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C^*-Extreme Points of the Generalized State Space of a Commutative C^*-Algebra

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C^*-EXTREME POINTS
OF THE GENERALIZED STATE SPACE OF A
COMMUTATIVE C^*-ALGEBRA

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The generalized state space of a commutative C^*-algebra, denoted $S_{\mathcal{H}}(C(X))$, is the set of positive unital maps from $C(X)$ to the algebra $\mathcal{B}(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$. C^*-convexity is one of several non-commutative analogs of convexity which have been discussed in this context. We show that a C^*-extreme point of $S_{\mathcal{H}}(C(X))$ satisfies a certain spectral condition on the operators in the range of an associated measure, which is a positive operator-valued measure on X. We then show that C^*-extreme maps from $C(X)$ into $\mathcal{K}^+$, the C^*-algebra generated by the compact and scalar operators, are multiplicative, generalizing a result of D. Farenick and P. Morenz.

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Contents

Chapter 1

Introduction

1.1 Positive and Completely Positive Maps

Positive maps play a key role in the study of C^*-algebras, and the GNS construction (see, for example, [3, Theorem I.9.6]) which builds a representation of a C^*-algebra from a positive linear functional, is of great importance in the subject. W. Forrest Stinespring's paper of 1955 [18], which generalizes the GNS construction to allow a completely positive unital map to be dilated to a representation and characterizes completely positive maps, is a significant extension of the ideas of Gelfand, Naimark, and Segal. Completely positive maps have been studied extensively since the publication of Stinespring's paper; recently they have received increased attention because of their importance in quantum computing. Section 1.3 presents a very brief explanation of the role of these maps in quantum information theory.

Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the bounded linear operators on $\mathcal{H}$. The set of unital positive linear functionals of a C^*-algebra $\mathcal{A}$ is called the state space of $\mathcal{A}$. The corresponding collection of unital completely positive maps from $\mathcal{A}$ to $\mathcal{B}(\mathcal{H})$ is called the generalized state space. Arveson solved several extremal problems in the space of completely positive maps; in particular, he characterized the extreme points of the generalized state space [1, Theorem 1.4.6]. As a result of this theorem, Arveson is able to demonstrate the existence of non-multiplicative extreme points in the generalized state space of a commutative C^*-algebra. In contrast, in the classical state space of a commutative C^*-algebra, extreme points are always multiplicative (see, for example [15]). Farenick and Morenz [6] examine another class of extreme points in the generalized state space, the C^*-extreme points. Their results show that, when $\mathcal{H}$ is finite dimensional, the C^*-extreme points of the generalized state space of a commutative C^*-algebra are necessarily multiplicative. Thus, in some respects, the C^*-extreme points of the generalized state space mirror the extreme points of the state space. Farenick and Morenz suggest that "the C^*-extreme points are the appropriate extreme points" [5, p. 1727] for the generalized state space. In Section 1.2, we will examine what is known about the relationship between several distinguished classes of points in the generalized state spaces, including pure and multiplicative maps in addition to extreme and C^*-extreme maps. Several other brands of noncommutative convexity have also been studied in the context of the generalized state spaces of a C^*-algebra; two of these will be discussed in Section 1.4.

In Chapter 2, we will examine operator-valued measures. The role of these operator-valued measures in the generalized state spaces of $\mathcal{B}(\mathcal{H})$ is comparable to that of the regular Borel measures in the state space of $C(X)$: each bounded linear map $\phi : C(X) \to \mathcal{B}(\mathcal{H})$ in the generalized state space of $C(X)$ can be expressed as integration against a unital positive operator-valued measure, which is uniquely determined by ϕ. We will exploit several useful properties of these operator-valued measures in the proof of Theorem 2.2.1, which gives a necessary condition for a map to be C^*-extreme in the generalized state space of $C(X)$. Examples will be given which demonstrate that this necessary condition is not sufficient, and that non-multiplicative C^*-extreme points exist when $\mathcal{H}$ has infinite dimension.

Theorem 2.2.1 will enable us to prove that C^*-extreme maps in the generalized state space of $C(X)$ which have their range in the unital C^*-algebra generated by the compact operators must be multiplicative. This result, Theorem 3.1.2, extends a result of Farenick and Morenz [6, Proposition 2.2], and leads to a complete description of the structure of these maps, and the support of the associated operator-valued measures.

