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George Avalos
University of Nebraska-Lincoln, gavalos@math.unl.edu

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Asymptotic Stability of a Fluid-Structure Semigroup

George Avalos
Department of Mathematics
University of Nebraska-Lincoln, 68588

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Abstract

The strong stability problem for a fluid-structure interactive partial differential equation (PDE) is considered. The PDE comprises a coupling of the linearized Stokes equations to the classical system of elasticity, with the coupling occurring on the boundary interface between the fluid and solid media. It is now known that this PDE may be modeled by a C_0-semigroup of contractions on an appropriate Hilbert space. However, because of the nature of the unbounded coupling between fluid and structure, the resolvent of the semigroup generator will *not* be a compact operator. In consequence, the classical solution to the stability problem, by means of the Nagy-Foias decomposition, will not avail here. Moreover, it is not practicable to write down explicitly the resolvent of the fluid-structure generator; this situation thus makes it problematic to use the wellknown semigroup stability result of Arendt-Batty and Lyubich-Phong. Instead, our proof of strong stability for the fluid-structure PDE will depend on the appropriate usage of a recently derived abstract stability result of Y. Tomilov.

1 Statement of the Problem

In this paper, we show how a recently derived abstract operator theoretic result can be used to ascertain the asymptotic decay of solutions for a so-called "transmission hyperbolic-parabolic problem". A simplified version of this model and its relevance to biological modeling is discussed in [11]; see also [7] and [8] for related partial differential equations (PDEs). Because of the non-compactness of the resolvent for the associated semigroup generator–see (5) below–the classical stability treatments involving the Nagy-Foias decomposition and Lasalle Invariance Principle are *not* applicable (see [10] and references therein). Nor does the resolvent of this fluid-structure semigroup admit an explicit, working expression which might allow an appeal to the now wellknown abstract stability results in [1] and [13]. Instead, we will use the recently derived stability result posted in [16] and [6] (see Theorem 2 below; see also a precursor of this result in [5]). This stability result of Y. Tomilov is formulated as a necessary resolvent criterion; however, to use this result one does not actually *need* to know what the resolvent looks like.

The methodology for the use of Tomilov's abstract stability criterion was first developed in [3], in the context of discerning strong stability for a given PDE dynamics. In fact, the game plan developed in [3], to infer the asymptotic decay of a given PDE, can be generally applied to obtain the asymptotic decay of those general PDE models under inserted dissipation. (Of course the details of proof will necessarily be intrinsic to the model under consideration.) We sketch the general approach here: Let $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ (Hilbert) be the infinitesimal generator of a C_0-semigroup of contractions $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ on $\mathbf{H}$. We assume that the generator models some PDE system under the influence of some (unbounded) dissipative mechanism, in which case the question of stability naturally arises.

To infer stability for a given C_0-semigroup of contractions $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$, one would generally perform the following sequence of steps: *(i)* Show that contraction semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ is "completely non-unitary". (We recall that a contraction C_0-semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$ is *completely non-unitary* (c.n.u.) if $\mathbf{H}$ has no nontrivial reducing subspace for $e^{\mathcal{A}(\cdot)}$ on which $e^{\mathcal{A}(\cdot)}$ is unitary.) *(ii)* Given arbitrary $f \in \mathbf{H}$, one subsequently considers the quantity

$$x(\alpha) \equiv \mathcal{R}(\alpha + i\beta; \mathcal{A})f,$$

where $\mathcal{R}(\lambda; \mathcal{A})$ is the resolvent operator of $\mathcal{A}$ corresponding to complex λ, $\alpha > 0$ and β is any element in $\mathbb{R}\diagdown\mathcal{S}$, where $\mathcal{S}$ is some (suitably chosen) set with zero Lebesgue measure. In applications, the function $x(\alpha)$ will be the solution of a β-parameterized steady state PDE; moreover, $\mathcal{S}$ will (essentially) be the eigenvalues of a particular elliptic operator. To infer strong stability of $\{e^{\mathcal{A}t}\}_{t\geq 0}$, we must show that

$$\lim_{\alpha\to 0^+} \sqrt{\alpha}x(\alpha) = 0 \quad \text{for all } \beta \in \mathbb{R}\diagdown\mathcal{S}. \tag{1}$$

(see [16]). To this end, the following steps *(iii)-(iv)*. *(iii)* Establish *a priori* bounds for the damping mechanism inherent in $x(\alpha)$. *(iv)* Use the result of *(iii)* to establish *a priori* bounds for $\sqrt{\alpha}x(\alpha)$, initially in a topology *lower* than that of $\mathbf{H}$. *(v)* Now use the bounds obtained in *(iii)* and *(iv)* to recover *a priori* bounds for $\sqrt{\alpha}x(\alpha)$ in the *full finite energy* topology $\mathbf{H}$. *(vi)* Use the *a priori* bounds in (iii) and (v) to show that the weak limit of $\sqrt{\alpha}x(\alpha)$ is actually a strong limit, with value zero. Of course each problem will have its own intrinsic set of details, but in principle, this ostensibly simple algorithm can be applied to any dissipative PDE system. In particular, this method can be applied to PDE systems in which there is no compactness of the underlying semigroup generator resolvent–a situation which will obtain for many coupled PDE systems where the coupling is accomplished via boundary interfaces–or for PDE's in which an explicit expression of the resolvent is not readily computable. *Once* the basic question of asymptotic decay is addressed for a given PDE model, then of course one can proceed to consider other control theoretic issues for the model.

We intend to use the aforesaid methodology to obtain the conclusion of strong decay for a particular fluid-structure PDE system. Because of a lack of compactness of the resolvent of the associated generator (see (5) below), this strong stability cannot be inferred by the classical Nagy-Foias approach; nor can one readily write down this resolvent, thereby precluding the use of the stability result of Arendt-Batty/Lyubich-Phong. So in answering the strong stability question for the fluid-structure PDE under present consideration, the appropriate use of Tomilov's stability criterion is indispensible.

We now describe this fluid-structure PDE: Let Ω_f and Ω_s be bounded open sets, with smooth boundaries Γ_f and Γ_s, respectively; these geometries are configured as in Figure 1: On the "fluid portion" Ω_f of the geometry, we define the following spaces:

$$\begin{aligned} Null(div) &= \left\{u \in [L^2(\Omega_f)]^3 : div\ u = 0\right\}; \\ V &= \left\{\phi \in [H^1(\Omega_f)]^3 \cap Null(div) : \phi|_{\Gamma_f} = 0\right\}. \end{aligned}$$

With respect to the "solid portion" Ω_s of the geometry, we define the following classic operators which mathematically realize the 3-D system of elasticity (see e.g., [9]):

1. For $w = [w_1, w_2, w_3]$, the *strain tensor* $\{\epsilon_{ij}\}$ is given by

$$\epsilon_{ij}(w) = \frac{1}{2}\left(\frac{\partial w_j}{\partial x_i} + \frac{\partial w_i}{\partial x_j}\right), \quad 1 \leq i, j \leq 3.$$

2. Subsequently, the *stress tensor* is described by means of Hooke's Law:

$$\sigma_{ij}(w) = \lambda\left(\sum_{k=1}^{3} \epsilon_{kk}(w)\right)\delta_{ij} + 2\mu\epsilon_{ij}(w), \ \ 1 \leq i, j \leq 3,$$

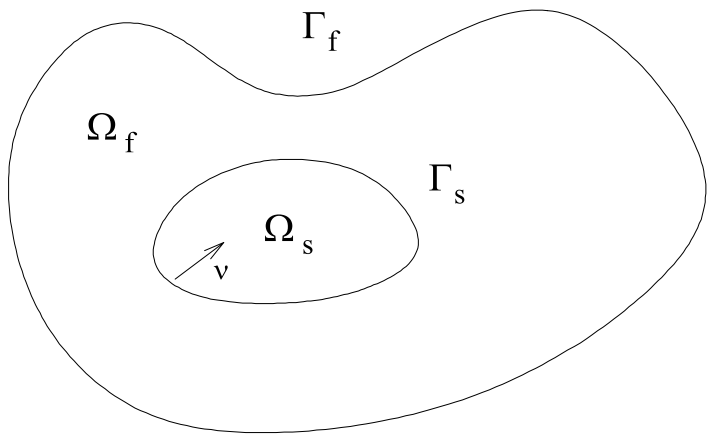

Figure 1: The Geometry of the Problem

where $\lambda \geq 0$ and $\mu > 0$ are the so-called *Lamé's coefficients* of the system. Moreover, δ_{ij} denotes as usual the Kronecker delta; i.e., $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$.

