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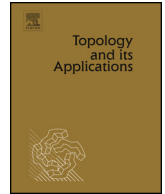
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The diffeology of Milnor's classifying space



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ABSTRACT

We define a diffeology on the Milnor classifying space of a diffeological group G , constructed in a similar fashion to the topological version using an infinite join. Besides obtaining the expected classification theorem for smooth principal bundles, we prove the existence of a diffeological connection on any principal bundle (with mild conditions on the bundles and groups), and apply the theory to some examples, including some infinite-dimensional groups, as well as irrational tori.

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1. Introduction

Let G be a topological group with a reasonable topology (Hausdorff, paracompact, and second-countable, say). Milnor [19] constructed a universal topological bundle $EG \rightarrow BG$ with structure group G satisfying:

- for any principal G -bundle $E \rightarrow X$ over a space (again, assume Hausdorff, paracompact, and second-countable) there is a continuous “classifying map” $F: X \rightarrow BG$ for which E is G -equivariantly homeomorphic to the pullback bundle F^*EG ,
- any continuous map $F: X \rightarrow BG$ induces a bundle F^*EG with F a classifying map, and
- any two principal G -bundles are isomorphic if and only if their classifying maps are homotopic.

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To push this classification into the realm of Lie groups and smoothness, one would need smooth structures on the spaces EG and BG . Many approaches exist, which extend to infinite-dimensional groups (see, for example, [20], [9, Theorem 44.24]). The main point here is that EG and BG are not typically manifolds, and so a more general smooth structure is required.

In this paper, we take the approach of diffeology. The language is quite friendly, allowing one to differentiate and apply other analytical tools with ease to infinite-dimensional groups such as diffeomorphism groups, including those of non-compact manifolds, as well as projective limits of groups, including some groups which appear naturally in the ILB setting of Omori [22], and may not exhibit atlases. Moreover, we can include in this paper interesting groups which are not typically considered as topological groups. For example, irrational tori (see Example 5.3) have trivial topologies (and hence have no atlas) and only constant smooth functions; however, they have rich diffeologies, which hence are ideal structures for studying the groups. Irrational tori appear in important applications, such as pre-quantum bundles on a manifold associated to non-integral closed 2-forms (see Subsection 5.6). Another benefit of using diffeology is that we can directly use the language to construct connection 1-forms on EG . Our main source for the preliminaries on diffeology is the book by Iglesias-Zemmour [8].

Our main results include Theorem 3.6, which states that there is a natural bijection between isomorphism classes of so-called D-numerable principal G -bundles over a Hausdorff, second-countable, smoothly paracompact diffeological space X , and smooth homotopy classes of maps from X to BG . This holds for any diffeological group G . As well, we prove Theorem 4.3, which states that if G is a regular diffeological Lie group, and X is Hausdorff and smoothly paracompact, then any D-numerable principal G -bundle admits a connection; in particular, such a bundle admits horizontal lifts of smooth curves. These theorems provide a method for constructing classifying spaces different, for example, to what Kriegl and Michor do in [9, Theorem 44.24] with $G = \text{Diff}(M)$ for M a compact smooth manifold, where they show that the space of embeddings of M into ℓ^2 yields a classifying space for G .

Our framework is applied to a number of situations. We show that EG is contractible (Proposition 5.1), which allows us to study the homotopy of BG (Proposition 5.2). We also study smooth homotopies between groups, and how these are reflected in classifying spaces and principal bundles (Subsection 5.2). We transfer the theory to general diffeological fibre bundles via their associated principal G -bundles (Corollary 5.9) and discuss horizontal lifts in this context (Proposition 5.11). We also transfer the theory to diffeological limits of groups, with an application to certain ILB principal bundles (Proposition 5.18). We show that a short exact sequence of diffeological groups induces a long exact sequence of diffeological homotopy groups of classifying spaces (Proposition 5.19), and apply this to a short exact sequence of pseudo-differential operators and Fourier integral operators (Example 5.21). Finally, we apply this theory to irrational torus bundles, which are of interest in geometric quantisation [26], [7], [8, Articles 8.40–8.42] and the integration of certain Lie algebroids [5].

This paper is organised as follows. Section 2 reviews necessary prerequisites on diffeological groups (including diffeological Lie groups and regular groups), internal tangent bundles, and diffeological fibre bundles. In Section 3 we construct the diffeological version of the Milnor classifying space, $EG \rightarrow BG$, and prove Theorem 3.6. In Section 4, we introduce the theory of connections from the diffeological point-of-view, and prove Theorem 4.3. Finally, in Section 5, we have our applications.

A few open questions are inspired. The conditions on the D-topology (in particular, Hausdorff and second-countable conditions, and sometimes smooth paracompactness as well) seem out-of-place in the general theory of diffeology. Even though these conditions are satisfied in our examples, in some sense, topological conditions and arguments should be *replaced* with diffeological conditions and arguments. This leads to the first question.

Question 1. Under what conditions is the D-topology of a diffeological space Hausdorff, second-countable, and smoothly paracompact? Can one weaken these conditions?

An answer to this would allow us to rephrase [Theorem 3.6](#) in a manner more natural to diffeology. A partial answer, for example, is to require that the smooth real-valued functions separate points: in this case, the weakest topology induced by the functions is Hausdorff, and this is a sub-topology of the D-topology. It follows that the D-topology is Hausdorff.

Another question is an obvious one:

Question 2. Given a diffeological group G , if $E \rightarrow X$ is a principal G -bundle satisfying necessary mild conditions, and the total space E is diffeologically contractible, is $E \rightarrow X$ **universal** in the sense that any principal G -bundle over a sufficiently nice diffeological space Y has a unique (up to smooth homotopy) classifying map to X ? (The answer is affirmative in the topological category.)

The theme of universality continues:

Question 3. Let G be a regular diffeological Lie group ([Definition 2.13](#)), and let $E \rightarrow X$ be a principal G -bundle in which the D-topology on X satisfies mild conditions. Is every diffeological connection, or even every connection 1-form, the pullback of the diffeological connection (or connection 1-form) constructed on $EG \rightarrow BG$ in [Theorem 4.9](#)?

Some of these questions have been addressed in [\[4\]](#). As mentioned above, Kriegl and Michor give a classifying space for the group $\text{Diff}(M)$ where M is a compact smooth manifold, and also show that this classifying space has a universal connection. However, the proof uses the fact that every such M has an isometric embedding into some Euclidean space, and we no longer have this advantage for general diffeological groups.

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2. Preliminaries

This section provides background on diffeological groups and fibre bundles. For a review of more basic properties of diffeological spaces, we refer to the book of Iglesias-Zemmour [\[8\]](#). In particular, products, sub-objects, quotients, and underlying D-topologies of diffeological spaces, as well as homotopies of maps between them, are used throughout this paper.

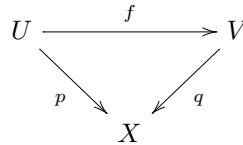
2.1. Diffeological groups and internal tangent bundles

In this subsection we review diffeological groups and their actions. We then introduce the internal tangent bundle of a diffeological space with the goal of obtaining a Lie algebra for certain diffeological groups admitting an exponential map. For more details on the basics of diffeological groups, see [\[8, Chapter 7\]](#). For more on internal tangent bundles, see [\[3\]](#). For more details on diffeological Lie groups, see [\[12\]](#) and [\[9\]](#) (although the latter reference deals with infinite-dimensional groups, not diffeological ones).

Definition 2.1 (*Diffeological groups and their actions*). A **diffeological group** is a group G equipped with a diffeology such that the multiplication map $m: G \times G \rightarrow G$ and the inversion map $\text{inv}: G \rightarrow G$ are smooth. A **diffeological group action** of G on a diffeological space X is a group action in which the map $G \times X \rightarrow X$ sending (g, x) to $g \cdot x$ is smooth. Fixing $g \in G$, denote by L_g left multiplication by g , and by R_g right multiplication by g . Denote by e the identity element of G .

In order to begin associating a Lie algebra to certain diffeological groups, we need to establish the theory of tangent spaces and bundles to a diffeological space.

Definition 2.2 (*Internal tangent space*). Let (X, \mathcal{D}) be a diffeological space, and fix $x \in X$. Let $\mathbf{Plots}_0(\mathcal{D}, x)$ be the category whose objects are plots $(p: U \rightarrow X) \in \mathcal{D}$ for which U is connected, $0 \in U$, and $p(0) = x$; and whose arrows from $p: U \rightarrow X$ to $q: V \rightarrow X$ are commutative triangles



where $f: U \rightarrow V$ is smooth and $f(0) = 0$. Let F be the forgetful functor from $\mathbf{Plots}_0(\mathcal{D}, x)$ to \mathbf{Open}_0 , the category whose objects are connected open subsets of Euclidean spaces containing 0 and whose arrows are smooth maps between these that fix 0. F sends a plot to its domain, and a commutative triangle as above to $f: U \rightarrow V$. Finally, let \mathbf{Vect} be the category of vector spaces with linear maps between them, and let $T_0: \mathbf{Open}_0 \rightarrow \mathbf{Vect}$ be the functor sending U to T_0U and $f: U \rightarrow V$ to $f_*|_0: T_0U \rightarrow T_0V$. Then define the **internal tangent space** T_xX to be the colimit of the functor $T_0 \circ F$. Denote by TX the set $\bigsqcup_{x \in X} T_xX$.

In order to equip TX with a suitable diffeology, we need a few more definitions.

Definition 2.3 (*Induced tangent maps*). Let (X, \mathcal{D}) be a diffeological space. For a plot $(p: U \rightarrow X) \in \mathbf{Plots}_0(\mathcal{D}, x)$, denote by p_* the map sending vectors in T_0U to $T_{p(0)}X$ given by the definition of a colimit. Extend this to more general plots $p: U \rightarrow X$ in \mathcal{D} as follows. Define $p_*: TU \rightarrow TX$ to be the map that sends $v \in T_uU$ to the element in $T_{p(u)}X$ given by $(p \circ \text{tr}_u)_*((\text{tr}_{-u})_*v)$, where tr_u is the translation in Euclidean space sending $U - u$ to U . If $c: \mathbb{R} \rightarrow X$ is a smooth curve, then by $\frac{dc}{dt}$ we mean $c_*\left(\frac{d}{dt}\right)$ where $\frac{d}{dt}$ is the constant section $\mathbb{R} \rightarrow T\mathbb{R}: t \mapsto (t, 1)$. We may also denote p_* by Tp , emphasising the functoriality of T (see [Remark 2.8](#)).

Definition 2.4 (*Diffeological vector spaces*). A **diffeological vector space** is a vector space V over \mathbb{R} equipped with a diffeology such that addition $+: V \times V \rightarrow V$ and scalar multiplication $\cdot: \mathbb{R} \times V \rightarrow V$ are smooth.

We use the terminology of [\[23\]](#), but continue to follow [\[3\]](#).

Definition 2.5 (*Diffeological pseudo-bundles*). A **(diffeological) vector pseudo-bundle** $\pi: E \rightarrow X$ is a pair of diffeological spaces E and X with a smooth surjection $\pi: E \rightarrow X$ between them such that for each $x \in X$ the fibre $\pi^{-1}(x)$ is a diffeological vector space. Moreover, fibrewise addition $+: E \times_X E \rightarrow E$, scalar multiplication $\cdot: \mathbb{R} \times E \rightarrow E$, and the zero section $X \hookrightarrow E$ are required to be smooth.

Remark 2.6. Let $\pi: E \rightarrow X$ be a smooth map between diffeological spaces such that for each x , the fibre $\pi^{-1}(x)$ is a diffeological vector space. Then there is a smallest diffeology on E that contains the original diffeology on E and so that π is a vector pseudo-bundle; see [\[3, Proposition 4.6\]](#).