1.2 Extreme, C^*-Extreme, Pure and Multiplicative Maps

Henceforth, let $\mathcal{A}$ be an arbitrary unital C^*-algebra, X a compact Hausdorff space, $C(X)$ the commutative unital C^*-algebra of continuous functions on X, and $\mathcal{H}$ a Hilbert space. Recall that we say $a \in \mathcal{A}$ is positive, and write $a \geq 0$, if a is self-adjoint and has non-negative spectrum (we will denote the spectrum of a by $\sigma(a)$). The positive elements play an important role in the study of C^*-algebras; for example, every element of a C^*-algebra can be written as a linear combination of four positive elements (see, for example, [3, p.7 and Corollary 1.4.2]). There is also a partial ordering of the self-adjoint elements of $\mathcal{A}$ determined by the positive elements: for self-adjoint $a, b \in \mathcal{A}$ we say $a \leq b$ if $b - a \geq 0$. Given the importance of positive elements in the study of C^*-algebras, it is not surprising that maps which preserve positivity (and hence order) are of great interest in the field. Denote the bounded linear operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. Given a bounded linear map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and $n \in \mathbb{N}$, define maps $\phi^{(n)} : M_n \to M_n(\mathcal{B}(\mathcal{H}))$ by

$$\phi^{(n)}([a_{ij}]) = [\phi(a_{ij})] \,.$$

Definition 1.2.1. Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be positive. If $\phi^{(n)}$ is positive, we say that ϕ is *n-positive.* If ϕ is n-positive for every n, we say that ϕ is *completely positive.*

It would be difficult to discuss completely positive maps without mentioning Stinespring's Theorem [18, Theorem 1]. This theorem characterizes completely positive maps, and generalizes the GNS construction.

Theorem 1.2.2. *[18, Theorem 1] For a unital C^*-algebra $\mathcal{A}$ and a Hilbert space $\mathcal{H}$, $\phi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ is a completely positive map if and only if there exist a Hilbert space $\mathcal{K}$, a bounded linear map $V : \mathcal{H} \longrightarrow \mathcal{B}(\mathcal{K})$, and a representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{K})$ such that $\phi = V^*\pi V$.*

If the map ϕ in Stinespring's Theorem is unital, then V is an isometry. Also, with the stipulation that $[\pi(\mathcal{A})V(H)] = \mathcal{K}$, the decomposition $\phi = V^*\pi V$ is unique up to unitary equivalence. In this case, we call $\phi = V^*\pi V$ a *minimal Stinespring decomposition.*

Definition 1.2.3. The *generalized state space* of $\mathcal{A}$ is

$$\{\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \,|\, \phi \text{ is completely positive and } \phi(1) = I\},$$

which we will denote by $S_{\mathcal{H}}(\mathcal{A})$, using the notation of [6].

For a commutative C^*-algebra, every positive map is also completely positive [18, Theorem 4] so that $S_{\mathcal{H}}(C(X)) = \{\phi : C(X) \to \mathcal{B}(\mathcal{H}) \,|\, \phi \text{ is positive and } \phi(1) = I\}$. If $\mathcal{H} = \mathbb{C}$, the generalized state space $S_{\mathbb{C}}(\mathcal{A})$ coincides with the classical state space of $\mathcal{A}$. Several special classes of maps in the generalized state space have been studied, including extreme and pure maps [1], C^*-extreme maps [6],[22], [7], and multiplicative maps. We begin with formal definitions of these types of maps, and a discussion of what is known about the relationships between them. To avoid possible confusion, we will refer to extreme points as "linear extreme". The definition of a linear extreme point is no doubt known to the reader, but is included here for comparison to C^*-extreme points, which are less likely to be familiar.

Definition 1.2.4. Let $\phi, \psi_1, \ldots, \psi_n \in S_{\mathcal{H}}(\mathcal{A})$ and $t_1, \ldots, t_n \in (0, 1)$ with $t_1 + \cdots + t_n = 1$. Then

$$\phi(f) = t_1\psi_1(f) + \cdots + t_n\psi_n(f)$$

is a *proper linear convex combination.* If, whenever ϕ is written as a proper linear convex combination of $\psi_1, ..., \psi_n$ it follows that $\phi = \psi_1 = \cdots = \psi_n$, then we call ϕ a *linear extreme point.*

Before defining C^*-extreme points, we need the following:

Definition 1.2.5. We say that $\phi, \psi \in S_{\mathcal{H}}(C(X))$ are *unitarily equivalent*, and write $\phi \sim \psi$, if there is a unitary $u \in \mathcal{B}(\mathcal{H})$ such that $\phi(f) = u^*\psi(f)u$ for every $f \in C(X)$.