Letting

$$\sigma(w) = (\sigma_{ij}(w))_{i,j=1}^{3}, \;\; \epsilon(w) = (\epsilon_{ij}(w))_{i,j=1}^{3},$$

then by virtue of Korn's inequality, $[H^1(\Omega_s)]^3$ may be endowed with the following inner-product, equivalent to the usual $[H^1(\Omega_s)]^3$-norm:

$$\begin{aligned} (w, \tilde{w})_{1,\Omega_s} &= (\epsilon(w), \sigma(\tilde{w}))_{\Omega_s} + (w, \tilde{w})_{\Omega_s}; \\ \|w\|_{1,\Omega_s}^2 &= (\epsilon(w), \sigma(w))_{\Omega_s} + \|w\|_{\Omega_s}^2 \end{aligned} \tag{2}$$

3. With this nomenclature, we denote the Hilbert space $\mathbf{H}$ (of wellposedness) as

$$\begin{aligned} \mathbf{H} &\equiv Null(div) \times [H^1(\Omega_s)]^3 \times [L^2(\Omega_s)]^3; \\ \left(\begin{bmatrix} u_0 \\ w_0 \\ w_1 \end{bmatrix}, \begin{bmatrix} \tilde{u}_0 \\ \tilde{w}_0 \\ \tilde{w}_1 \end{bmatrix} \right)_{\mathbf{H}} &\equiv (u_0, \tilde{u}_0)_{\Omega_f} + (\epsilon(w_0), \sigma(\tilde{w}_0))_{\Omega_s} + (w_0, \tilde{w}_0)_{\Omega_s} + (w_1, \tilde{w}_1)_{\Omega_s}. \end{aligned}$$

(Here, $(\cdot,\cdot)_{\Omega_f}$ and $(\cdot,\cdot)_{\Omega s}$ denote the respective L^2-norms on the two geometries.)

We will discern strong stability properties of functions $[u(t), w(t), w_t(t)] \in C([0,T]; \mathbf{H})$ which solve the following problem:

$$\begin{cases} (u_t, \phi)_{\Omega_f} + (\nabla u, \nabla \phi)_{\Omega_f} - \langle \sigma(w) \cdot \nu, \phi \rangle_{\Gamma_s} = 0 \;\text{ on } (0,\infty), \text{ for all } \phi \in V; \\ div u = 0 \;\text{ in } (0,\infty) \times \Omega_f \\ u|_{\Gamma_f} = 0 \;\text{ on } (0,\infty) \times \Gamma_f \\ \\ w_{tt} - div\sigma(w) + w = 0 \;\text{ in } (0,\infty) \times \Omega_s \\ w_t|_{\Gamma_s} = u|_{\Gamma_s} \;\text{ on } (0,\infty) \times \Gamma_s \\ \\ [u(0), w(0), w_t(0)] = [u_0, w_0, w_1] \in \mathbf{H}. \end{cases} \tag{3}$$

Above, the divergence of the stress tensor is defined in the usual way; i.e.,

$$(div\sigma(w))_i = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}\left(\sigma_{ij}(w)\right), \quad 1 \le i \le 3;$$

see e.g., [15].

Remark 1 *The fluid variational relation in (3) is the weak formulation of the following coupled Stokes flow-elasticity system, with inhomogeneous Neumann boundary data:*

$$\begin{cases} u_t - \Delta u + \nabla p = 0 \quad on\ (0,\infty)\times\Omega_f; \\ divu = 0 \quad in\ (0,\infty)\times\Omega_f \\ u|_{\Gamma_f} = 0 \quad on\ (0,\infty)\times\Gamma_f \\ \\ \dfrac{\partial u}{\partial \nu} = \sigma(w)\cdot\nu - p\nu \quad in\ (0,\infty)\times\Gamma_s \\ w_{tt} - div\sigma(w) + w = 0 \quad in\ (0,\infty)\times\Omega_s \\ w_t|_{\Gamma_s} = u|_{\Gamma_s} \quad on\ (0,\infty)\times\Gamma_s \\ \\ [u(0), w(0), w_t(0)] = [u_0, w_0, w_1] \in \mathbf{H} \end{cases} \tag{4}$$

(Here, p is the associated pressure of the weak solution u). In fact, because of the "hidden regularity" enjoyed by the displacement w–i.e., $\sigma(w)\cdot\nu|_{\Gamma_s} \in L^2(0,T;[H^{-\frac{1}{2}}(\Gamma_s)]^3)$ (see [4])–one can justify that weak solutions of (3) are classical solutions of (4), in the sense of distributions.

Because of the recent wellposedness result in [4], we have continuity of the solution map in the space $\mathbf{H}$ of wellposedness; i.e.,

$$[u_0, w_0, w_1] \in \mathbf{H} \Rightarrow [u(\cdot), w(\cdot), w_t(\cdot)] \in C([0,T];\mathbf{H}).$$

In fact, this problem admits (a nonpedestrian) semigroup formulation: To wit, as in [4] we define the operator $A : V \times [H^{-\frac{1}{2}}(\Gamma_s)]^3 \to V'$ by

$$\langle A(u,z), \phi\rangle_{V'\times V} = (\nabla u, \nabla\phi)_{\Omega_f} - \langle z, \phi\rangle_{\Gamma_s} \quad \text{for all } \phi \in V,$$

where $\langle\cdot,\cdot\rangle_{\Gamma_s}$ denotes the duality pairing between $[H^{-\frac{1}{2}}(\Gamma_s)]'$ and its topological dual. Subsequently, we define the operator $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ by

$$\begin{aligned} \mathcal{A}\begin{bmatrix} u_0 \\ w_0 \\ w_1 \end{bmatrix} &= \begin{bmatrix} -A(u_0, \sigma(w_0)\cdot\nu) \\ w_1 \\ div(w_0) - w_0 \end{bmatrix}, \\ D(\mathcal{A}) &= \left\{[u_0, w_0, w_1] \in \mathbf{H} : u_0 \in V,\ \ [A(u_0, \sigma(w_0)), w_1, div\sigma(w_0)] \in \mathbf{H},\ \ u_0|_{\Gamma_s} = w_1|_{\Gamma_s}\right\} \end{aligned} \tag{5}$$

(Note that as $div\sigma(w_0) \in [L^2(\Omega_s)]^3$ in the definition of $D(\mathcal{A})$, then $\sigma(w_0)\cdot\nu$ is well-defined in $[H^{-\frac{1}{2}}(\Gamma_s)]^3$; see e.g., Théorèm 1, p. 307 of [2]. Thus the first component of the operator $\mathcal{A}$ is well-defined.