Definition 2.7 (*Internal tangent bundle*). Define the **internal tangent bundle** of a diffeological space X to be the set $TX := \bigsqcup_{x \in X} T_xX$ equipped with the smallest diffeology $T\mathcal{D}$ such that

- (1) TX is a vector pseudo-bundle,
- (2) for each plot $p: U \rightarrow X$, the induced map $Tp: TU \rightarrow TX$ is smooth.

Remark 2.8. T is a functor, sending diffeological spaces (X, \mathcal{D}) to $(TX, T\mathcal{D})$. Consequently, we get the chain rule: given smooth maps $f: X \rightarrow Y$ and $g: Y \rightarrow Z$, we have that $T(g \circ f) = Tg \circ Tf$. Moreover, T respects products: let X and Y be diffeological spaces. Then $T(X \times Y) \cong TX \times TY$ [3, Proposition 4.13].

We now arrive at an important generalisation of what is very well known in standard Lie group theory.

Theorem 2.9 (*Tangent bundle of a diffeological group*). *Let G be a diffeological group with identity element e . Then, TG is isomorphic as a vector pseudo-bundle to $G \times T_eG$.*

Proof. This is [3, Theorem 4.15]. The diffeomorphism is given by sending $v \in T_gG$ to $(g, (L_{g^{-1}})_*v)$. \square

Diffeological groups admit adjoint actions of G on T_eG .

Definition 2.10 (*Adjoint action*). Let G be a diffeological group, and let C be the (smooth) **conjugation action**: $C: G \times G \rightarrow G: (g, h) \mapsto ghg^{-1}$. Define the **adjoint action** of G on T_eG to be the (smooth) action $g \cdot \xi := TC_g(\xi)$ for all $\xi \in T_eG$.

This does not immediately give us a well-defined infinitesimal adjoint action (*i.e.* Lie bracket) on T_eG . This issue is resolved by Leslie in [12] by considering the following type of diffeological group.

Definition 2.11 (*Diffeological Lie group*). A diffeological group G is a **diffeological Lie group** if

- (1) for every $\xi \in T_eG$ there exists a smooth real-valued linear functional $\ell: T_eG \rightarrow \mathbb{R}$ satisfying $\ell(\xi) \neq 0$,
- (2) the plots of T_eG smoothly factor through a map $\varphi: V \rightarrow T_eG$, where V is an open subset of a complete Hausdorff locally convex topological vector space.

Remark 2.12 (*T_eG is a Lie algebra*). Let G be a diffeological Lie group. Then T_eG is a Lie algebra under the infinitesimal adjoint action ([12, Theorem 1.14]). In this case, we denote T_eG by \mathfrak{g} .

Finally, in order to associate curves in G with curves in \mathfrak{g} , we need to introduce the notion of a *regular* Lie group, following [12], [9, Section 38], and [22]. This correspondence is important when obtaining horizontal lifts of fibre bundles from connection 1-forms later in the paper.

Definition 2.13. A diffeological Lie group G is **regular** if there is a smooth map

$$\exp: C^\infty([0, 1], \mathfrak{g}) \rightarrow C^\infty([0, 1], G)$$

sending a smooth curve $\xi(t)$ to a smooth curve $g(t)$ such that $g(t)$ is the unique solution of the differential equation

$$\begin{cases} g(0) = e, \\ (R_{g(t)^{-1}})_* \frac{dg(t)}{dt} = \xi(t). \end{cases}$$

Remark 2.14. Unless specified otherwise, we take the subset diffeology on $[0, 1] \subseteq \mathbb{R}$ throughout the paper.

Remark 2.15. Not all diffeological groups are regular. For example, the diffeological group $\text{Diff}(\mathbb{R})$ (see [10, Example 1.21]) or $\text{Diff}_+(0, 1)$ (see [15] or [17, Subsection 2.5.2]).

2.2. Diffeological fibre bundles

We now review diffeological fibre bundles and principal G -bundles; for more details, see [8]. Instead of using the original definition of a diffeological fibre bundle given in [6], we take the equivalent definition below (see [8, Article 8.9]). Similarly, we take an equivalent definition for a principal G -bundle (see [8, Article 8.13]).

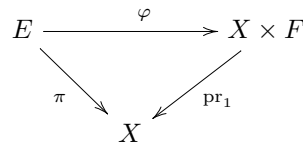
Definition 2.16 (*Diffeological fibre bundles*). Let $\pi: E \rightarrow X$ be a smooth surjective map of diffeological spaces.

- (1) Define the **pullback** of $\pi: E \rightarrow X$ by a smooth map $f: Y \rightarrow X$ of diffeological spaces to be the set

$$f^*E := \{(y, e) \in Y \times E \mid f(y) = \pi(e)\}$$

equipped with the subset diffeology induced by the product. It comes with the two smooth maps $\tilde{f}: f^*E \rightarrow E$ and $\pi_f: f^*E \rightarrow Y$ induced by the projection maps, the latter of which is also surjective.

- (2) $\pi: E \rightarrow X$ is **trivial** with **fibre** F if there is a diffeological space F and a diffeomorphism $\varphi: E \rightarrow X \times F$ making the following diagram commute.



- (3) $\pi: E \rightarrow X$ is **locally trivial** with fibre F if there exists an open cover $\{U_\alpha\}_\alpha$ in the D-topology on X such that for each α , the pullback of $\pi: E \rightarrow X$ to U_α via the inclusion map is trivial with fibre F . The collection $\{(U_\alpha, \varphi_\alpha)\}_\alpha$, where the diffeomorphism $\varphi_\alpha: E|_{U_\alpha} \rightarrow U_\alpha \times F$ is as above, make up a **local trivialisation** of π .
- (4) We say that $\pi: E \rightarrow X$ is a **diffeological fibration** or **diffeological fibre bundle** if for every plot $p: U \rightarrow X$, the pullback bundle $p^*E \rightarrow U$ is locally trivial.
- (5) Let G be a diffeological group. We say that a diffeological fibre bundle $\pi: E \rightarrow X$ is a **principal G -bundle** if there is a smooth action of G on E for which every plot $p: U \rightarrow X$ induces a pullback bundle $p^*E \rightarrow U$ that is **locally equivariantly trivial**. That is, for any $u \in U$ there exists an open neighbourhood V of u , a plot $q: V \rightarrow E$ satisfying $\pi \circ q = p|_V$, and an equivariant diffeomorphism $\psi: V \times G \rightarrow p|_V^*E$ sending (v, g) to $(v, g \cdot q(v))$. Furthermore, a principal G -bundle itself is **locally equivariantly trivial** if it admits a local trivialisation $\{U_\alpha, \varphi_\alpha\}_\alpha$ in which each φ_α is G -equivariant.
- (6) We say that a diffeological fibre bundle is **weakly D-numerable** if there exists a local trivialisation $\{U_i\}_{i \in \mathbb{N}}$ of X and a pointwise-finite smooth partition of unity $\{\zeta_i\}_{i \in \mathbb{N}}$ of X such that $\zeta_i^{-1}((0, 1]) \subseteq U_i$ for each i . We say that a weakly D-numerable diffeological fibre bundle is **D-numerable** if the local trivialisation can be chosen to be locally finite and the smooth partition of unity subordinate to $\{U_i\}$; that is, $\text{supp}(\zeta_i) \subseteq U_i$ for each $i \in \mathbb{N}$.

The point of having weak D-numerability versus D-numerability is that *a priori*, as we will see below, the Milnor construction as a bundle is not necessarily D-numerable, only weakly D-numerable.

Remark 2.17. Diffeological fibre bundles pull back to diffeological fibre bundles. Moreover, trivial bundles pull back to trivial bundles. Consequently, (weakly) D-numerable bundles pull back to (weakly) D-numerable bundles. Similar statements hold for principal G -bundles.

Remark 2.18. Not every diffeological fibre bundle is locally trivial. The 2-torus modulo the irrational Kronecker flow is such an example [8, Article 8.38].

Remark 2.19. A weakly D-numerable principal diffeological fibre bundle with Hausdorff, smoothly paracompact base is D-numerable. A locally trivial diffeological fibre bundle over a Hausdorff, second-countable, smoothly paracompact base is also D-numerable. We include the definition of smooth paracompactness below for completeness.

Definition 2.20 (*Smooth paracompactness*). A diffeological space X is **smoothly paracompact** if the D-topology of X is paracompact (any open cover admits a locally finite open refinement), and any open cover of X admits a *smooth* partition of unity subordinate to it.

3. Milnor’s classifying space

We begin this section with the construction of a classifying space of a diffeological group G , completely in the diffeological category (Proposition 3.4). Under certain topological constraints on a diffeological space X , we get the desired natural bijection between homotopy classes of diffeologically smooth maps $X \rightarrow BG$ and D-numerable principal G -bundles over X (see Theorem 3.6).

3.1. The Milnor construction

We construct Milnor’s classifying space in the diffeological category for a diffeological group G . This results in a principal G -bundle $\pi: EG \rightarrow BG$ that is at least weakly D-numerable.

Definition 3.1 (*Join operation*). Let $\{X_i\}_{i \in \mathbb{N}}$ be a family of diffeological spaces. Define the **join** $\star_{i \in \mathbb{N}} X_i$ of this family as follows. Take the subset S of the product $(\prod_{i \in \mathbb{N}} [0, 1]) \times (\prod_{i \in \mathbb{N}} X_i)$ consisting of elements $(t_i, x_i)_{i \in \mathbb{N}}$ in which only finitely many of the t_i are non-zero, and $\sum_{i \in \mathbb{N}} t_i = 1$. Equip S with the subset diffeology induced by the product diffeology. Let \sim be the equivalence relation on S given by: $(t_i, x_i)_{i \in \mathbb{N}} \sim (t'_i, x'_i)_{i \in \mathbb{N}}$ if

- (1) $t_i = t'_i$ for each $i \in \mathbb{N}$, and
- (2) if $t_i = t'_i \neq 0$, then $x_i = x'_i$.

Then the join is the quotient S/\sim equipped with the quotient diffeology. We denote elements of $\star_{i \in \mathbb{N}} X_i$ by $(t_i x_i)$.

Definition 3.2 (*Classifying space*). Let G be a diffeological group. Define $EG = \star_{i \in \mathbb{N}} G$. There is a natural smooth action of G on EG given by $h \cdot (t_i g_i) = (t_i g_i h^{-1})$ induced by the diagonal action of G on $G^{\mathbb{N}}$ and the trivial action on $[0, 1]^{\mathbb{N}}$. Denote the quotient EG/G by BG and elements of BG by $[t_i g_i]$. This is the **(diffeological) Milnor classifying space** of G . Since we will make use of it, denote by S_G the subset of $[0, 1]^{\mathbb{N}} \times G^{\mathbb{N}}$ corresponding to the set S in Definition 3.1.

