositions are "unconditionally true."

Neither in logic nor in mathematics is any mistake or unexpected discovery possible (Wittgenstein, 1922:95, 133, and 129).

Their validity *a priori* is connected with the capability of conceiving a world: "What makes logic a priori is the *impossibility* of illogical thought" (Wittgenstein, 1922:95).

In summary, mathematical and logical propositions are similar in the following respects: (1) they share the feature of being different from the propositions of natural science: neither mathematical nor logical propositions picture facts; (2) they both show the logical structure of the world; and (3) they are either "obviously" or "unconditionally" true.

On the other hand, they are different in two says: (1) mathematical propositions are equations and not tautologies, and (2) their peculiar function is to allow the transition from a set of expressions not belonging to mathematics to another set of expressions not belonging to mathematics.

In this paper the early view Wittgenstein (1922) held on mathematics will not be criticized. The purpose, rather, is to outline a comparison between his earlier and later positions.

Nonetheless, some directions along which a critical analysis would be developed are mentioned:

1. The characterization of mathematical truths as equations always obviously true does not seem to cover the whole field of mathematically true propositions.

2. The account of identity, which constitutes the basis of the early theory of mathematics, leaves unsolved the problem of the status of identity itself: its not being "an essential constituent of conceptual notation," if can be understood with regard to the objects, whose difference or equality might be solved through a less rich and various use of signs, seems to be compatible with an account of mathematics as constituted only of propositions employing the sign of identity, only on the basis of the assumption of an eventual total superfluity of mathematics itself.

3. Finally, the relation between logic and mathematics cannot be interpreted as a relation of mere inclusion of the second in the first, as a logicist reading of the first work (Wittgenstein, 1922) would suggest: some difference between the two is stressed and the common characteristic of showing the logical structure of the world is asserted in a rather vague way.

Let us come back now to the main aim of this exposition. It should be clear enough at this point why Wittgenstein's early conception of mathematics (1922) is characterized as a "mathematics of necessity."

If that account of mathematics is examined, the idea of a presupposed relation of kinship linking logic and mathematics is easily envisioned. One of the essential features that distinguishes this relation is the *a priori* necessary truth of this set of propositions.

Furthermore, their necessity wholly derives from their peculiar link to the world, *i.e.,* their lacking any pictorial connection with states of affairs, and their showing the logical structure of the world.

What then is Wittgenstein's (1956) account of mathematical propositions, of their form, of their relation to logic, and finally of their truth in the later development?

First, there is no explicit restriction of mathematical propositions to equations, but despite this the points of contact seem numerous: the examples of mathematical propositions are generally equations, the discussion related to the notion of "calculations" is developed through the presentation of equations, and the discussion of the concept of "proof" relies again on inferences whose steps are justified on the basis of the equivalence of the various lines.

Furthermore, mathematical propositions are used for the transition from sets of true propositions to different sets of true propositions.

Yet despite these similarities with the early writing (Wittgenstein, 1922), the fact that no restriction on the form of mathematical propositions is given is not wholly irrelevant: mathematics is "a MOTLEY of techniques of proof," and in addition is regarded as not being "a sharply delimited concept" (Wittgenstein, 1956:84e and 155e).

Perhaps Wittgenstein was still convinced of the possibility of reducing all mathematical truths to the form of equation, but it seems that his first concern was to give the most general and comprehensive account of all the various branches of this "science" ("bird's eye view"), rather than to synthesize most of its features into a small number of characterizations ("the general form of the proposition").

Let us now consider in the second work, what in the first seemed to be the central point of Wittgenstein's conception of mathematics, namely its relation to logic.

First, the differentiation of logical and mathematical propositions from empirical propositions remains, though the grounds of the difference have changed. Wittgenstein's approach to this topic was on the line of an epistemological analysis, *i.e.,* he considered the different quality of the certainty that distinguishes our conviction of a mathematical truth from our conviction of an empirical truth.

The inexorability of mathematical and logical inferences, the way that we do not even imagine to call in question the certainty of their conclusions, is peculiar to them, strictly connected with their function in our life, and independent of experience. This, on the contrary, supports the truth of those empirical propositions we regard as true (Wittgenstein, 1956: 24e and 27e): "Why are the Newtonian laws not axioms of mathematics? Because we could quite well imagine things being otherwise. . . . A proposition which it is supposed to be possible to imagine as other than true has a different *function* from one for which this does not hold" (Wittgenstein, 1956:114e).

Through the discussion of "experiment" and repeated assertion that the result of each mathematical proof is not unexpected, Wittgenstein stressed the "*fundamental* difference, together with an apparent similarity, between the roles of an arithmetical proposition and an empirical proposition" (Wittgenstein, 1956:32e).