It is shown in [4] that $\mathcal{A}$ generates a C_0-semigroup of contractions $\left\{e^{\mathcal{A}t}\right\}_{t\ge 0}$ on $\mathbf{H}$. Thus the weak solution to (3) is given by

$$\begin{bmatrix} u(t) \\ w(t) \\ w_t(t) \end{bmatrix} = e^{\mathcal{A}t}\begin{bmatrix} u_0 \\ w_0 \\ w_1 \end{bmatrix} \in C([0,T];\mathbf{H}).$$

From (3), we readily see that this semigroup is dissipative: For, if in (3) we take $\phi \equiv u$, and multiply the elastic equation by w_t, we eventually obtain, for all $0 \leq s < t$, the relation

$$\left\| \begin{bmatrix} u(s) \\ w(s) \\ w_t(s) \end{bmatrix} \right\|_{\mathbf{H}}^2 = \left\| \begin{bmatrix} u(t) \\ w(t) \\ w_t(t) \end{bmatrix} \right\|_{\mathbf{H}}^2 + 2\int_s^t |\nabla u|^2 \, d\tau. \tag{6}$$

This dissipation naturally gives rise to the question of strong stability: recall that a C_0-semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0} \subset \mathcal{L}(\mathbf{H})$ is said to be *strongly stable* if for every $\mathbf{x} \in \mathbf{H}$, $\lim_{t\to\infty} e^{\mathcal{A}t}\mathbf{x} = 0$. Moreover, since one can readily infer that zero is not an eigenvalue of $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$, as defined in (5), the question of strong stability for the fluid-structure dynamics is an unambiguous one, since there will be no complication presented by "steady states".

As we said at the outset, the definition of the domain $D(\mathcal{A})$ means that the resolvent $\mathcal{R}(\lambda; \mathcal{A})$ is not compact as a mapping into $\mathbf{H}$. Nor does $\mathcal{R}(\lambda; \mathcal{A})$ admit of an explicit representation. Thus, the method of solution for the stability problem, outlined respectively in [10] and [1] (and [13]), is not applicable. Instead, we will appeal to the following operator theoretic result:

Theorem 2 *(see See Theorem 8.7 of [6]; see also p. 75-76 of [16]) Let $\mathcal{A}$ generate a C_0-semigroup of completely non-unitary contractions on a Hilbert space $\mathbf{H}$. If there exists a dense set $M \subset \mathbf{H}$ such that*

$$\lim_{\alpha\to 0^+} \sqrt{\alpha}\mathcal{R}(\alpha + i\beta; \mathcal{A})x = 0 \quad \textit{for every } x \in M \textit{ and almost every } \beta \in \mathbb{R},$$

then the semigroup is strongly stable.

Through the agency of this abstract result, we will establish the asymptotic decay of weak solutions to (3):

Theorem 3 *The fluid-structure semigroup $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ generated by $\mathcal{A} : D(\mathcal{A}) \subset \mathbf{H} \to \mathbf{H}$ (as defined in (5)) is strongly stable.*

2 Proof of Theorem 3

2.1 A Preliminary Result

The proof will follow the algorthim devised in [3]. In what follows, we will have need of the following elliptic operator $\mathbf{\mathring{A}}$: $D(\mathbf{\mathring{A}}) \subset [L^2(\Omega_s)]^3 \to [L^2(\Omega_s)]^3$, defined on the solid portion of the geometry Ω_s:

$$\mathbf{\mathring{A}}\omega = -div\sigma(\omega) + \omega; \quad D(\mathbf{\mathring{A}}) = \left[H^2(\Omega_s) \cap H_0^1(\Omega_s)\right]^3. \tag{7}$$

By Korn's inequality, $\mathbf{\mathring{A}}$ is positive definite and self-adjoint, with compact resolvent.

To justify the invocation of Theorem 2, we must first show the following:

Proposition 4 *The contraction semigroup $\{e^{\mathcal{A}t}\} \subset \mathcal{L}(\mathbf{H})$ of the generator defined in (5) is completely non-unitary.*

Proof of Proposition 4: Let $\mathbf{H}_u$ denote a subspace of $\mathbf{H}$ on which $\{e^{\mathcal{A}t}\}$ is unitary. Then by Stone's Theorem $i\mathcal{A}|_{\mathbf{H}_u}$ is *self-adjoint*. Thus, if nonzero λ is a (real) eigenvalue of $i\mathcal{A}|_{\mathbf{H}_u}$, corresponding to eigenfunction $[u_0, w_0, w_1]$ in $\mathbf{H}_u$, we have from (5) the following relations:

$$\begin{aligned}
-\left(\nabla \operatorname{Re} u_0, \nabla\phi\right)_{\Omega_f} + \langle\sigma(\operatorname{Re} w_0)\cdot\nu, \phi\rangle_{\Gamma_s} &= \lambda\left(\operatorname{Im} u_0, \phi\right)_{\Omega_f} \quad \text{for all } \phi\in V; && (8)\\
\left(\nabla \operatorname{Im} u_0, \nabla\phi\right)_{\Omega_f} - \langle\sigma(\operatorname{Im} w_0)\cdot\nu, \phi\rangle_{\Gamma_s} &= \lambda\left(\operatorname{Re} u_0, \phi\right)_{\Omega_f} \quad \text{for all } \phi\in V; && (9)\\
\operatorname{Re} w_1 &= \lambda \operatorname{Im} w_0 \ \text{ and } \ \operatorname{Im} w_1 = -\lambda \operatorname{Re} w_0; && (10)\\
div(\sigma(\operatorname{Re} w_0)) - \operatorname{Re} w_0 &= \lambda \operatorname{Im} w_1; && (11)\\
-div(\sigma(\operatorname{Im} w_0)) + \operatorname{Im} w_0 &= \lambda \operatorname{Re} w_1. && (12)
\end{aligned}$$

We now: (i) take $\phi \equiv -\operatorname{Re} u_0$ in (8); (ii) take $\phi \equiv \operatorname{Im} u_0$ in (9); (iii) multiply both sides of (11) by $-\operatorname{Re} w_1$ and integrate; (iv) multiply both sides of (12) by $\operatorname{Im} w_1$ and integrate. Upon an addition of these relations, we then have,

$$\|\nabla \operatorname{Re} u_0\|_{\Omega_f}^2 + \|\nabla \operatorname{Im} u_0\|_{\Omega_f}^2 = 0 \qquad (13)$$

(in obtaining this relation, we have also implicitly used (10) and the fact that $u_0|_{\Gamma_s} = w_1|_{\Gamma_s}$). By Poincaré's inequality, we have then that

$$\operatorname{Re} u_0 = \operatorname{Im} u_0 = 0. \qquad (14)$$

In turn, from (8) and (9) and the definition of $D(\mathcal{A})$ we have that

$$\begin{aligned}
\sigma(\operatorname{Re} w_0)\cdot\nu &= 0 \ \text{ on } \Gamma_s;\\
\sigma(\operatorname{Im} w_0)\cdot\nu &= 0 \ \text{ on } \Gamma_s. && (15)
\end{aligned}$$

In turn, since $w_1|_{\Gamma_s} = 0$ from the definition of $D(\mathcal{A})$, then using (10), (11) and (15), we have that $\operatorname{Re} w_0$ satisfies

$$\begin{aligned}
\left(\lambda^2 - \mathring{\mathbf{A}}\right)\operatorname{Re} w_0 &= 0 \ \text{ in } \Omega_s;\\
\sigma(\operatorname{Re} w_0)\cdot\nu &= 0 \ \text{ on } \Gamma_s,
\end{aligned}$$

where $\mathring{\mathbf{A}}$: $D(\mathring{\mathbf{A}}) \subset [L^2(\Omega_s)]^3 \to [L^2(\Omega_s)]^3$ is as defined in (7). From elliptic theory we have consequently that $\operatorname{Re} w_0 = 0$. In turn, from (10) we have that $\operatorname{Im} w_1 = 0$. In the same way, $\operatorname{Im} w_0 = 0$ and $\operatorname{Re} w_1 = 0$. These consequences and (14) now complete the proof of Proposition 4. □

As $\left\{e^{\mathcal{A}t}\right\}_{t\geq 0}$ is c.n.u., we can now attempt to apply Tomilov's resolvent criterion. In fact, we shall eventually invoke Theorem 2 with therein, $M = \mathbf{H}$ and $\beta \in \mathbb{R}\diagdown\mathcal{S}$, where

$$\mathcal{S} = \left\{\beta\in\mathbb{R} : \beta^2 \text{ is an eigenvalue of } \mathring{\mathbf{A}} : D(\mathring{\mathbf{A}}) \subset [L^2(\Omega_s)]^3 \to [L^2(\Omega_s)]^3\right\} \qquad (16)$$

(so $\mathcal{S}$ is a countable set).