Proposition 3.3. *The action of G on EG is smooth.*

Proof. We want to show that the map $a_E: G \times EG \rightarrow EG$ sending $(h, (t_i g_i))$ to $(t_i g_i h^{-1})$ is smooth. Let $p: U \rightarrow G \times EG$ be a plot. It is enough to show that $a_E \circ p$ locally lifts to a plot of S_G via the quotient map $\pi: S_G \rightarrow EG$. Let pr_i ($i = 1, 2$) be the natural projection maps on $G \times EG$. Then there exist an open cover $\{U_\alpha\}$ of U and for each α a plot $q_\alpha: U_\alpha \rightarrow S_G$ such that

$$\text{pr}_2 \circ p|_{U_\alpha} = \pi \circ q_\alpha.$$

Let $a: G \times S_G \rightarrow S_G$ be the smooth map sending $(h, (t_i, g_i))$ to $(t_i, g_i h^{-1})$. Then,

$$a_E \circ p|_{U_\alpha} = \pi \circ a(q_\alpha, \text{pr}_1 \circ p|_{U_\alpha}),$$

where the right-hand side is a plot of EG . \square

Proposition 3.4 (*EG → BG is principal*). *EG → BG is a weakly D-numerable principal G-bundle.*

Proof. Define for each $j \in \mathbb{N}$ the function $s_j: EG \rightarrow [0, 1]$ by $s_j(t_i g_i) := t_j$. These are diffeologically smooth, hence continuous with respect to the D-topology. For each $j \in \mathbb{N}$ define the open set

$$V_j := s_j^{-1}(0, 1] = \{(t_i g_i) \mid t_j > 0\}.$$

Since for any point $(t_i g_i) \in EG$ we have that $\sum t_i = 1$, it follows that $\{V_j\}$ forms an open cover of EG . Since each V_j is G -invariant, setting $U_j := \pi(V_j)$ we get an open covering $\{U_j\}$ of BG .

Define for each $j \in \mathbb{N}$ the map $\varphi_j: V_j \rightarrow G \times U_j$ by

$$\varphi_j(t_i g_i) := (g_j, [t_i g_i g_j^{-1}]).$$

This map is well-defined, smooth, and G -equivariant where G acts trivially on BG and via $k \cdot g = gk^{-1}$ on itself. Moreover, φ_j is invertible with smooth inverse:

$$\varphi_j^{-1}(k, [t_i g_i]) = (t_i g_i k)$$

where we set $g_j = e$. Thus, $\{(U_j, \varphi_j)\}$ is a local trivialisation of $EG \rightarrow BG$. It follows that $EG \rightarrow BG$ is a locally trivial diffeological principal G -bundle.

Finally, for each $(t_i g_i) \in EG$,

$$\sum_{j \in \mathbb{N}} s_j(t_i g_i) = \sum_{j \in \mathbb{N}} t_j = 1,$$

and each s_j is G -invariant and so descends to a smooth map $\zeta_j: BG \rightarrow [0, 1]$ with $\zeta_j^{-1}(0, 1] = U_j$. The collection $\{\zeta_j\}$ is a pointwise-finite smooth partition of unity on BG , and so we have shown that $\pi: EG \rightarrow BG$ is weakly D-numerable. \square

3.2. Classifying bundles

The main result concerning a classifying space of a diffeological group G is that it classifies principal G -bundles, up to isomorphism. To establish this, we follow the topological presentation by tom Dieck (see [25, Sections 14.3, 14.4]). While many of the proofs only require slight modifications to ensure smoothness, some such as the proof to Proposition 3.12 requires the development of some diffeological theory related to the D-topology and homotopy, which appears in Lemmas 3.9, 3.10, and 3.11. To make the classification result precise, we introduce the following notation.

Definition 3.5 ($\mathcal{B}_G(\cdot)$ and $[\cdot, BG]$). Let G be a diffeological group, and let X be diffeological space. Denote by $\mathcal{B}_G(X)$ the set of isomorphism classes of D-numerable principal G -bundles over X , and denote by $[X, BG]$ the set of smooth homotopy classes of smooth maps $X \rightarrow BG$. Given another diffeological space Y and a smooth map $\varphi: X \rightarrow Y$, define $\mathcal{B}(\varphi)$ to be the pullback $\varphi^*: \mathcal{B}_G(Y) \rightarrow \mathcal{B}_G(X)$, and $[\varphi, BG]$ to be the pullback $\varphi^*: [Y, BG] \rightarrow [X, BG]$.

The main theorem of this section is the following.

Theorem 3.6 ($\mathcal{B}_G(\cdot)$ and $[\cdot, BG]$ are naturally isomorphic). *Let $\mathbf{Diffeol}_{\text{HSP}}$ be the full subcategory of $\mathbf{Diffeol}$ consisting of Hausdorff, second-countable, smoothly paracompact diffeological spaces, and let G be a diffeological group. Then, $\mathcal{B}_G(\cdot)$ and $[\cdot, BG]$ are naturally isomorphic functors from $\mathbf{Diffeol}_{\text{HSP}}$ to \mathbf{Set} .*

To prove this, we begin by constructing a correspondence $[X, BG] \rightarrow \mathcal{B}_G(X)$.

Proposition 3.7 (Pullback bundles). *Let X be a diffeological space, and let G be a diffeological group. For any smooth map $F: X \rightarrow BG$ the pullback bundle F^*EG is weakly D-numerable. Additionally, if X is Hausdorff and smoothly paracompact, then F^*EG is D-numerable.*

Proof. Let $F: X \rightarrow BG$ be a smooth map. Let the collection $\{\zeta_j: BG \rightarrow [0, 1]\}_{j \in \mathbb{N}}$ be the partition of unity in the proof of Proposition 3.4. For each $j \in \mathbb{N}$, define $\xi_j: X \rightarrow [0, 1]$ by $\xi_j := \zeta_j \circ F$. Then $\{\xi_j\}$ is a pointwise-finite smooth partition of unity, and we have an open cover of X given by open sets $W_j := \xi_j^{-1}(0, 1]$. We now show that F^*EG is trivial over each W_j .

Recall

$$F^*EG = \{(x, (t_i g_i)) \in X \times EG \mid F(x) = [t_i g_i]\},$$

where for all $k \in G$ we have the smooth action $k \cdot (x, (t_i g_i)) = (x, (t_i g_i k^{-1}))$. Define $\tilde{F}: F^*EG \rightarrow EG$ to be the second projection map. Define $\psi_j: \pi^{-1}(W_j) \rightarrow G \times W_j$ by

$$\psi_j(x, (t_i g_i)) := (g_j, x).$$

Then ψ_j is well-defined, smooth, G -equivariant, and has a smooth inverse. It follows that F^*EG is weakly D-numerable.

If X is Hausdorff and smoothly paracompact, we can choose an appropriate open refinement of $\{W_j\}$ and smooth partition of unity subordinate to it (see, for example, [21, Lemma 41.6]). This completes the proof. \square

To ensure that the correspondence $[X, BG] \rightarrow \mathcal{B}_G(X)$ is well-defined, we must show that smoothly homotopic maps yield isomorphic bundles. This is Proposition 3.12. To prove this, we need to add another topological constraint, second-countability, and establish a series of lemmas. Also, we need to consider the D-topology of a product of diffeological spaces. Generally, the D-topology induced by the product diffeology contains the product topology. However, we have the following result of Christensen, Sinnamon, and Wu in [2, Lemma 4.1].

Lemma 3.8. *Let X and Y be diffeological spaces such that the D-topology of Y is locally compact. Then the D-topology of the product $X \times Y$ is equal to the product topology.*

Lemma 3.9. *Let X be a Hausdorff, paracompact, second-countable diffeological space, and let $\pi: E \rightarrow X \times \mathbb{R}$ be a locally trivial diffeological fibre bundle. Then for any $a < b$, there exists a countable locally finite open cover \mathcal{U} of X such that π is trivial over $U \times [a, b]$ for any $U \in \mathcal{U}$. In the case of a principal bundle, the trivialisation can be chosen to be equivariant.*

Proof. Since \mathbb{R} is locally compact, the D-topology on $X \times \mathbb{R}$ is equal to the product topology by Lemma 3.8. Fix a local trivialisation (equivariant in the case of a principal bundle) \mathcal{V} of π , and fix $x \in X$. Since $[a, b]$ is compact there exist $k > 0$, for each $i = 1, \dots, k$ an open interval (s_i, t_i) such that $\bigcup_i (s_i, t_i) \supseteq [a, b]$, and for each i an open neighbourhood U_i of x such that $U_i \times (s_i, t_i)$ is contained in an element of \mathcal{V} .

Let $U = \bigcap_i U_i$. By [8, Note 1 of Article 8.16 and Lemma 1 of Article 8.19], π is trivial over $U \times [a, b]$. Now, take such an open neighbourhood U for each $x \in X$; this is an open cover of X . We now take an appropriate refinement of this cover to obtain \mathcal{U} . \square

It is standard that given a normal topological space, one can find a continuous function that separates disjoint closed sets. We need a smooth version of this fact.

Lemma 3.10. *Let X be a smoothly paracompact diffeological space, and let A and B be disjoint closed subsets of X . Then there exists a smooth function $f: X \rightarrow [0, 1]$ such that $f|_A \equiv 0$ and $f|_B \equiv 1$.*

Proof. Consider the open cover $\{X \setminus A, X \setminus B\}$. It admits a smooth partition of unity $\{\zeta_{X \setminus A}, \zeta_{X \setminus B}\}$. Let $f = \zeta_{X \setminus A}$.¹ \square

Lemma 3.11. *Let X be a Hausdorff, second-countable, smoothly paracompact diffeological space, and let $\pi: E \rightarrow X \times \mathbb{R}$ be a locally trivial diffeological fibre bundle. Then $E|_{X \times \{0\}}$ is bundle-diffeomorphic to $E|_{X \times \{1\}}$. In the case of a principal bundle, the bundle-diffeomorphism can be chosen to be equivariant.*

Proof. By Lemma 3.9, there exists a locally finite open cover $\{U_i\}_{i \in \mathbb{N}}$ of X such that π is trivial over $U_i \times [0, 1]$ for each i . Let $\{(U_i \times [0, 1], \psi_i)\}$ be the corresponding local trivialisation (equivariant in the case of a principal bundle) for π restricted over $X \times [0, 1]$. Since X is Hausdorff and paracompact, there is a locally finite open refinement $\{V_i\}_{i \in \mathbb{N}}$ of $\{U_i\}$ such that $\overline{V_i} \subseteq U_i$ for each i . By Lemma 3.10 there is a family of smooth maps $\{b_i: X \rightarrow [0, 1]\}_{i \in \mathbb{N}}$ such that $b_i|_{V_i} = 1$ and $\text{supp}(b_i) \subseteq U_i$ for each i .

Fix i , and denote by F the fibre of π . Let $r_i: X \times [0, 1] \rightarrow X \times [0, 1]$ be the smooth map sending (x, t) to $(x, t + (1 - t)b_i(x))$ and let $R_i: E \rightarrow E$ be the map equal to the identity over the complement of $U_i \times [0, 1]$, and such that for all $(x, t, a) \in U_i \times [0, 1] \times F$,

$$\psi_i \circ R_i \circ \psi_i^{-1}(x, t, a) = (x, t + (1 - t)b_i(x), a).$$

The pair (r_i, R_i) form a smooth bundle map over $X \times [0, 1]$, whose restriction to $E|_{X \times \{1\}}$ is the identity map. Moreover, the restriction of R_i to $E|_{X \times \{0\}}$ is a diffeomorphism onto its image. In the case of a principal bundle, this is equivariant.