In effect, form and empirical content are not features that characterize propositions as empirical or non-empirical (Wittgenstein, 1956:13e and 75e).

Calculation, the main mathematical activity with which Wittgenstein dealt, is not an experiment when it is normally performed: there is no expectancy of unpredictable results. Yet there may be a situation in which the same expressions that in a normal activity of calculating are mathematical expressions can turn into the steps of an experiment. The analogy with the action of "lighting the stove in the morning" seems very enlightening. Indeed this action is surely performed, under normal circumstances, with a specific purpose, but nothing in the action itself can forbid the performance of this same action for an experiment. Analogously, calculation is normally one of the mathematical activities performed with propositions, whose characteristic is that of following certain rules in such a way that nothing unexpected, uncertain, or ambiguous can result. But, nothing in the propositions rules out the possibility of using them with the aim of performing an experiment, *e.g.,* when "the teacher makes the pupil do a calculation in order to see whether he can calculate" (Wittgenstein, 1956:95e, 97e, 98e, and 99e).

What is then the root of cogency with which mathematical proofs, calculations, and measurements convince us of the truth of their results?

The analysis of logical inference reveals both similarity and difference between Wittgenstein's early (1922) and late (1956) points of view.

In fact, if on one side mathematics and logic are still regarded as fields of linguistic expression in which absolute certainty, "peculiar solidity," and "unassailable position," are warranted, on the other side is a severe criticism of the justification of these characteristics in terms of any metaphysical correspondence with a mysterious structure, essence, or truth in reality (Wittgenstein, 1956:5e and 47e).

In particular, Wittgenstein denied the myth present in his first account, namely that "logic is a kind of ultra-physics, the description of the 'logical structure' of the world, which we perceive through a kind of ultra-experience."

In opposition to the theory of logic as "showing" the necessary *a priori* structure of the world, and analogously to the theory of logic as "grammar," *i.e.,* as the set of rules of every language-game connected to a form of life (Wittgenstein, 1968), in his late account the explanation of the cogency of logical inferences, through which mathematical truths are derived, is the function that our use of these expressions gave to them.

The starting point is the acknowledgment of mathematics as one of the many activities of our life and only from the peculiar role it has in our life, does the necessity of logical inferences derive (Wittgenstein, 1956:3e and 37e).

Therefore, the "hardness of the logical *must*" is wholly dependent upon our conventional, agreed conception of what line logical thinking should follow; the "inexorability" of logic is nothing but the actual fact that *we* are inexorable in applying these laws. And this is only because, in our activities, this particular way of inferring, counting, and calculating resulted to be the most useful, and the fittest to our practical purposes (Wittgenstein, 1956:35e, 37e, and 107e).

A calculation procedure that always gave different results in different situations or repetitions of the same calculi, would be of less use in our life.

With the shifting of the foundations of logic and correctness of inferring from the ideal correspondence with the structure of the world to our practical obedience to the logical rules, the notions of truth and of right and wrong change deeply: "But is this counting only a *use*, then; isn't there also some truth corresponding to this sequence?" "The *truth* is that counting has proved to pay" "and of course there is such a thing as right and wrong in passing from one measure to the other; but what is the reality that 'right' accords with here? Presumably a *convention,* or a *use,* and perhaps our practical requirements" (Wittgenstein, 1956:3–4e and 6e).

Wittgenstein's later philosophy of mathematics is much less simple than it could seem at first sight. In fact, if in a certain sense it may seem that any sort of foundation of mathematical truth relies on actual use of mathematical procedures, and that in turn the only root of their correctness is previous conventional agreement on this use itself, such that a different historical development could have produced a different set of mathematical truths and procedures, on the other side such a wholly and merely "conventionalistic" interpretation of Wittgenstein's later point of view should not be supported.

In fact, Wittgenstein's appeal to convention and use as the grounds for logical and mathematical necessity is quite explicit, but Wittgenstein's notion of convention, contrary to Dummet's (1966) reading of it, is not the result of either a subjective or an arbitrary decision. Convention is not an abstract and theoretical choice of some rules as those which we agree to follow before starting the game. The convention, the use from which the rules of our language-game derive, are actually produced within the form of life in which that language-game is employed. The influence that the form of life exerts on the prevalence of one or the other rule is a determining factor.

Nonetheless, there still remains a certain obscurity in some of Wittgenstein's remarks as for the process of production of this convention itself. The problem is that of understanding whether the pragmatic and behavioristic foundation of all linguistic activities was able, in Wittgenstein's account, to justify thoroughly the actual working of language. That is, through the new notions of "form of life" and "language-game" did he really succeed in ruling out any appeal to supposedly deeper, undefinable roots of our agreement on what we call "logical truth"?

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