2.2 Proof proper of Theorem 3

Step 1 (A priori bounds for the damping mechanism)

With $\lambda = \alpha + i\beta$, where $\beta \in \mathbb{R}\diagdown\mathcal{S}$, we look at the resolvent equation

$$(\lambda I - \mathcal{A})\begin{bmatrix} u_0\\ w_0\\ w_1\end{bmatrix} = \begin{bmatrix} f_0\\ g_0\\ g_1\end{bmatrix} \in \mathbf{H}. \qquad (17)$$

Since $\beta = 0$ is an easy case, as there is then no coupling between real and imaginary parts, we also assume throughout that $\beta \neq 0$. By Theorem 2, it is enough to show that

$$\lim_{\alpha \to 0^+} \sqrt{\alpha} \, \|[u_0(\alpha + i\beta), w_0(\alpha + i\beta), w_1(\alpha + i\beta)]\|_{\mathbf{H}} = 0. \tag{18}$$

Componentwise, (17) gives the following relations:

$$\begin{aligned}
\lambda \,(u_0, \phi)_{\Omega_f} + (\nabla u_0, \nabla \phi)_{\Omega_f} - \langle \sigma(w_0) \cdot \nu, \phi \rangle_{\Gamma_s} &= (f_0, \phi)_{\Omega_f} \quad \text{for every } \phi \in V; \\
\alpha \operatorname{Re} w_0 - \beta \operatorname{Im} w_0 - \operatorname{Re} w_1 &= \operatorname{Re} g_0 \in [H^1(\Omega_s)]^3; \\
\alpha \operatorname{Im} w_0 + \beta \operatorname{Re} w_0 - \operatorname{Im} w_1 &= \operatorname{Im} g_0 \in [H^1(\Omega_s)]^3; \\
\lambda w_1 + w_0 - div \sigma(w_0) &= g_1 \in [L^2(\Omega_s)]^3.
\end{aligned} \tag{19}$$

Subsequently distinguishing real and imaginary parts gives then

$$\begin{aligned}
(\alpha \operatorname{Re} u_0 - \beta \operatorname{Im} u_0, \phi)_{\Omega_f} + (\nabla \operatorname{Re} u_0, \nabla \phi)_{\Omega_f} - \langle \sigma(\operatorname{Re} w_0) \cdot \nu, \phi \rangle_{\Gamma_s} &= (\operatorname{Re} f_0, \phi)_{\Omega_f} \quad \text{for every } \phi \in V; && (20) \\
(\alpha \operatorname{Im} u_0 + \beta \operatorname{Re} u_0, \phi)_{\Omega_f} + (\nabla \operatorname{Im} u_0, \nabla \phi)_{\Omega_f} - \langle \sigma(\operatorname{Im} w_0) \cdot \nu, \phi \rangle_{\Gamma_s} &= (\operatorname{Im} f_0, \phi)_{\Omega_f} \quad \text{for every } \phi \in V; && (21) \\
(\alpha^2 + 1) \operatorname{Re} w_0 - 2\alpha\beta \operatorname{Im} w_0 - \beta^2 \operatorname{Re} w_0 - div \sigma(\operatorname{Re} w_0) &= \operatorname{Re} g_1 + \alpha \operatorname{Re} g_0 - \beta \operatorname{Im} g_0; && (22) \\
(\alpha^2 + 1) \operatorname{Im} w_0 + 2\alpha\beta \operatorname{Re} w_0 - \beta^2 \operatorname{Im} w_0 - div \sigma(\operatorname{Im} w_0) &= \operatorname{Im} g_1 + \alpha \operatorname{Im} g_0 + \beta \operatorname{Re} g_0. && (23)
\end{aligned}$$

We now multiply (22) by $-\beta \operatorname{Im} w_0$, multiply (23) by $\beta \operatorname{Re} w_0$, and integrate the two subsequent relations. Integrating by parts and adding the two gives

$$2\alpha\beta^2 \|\operatorname{Im} w_0\|^2_{\Omega_s} + 2\alpha\beta^2 \|\operatorname{Re} w_0\|^2_{\Omega_s} - \beta \langle \sigma(\operatorname{Re} w_0) \cdot \nu, \operatorname{Im} w_0 \rangle_{\Gamma_s} + \beta \langle \sigma(\operatorname{Im} w_0) \cdot \nu, \operatorname{Re} w_0 \rangle_{\Gamma_s} = F^{(1)}_\alpha, \tag{24}$$

where

$$F^{(1)}_\alpha = -\beta \left(\operatorname{Re} g_1 + \alpha \operatorname{Re} g_0 - \beta \operatorname{Im} g_0, \operatorname{Im} w_0\right)_{\Omega_s} + \beta \left(\operatorname{Im} g_1 + \alpha \operatorname{Im} g_0 + \beta \operatorname{Re} g_0, \operatorname{Re} w_0\right)_{\Omega_s}. \tag{25}$$

Using the second and third relations in (19) to rewrite the boundary terms in (24), we have then

$$\begin{aligned}
& 2\alpha^2\beta^2 \|\operatorname{Im} w_0\|^2_{\Omega_s} + 2\alpha^2\beta^2 \|\operatorname{Re} w_0\|^2_{\Omega_s} \\
& + \alpha \langle \sigma(\operatorname{Re} w_0) \cdot \nu, \operatorname{Re} w_1 - \alpha \operatorname{Re} w_0 + \operatorname{Re} g_0 \rangle_{\Gamma_s} + \alpha \langle \sigma(\operatorname{Im} w_0) \cdot \nu, \operatorname{Im} w_1 - \alpha \operatorname{Im} w_0 + \operatorname{Im} g_0 \rangle_{\Gamma_s} \\
= \; & \alpha F^{(1)}_\alpha.
\end{aligned} \tag{26}$$

Moreover, we take $\phi \equiv \alpha \operatorname{Re} u_0$ in (20); we take $\phi \equiv \alpha \operatorname{Im} u_0$ in (21). Integrating in space and adding the subsequent relations, we then obtain,

$$\begin{aligned}
& \alpha^2 \|\operatorname{Re} u_0\|^2_{\Omega_f} + \alpha^2 \|\operatorname{Im} u_0\|^2_{\Omega_f} + \alpha \|\nabla \operatorname{Re} u_0\|^2_{\Omega_f} + \alpha \|\nabla \operatorname{Im} u_0\|^2_{\Omega_f} \\
& - \alpha \langle \sigma(\operatorname{Re} w_0) \cdot \nu, \operatorname{Re} u_0 \rangle_{\Gamma_s} - \alpha \langle \sigma(\operatorname{Im} w_0) \cdot \nu, \operatorname{Im} u_0 \rangle_{\Gamma_s} = F^{(2)}_\alpha,
\end{aligned} \tag{27}$$

where

$$F^{(2)}_\alpha = \alpha \left(\operatorname{Re} f_0, \operatorname{Re} u_0\right)_{\Omega_f} + \alpha \left(\operatorname{Im} f_0, \operatorname{Im} u_0\right)_{\Omega_f}. \tag{28}$$