Let r be the composition of the maps r_i , taken in order: $r = \dots \circ r_2 \circ r_1$. This is well-defined since $r_i(x) = x$ for all but finitely many i . Similarly, define R to be the composition of all R_i . The pair (r, R) is a smooth bundle map, whose restriction to $E|_{X \times \{1\}}$ is the identity map. Moreover, the restriction of R to $E|_{X \times \{0\}}$ is a diffeomorphism onto $E|_{X \times \{1\}}$. Again, in the case of a principal bundle, this is also equivariant. \square

Proposition 3.12 (*Homotopic maps and isomorphic bundles*). *Let X and Y be diffeological spaces, and assume that X is Hausdorff, second-countable, and smoothly paracompact. Let $f_i: X \rightarrow Y$ ($i = 0, 1$) be smooth maps with a smooth homotopy $H: X \times \mathbb{R} \rightarrow Y$ between them, and let $\pi: E \rightarrow Y$ be a diffeological fibre bundle. If $H^*E \rightarrow X$ is locally trivial, then the pullback bundles $f_i^*E \rightarrow X$ ($i = 0, 1$) are bundle-diffeomorphic. In the case of a principal bundle, the bundle-diffeomorphism can be chosen to be equivariant.*

Proof. Assume that $H^*E \rightarrow X$ is locally trivial. Then there is a bundle-diffeomorphism between $H^*E|_{X \times \{0\}}$ and $H^*E|_{X \times \{1\}}$ by Lemma 3.11. But these two restricted bundles are exactly bundle-diffeomorphic to f_0^*E and f_1^*E , respectively. \square

¹ Thanks to Dan Christensen for this proof, which is much simpler than the original.

Remark 3.13. Iglesias-Zemmour proves the above proposition for diffeological fibre bundles admitting a diffeological connection; see [Definition 4.1](#), [\[8, Article 8.32 and Article 8.34\]](#).

We now have a well-defined correspondence $[X, BG] \rightarrow \mathcal{B}_G(X)$. To construct an inverse correspondence, we must first define a “classifying map” $X \rightarrow BG$ for a principal G -bundle $E \rightarrow X$. In order to show that the classifying map is smooth, we require that E be weakly D-numerable with a Hausdorff and smoothly paracompact base (in which case, E is D-numerable).

Proposition 3.14 (*Classifying maps*). *If $\pi: E \rightarrow X$ is a weakly D-numerable principal G -bundle in which X is Hausdorff and smoothly paracompact, then there is a smooth G -equivariant map $\tilde{F}: E \rightarrow EG$ which descends to a smooth map $F: X \rightarrow BG$, called a **classifying map** of π , such that E is isomorphic as a principal G -bundle to F^*EG .*

Proof. Let $\pi: E \rightarrow X$ be a weakly D-numerable principal G -bundle in which X is Hausdorff and smoothly paracompact (and hence π is in fact D-numerable by [Remark 2.19](#)), and fix a locally finite local trivialisation $\{(W_j, \psi_j)\}_{j \in \mathbb{N}}$ and a smooth partition of unity $\{\xi_j\}_{j \in \mathbb{N}}$ subordinate to $\{W_j\}$. Define a map $\tilde{F}: E \rightarrow EG$ by

$$\tilde{F}(y) := ((\xi_i \circ \pi)(y))(\text{pr}_2 \circ \psi_i(y)).$$

Since ξ_j has support contained in W_j , \tilde{F} is well-defined.

To show that \tilde{F} is smooth, fix a plot $p: U \rightarrow E$. It is enough to show that $\tilde{F} \circ p$ locally lifts to a plot of S_G . Fix $u \in U$. Since $\{W_j\}$ is a locally finite open cover, there is an open neighbourhood B of $p(u)$ such that B intersects only finitely many of the open sets $\pi^{-1}(W_j)$; without loss of generality, assume that these are W_1, \dots, W_k . Next, of these, only $l \leq k$ contain $p(u)$. Again, without loss of generality, assume that these are W_1, \dots, W_l . Since $\{\xi_j\}$ is subordinate to $\{W_j\}$, we have that the closed set $\bigcup_{j=1}^{k-l} \text{supp}(\xi_{l+j})$ is disjoint from $p(u)$, which itself is closed since X is Hausdorff. Furthermore, since X is also paracompact, it is normal, and so shrink B so that it only intersects W_{l+1}, \dots, W_k outside of each $\text{supp}(\xi_j)$ ($j = l + 1, \dots, k$).

Let $p_B := p|_{p^{-1}(B)}$ and let $\rho: S_G \rightarrow EG$ be the quotient map. Then, $\tilde{F} \circ p_B = \rho \circ \sigma_B$ where $\sigma_B: p^{-1}(B) \rightarrow S_G$ is defined to be the smooth map $\sigma_B(u) = (t_i(u), g_i(u))$ where $g_i(u) = e$ if $i > l$, $g_i(u) = \text{pr}_2(\psi_i(p_B(u)))$ if $i = 1, \dots, l$, and $t_i(u) = \xi_i(\pi(p_B(u)))$ for each i . It follows that \tilde{F} is a smooth map into EG . It is also G -equivariant, and so descends to a smooth map $F: X \rightarrow BG$.

To show that E and F^*EG are isomorphic, define a map $\Phi: E \rightarrow F^*EG$ by $\Phi(y) := (\pi(y), \tilde{F}(y))$. Then, Φ is a well-defined smooth bijection. Fixing $x \in X$, let B be an open neighbourhood of x intersecting only finitely many W_j . Then $\Phi^{-1}|_{\pi^{-1}(B)}(x, (t_i g_i)) = \psi_j^{-1}(x, g_j)$ for any j such that $x \in W_j$, which is smooth, and hence Φ is a diffeomorphism. G -equivariance of Φ follows from the G -equivariance of \tilde{F} . \square

Remark 3.15. If $\pi: E \rightarrow X$ is a principal G -bundle that has a smooth classifying map $F: X \rightarrow BG$, then it follows that F^*EG is weakly D-numerable, and so by definition of a classifying map, $\pi: E \rightarrow X$ is also weakly D-numerable.

To show that our inverse correspondence $\mathcal{B}_G(X) \rightarrow [X, BG]$ is well-defined, we need to show that isomorphic bundles yield smoothly homotopic classifying maps. This is a consequence of the following proposition.

Proposition 3.16 (*Homotopy equivalence of maps to EG*). *Let G be a diffeological group, $E \rightarrow X$ a principal G -bundle, and $f, h: E \rightarrow EG$ two G -equivariant smooth maps. Then, f and h are smoothly G -equivariantly homotopic, and so descend to smoothly homotopic maps $X \rightarrow BG$.*

Proof. Denote the images of f and h by

$$f(y) = (s_1(y)f_1(y), s_2(y)f_2(y), \dots) \text{ and } h(y) = (t_1(y)h_1(y), t_2(y)h_2(y), \dots)$$

where $f_i(y), h_i(y) \in G$ and $s_i(y), t_i(y) \in [0, 1]$. We will construct an homotopy H such that $H(y, 0) = f(y)$ and $H(y, 1) = h(y)$ for all $y \in E$. The construction will be a concatenation of homotopies that we construct now. Throughout these, we use a smooth function $b: \mathbb{R} \rightarrow [0, 1]$ such that $b(\tau) = 0$ for all $\tau \leq \varepsilon$, $b(\tau) = 1$ for all $\tau \geq 1 - \varepsilon$, and $\frac{db}{d\tau} \geq 0$, for some fixed $\varepsilon \in (0, 1/2)$.

Let $F_1: E \times [0, 1] \rightarrow EG$ be defined by

$$F_1(y, \tau) := (s_1(y)f_1(y), b(\tau)s_2(y)f_2(y), (1 - b(\tau))s_2(y)f_2(y), \\ b(\tau)s_3(y)f_3(y), (1 - b(\tau))s_3(y)f_3(y), \dots).$$

Then F_1 is smooth, and satisfies

$$F_1(y, 0) = (s_1(y)f_1(y), 0, s_2(y)f_2(y), 0, \dots) \text{ and} \\ F_1(y, 1) = (s_1(y)f_1(y), s_2(y)f_2(y), 0, s_3(y)f_3(y), 0, \dots).$$

Similarly, for each $n > 1$, let $F_n: E \times [0, 1] \rightarrow EG$ be defined by

$$F_n(y, \tau) := (s_1(y)f_1(y), \dots, s_n(y)f_n(y), b(\tau)s_{n+1}(y)f_{n+1}(y), \\ (1 - b(\tau))s_{n+1}(y)f_{n+1}(y), b(\tau)s_{n+2}(y)f_{n+2}(y), \dots).$$

Each F_n is smooth, and $F_n(y, 0) = F_{n-1}(y, 1)$ for all $n > 1$.

Define $F: E \times [0, \infty) \rightarrow EG$ by $F(y, \tau) = F_{\lfloor \tau+1 \rfloor}(y, \tau - \lfloor \tau \rfloor)$, where $\lfloor \cdot \rfloor$ is the floor function. Since each F_n is constant near $\tau = 0$ and $\tau = 1$ for fixed y , it follows that F is smooth. Let $c: [0, 1) \rightarrow [0, \infty)$ be an increasing diffeomorphism ($x \mapsto \tan(\frac{\pi}{2}x)$, say), and define $F': E \times [0, 1) \rightarrow EG$ as the smooth composition $F \circ (\text{id}_E \times c)$. Define $F'(y, 1) := f(y)$. This yields a smooth extension $F': E \times [0, 1] \rightarrow EG$.

Similarly, for each $n > 0$, define smooth maps $S_n: E \times [0, 1] \rightarrow EG$ by

$$S_n(y, \tau) := (t_1(y)h_1(y), \dots, t_{n-1}(y)h_{n-1}(y), b(\tau)t_n(y)h_n(y), \\ (1 - b(\tau))t_n(y)h_n(y), b(\tau)t_{n+1}(y)h_{n+1}(y), \dots).$$

We have $S_n(y, 0) = S_{n-1}(y, 1)$ for each $n > 1$. Defining S and S' similar to F and F' , by setting $S'(y, 1) = h(y)$ we obtain a smooth extension $S': E \times [0, 1] \rightarrow EG$.

Finally, define $T: E \times [0, 1] \rightarrow EG$ by

$$T(y, \tau) := ((1 - b(\tau))s_1(y)f_1(y), b(\tau)t_1(y)h_1(y), \\ (1 - b(\tau))s_2(y)f_2(y), b(\tau)t_2(y)h_2(y), \dots).$$

Then T is smooth, $T(y, 0) = F'(y, 0)$, and $T(y, 1) = S'(y, 0)$.

We now define the smooth homotopy $H: E \times [0, 1] \rightarrow EG$ between f and h by

$$H(y, \tau) := \begin{cases} F'(y, 1 - b(3\tau)) & \text{for } \tau \in [0, \frac{1}{3}], \\ T(y, b(3\tau - 1)) & \text{for } \tau \in [\frac{1}{3}, \frac{2}{3}], \\ S'(y, b(3\tau - 2)) & \text{for } \tau \in [\frac{2}{3}, 1]. \end{cases}$$

The G -equivariance of H is clear. This completes the proof. \square

We are now ready to prove the main result of this section.

Proof of Theorem 3.6. The fact that $\mathcal{B}_G(\cdot)$ and $[\cdot, BG]$ are functors is clear. Fix a diffeological space from $\mathbf{Diffeol}_{\text{HSP}}$. It follows from Proposition 3.7 and Proposition 3.12 that there is a map α from the set $[X, BG]$ to $\mathcal{B}_G(X)$. It follows from Proposition 3.14 and Proposition 3.16 that α has an inverse, and hence $\mathcal{B}_G(X)$ is in bijection with $[X, BG]$. Finally, this bijection is natural: given two diffeological spaces X and Y from $\mathbf{Diffeol}_{\text{HSP}}$ and a smooth map $\varphi: X \rightarrow Y$, we have that $\mathcal{B}(\varphi)$ and $[\varphi, BG]$ commute with these bijections. \square

4. Diffeological connections and connection 1-forms

In [8, Article 8.32] Iglesias-Zemmour gives a definition of a connection on a principal G -bundle in terms of paths on the total space, generalising the classical notion for principal bundles with structure group a finite-dimensional Lie group. From this definition one obtains the usual properties that one expects from a connection (see Remark 4.2). The purpose of this section is to prove Theorem 4.3: that any principal G -bundle satisfying mild conditions admits one of these connections provided G is a regular diffeological Lie group. We do this by way of constructing a connection 1-form on the G -bundle $EG \rightarrow BG$, and showing that this induces a connection in the sense of Iglesias-Zemmour. Since connections pull back, we obtain our result.