Definition 1.2.6. Let $\phi, \psi_1, \ldots \psi_n \in S_{\mathcal{H}}(C(X))$ and $t_1, \ldots t_n \in \mathcal{B}(\mathcal{H})$ be invertible with $t_1^*t_1 + \ldots + t_n^*t_n = I$. Then

$$\phi(f) = t_1^*\psi_1(f)t_1 + \ldots + t_n^*\psi_n(f)t_n$$

is called a *proper C^*-convex combination.* For convenience, we write $\phi = t_1^*\psi_1 t_1 + \cdots + t_n^*\psi_n t_n$. We call a map $\phi \in S_{\mathcal{H}}(C(X))$ *C^*-extreme* if, whenever ϕ is written as a proper C^*-convex combination of $\psi_1, \ldots, \psi_n$, then $\psi_j \sim \phi$ for each $j = 1, \ldots, n$.

Note that in the definition of C^*-extreme points, it is sufficient to take $n = 2$ [22, Proposisition 2.1.2]. The requirement that the C^*-convex coefficients t_j be invertible is analogous to the restriction that the convex coefficients in a proper linear convex combination be non-zero [6, p. 1726]. In addition, observe that for a non-trivial projection $P \in \phi(\mathcal{A})'$

$$\phi = P\phi P + P^\perp \phi P^\perp,$$

so that allowing the C^*-convex coefficients to be non-invertible would mean that no reducible maps could be C^*-extreme. In addition to linear extreme and C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$, we will also be interested in the generalized states which are *pure.*

Definition 1.2.7. [1, p. 160] A completely positive map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is called *pure* if whenever $\psi \leq \phi$, $\psi = t\phi$ for some $t \in [0,1]$. (Note that these maps need not be unital.)

The relationships between linear extreme, C^*-extreme, pure and multiplicative maps in $S_{\mathcal{H}}(\mathcal{A})$ are summarized in Table 1, below. Explanations and examples follow.

$S_{\mathbb{C}}(C(X))$	extreme	$=$	C^*-extreme	$=$	pure	$=$	multiplicative
$S_{\mathbb{C}}(\mathcal{A})$	extreme	$=$	C^*-extreme	$=$	pure	$\supsetneq$	multiplicative
$S_{\mathbb{C}^n}(C(X))$	extreme	$\supsetneq$	C^*-extreme	$=$	multiplicative		
$S_{\mathbb{C}^n}(\mathcal{A})$	extreme	$\supsetneq$	C^*-extreme	$\supsetneq$	pure		
			C^*-extreme	$\supsetneq$	multiplicative		
$\phi : C(X) \to \mathcal{K}^+$	extreme	$\supsetneq$	C^*-extreme	$=$	multiplicative		
$S_{\mathcal{H}}(C(X))$	extreme	$\natural$	C^*-extreme	$\supsetneq$	multiplicative		
$S_{\mathcal{H}}(\mathcal{A})$	extreme	$\natural$	C^*-extreme	$\supsetneq$	pure		
			C^*-extreme	$\supsetneq$	multiplicative		

Table 1

the symbol $\natural$ indicates that the relationship is unknown

Recall that two representations of a C^*-algebra are *disjoint* if they have no unitarily equivalent sub-representations. Similarly, two unital completely positive maps ϕ_1, ϕ_2 is called disjoint if π_1, π_2 are disjoint representations, where $\phi_j = v_j^* \pi_j v_j$ is a minimal Stinespring representation for each j[6, p. 1731]. Farenick and Morenz have proven that for any choice of $\mathcal{A}$ or $\mathcal{H}$, every $*$-representation is both linear extreme and C^*-extreme [6, Proposition 1.2]. In addition, Zhou [22, Proposition 3.4.1] has shown that, regardless of the choice of $\mathcal{H}$ or $\mathcal{A}$, any direct sum of disjoint pure maps is C^*-extreme, so that, in particular, any pure map is C^*-extreme. Note that, by a theorem of Arveson [1, Corollary 1.4.3], pure generalized states are exactly those elements of $S_{\mathcal{H}}(\mathcal{A})$ whose minimal Stinespring dilation is an irreducible representation. As all irreducible $*$-representations of a commutative C^*-algebra are one dimensional, there are no pure maps in $S_{\mathcal{H}}(C(X))$ when $\dim \mathcal{H} > 2$.