Adding the relations (26) and (27) and using the boundary condition $w_1|_{\Gamma_s} = u_0|_{\Gamma_s}$, we have:

Proposition 5 *The fluid component $u_0(\alpha + i\beta)$ of the resolvent relation (17) satisfies the following:*

$$\begin{aligned}
& \alpha \|\nabla \operatorname{Im} u_0\|^2_{\Omega_f} + \alpha \|\nabla \operatorname{Re} u_0\|^2_{\Omega_f} + 2\alpha^2\beta^2 \|\operatorname{Im} w_0\|^2_{\Omega_s} + 2\alpha^2\beta^2 \|\operatorname{Re} w_0\|^2_{\Omega_s} + \alpha^2 \|\operatorname{Re} u_0\|^2_{\Omega_f} + \alpha^2 \|\operatorname{Im} u_0\|^2_{\Omega_f} \\
= \; & \alpha \left(\langle \sigma(\operatorname{Re} w_0) \cdot \nu, \alpha \operatorname{Re} w_0 - \operatorname{Re} g_0 \rangle_{\Gamma_s} + \langle \sigma(\operatorname{Im} w_0) \cdot \nu, \alpha \operatorname{Im} w_0 - \operatorname{Im} g_0 \rangle_{\Gamma_s} \right) + \alpha F^{(1)}_\alpha + F^{(2)}_\alpha,
\end{aligned} \tag{29}$$

where the $F^{(i)}_\alpha$ are as given in (25) and (28).

We proceed now to estimate the first term on the right hand side of (29). To this end, we can refer to the abstract trace result in Théoèrem 1, p. 307 of [2], in order to justify the following $\left[H^{-\frac{1}{2}}(\Gamma_s)\right]^3$-estimate:

$$\begin{aligned}
& \|\sigma(\operatorname{Re} w_0)\cdot\nu\|_{-\frac{1}{2},\Gamma_s} \\
\leq\; & C\left(\|\operatorname{Re} w_0\|_{1,\Omega_s} + \|div\sigma(\operatorname{Re} w_0)\|_{\Omega_s}\right) \\
=\; & C\left(\|\operatorname{Re} w_0\|_{1,\Omega_s} + \left\|(\alpha^2+1)\operatorname{Re} w_0 - 2\alpha\beta\operatorname{Im} w_0 - \beta^2\operatorname{Re} w_0 - \operatorname{Re} g_1 - \alpha\operatorname{Re} g_0 + \beta\operatorname{Im} g_0\right\|_{\Omega_s}\right), \qquad (30)
\end{aligned}$$

where in the last step we have also used the relation (22).

Moreover, we also invoke the following basic result from semigroup theory: *Given Banach space X, if $\mathcal{A}: D(\mathcal{A}) \subset X \to X$ is the infinitesimal generator of a contraction semigroup, then for all $\lambda = \alpha + i\beta$, with $\alpha > 0$, we have the estimate*

$$\|\mathcal{R}(\lambda;\mathcal{A})\|_X \leq \frac{1}{\alpha} \qquad (31)$$

(see; e.g., p. 11 of [14]). Using (30), the Sobolev Trace Theorem, and (31), we have now

$$\begin{aligned}
& \left|\alpha\left\langle\sigma(\operatorname{Re} w_0)\cdot\nu, \alpha\operatorname{Re} w_0 - \operatorname{Re} g_0\right\rangle_{\Gamma_s}\right| \leq \alpha\|\sigma(\operatorname{Re} w_0)\cdot\nu\|_{-\frac{1}{2},\Gamma_s}\|\alpha\operatorname{Re} w_0 - \operatorname{Re} g_0\|_{\frac{1}{2},\Gamma_s} \\
\leq\; & \alpha C\left(\|\operatorname{Re} w_0\|_{1,\Omega_s} + \left\|(\alpha^2+1)\operatorname{Re} w_0 - 2\alpha\beta\operatorname{Im} w_0 - \beta^2\operatorname{Re} w_0 - \operatorname{Re} g_1 - \alpha\operatorname{Re} g_0 + \beta\operatorname{Im} g_0\right\|_{\Omega_s}\right) \\
& \quad \times\|\alpha\operatorname{Re} w_0 - \operatorname{Re} g_0\|_{\frac{1}{2},\Gamma_s} \\
\leq\; & C_\beta\|[f_0,g_0,g_1]\|_{\mathbf{H}}^2. \qquad (32)
\end{aligned}$$

In the exact same way, we have

$$\begin{aligned}
& \|\sigma(\operatorname{Im} w_0)\cdot\nu\|_{-\frac{1}{2},\Gamma_s} \\
\leq\; & C\left(\|\operatorname{Im} w_0\|_{1,\Omega_s} + \|div\sigma(\operatorname{Im} w_0)\|_{\Omega_s}\right) \\
=\; & C\left(\|\operatorname{Im} w_0\|_{1,\Omega_s} + \left\|(\alpha^2+1)\operatorname{Im} w_0 + 2\alpha\beta\operatorname{Re} w_0 - \beta^2\operatorname{Im} w_0 - \operatorname{Im} g_1 - \alpha\operatorname{Im} g_0 - \beta\operatorname{Re} g_0\right\|_{\Omega_s}\right); \qquad (33)
\end{aligned}$$

which along with (31) gives rise to

$$\left|\alpha\left\langle\sigma(\operatorname{Im} w_0)\cdot\nu, \alpha\operatorname{Im} w_0 - \operatorname{Im} g_0\right\rangle_{\Gamma_s}\right| \leq C_\beta\|[f_0,g_0,g_1]\|_{\mathbf{H}}^2. \qquad (34)$$

Combining (29) with (32), (34) and the resolvent estimate (31), we have finally the following estimate for the gradient of the fluid component:

$$\begin{aligned}
& \alpha\|\nabla\operatorname{Im} u_0\|_{\Omega_f}^2 + \alpha\|\nabla\operatorname{Re} u_0\|_{\Omega_f}^2 + 2\alpha^2\beta^2\|\operatorname{Im} w_0\|_{\Omega_s}^2 + 2\alpha^2\beta^2\|\operatorname{Re} w_0\|_{\Omega_s}^2 + \alpha^2\|\operatorname{Re} u_0\|_{\Omega_f}^2 + \alpha^2\|\operatorname{Im} u_0\|_{\Omega_f}^2 \\
\leq\; & C_\beta\|[f_0,g_0,g_1]\|_{\mathbf{H}}^2. \qquad (35)
\end{aligned}$$

This estimate and Poincaré's inequality gives now the following *a priori* bound for the fluid component of (17):

Lemma 6 *Given initial data $[f_0,g_0,g_1] \in \mathbf{H}$, the fluid variable of the quantity in (17), for all $\beta \in \mathbb{R}$, satisfies the estimate*

$$\sqrt{\alpha}\|u_0\|_{\Omega_f} + \sqrt{\alpha}\|\nabla u_0\|_{\Omega_f} \leq C_\beta\|[f_0,g_0,g_1]\|_{\mathbf{H}}, \qquad (36)$$

where C_β is independent of α (small).

Step 2 (a priori bounds in a lower topology).