Definition 4.1 (*Diffeological connections*). Let G be a diffeological group, and let $\pi: E \rightarrow X$ be a principal G -bundle. Denote by $\text{Paths}_{\text{loc}}(E)$ the diffeological space of **local paths**

$$\text{Paths}_{\text{loc}}(E) := \{\gamma \in C^\infty((a, b), E) \mid (a, b) \subseteq \mathbb{R}\}$$

equipped with the subset diffeology induced by the standard functional diffeology *on a diffeology* (see [8, Article 1.63]). We recall what this standard functional diffeology is: a parametrisation $p: U \rightarrow \mathcal{D}$ is a plot if and only if for each $u \in U$ and $v \in \text{dom}(p(u))$ there exist open neighbourhoods U' of u and V' of v such that for each $u' \in U'$,

- (1) $V' \subseteq \text{dom}(p(u'))$, and
- (2) the map $\Psi: U' \times V' \rightarrow X: (u', v') \mapsto p(u')(v')$ is in \mathcal{D} .

Denote by $\text{tbPaths}_{\text{loc}}(E)$ the **tautological bundle of local paths**, equipped with the subset diffeology induced by $\text{Paths}_{\text{loc}}(E) \times \mathbb{R}$:

$$\text{tbPaths}_{\text{loc}}(E) := \{(\gamma, t) \in \text{Paths}_{\text{loc}}(E) \times \mathbb{R} \mid t \in \text{dom}(\gamma)\}.$$

A **diffeological connection** is a smooth map $\theta: \text{tbPaths}_{\text{loc}}(E) \rightarrow \text{Paths}_{\text{loc}}(E)$ satisfying the following properties for any $(\gamma, t_0) \in \text{tbPaths}_{\text{loc}}(E)$:

- (1) the domain of γ equals the domain of $\theta(\gamma, t_0)$,
- (2) $\pi \circ \gamma = \pi \circ \theta(\gamma, t_0)$,
- (3) $\theta(\gamma, t_0)(t_0) = \gamma(t_0)$,
- (4) $\theta(g \cdot \gamma, t_0) = g \cdot \theta(\gamma, t_0)$ for all $g \in G$,
- (5) $\theta(\gamma \circ f, s) = \theta(\gamma, f(s)) \circ f$ for any smooth map f from an open subset of \mathbb{R} into $\text{dom}(\gamma)$,
- (6) $\theta(\theta(\gamma, t_0), t_0) = \theta(\gamma, t_0)$.

Remark 4.2. Diffeological connections satisfy many of the usual properties that classical connections on a principal G -bundle (where G is a finite-dimensional Lie group) enjoy; in particular, they admit unique horizontal lifts of paths into the base of a principal bundle [8, Article 8.32], and they pull back by smooth maps [8, Article 8.33].

We now state the main purpose of this section.

Theorem 4.3 (*Diffeological connections on principal G -bundles*). *Let G be a regular diffeological Lie group, and let X be a Hausdorff smoothly paracompact diffeological space. Then any weakly D -numerable principal G -bundle $E \rightarrow X$ admits a diffeological connection. Consequently, for this diffeological connection, any smooth curve into X has a unique horizontal lift to E .*

To prove this, we begin by constructing a connection 1-form on $EG \rightarrow BG$.

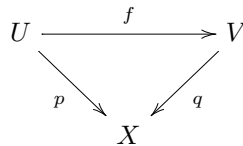
Definition 4.4 (*Connection 1-form*). Let G be a diffeological group, and let $E \rightarrow X$ be a principal G -bundle. A **connection 1-form** on E is a G -equivariant smooth fibrewise linear map $\omega: TE \rightarrow T_eG$ (with respect to the adjoint action on T_eG) such that for any $y \in E$ and $\xi \in T_eG$, we have $\omega(\xi_E|_y) = \xi$ where

$$\xi_E|_y := \left. \frac{d}{dt} \right|_{t=0} (g(t) \cdot y)$$

in which $g(t)$ is a smooth curve in G such that $g(0) = e$ and $\dot{g}(0) = \xi$.

To make clear the connection between ordinary diffeological differential 1-forms and smooth fibrewise linear maps as defined above, we present the following definition and proposition.

Definition 4.5. Denote by **DVB** the category of diffeological vector pseudo-bundles with smooth fibrewise linear maps between them. For a diffeological space (X, \mathcal{D}) denote by $\text{Plots}(\mathcal{D})$ the category with objects plots in \mathcal{D} and arrows commutative triangles



in which f is smooth. Let V be a diffeological vector space. Denote $\Omega^1(X; V)$ to be the **V -valued differential 1-forms on X** , defined to be the limit

$$\lim_{\text{Plots}(\mathcal{D})} \text{Hom}_{\mathbf{DVB}}(T \circ F(\cdot), X \times V \rightarrow X).$$

Here, $X \times V \rightarrow X$ is the trivial V -bundle over X , T is the tangent functor, and F the forgetful functor sending plots to their domains and commutative triangles to the corresponding maps between Euclidean open sets.

Proposition 4.6. *Let (X, \mathcal{D}) be a diffeological space and V a diffeological vector space. Then there is a natural identification between smooth fibrewise \mathbb{R} -linear maps $TX \rightarrow V$ and V -valued 1-forms $\Omega^1(X; V)$.²*

Proof. The set of smooth fibrewise \mathbb{R} -linear maps $TX \rightarrow V$ is exactly

$$\text{Hom}_{\mathbf{DVB}}(TX, X \times V \rightarrow X).$$

But TX is the colimit $\text{colim}(T \circ F)$ in **DVB** over the category $\text{Plots}(\mathcal{D})$ where T and F are as in [Definition 4.5](#); see [\[3, Theorem 4.17\]](#). But then,

² Proof suggested by Daniel Christensen via private communication.

$$\text{Hom}_{\text{DVB}}(TX, X \times V \rightarrow X) = \lim_{\text{Plots}(\mathcal{D})} \text{Hom}_{\text{DVB}}(T \circ F(\cdot), X \times V \rightarrow X);$$

see, for example, [1, Corollary 5.29]. \square

Remark 4.7. Ordinary differential 1-forms on a diffeological space X are, by definition, equal to

$$\lim_{\text{Plots}(\mathcal{D})} \text{Hom}_{\text{DVB}}(T \circ F(\cdot), X \times \mathbb{R} \rightarrow X)$$

Returning to connection 1-forms, the most basic connection 1-form is the Maurer–Cartan form on a diffeological group.

Lemma 4.8 (Maurer–Cartan form). *Let G be a diffeological group. Then G has a **Maurer–Cartan form** α ; that is, a smooth fibrewise-linear map $\alpha: TG \rightarrow T_eG$ sending $v \in T_gG$ to $(L_{g^{-1}})_*v$. It is G -equivariant with respect to the adjoint action on T_eG and the left action on TG defined by $h \cdot v := (R_{h^{-1}})_*v$.*

Proof. This is immediate from the trivialisation of TG given in Theorem 2.9. \square

Since EG is constructed out of an infinite product of the group G , we can take the infinite sum of the Maurer–Cartan form, with coefficients t_j ; since only finitely many of the t_j are non-zero, the sum converges. The goal is to show that the result is smooth.

Theorem 4.9 (Connection on EG). *Let G be a diffeological group. Then the principal G -bundle $EG \rightarrow BG$ admits a connection 1-form ω .*

Proof of Theorem 4.9. Let α be the Maurer–Cartan form on G . Define $\tilde{\omega}$ to be the smooth map $\tilde{\omega}: TS_G \rightarrow T_eG$ given by

$$\tilde{\omega}|_{(g_i, t_i)} = \sum_{i \in \mathbb{N}} t_i \text{pr}_{g_i}^* \alpha$$

where $\text{pr}_{g_i}: S_G \rightarrow G$ is the projection map onto the i th copy of G . Then $\tilde{\omega}$ is a well-defined connection 1-form on the bundle $S_G \rightarrow S_G/G =: EG$.

We need to show that $\tilde{\omega}$ descends to a form ω on EG . To do this, we use the following fact (see [8, Article 6.38], noting that the proof goes through with our definition of 1-form, independent of the fact that the forms in the proof are real-valued): if $\rho: S_G \rightarrow EG$ is the quotient map, then a form μ on S_G is equal to $\rho^*\nu$ for some form ν on EG if and only if for any two plots $p_1, p_2: U \rightarrow S_G$ satisfying $\rho \circ p_1 = \rho \circ p_2$, we have $p_1^*\mu = p_2^*\mu$. Fix two such plots p_1 and p_2 , and fix $u \in U$ and $v \in T_uU$. It is enough for us to show

$$v \lrcorner (p_1^*\tilde{\omega} - p_2^*\tilde{\omega}) = 0. \tag{4.1}$$

There are only finitely many $j \in \mathbb{N}$ such that $\text{pr}_{t_j}(p_1(u)) \neq 0$ where pr_{t_j} is the j th projection from S_G onto $[0, 1]$. Hence there are only finitely many open sets

$$\tilde{V}_j = \{(g_i, t_i) \mid t_j \neq 0\} \subseteq S_G$$

containing $p_1(u)$. Thus, the intersection \tilde{V} of all such \tilde{V}_j is open, and its pre-image $W := p_1^{-1}(\tilde{V}) \subseteq U$ is open. Moreover, it follows from $\rho \circ p_1 = \rho \circ p_2$ that $W = p_2^{-1}(\tilde{V})$.

Let $c: \mathbb{R} \rightarrow U$ be a curve such that $c(0) = u$ and $\dot{c}(0) = v$. Then for all τ in $c^{-1}(W)$ we have $\text{pr}_{g_j} \circ p_1 \circ c(\tau) = \text{pr}_{g_j} \circ p_2 \circ c(\tau)$ for all j such that $t_j \neq 0$. It follows that

$$\alpha((\text{pr}_{g_j})_*(p_1)_*v) = \alpha((\text{pr}_{g_j})_*(p_2)_*v).$$

Equation (4.1) follows. \square

We have the following corollary:

Corollary 4.10. *Let G be a diffeological group. Then any principal G -bundle $E \rightarrow X$ admitting a smooth classifying map has a connection 1-form. In particular, if $E \rightarrow X$ is weakly D -numerable over a Hausdorff smoothly paracompact diffeological space X , then it admits a connection 1-form.*

Proof. This follows from the existence of a G -equivariant map $E \rightarrow EG$ (Proposition 3.14). \square

In order to continue, we need the following basic lemma about differentiating plots.

Lemma 4.11. *Let (X, \mathcal{D}) be a diffeological space, and let $T: \mathcal{D} \rightarrow T\mathcal{D}$ be the map sending a plot p to its induced plot Tp . Then T is a smooth map.*

Proof. Fix a plot $p: U \rightarrow \mathcal{D}$ in the standard functional diffeology. Using the same notation as in Definition 4.1, by the definition of the diffeology on TX and Remark 2.8 the map $T\Psi: TU' \times TV' \rightarrow TX$ is a plot of TX . Restricting the first coordinate of $T\Psi$ to the zero-section of TU' we get the smooth map $U' \times TV' \rightarrow TX: (u', w) \mapsto T\Psi|_{(u', v')}(0, w)$ where v' is the foot-point of w . Since this is smooth, by the definition of the functional diffeology we have that the map $U' \rightarrow C^\infty(TV', TX)$ sending u' to $Tp(u')$ is a plot of $T\mathcal{D}$. \square

Now the main obstacle in proving Theorem 4.3 is to show that a connection 1-form on a principal G -bundle yields a diffeological connection; in particular, that one obtains a smooth map from the tautological bundle to the local paths of the total space. This is the content of the following proposition (cf. [16, Section 2.1]).