Now consider the case $\mathcal{H} = \mathbb{C}$ (the first row of Table 1). If $\phi \in S_{\mathbb{C}}(C(X))$, then the linear extreme, pure, and multiplicative maps coincide [15, Lemma 3.4.6, Theorem 3.4.7]. For any C^*-algebra $\mathcal{A}$, when $\mathcal{H} = \mathbb{C}$, the coefficients of a C^*-convex combination will be scalars, so that linear and C^*-convex combinations are the same. Also, if maps in $S_{\mathbb{C}}(\mathcal{A})$ are unitarily equivalent, they must be equal (as the only unitary operators are multiplication by a scalar of modulus one). Thus, in $S_{\mathbb{C}}(\mathcal{A})$ the linear extreme and C^*-extreme maps are identical. In addition, Farenick and Morenz [6, Theorem 2.1] prove that when $\mathcal{H}$ is finite dimensional, any C^*-extreme map is a direct sum of pure maps. Thus, when $\mathcal{H} = \mathbb{C}$, C^*-extreme maps are also pure. To see that the multiplicative maps of $S_{\mathbb{C}}(\mathcal{A})$ are properly contained in the pure maps, we consider the following example.

Example 1.2.8. Define a map $\phi : M_2 \to \mathbb{C}$ by

$$\phi(a_{ij}) = a_{11}.$$

It is evident that ϕ is not multiplicative. The identity representation of M_2 is a minimal Stinespring dilation of ϕ, and is irreducible, thus ϕ is pure [1, Corollary 1.4.3].

We have now established all the containments and equalities in the first row of Table 1. If $\mathcal{H}$ is finite dimensional (middle row of Table 1), or if $\phi : \mathcal{A} \to \mathcal{K}^+$ (bottom row), Farenick and Morenz have shown that every C^*-extreme point is also linear extreme [6, Proposition 1.1]. In this case, however, there are linear extreme points of the generalized state space which are not C^*-extreme as we will see in Example 1.2.11. A theorem of Arveson characterizes the linear extreme maps in $S_{\mathbb{C}^n}(C(X))$; before stating the theorem we give the following definition:

Definition 1.2.9. [1, p. 165] Let $M_1, \dots, M_n$ be subspaces of $\mathcal{H}$. If whenever $T_1, \dots, T_n$ are operators for which M_j contains the ranges of both T_j and T_j^* for each j and

$$T_1 + \cdots + T_n = 0$$

it follows that

$$T_1 = \cdots = T_n = 0$$

then we call the subspaces $M_1, \dots, M_n$ *weakly independent*.

Arveson offers an example of weakly independent spaces: if x, y are linearly independent vectors, and $z = x + y$, then the 1-dimensional spaces $\mathrm{sp}\{x\}$, $\mathrm{sp}\{y\}$, and $\mathrm{sp}\{z\}$ are weakly independent, although they are not linearly independent. In fact, the ranges of the rank-1 projections xx^*, yy^* and zz^* will be weakly independent as long as the vectors x, y, and z are pairwise linearly independent. (Here, the rank one operator xx^* is given by $xx^*(v) = \langle v, x\rangle x$, for any vector $v \in \mathcal{H}$.)

Theorem 1.2.10. *[1, Theorem 1.4.10] A map ϕ in $\{\phi : C(X) \to M_n \,|\, \phi(1) = S\}$ is linear extreme if and only if*

$$\phi(f) = f(x_1)P_1 + \cdots + f(x_k)P_k,$$

where $x_1, \dots, x_k$ are distinct points of X and $P_1, \dots, P_n$ are positive operators with weakly independent ranges and $P_1 + \cdots + P_k = S$.

The following example demonstrates the construction of an extreme point of $S_{\mathbb{C}^n}(C(X))$ which is not C^*-extreme.