The estimate (36) can in turn be used to derive the following:

Lemma 7 *For $\alpha > 0$ and $\beta \in \mathbb{R}\diagdown\mathcal{S}$, the elastic component* $[\operatorname{Re} w_0, \operatorname{Im} w_0]$ *of (17) obeys the following estimate:*

$$\left\| \left[\sqrt{\alpha}\operatorname{Re} w_0, \sqrt{\alpha}\operatorname{Im} w_0\right] \right\|_{\Omega_s} \leq C_\beta \left\| [f_0, g_0, g_1] \right\|_{\mathbf{H}}, \tag{37}$$

where the constant C is independent of α (small).

Proof of Lemma 7: We define the elliptic operator $D : [L^2(\Gamma_s)]^3 \to [L^2(\Omega_s)]^3$ by $Df = g$ if and only if g satisfies

$$\begin{aligned} -div\sigma(g) + g &= 0 \ \text{ on } \Omega_s \\ g|_{\Gamma_s} &= f \ \text{ on } \Gamma_s. \end{aligned}$$

By elliptic theory, see e.g., [12], we have $D \in \mathcal{L}([L^2(\Gamma_s)]^3, [H^{\frac{1}{2}}(\Omega_s)]^3)$. Accordingly, we have for any smooth enough function ω on Ω_s,

$$-div\sigma(\omega) + w = \mathring{\mathbf{A}}\omega - \mathring{\mathbf{A}}D(\omega|_{\Gamma_s}), \tag{38}$$

where $\mathring{\mathbf{A}} : D(\mathring{\mathbf{A}}) \subset [L^2(\Omega_s)]^3 \to [L^2(\Omega_s)]^3$ is the elliptic operator defined in (7) (of course, the equality here is taken in $[D(\mathring{\mathbf{A}})]'$).

Applying the expression (38) into (22), we have

$$\left(\beta^2 - \mathring{\mathbf{A}}\right)\operatorname{Re} w_0 = -\mathring{\mathbf{A}}D(\operatorname{Re} w_0|_{\Gamma_s}) + \alpha^2 \operatorname{Re} w_0 - 2\alpha\beta \operatorname{Im} w_0 - (\operatorname{Re} g_1 + \alpha \operatorname{Re} g_0 - \beta \operatorname{Im} g_0).$$

Since $\beta \in \mathbb{R}\diagdown\mathcal{S}$, we can multiply both sides of this relation by $\alpha\mathcal{R}(\beta^2;\mathring{\mathbf{A}})\operatorname{Re} w_0$. Doing so and subsequently integrating, we obtain

$$\begin{aligned} \alpha \left\|\operatorname{Re} w_0\right\|_{\Omega_s}^2 \ = \ & -\alpha\left(\mathcal{R}(\beta^2;\mathring{\mathbf{A}})\mathring{\mathbf{A}}D(\operatorname{Re} w_0|_{\Gamma_s}), \operatorname{Re} w_0\right)_{\Omega_s} + \alpha\left(\alpha^2 \operatorname{Re} w_0 - 2\alpha\beta \operatorname{Im} w_0, \mathcal{R}(\beta^2;\mathring{\mathbf{A}})\operatorname{Re} w_0\right)_{\Omega_s} \\ & -\alpha\left(\operatorname{Re} g_1 + \alpha \operatorname{Re} g_0 - \beta \operatorname{Im} g_0, \mathcal{R}(\beta^2;\mathring{\mathbf{A}})\operatorname{Re} w_0\right)_{\Omega_s}. \end{aligned} \tag{39}$$

To handle the first term on the right hand side of (39), we use again the third relation in (19) to have

$$-\mathcal{R}(\beta^2;\mathring{\mathbf{A}})\mathring{\mathbf{A}}D(\operatorname{Re} w_0|_{\Gamma_s}) = \frac{1}{\beta}\mathcal{R}(\beta^2;\mathring{\mathbf{A}})\mathring{\mathbf{A}}D([\alpha \operatorname{Im} w_0 - \operatorname{Im} w_1 - \operatorname{Im} g_0]_{\Gamma_s}).$$

Using $w_1|_{\Gamma_s} = u_0|_{\Gamma_s}$ and the resolvent estimate (31), we have then

$$\left\|\mathcal{R}(\beta^2;\mathring{\mathbf{A}})\mathring{\mathbf{A}}D(\operatorname{Re} w_0|_{\Gamma_s})\right\|_{\Omega_s} \leq C_\beta\left(\|\nabla u_0\|_{\Omega_f} + \|[f_0, g_0, g_1]\|_{\mathbf{H}}\right). \tag{40}$$

Applying this estimate to the right hand side of (39), followed by use of the estimates (31) and $ab \leq \delta a^2 + C_\delta b^2$, give now

$$\begin{aligned} \alpha \left\|\operatorname{Re} w_0\right\|_{\Omega_s}^2 &\leq \alpha \left\|\mathcal{R}(\beta^2;\mathring{\mathbf{A}})\mathring{\mathbf{A}}D(\operatorname{Re} w_0|_{\Gamma_s})\right\|_{\Omega_s} \|\operatorname{Re} w_0\|_{\Omega_s} + C_\beta \|[f_0, g_0, g_1]\|_{\mathbf{H}}^2 \\ &\leq \alpha C_\beta \left(\|\nabla u_0\|_{\Omega_f} + \|[f_0, g_0, g_1]\|_{\mathbf{H}}\right) \|\operatorname{Re} w_0\|_{\Omega_s} + C_\beta \|[f_0, g_0, g_1]\|_{\mathbf{H}}^2 \\ &\leq \alpha\delta \|\operatorname{Re} w_0\|_{\Omega_s}^2 + C_\delta \left\|\sqrt{\alpha}\nabla u_0\right\|_{\Omega_f}^2 + C_\beta \|[f_0, g_0, g_1]\|_{\mathbf{H}}^2. \end{aligned}$$

Invoking Lemma 6 and taking $\delta < 1$, we have now

$$\alpha \left\|\operatorname{Re} w_0\right\|_{\Omega_s}^2 \leq C_{\beta,\delta} \|[f_0, g_0, g_1]\|_{\mathbf{H}}^2. \tag{41}$$

By the means just employed, we can use (23) to obtain also

$$\alpha \left\| \operatorname{Im} w_0 \right\|_{\Omega_s}^2 \leq C_{\beta,\delta} \left\| [f_0, g_0, g_1] \right\|_{\mathbf{H}}^2 , \tag{42}$$

which concludes the proof of Lemma 7. □

Step 3 (a priori bounds in finite energy topology) Multiplying (22) by $\alpha \operatorname{Re} w_0$, and subsequently integrating gives

$$\begin{aligned}
&\alpha \left(\sigma(\operatorname{Re} w_0), \epsilon(\operatorname{Re} w_0) \right)_{\Omega_s} + \alpha \left(\operatorname{Re} w_0, \operatorname{Re} w_0 \right)_{\Omega_s} = \\
&-\alpha \left\langle \sigma(\operatorname{Re} w_0) \cdot \nu, \operatorname{Re} w_0 \right\rangle_{\Gamma_s} + \alpha\beta^2 \left\| \operatorname{Re} w_0 \right\|_{\Omega_s}^2 \\
&-\alpha \left(\alpha^2 \operatorname{Re} w_0 - 2\alpha\beta \operatorname{Im} w_0 - \operatorname{Re} g_1 - \alpha \operatorname{Re} g_0 + \beta \operatorname{Im} g_0, \operatorname{Re} w_0 \right)_{\Omega_s} .
\end{aligned} \tag{43}$$