Proposition 4.12 (Connection 1-forms induce diffeological connections). *Let G be a regular diffeological Lie group, and let $\pi: E \rightarrow X$ be a principal G -bundle. Then a connection 1-form ω on π induces a diffeological connection on π .*

Proof. Fix a smooth curve $\gamma: (a, b) \rightarrow E$ and a point $t_0 \in (a, b)$. Our first goal is to obtain a smooth curve $g: (a, b) \rightarrow G$ such that $g(t_0) = e$ and the smooth curve $t \mapsto g(t) \cdot \gamma(t)$ satisfies

$$\omega\left(\frac{d(g \cdot \gamma)(t)}{dt}\right) = 0.$$

We will denote $g(t) \cdot \gamma(t)$ by $\theta(\gamma, t_0)(t)$. By the chain rule (see Remark 2.8) the derivative of $\theta(\gamma, t_0)$ is $g \cdot \dot{\gamma} + \dot{g} \cdot \gamma$. Applying ω to this, we obtain the differential equation

$$\omega(\dot{\gamma})(t) = -\text{Ad}_{g^{-1}}\dot{g}|_{\gamma(t)}.$$

Choose $\varepsilon > 0$ so that $a < t_0 - \varepsilon$ and $t_0 + \varepsilon < b$. After composing with an appropriate translation and dilation, it follows from the regularity of G that there is a smooth solution $[t_0 - \varepsilon, t_0 + \varepsilon] \rightarrow G$; in fact, uniqueness of solutions implies that applying this procedure to each $t_0 \in (a, b)$ will yield a smooth curve $g: (a, b) \rightarrow G$ as required. We thus have proved the existence of $\theta: \text{tbPaths}_{\text{loc}}(E) \rightarrow \text{Paths}_{\text{loc}}(E)$.

To show that θ is smooth, note that the map $\gamma \mapsto \dot{\gamma}$ is smooth by Lemma 4.11, as well as ω and the exponential map in the definition of the regularity of G . Finally, the translations and dilations are smooth, and since θ is a composition of all of these things, smoothness follows.

It is an easy exercise to check that θ satisfies the six conditions in [Definition 4.1](#). This completes the proof. \square

Proof of Theorem 4.3. By [Corollary 4.10](#) the principal G -bundle $\pi: E \rightarrow X$ has a connection 1-form. By [Proposition 4.12](#) this induces a diffeological connection on π . In particular, we obtain horizontal lifts of smooth curves into X (see [Remark 4.2](#)). \square

Remark 4.13. We end this section with the following note. Let G be a *regular* diffeological Lie group. By [Proposition 4.12](#) we have a diffeological connection on any principal G -bundle. By [Remark 3.13](#), [Proposition 3.12](#) holds without any assumption on the topology of the base. Since [Proposition 3.16](#) also has no conditions on the topology of the base, it follows that if you have two principal G -bundles (with no assumptions on them) $E \rightarrow X$ and $E' \rightarrow X$ with classifying maps $F: X \rightarrow BG$ and $F': X \rightarrow BG$, then E and E' are isomorphic bundles if and only if F and F' are smoothly homotopic.

5. Applications

In this section, we apply the theory developed in the previous sections to various situations. Many of these are motivated by applications found in [\[8\]](#), [\[9\]](#), [\[16\]](#), [\[18\]](#), [\[24\]](#), and [\[25\]](#).

5.1. Contractibility of EG & homotopy of BG

In this subsection, we show that EG is smoothly contractible, which allows us to compute the (diffeological) homotopy groups of BG in terms of those of G . Again, we look to the topological proof found in [\[25\]](#) for a template.

Proposition 5.1 (*EG is contractible*). *Let G be a diffeological group. Then EG is smoothly contractible.*

Proof. Let $H_1: EG \times [0, 1] \rightarrow EG$ be a smooth homotopy given by

$$H_1((t_i g_i), 0) = (t_i g_i) \text{ and}$$

$$H_1((t_i g_i), 1) = (t_1 g_1, 0, t_2 g_2, t_3 g_3, \dots),$$

using the same notation as that in the proof of [Proposition 3.16](#). This exists by [Proposition 3.16](#). Let b be the smooth function from the same proof and let $H_2: EG \times [0, 1] \rightarrow EG$ be the smooth map

$$H_2((t_i g_i), \tau) = ((1 - b(\tau))t_1 g_1, b(\tau)e, (1 - b(\tau))t_2 g_2, (1 - b(\tau))t_3 g_3, \dots).$$

Concatenating H_1 and H_2 yields a smooth homotopy. Since

$$H_2((t_i g_i), 1) = (0, e, 0, 0, \dots),$$

which is constant on EG , it follows that EG is smoothly contractible. \square

Proposition 5.2 (*Homotopy of BG*). *Let G be a diffeological group. Then for each $k > 0$, we have $\pi_k(BG) \cong \pi_{k-1}(G)$.*

Proof. This is immediate from the fact that $EG \rightarrow BG$ is a principal G -bundle, [Proposition 5.1](#), and the long exact sequence of (diffeological) homotopy groups; see [\[8, Article 8.21\]](#). \square

Example 5.3 (*Homotopy of BT_α*). Fix an irrational number α . Let G be the irrational torus $T_\alpha := \mathbb{R}/\mathbb{Z}^2$, where \mathbb{Z}^2 acts on \mathbb{R} by $(m, n) \cdot x = x + m + n\alpha$. By the long exact sequence of (diffeological) homotopy groups [8, Article 8.21], it is immediate that $\pi_0(T_\alpha) = \{T_\alpha\}$, $\pi_1(T_\alpha) = \mathbb{Z}^2$, and $\pi_k(T_\alpha) = 0$ for $k > 1$. It follows from Proposition 5.2 that $\pi_0(BT_\alpha) = \{BT_\alpha\}$, $\pi_1(BT_\alpha) = 0$, $\pi_2(BT_\alpha) = \mathbb{Z}^2$, and $\pi_k(BT_\alpha) = 0$ for all $k > 2$. \square

5.2. Smooth homotopies of groups and bundles

Here we look at how smooth group homomorphisms between diffeological groups induce smooth maps between the classifying spaces, with applications to smooth strong deformation retractions.

Proposition 5.4 (*Smooth maps between diffeological groups*). Let G and H be diffeological groups, and let $\varphi: G \rightarrow H$ be a smooth map between them. Then φ induces a smooth map $\tilde{\Phi}: EG \rightarrow EH$ defined as

$$\tilde{\Phi}(t_i g_i) = (t_i \varphi(g_i)).$$

Moreover, if φ is a smooth homomorphism, $\tilde{\Phi}$ descends to a smooth map $\Phi: BG \rightarrow BH$.

Proof. The proof follows immediately from the definitions. \square

Remark 5.5. If $\varphi_\tau: G \rightarrow H$ is a smooth family of maps between diffeological groups G and H , then a similar proof to the above yields a smooth family of maps $\tilde{\Phi}_\tau: EG \rightarrow EH$ defined in the obvious way, and in the case of a smooth family of group homomorphisms, we obtain a smooth family of maps $\Phi_\tau: BG \rightarrow BH$.

Corollary 5.6 (*Smooth deformation retracts of bundles*). Let G be a diffeological group, and let $\Phi: G \times [0, 1] \rightarrow G$ be a smooth strong deformation retraction of G onto a subgroup H such that for each $\tau \in [0, 1]$, the map $\Phi(\cdot, \tau): G \rightarrow G$ is a group homomorphism. Then, up to isomorphism, there is a smooth strong deformation retract of any weakly D-numerable principal G -bundle over a Hausdorff smoothly paracompact base to a principal H -bundle over the same base.

Proof. By Proposition 5.4 and Remark 5.5, we obtain a smooth strong deformation retract $\tilde{\Phi}_\tau$ of EG onto EH , which descends to a smooth strong deformation retract Φ_τ of BG onto BH . Let $E \rightarrow X$ be a weakly D-numerable principal G -bundle over a Hausdorff smoothly paracompact base X . By Proposition 3.14, there is a smooth map $\tilde{F}: E \rightarrow EG$ which descends to a smooth classifying map $F: X \rightarrow BG$ for which E is isomorphic as a principal G -bundle to F^*EG . Composing F with Φ_τ , we obtain a smooth strong deformation retract of F^*EG onto $(\Phi_0 \circ F)^*EH$, where at any τ we have

$$(\Phi_\tau \circ F)^*EG = \{(x, (t_i g_i)) \mid \Phi_\tau \circ F(x) = [t_i g_i]\}. \quad \square$$

Example 5.7 ($E(\text{Diff}(\mathbb{R}^n; 0))$). Let $G = \text{Diff}(\mathbb{R}^n; 0)$ be the diffeological group of diffeomorphisms of \mathbb{R}^n that fix the origin. Define $\Phi: G \times [0, 1] \rightarrow G$ by

$$\Phi(\varphi, \tau) = \begin{cases} m_{1/\tau} \circ \varphi \circ m_\tau & \text{if } \tau \neq 0, \\ d\varphi|_0 & \text{if } \tau = 0, \end{cases}$$

where $m_\tau: \mathbb{R}^n \rightarrow \mathbb{R}^n$ is scalar multiplication by τ (which is smooth). By definition of the functional diffeology [8, Article 1.57], it is an easy exercise to check that this is a smooth strong deformation retract of G onto $\text{GL}(n; \mathbb{R})$. Moreover, the chain rule shows that $\Phi(\cdot, \tau)$ is a group homomorphism for each τ . It follows from Proposition 5.4 that EG has a smooth strong deformation retract onto $E\text{GL}(n; \mathbb{R})$, and this

descends to a smooth strong deformation retract of BG onto $BGL(n; \mathbb{R})$. By [Corollary 5.6](#) we have that any weakly D-numerable principal G -bundle over a Hausdorff smoothly paracompact base has a smooth strong deformation retract onto a principal $GL(n; \mathbb{R})$ -bundle over the same base. \square

Example 5.8 ($E(\text{Diff}^+(\mathbb{S}^2))$). Let $G = \text{Diff}^+(\mathbb{S}^2)$ be the diffeological group of orientation-preserving diffeomorphisms of \mathbb{S}^2 (equipped with the functional diffeology). By the main result of [\[13\]](#) there is a smooth strong deformation retraction $\kappa: G \times [0, 1] \rightarrow G$ onto $\text{SO}(3)$. By [Proposition 5.4](#) and [Remark 5.5](#), κ induces a smooth strong deformation retraction from EG to $E(\text{SO}(3))$. Unfortunately, the smooth strong deformation retraction in [\[13\]](#) does not give a group homomorphism from G to $\text{SO}(3)$, and so the deformation retraction does not descend to BG and $BSO(3)$. \square

5.3. Associated fibre bundles

Let G be a diffeological group, and let $\pi: E \rightarrow X$ be a principal G -bundle. Let F be a diffeological space admitting a smooth action of G . Then we may construct the **associated bundle to π with fibre F** , denoted $\tilde{\pi}: \tilde{E} \rightarrow X$, as follows: let G act diagonally on $E \times F$; then define \tilde{E} as the quotient $E \times_G F := (E \times F)/G$. Any diffeological fibre bundle with structure group G can be constructed as an associated fibre bundle of some principal G -bundle. See [\[8, Article 8.16\]](#) for more details. Note that if a fibre bundle is (weakly) D-numerable any associated principal G -bundle is (weakly) D-numerable as well. We have the following corollary to [Theorem 3.6](#):

Corollary 5.9 (*Diffeological fibre bundles*). Let $\mathcal{B}_G^F(\cdot)$ be the functor from $\mathbf{Diffeol}_{\text{HSP}}$ to \mathbf{Set} sending an object X to the set of isomorphism classes of locally trivial diffeological fibre bundles with fibre a fixed G -space F . Then there is a natural surjection from $[\cdot, BG]$ to $\mathcal{B}_G^F(\cdot)$.