Using the third resolvent relation in (19), we have

$$\alpha \left\langle \sigma(\operatorname{Re} w_0) \cdot \nu, \operatorname{Re} w_0 \right\rangle_{\Gamma_s} = \frac{\alpha}{\beta} \left\langle \sigma(\operatorname{Re} w_0) \cdot \nu, \operatorname{Im} u_0 - \alpha \operatorname{Im} w_0 + \operatorname{Im} g_0 \right\rangle_{\Gamma_s} ; \tag{44}$$

applying the estimates in (30), (36) (37) and (31), we have then

$$\begin{aligned}
&\left| \alpha \left\langle \sigma(\operatorname{Re} w_0) \cdot \nu, \operatorname{Re} w_0 \right\rangle_{\Gamma_s} \right| \leq \frac{\alpha}{|\beta|} \left\| \sigma(\operatorname{Re} w_0) \cdot \nu \right\|_{-\frac{1}{2},\Gamma_s} \left\| \operatorname{Im} u_0 - \alpha \operatorname{Im} w_0 + \operatorname{Im} g_0 \right\|_{\frac{1}{2},\Gamma_s} \\
\leq\ & \frac{\alpha}{|\beta|} \left(\left\| \operatorname{Re} w_0 \right\|_{1,\Omega_s} + \left\| (1+\alpha^2) \operatorname{Re} w_0 - 2\alpha\beta \operatorname{Im} w_0 - \beta^2 \operatorname{Re} w_0 - \operatorname{Re} g_1 - \alpha \operatorname{Re} g_0 + \beta \operatorname{Im} g_0 \right\|_{\Omega_s} \right) \\
& \qquad \times \left\| \operatorname{Im} u_0 - \alpha \operatorname{Im} w_0 + \operatorname{Im} g_0 \right\|_{\frac{1}{2},\Gamma_s} \\
\leq\ & \alpha C_\beta \left\| \operatorname{Re} w_0 \right\|_{1,\Omega_s} \left\| \nabla u_0 \right\|_{\Omega_f} + \alpha C_\beta \left\| \operatorname{Re} w_0 \right\|_{\Omega_s} \left\| \nabla u_0 \right\|_{\Omega_f} + C_\beta \left\| [f_0, g_0, g_1] \right\|_{\mathbf{H}}^2 \\
\leq\ & \delta\alpha \left\| \operatorname{Re} w_0 \right\|_{1,\Omega_s}^2 + C_{\beta,\delta} \left\| \sqrt{\alpha} \nabla u_0 \right\|_{\Omega_f}^2 + C_\beta \left\| [f_0, g_0, g_1] \right\|_{\mathbf{H}}^2 .
\end{aligned}$$

Applying this estimate to the right hand side of (43) and subsequently invoking estimate (37) gives now,

$$\alpha \left(\sigma(\operatorname{Re} w_0), \epsilon(\operatorname{Re} w_0) \right)_{\Omega_s} + \alpha \left(\operatorname{Re} w_0, \operatorname{Re} w_0 \right)_{\Omega_s} \leq C_{\beta,\delta} \left\| [f_0, g_0, g_1] \right\|_{\mathbf{H}}^2 . \tag{45}$$

The analogous steps will give us a priori energy bounds for $\operatorname{Im} w_0$. That is, we can multiply both sides of (23) by $\alpha \operatorname{Im} w_0$ to obtain the relation

$$\begin{aligned}
&\alpha \left(\sigma(\operatorname{Im} w_0), \epsilon(\operatorname{Im} w_0) \right)_{\Omega_s} + \alpha \left(\operatorname{Im} w_0, \operatorname{Im} w_0 \right)_{\Omega_s} = \\
&-\alpha \left\langle \sigma(\operatorname{Im} w_0) \cdot \nu, \operatorname{Im} w_0 \right\rangle_{\Gamma_s} + \alpha\beta^2 \left\| \operatorname{Im} w_0 \right\|_{\Omega_s}^2 \\
&-\alpha \left(\alpha^2 \operatorname{Im} w_0 + 2\alpha\beta \operatorname{Im} w_0 - \operatorname{Im} g_1 - \alpha \operatorname{Im} g_0 - \beta \operatorname{Re} g_0, \operatorname{Im} w_0 \right)_{\Omega_s} .
\end{aligned} \tag{46}$$

Subsequently, we can estimate the right hand side of this expression by using the second resolvent relation in (19), and then (33), (36) (37) and (31), so as to have

$$\alpha \left(\sigma(\operatorname{Im} w_0), \epsilon(\operatorname{Im} w_0) \right)_{\Omega_s} + \alpha \left(\operatorname{Im} w_0, \operatorname{Im} w_0 \right)_{\Omega_s} \leq C_{\beta,\delta} \left\| [f_0, g_0, g_1] \right\|_{\mathbf{H}}^2 . \tag{47}$$

Combining (45) and (47), with the equivalent H^1-norm given in (2), now establishes the following:

Proposition 8 *For $\alpha > 0$, the elastic component $[\operatorname{Re} w_0, \operatorname{Im} w_0]$ of (17) obeys the following estimate:*

$$\left\| \left[\sqrt{\alpha} \operatorname{Re} w_0, \sqrt{\alpha} \operatorname{Im} w_0 \right] \right\|_{1,\Omega_s} \leq C_\beta \left\| [f_0, g_0, g_1] \right\|_{\mathbf{H}} , \tag{48}$$

where the constant C is independent of α (small).

Step 4. Conclusion of the proof of Theorem 3.

We first note that the *a priori* bounds and relations we have obtained will imply that $\sqrt{\alpha}\,\mathrm{Re}\,u_0$ and $\sqrt{\alpha}\,\mathrm{Im}\,u_0$ each converge to zero strongly in $\left[H^1(\Omega_f)\right]^3$. In fact, from the *a priori* relation in (29), we have, after using (30), (33) and (31), the estimate

$$\begin{aligned}
&\alpha \left\|\nabla \mathrm{Im}\,u_0\right\|_{\Omega_f}^2 + \alpha \left\|\nabla \mathrm{Re}\,u_0\right\|_{\Omega_f}^2 \\
\leq\ & \alpha \left(\left\|\sigma(\mathrm{Re}\,w_0)\cdot\nu\right\|_{-\frac{1}{2},\Gamma_s} \left\|\alpha\,\mathrm{Re}\,w_0 - \mathrm{Re}\,g_0\right\|_{\frac{1}{2},\Gamma_s} + \left\|\sigma(\mathrm{Im}\,w_0)\cdot\nu\right\|_{-\frac{1}{2},\Gamma_s} \left\|\alpha\,\mathrm{Im}\,w_0 - \mathrm{Im}\,g_0\right\|_{\frac{1}{2},\Gamma_s} \right) \\
&\qquad + \left|\alpha F_\alpha^{(1)} + F_\alpha^{(2)}\right| \\
\leq\ & \alpha C \Big(\left\|\mathrm{Re}\,w_0\right\|_{1,\Omega_s} + \left\|(\alpha^2+1)\,\mathrm{Re}\,w_0 - 2\alpha\beta\,\mathrm{Im}\,w_0 - \beta^2\,\mathrm{Re}\,w_0 - \mathrm{Re}\,g_1 - \alpha\,\mathrm{Re}\,g_0 + \beta\,\mathrm{Im}\,g_0\right\|_{\Omega_s} \\
&\quad + \left\|\mathrm{Im}\,w_0\right\|_{1,\Omega_s} + \left\|(\alpha^2+1)\,\mathrm{Im}\,w_0 + 2\alpha\beta\,\mathrm{Re}\,w_0 - \beta^2\,\mathrm{Im}\,w_0 - \mathrm{Im}\,g_1 - \alpha\,\mathrm{Im}\,g_0 - \beta\,\mathrm{Re}\,g_0\right\|_{\Omega_s} \Big) \\
&\quad \times \left\|[f_0, g_0, g_1]\right\|_{\mathbf{H}} + \left|\alpha F_\alpha^{(1)} + F_\alpha^{(2)}\right|,
\end{aligned}$$

where the $F_\alpha^{(i)}$ are as defined in (25) and (28), respectively. Letting $\alpha \downarrow 0$, we have after using the estimates (36) and (48),

$$\begin{aligned}
\lim_{\alpha\to 0^+} \sqrt{\alpha}\,\mathrm{Re}\,u_0 &= 0 \ \text{ in } \left[H^1(\Omega_f)\right]^3; \\
\lim_{\alpha\to 0^+} \sqrt{\alpha}\,\mathrm{Im}\,u_0 &= 0 \ \text{ in } \left[H^1(\Omega_f)\right]^3
\end{aligned} \tag{49}$$

(here, we also implicitly used Poincaré's inequality).