Using the same notation as above, let θ be a diffeological connection on E induced by a connection 1-form ω on $\pi: E \rightarrow X$ as in [Proposition 4.12](#). Now $\omega \oplus 0$ is a T_eG -valued 1-form on $E \times F$, and one can check using the fact that π is D-numerable that ω descends to a T_eG -valued 1-form $\omega_{\tilde{E}}$ on \tilde{E} , which we call a **connection 1-form** on \tilde{E} .

Definition 5.10 (*Horizontal lifts on associated bundles*). Let $\tilde{\pi}: \tilde{E} \rightarrow X$ be a diffeological fibre bundle, let $c: (a, b) \rightarrow X$ be a smooth curve, and fix $z \in \tilde{E}$ and $t_0 \in (a, b)$ such that $\tilde{\pi}(z) = c(t_0)$. A curve $c_{\tilde{E}}: (a, b) \rightarrow \tilde{E}$ is a **horizontal lift** of c through z if

- (1) $\tilde{\pi} \circ c_{\tilde{E}} = c$,
- (2) $c_{\tilde{E}}(t_0) = z$,
- (3) $\omega_{\tilde{E}}\left(\frac{d}{dt}c_{\tilde{E}}(t)\right) = 0$.

Proposition 5.11 (*Horizontal lifts on associated bundles exist*). Let G be a regular diffeological Lie group, F a diffeological space with a fixed smooth action of G , X a Hausdorff smoothly paracompact diffeological space, and $\pi: E \rightarrow X$ a weakly D-numerable principal G -bundle. Equip E with the connection 1-form ω as in [Corollary 4.10](#). Then for any smooth curve $c: (a, b) \rightarrow X$, and any $z \in \tilde{E}$, there is a horizontal lift $c_{\tilde{E}}: (a, b) \rightarrow \tilde{E}$ through z .

Proof. Take $c_{\tilde{E}}(t) := \rho_{\tilde{E}}(\theta(c, t_0)(t), z')$, where $\rho_{\tilde{E}}: E \times F \rightarrow \tilde{E}$ is the quotient map and $z' \in F$ such that $\rho_{\tilde{E}}(\theta(c, t_0)(t_0), z') = z$. This definition is independent of the choice of z' . This satisfies all of properties required in [Definition 5.10](#). \square

Remark 5.12. We do not mention uniqueness of horizontal lifts in Proposition 5.11. This requires the uniqueness of solutions to the initial-value problem given by the differential equation in Item (3) with initial condition as given in Item (2), both in Definition 5.10. It is not immediately clear under what conditions this would hold on a general diffeological space.

Example 5.13 (*Diff(F)-bundles*). Let $\pi: \tilde{E} \rightarrow X$ be a diffeological fibre bundle with fibre F and structure group $G = \text{Diff}(F)$. Define the set \mathcal{FR} to be all commutative diagrams

$$\begin{array}{ccc} F & \xrightarrow{f} & \tilde{E} \\ \downarrow & & \downarrow \pi \\ \{*\} & \longrightarrow & X \end{array}$$

where f is a diffeomorphism onto a fibre of \tilde{E} . Equip \mathcal{FR} with the subset diffeology induced by $C^\infty(F, \tilde{E})$. One may think of this as a “non-linear frame bundle” of the bundle \tilde{E} . From above, we have that $\mathcal{FR} \times_{\text{Diff}(F)} F \cong \tilde{E}$, and by Corollary 5.9, we have that there is a natural surjection from smooth homotopy classes of maps $X \rightarrow B\text{Diff}(F)$ to isomorphism classes of such bundles \tilde{E} , provided that the bundles are locally trivial, and X has Hausdorff, second-countable, and smoothly paracompact topology.

If F is a smooth manifold, then $\text{Diff}(F)$ is a regular diffeological Lie group [9, Theorem 43.1], and so provided that $\tilde{E} \rightarrow X$ is a locally trivial principal $\text{Diff}(F)$ -bundle, and the topology on X is Hausdorff, second-countable, and smoothly paracompact, then by Proposition 5.11 \tilde{E} has a diffeological connection, which in turn gives us horizontal lifts of curves. \square

5.4. *Limit of groups & ILB principal bundles*

We consider in this subsection limits of diffeological group, and in particular infinite-dimensional groups. We rely heavily on [22] for terminology.

Fix a small category J , which we will think of as our “index category”. Let **DGroup** be the category of diffeological groups with smooth group homomorphisms between them, and let $F: J \rightarrow \mathbf{DGroup}$ be a functor. Denote by G_j the image $F(j)$ for each object j of J , and by φ_f the image $F(f)$ for each arrow f in J . Let $G = \lim F$ be the limit taken in the category of diffeological spaces; in particular, there is a smooth map $\varphi_j: G \rightarrow G_j$ for each object j in J such that if $f: j_1 \rightarrow j_2$ is an arrow in J , then $\varphi_f \circ \varphi_{j_1} = \varphi_{j_2}$.

Proposition 5.14 (*Limit of diffeological groups*). *Let F be the functor above, and define $G = \lim F$. Then G is a diffeological group.*

Proof. Recall that both the category of diffeological spaces and the category of groups are complete, and limits are constructed on the set-theoretical level:

$$G = \left\{ (g_j) \in \prod_{j \in J_0} G_j \mid \forall (f: j_1 \rightarrow j_2) \in J_1, \varphi_f \circ \text{pr}_{j_1}((g_j)) = \text{pr}_{j_2}((g_j)) \right\}$$

where J_0 is the set of objects of J , J_1 the set of arrows, and pr_j is the j th projection onto G_j . The diffeology is given by the intersection of all pullback diffeologies on G via the maps $\varphi_j := \text{pr}_j$. The group multiplication on G is given by the coordinate-wise product: $(g_j)(h_j) := (g_j h_j)$. It follows that the maps φ_j are group homomorphisms, from which it follows that multiplication and inversion are smooth maps $G \times G \rightarrow G$ and $G \rightarrow G$, respectively. Hence, G is a diffeological group. \square

Remark 5.15. Let G be the diffeological group constructed as a limit above. Then for every object j of J , from [Proposition 5.4](#) we obtain a commutative diagram:

$$\begin{array}{ccc}
 EG & \xrightarrow{\tilde{\Phi}_j} & EG_j \\
 \downarrow & & \downarrow \\
 BG & \xrightarrow{\Phi_j} & BG_j.
 \end{array}$$

Moreover, for any arrow $f: j_1 \rightarrow j_2$ of J , we obtain a commutative diagram:

$$\begin{array}{ccccc}
 EG & & & & EG_{j_1} \\
 \downarrow & \searrow & & \swarrow & \downarrow \\
 & EG_{j_2} & & & \\
 & \downarrow & & & \\
 & BG_{j_2} & & & \\
 \downarrow & \swarrow & & \searrow & \downarrow \\
 BG & & & & BG_{j_1}
 \end{array}$$

where all of the maps are the induced ones via [Proposition 5.4](#).

We now apply the theory developed to ILB-manifolds and ILB-bundles. Technically, our definition of an ILB-manifold below is only a special case of the definition given in [\[22, Definition I.1.9\]](#); however, it is sufficient for our needs. For this application to make sense, we must first note that infinite-differentiability in the Fréchet sense is equivalent to smoothness in the diffeological sense when we equip a Fréchet space/manifold with the diffeology comprising Fréchet infinitely-differentiable parametrisations. Moreover, Fréchet spaces form a full subcategory of diffeological spaces under this identification. See, for example, [\[14, Theorem 3.1.1\]](#), [\[11\]](#). Note that the limits used in the following definitions can be taken in either the Fréchet category or the diffeological category: in our cases below they coincide.

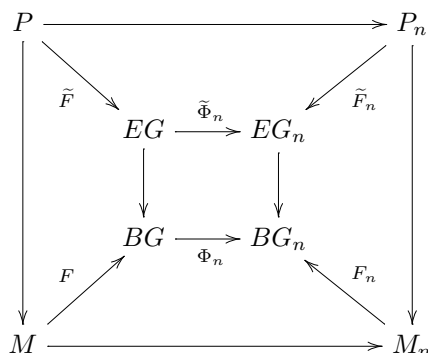
Definition 5.16 (*ILB-manifolds \mathcal{E} bundles*).

- (1) An **ILB-manifold** M is the limit in the category of Fréchet manifolds of a family of Banach manifolds $\{M_n\}_{n \in \mathbb{N}}$ in which there is a smooth and dense inclusion $M_{n+1} \hookrightarrow M_n$ for each n , and such that there exists a Banach atlas on M_0 that restricts to an atlas on M_n for each n .
- (2) An **ILB-map** f between two ILB-manifolds M and N is a smooth map $f: M \rightarrow N$ along with a family of smooth maps $\{f_n: M_n \rightarrow N_n\}$ such that f and all f_n commute with all inclusions maps $M \hookrightarrow M_{n+1} \hookrightarrow M_n$ and $N \hookrightarrow N_{n+1} \hookrightarrow N_n$.
- (3) An **ILB-principal bundle** is an ILB-map between two ILB-manifolds $(\pi: P \rightarrow M, \pi_n: P_n \rightarrow M_n)$ such that for each n , the map $\pi_n: P_n \rightarrow M_n$ is a principal G_n -bundle where G_n is a Banach Lie group.
- (4) An ILB-map F between two ILB-principal bundles $\pi: P \rightarrow M$ and $\pi': P' \rightarrow M'$ is an **ILB-bundle map** if $F_n: P_n \rightarrow P'_n$ is a (G_n -equivariant) bundle map for each n . Note that F induces an ILB-map $M \rightarrow M'$. An ILB-bundle map is an **ILB-bundle isomorphism** if it has an inverse ILB-bundle map.

It follows from the above definitions that given an ILB-principal bundle $\pi: P \rightarrow M$, the structure groups $\{G_n\}$ of the principal bundles $\{\pi_n: P_n \rightarrow M_n\}$ satisfy: G_{n+1} is smoothly and densely included in G_n for each n . Hence $\pi: P \rightarrow M$ is a principal G -bundle where $G = \lim G_n$. An ILB-bundle map from $\pi: P \rightarrow M$ to $\pi': P' \rightarrow M'$ is thus necessarily G -equivariant.

Definition 5.17 (*ILB-classifying maps & homotopies*).

- (1) Given an ILB-principal bundle $\pi: P \rightarrow M$ with structure group G , an **ILB-classifying map** $F: M \rightarrow BG$ is a classifying map for π such that there is a classifying map $F_n: M_n \rightarrow BG_n$ for each n , the following diagram commutes for all n ,



and these maps commute with the inclusion maps $EG \hookrightarrow EG_{n+1} \hookrightarrow EG_n$ and $BG \hookrightarrow BG_{n+1} \hookrightarrow BG_n$ as in [Remark 5.15](#).

- (2) An **ILB-homotopy** between two ILB-classifying maps $F, F': M \rightarrow BG$ for ILB-principal bundles $P \rightarrow M$ and $P' \rightarrow M$ each with structure group G is a family of smooth homotopies $H_n: M_n \times [0, 1] \rightarrow BG_n$ such that $H_n(\cdot, 0) = F_n$ and $H_n(\cdot, 1) = F'_n$ for each n , and a smooth homotopy $H: M \times [0, 1] \rightarrow BG$ such that $H(\cdot, 0) = F$ and $H(\cdot, 1) = F'$, and finally each H_n and H commute with the maps in [Remark 5.15](#).

Let $\pi: P \rightarrow M$ be an ILB-principal bundle, and assume that each M_n is a Hilbert manifold (in this case, we say that M is an **ILH-manifold**). Hilbert manifolds are smoothly paracompact [[9, Corollary 16.16](#)], and (smooth) partitions of unity pull back. Hence, a smooth partition of unity subordinate to a local trivialisation of $\pi_0: P_0 \rightarrow M_0$ induces similar partitions of unity for each $\pi_n: P_n \rightarrow M_n$, and on $\pi: P \rightarrow M$. Thus, in this case, each $\pi_n: P_n \rightarrow M_n$, as well as $\pi: P \rightarrow M$, is D-numerable.

Given an ILB-principal bundle $\pi: P \rightarrow M$ with ILB group G and ILH base as above, by [Proposition 3.14](#) there is a classifying map $F: M \rightarrow BG$ and so $\pi: P \rightarrow M$ is isomorphic to F^*EG . Moreover, for each n , there is a classifying map $F_n: M_n \rightarrow BG_n$ and so P_n is isomorphic to $F_n^*EG_n$. It follows from [Remark 5.15](#) that we obtain an ILB-classifying map for $\pi: P \rightarrow M$. From [Theorem 3.6](#) and [Remark 5.15](#) we obtain the following proposition:

Proposition 5.18. *Let $\pi: P \rightarrow M$ be an ILB-principal G -bundle in which $M = \lim M_n$ is an ILH-manifold. Then π has an ILB-classifying map. Moreover, smoothly homotopic ILB-classifying maps yield isomorphic ILB-principal bundles yield ILB-homotopic ILB-classifying maps.*

5.5. Group extensions

Fix a diffeological extension of diffeological groups (see [[8, Article 7.3](#)]):

$$1 \rightarrow G \rightarrow G' \rightarrow G'' \rightarrow 1. \tag{5.1}$$

By [Proposition 5.4](#), we obtain the following commutative diagram of diffeological spaces.

$$\begin{array}{ccccc}
 EG & \longrightarrow & EG' & \longrightarrow & EG'' \\
 \downarrow & & \downarrow & & \downarrow \\
 BG & \longrightarrow & BG' & \longrightarrow & BG''
 \end{array}$$

Since diffeological extensions are examples of principal bundles, [\(5.1\)](#) induces a long exact sequence of diffeological homotopy groups (see [\[8, Article 8.21\]](#)). The following result thus is a consequence of [Proposition 5.2](#).

Proposition 5.19 (*Long exact sequence of classifying spaces*). *Given a diffeological extension of diffeological groups as in [\(5.1\)](#), we obtain a long exact sequence*

$$\cdots \rightarrow \pi_{n+1}(BG'') \rightarrow \pi_n(BG) \rightarrow \pi_n(BG') \rightarrow \pi_n(BG'') \rightarrow \pi_{n-1}(BG) \rightarrow \dots$$

Example 5.20 (\mathbb{R}/\mathbb{Q}). Consider the rational numbers \mathbb{Q} as a diffeologically discrete subgroup of \mathbb{R} . We have the short exact sequence of diffeological groups

$$1 \rightarrow \mathbb{Q} \rightarrow \mathbb{R} \rightarrow \mathbb{R}/\mathbb{Q} \rightarrow 1.$$

From the above, we get a long exact sequence of homotopy groups:

$$\cdots \rightarrow \pi_{n+1}(B(\mathbb{R}/\mathbb{Q})) \rightarrow \pi_n(B\mathbb{Q}) \rightarrow \pi_n(B\mathbb{R}) \rightarrow \pi_n(B(\mathbb{R}/\mathbb{Q})) \rightarrow \pi_{n-1}(B\mathbb{Q}) \rightarrow \dots$$

It follows from this long exact sequence and [Proposition 5.2](#) that

$$\begin{aligned}
 \pi_0(B(\mathbb{R}/\mathbb{Q})) &= \{B(\mathbb{R}/\mathbb{Q})\}, \\
 \pi_1(B(\mathbb{R}/\mathbb{Q})) &\cong 1, \\
 \pi_2(B(\mathbb{R}/\mathbb{Q})) &\cong \mathbb{Q}, \text{ and} \\
 \pi_k(B(\mathbb{R}/\mathbb{Q})) &\cong 1 \text{ for all } k \geq 3.
 \end{aligned}$$

Of course, we could have used other easier means of computing these, but the point of this example is to illustrate what one could do with [Proposition 5.19](#). \square

Using short exact sequences and group extensions is very frequent in the study of infinite-dimensional Lie groups for constructing new groups. For example:

Example 5.21 (*Diff(M)-pseudo-differential operators*). Consider the $\text{Diff}(M)$ -pseudo-differential operators $FIO_{\text{Diff}}^{0,*}$ described in [\[18\]](#). Given a Hermitian bundle $E \rightarrow M$ over a closed manifold M , the group $FIO_{\text{Diff}}^{0,*}(M, E)$ is a group of Fourier integral operators acting on smooth sections of E , which can be seen as a central extension of the group of diffeomorphisms $\text{Diff}(M)$ by the group of 0-order invertible pseudo-differential operators acting on smooth sections of E , with exact sequence:

$$0 \rightarrow PDO^{0,*}(M, E) \rightarrow FIO_{\text{Diff}}^{0,*}(M, E) \rightarrow \text{Diff}(M) \rightarrow 0.$$

We obtain a long exact sequence of homotopy groups:

$$\begin{aligned} \cdots \rightarrow \pi_{n+1}(B\text{Diff}(M)) \rightarrow \pi_n(B(PDO^{0,*}(M, E))) \rightarrow \pi_n(B(FIO_{\text{Diff}}^{0,*}(M, E))) \rightarrow \\ \pi_n(B\text{Diff}(M)) \rightarrow \pi_{n-1}(B(PDO^{0,*}(M, E))) \rightarrow \dots \quad \square \end{aligned}$$

5.6. Irrational torus bundles

We now connect what we have done to the classical theory of Weil, in which circle (or complex line) bundles with connection over a fixed simply-connected manifold are classified by their curvatures, and conversely every integral 2-form on the manifold induces a circle bundle with connection whose curvature is that 2-form. This is extended to non-integral 2-forms in [7], in which the circle bundles are replaced with irrational torus bundles. See also the more general theory on diffeological spaces in [8].

Let T_α be the irrational torus, defined in Example 5.3. It is a regular diffeological Lie group (indeed, the quotient map $\mathbb{R} \rightarrow T_\alpha$ is the exponential map). Let ω be the connection 1-form on ET_α constructed in Theorem 4.9, and let $\pi: ET_\alpha \rightarrow BT_\alpha$ be the projection map.

Lemma 5.22. *The 2-form $d\omega$ is basic; that is, there exists a unique T_eG -valued 2-form Ω on BT_α such that $\pi^*\Omega = d\omega$.*

Remark 5.23. This statement in fact holds for any abelian diffeological group G . We prove this more general statement below.

Proof. We again use the following fact (see [8, Article 6.38]): $d\omega$ is the pullback of a form Ω on BG if and only if for any two plots $p_1, p_2: U \rightarrow EG$ satisfying $\pi \circ p_1 = \pi \circ p_2$, we have $p_1^*d\omega = p_2^*d\omega$. Fix two such plots p_1 and p_2 . Let $V_j \subseteq EG$ be the open set

$$V_j = \{(t_i g_i) \mid t_j \neq 0\}.$$

We have $p_1^{-1}(V_j) = p_2^{-1}(V_j) =: V$. The projection $\text{pr}_{g_j}: V_j \rightarrow G$ sending $(t_i g_i)$ to g_j is well-defined on V_j , and so we have a smooth map $\gamma: V \rightarrow G$ sending u to $\text{pr}_{g_j}(p_2(u))^{-1} \text{pr}_{g_j}(p_1(u))$. Thus, $\gamma(u) \cdot p_2(u) = p_1(u)$ for all $u \in V$.

Fix $u \in V$ and $v \in T_u V$. It follows from the chain rule (Remark 2.8) that

$$(p_1)_* v \lrcorner d\omega = (p_2)_* v \lrcorner (\gamma(u)^* d\omega) + \eta \lrcorner d\omega,$$

where $\eta = \frac{d}{dt} \Big|_{t=0} (\gamma(c(t)) \cdot p_1(u))$ for some smooth curve $c: (-\varepsilon, \varepsilon) \rightarrow V$ such that $c(0) = u$ and $\dot{c}(0) = v$ ($\varepsilon > 0$). Since ω is invariant, we have $\gamma(u)^* d\omega = d\omega$.

Since j , u , and v are arbitrary, to complete the proof, we only need to show that $\eta \lrcorner d\omega = 0$. Note that this is equal to the contraction of $d\omega$ by the map $g_{EG}: (-\varepsilon, \varepsilon) \rightarrow \text{Diff}(EG)$ induced by the curve $g := \gamma \circ c$ in G (see [8, Article 6.57]). By the Cartan–Lie formula ([8, Article 6.72]), we have

$$\eta \lrcorner d\omega = \mathcal{L}_g(\omega) - d(\eta \lrcorner \omega). \quad (5.2)$$

By definition of ω we know $\eta \lrcorner \omega = \dot{g}(0)$, and so the right term of the right-hand side of (5.2) vanishes. The left term of the right-hand side of (5.2) vanishes since ω is invariant (see [8, Article 6.55] for a definition of the Lie derivative).

Uniqueness of Ω follows from the fact that π^* is injective on forms. This finishes the proof. \square

We refer to Ω in Lemma 5.22 as the **curvature form** of ω . If X is any Hausdorff smoothly paracompact diffeological space, and $F: X \rightarrow BT_\alpha$ a smooth function, we obtain the **curvature form** $F^*\Omega$ of the connection 1-form $\tilde{F}^*\omega$ on F^*EG by Proposition 3.14. The 2-form $F^*\Omega$ satisfies

$$\text{pr}_1^* F^* \Omega = \tilde{F}^* d\omega$$

where $\text{pr}_1 : F^* EG \rightarrow X$ is the pullback bundle.

Let X be a connected diffeological space. In [8, Article 8.40], Iglesias-Zemmour proves that every principal T_α -bundle which can be equipped with a connection 1-form induces a unique class in $H^1(\text{Paths}(X), T_\alpha)$, its **characteristic class**. If additionally X Hausdorff and smoothly paracompact, then it follows from Proposition 3.14 that *all* weakly D-numerable T_α -bundles over X induce a unique class in $H^1(\text{Paths}(X), T_\alpha)$.

In [8, Article 8.42], Iglesias-Zemmour shows a converse for the simply-connected case: if X is a simply-connected (and hence connected) diffeological space, and μ is a non-zero closed 2-form on X , then there is a principal T_α -bundle on X whose curvature is μ , where T_α is the **torus of periods** of μ . If X is additionally Hausdorff and smoothly paracompact, and the T_α -bundle constructed is weakly D-numerable, then $\mu = F^* \Omega$, where F is the classifying map of the T_α -bundle, and Ω is the curvature form on BT_α .

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