Next, we use the elliptic operator defined in (7) so as to rewrite the relation in (22) as

$$\left(\beta^2 - \mathring{\mathbf{A}}\right)\mathrm{Re}\,w_0 = -\mathring{\mathbf{A}}D(\mathrm{Re}\,w_0|_{\Gamma_s}) + \alpha^2\,\mathrm{Re}\,w_0 - 2\alpha\beta\,\mathrm{Im}\,w_0 - \mathrm{Re}\,g_1 - \alpha\,\mathrm{Re}\,g_0 + \beta\,\mathrm{Im}\,g_0.$$

Using the fact that $\beta \in \mathbb{R}\diagdown\mathcal{S}$ and the third relation in (19) we have now

$$\begin{aligned}
\sqrt{\alpha}\,\mathrm{Re}\,w_0 &= -\frac{\sqrt{\alpha}}{\beta}\mathcal{R}(\beta^2;\mathring{\mathbf{A}})\mathring{\mathbf{A}}D([\mathrm{Im}\,u_0 - \alpha\,\mathrm{Im}\,w_0 + \mathrm{Im}\,g_0]_{\Gamma_s}) \\
&\quad + \sqrt{\alpha}\mathcal{R}(\beta^2;\mathring{\mathbf{A}})\left[\alpha^2\,\mathrm{Re}\,w_0 - 2\alpha\beta\,\mathrm{Im}\,w_0 - \mathrm{Re}\,g_1 - \alpha\,\mathrm{Re}\,g_0 + \beta\,\mathrm{Im}\,g_0\right].
\end{aligned}$$

To estimate the right hand side of this expression: we use the fact that $D \in \mathcal{L}([L^2(\Gamma_s)]^3, \left[L^2(\Omega_s)\right]^3)$, Sobolev Trace Theory and the estimate (31), so as to have

$$\left\|\sqrt{\alpha}\,\mathrm{Re}\,w_0\right\|_{\Omega_s} \leq C_\beta\sqrt{\alpha}\left\|\nabla u_0\right\|_{1,\Omega_f} + \sqrt{\alpha}C_\beta\left\|[f_0, g_0, g_1]\right\|_{\mathbf{H}}.$$

Taking $\alpha \downarrow 0$ and invoking (49), we obtain

$$\lim_{\alpha\to 0^+}\sqrt{\alpha}\,\mathrm{Re}\,w_0 = 0 \ \ \textit{strongly} \text{ in } \left[L^2(\Omega_s)\right]^3. \tag{50}$$

Using in the same way the relation

$$\left(\beta^2 - \mathring{\mathbf{A}}\right)\mathrm{Im}\,w_0 = -\mathring{\mathbf{A}}D(\mathrm{Im}\,w_0|_{\Gamma_s}) + \alpha^2\,\mathrm{Im}\,w_0 + 2\alpha\beta\,\mathrm{Re}\,w_0 - \mathrm{Im}\,g_1 - \alpha\,\mathrm{Im}\,g_0 - \beta\,\mathrm{Re}\,g_0,$$

from (23), we will have

$$\lim_{\alpha\to 0^+}\sqrt{\alpha}\,\mathrm{Im}\,w_0 = 0 \ \ \textit{strongly} \text{ in } \left[L^2(\Omega_s)\right]^3. \tag{51}$$

Combining (50) and (51) with the second and third relations of (19) give, in turn,

$$\begin{aligned}\lim_{\alpha\to 0^+}\sqrt{\alpha}\,\mathrm{Re}\,w_1 &= \sqrt{\alpha}(-\beta\,\mathrm{Im}\,w_0+\alpha\,\mathrm{Re}\,w_0-\mathrm{Re}\,g_0)=0\ \ \textit{strongly}\ \text{in}\ \left[L^2(\Omega_s)\right]^3;\\ \lim_{\alpha\to 0^+}\sqrt{\alpha}\,\mathrm{Im}\,w_1 &= \sqrt{\alpha}(-\beta\,\mathrm{Re}\,w_0+\alpha\,\mathrm{Im}\,w_0-\mathrm{Im}\,g_0)=0\ \ \textit{strongly}\ \text{in}\ \left[L^2(\Omega_s)\right]^3.\end{aligned} \tag{52}$$

Finally, we appeal to the elastic energy relation (43). Estimating this via (31), (30), and the third relation of (19), we have

$$\begin{aligned}&\alpha\left(\epsilon(\mathrm{Re}\,w_0),\epsilon(\mathrm{Re}\,w_0)\right)+\alpha\left(\mathrm{Re}\,w_0,\mathrm{Re}\,w_0\right)_{\Omega_s}\\ \leq\ &\alpha\left\|\sigma(\mathrm{Re}\,w_0)\cdot\nu\right\|_{-\frac{1}{2},\Gamma_s}\left\|\mathrm{Re}\,w_0\right\|_{\frac{1}{2},\Gamma_s}+\alpha\beta^2\left\|\mathrm{Re}\,w_0\right\|^2_{\Omega_s}+\alpha\left\|\mathrm{Re}\,w_0\right\|_{\Omega_s}\left\|[f_0,g_0,g_1]\right\|_{\mathbf{H}}\\ \leq\ &\sqrt{\alpha}C\left(\left\|\mathrm{Re}\,w_0\right\|_{1,\Omega_s}+\left\|(\alpha^2+1)\,\mathrm{Re}\,w_0-2\alpha\beta\,\mathrm{Im}\,w_0-\beta^2\,\mathrm{Re}\,w_0-\mathrm{Re}\,g_1-\alpha\,\mathrm{Re}\,g_0+\beta\,\mathrm{Im}\,g_0\right\|_{\Omega_s}\right)\\ &\times\sqrt{\alpha}\left\|\frac{1}{\beta}(\mathrm{Im}\,u_0-\alpha\,\mathrm{Im}\,w_0+\mathrm{Im}\,g_0)\right\|_{\frac{1}{2},\Gamma_s}+\alpha\beta^2\left\|\mathrm{Re}\,w_0\right\|^2_{\Omega_s}+\alpha\left\|\mathrm{Re}\,w_0\right\|_{\Omega_s}\left\|[f_0,g_0,g_1]\right\|_{\mathbf{H}}.\end{aligned}$$

Letting α tend to zero on both sides of the inequality, while using (48), (49) and (50) and (31), we have finally

$$\lim_{\alpha\to 0^+}\sqrt{\alpha}\,\mathrm{Re}\,w_0=0\ \ \textit{strongly}\ \text{in}\ \left[H^1(\Omega_s)\right]^3. \tag{53}$$

We can deal in the same way with the elastic energy relation (46, so as to have

$$\lim_{\alpha\to 0^+}\sqrt{\alpha}\,\mathrm{Im}\,w_0=0\ \ \textit{strongly}\ \text{in}\ \left[H^1(\Omega_s)\right]^3. \tag{54}$$

The relations (49), (52), (53) and (54) now establish the limit (18). The proof of Theorem 3 is now complete upon application of Tomilov's resolvent criterion for strong stability; namely, Theorem 2. □

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