Example 1.2.11. Let $X = \{x_1, x_2, x_3\}$, and $\{e_1, e_2\}$ be the standard basis for $\mathbb{C}^2$ and $e_3 = \frac{1}{\sqrt{2}}(e_1 + e_2)$. Then the rank one operators $e_1e_1^*, e_2e_2^*$, and $e_3e_3^*$ have weakly independent ranges. Define a map

$$\psi : C(X) \to M_2$$

by

$$\psi(\delta_j) = e_j e_j^*$$

where $\delta_j = \chi_{\{x_j\}}$ is the characteristic function of the singleton set $\{x_j\}$. Theorem 1.2.10 guarantees that the map ψ is an extreme point in $\{\phi : C(X) \to M_2 \,|\, \phi(1) = S\}$, where $S := e_1e_1^* + e_2e_2^* + e_3e_3^*$. However, ψ is not unital. Note that

$$S = I + e_3e_3^*,$$

so that S is an invertible operator. Thus we may define a map ϕ by

$$\phi(f) = S^{-\frac{1}{2}}\psi(f)S^{-\frac{1}{2}}.$$

Setting

$$P_j = S^{-\frac{1}{2}}(e_je_j^*)S^{-\frac{1}{2}} = (S^{-\frac{1}{2}}e_j)(S^{-\frac{1}{2}}e_j)^*$$

we see that each P_j is positive and that

$$P_1 + P_2 + P_3 = I$$

so that $\phi \in S_{\mathbb{C}^2}(C(X))$, i.e., ϕ is unital. Since e_1, e_2, and e_3 are pairwise linearly independent and S is invertible, the vectors $S^{-\frac{1}{2}}e_1, S^{-\frac{1}{2}}e_2$, and $S^{-\frac{1}{2}}$ are also pairwise linearly independent. The ranges of P_1, P_2, and P_3 are therefore weakly independent, so that ϕ is an extreme point of $S_{\mathbb{C}^2}(M_2)$. However, ϕ is not multiplicative, since P_1, P_2, and P_3 are not orthogonal projections.

Farenick and Morenz have shown that a map $\phi \in S_{\mathbb{C}^n}(C(X))$ is C^*-extreme if and only if it is multiplicative [6, Proposition 2.2], or equivalently, if the operators $P_1, \ldots, P_n$ of Theorem 1.2.10 are projections with orthogonal ranges. We will prove, in Theorem 3.1.2, that C^*-extreme maps with range in $\mathcal{K}^+$, the algebra generated by the compact operators and I, must also be multiplicative, thus extending this result of Farenick and Morenz. Also, Farenick and Zhou have given a structure theorem for C^*-extreme maps in $S_{\mathbb{C}^n}(\mathcal{A})$ [7, Theorem 2.1], which is stated below as Theorem 1.2.12. Example 3.2.1, due to Zhou, exhibits a map in $S_{\mathbb{C}^2}(M_2 \oplus M_1)$ which is C^*-extreme but not multiplicative. Farenick and Zhou define a *nested sequence of compressions* of a representation π to be a sequence of unital completely positive maps ϕ_j such that $\phi_1 = w_0^* \pi w_0$ is a minimal Stinespring decomposition, and for each j, $\phi_{j+1} = w_j^* \phi_j w_j$, where each w_j is an isometry.

Theorem 1.2.12. *[7, Theorem 2.1] Let $\mathcal{H}$ be a finite dimensional Hilbert space and $\mathcal{A}$ a unital C^*-algebra. Then $\phi \in S_{\mathcal{H}}(A)$ is C^*-extreme if and only if there are pairwise non-equivalent irreducible representations $\pi_1, \ldots, \pi_k$ of $\mathcal{A}$ and nested sequences of compressions $\phi_j^{\pi_i}$ of each representation π_i so that ϕ is unitarily equivalent to the direct sum*

$$\sum_{i=1}^{k}{}^{\oplus} \left(\sum_{j=1}^{n_i}{}^{\oplus} \phi_j^{\pi_i} \right)$$

of pure unital completely positive maps $\phi_j^{\pi_i}$.

When $\mathcal{H}$ has infinite dimension (final row of Table 1), C^*-extreme maps in $S_{\mathcal{H}}(C(X))$ need not be multiplicative; Examples 2.3.5 and 2.3.6, due to Farenick and Morenz, of non-multiplicative C^*-extreme maps will be given in Chapter 2. Lastly, in the final row of Table 1, the following question remains unanswered; it was interest in this question which prompted the author to pursue the study of C^*-convexity.

Question 1.2.13. When $\mathcal{H}$ is infinite dimensional and ϕ is a C^*-extreme point of $S_{\mathcal{H}}(\mathcal{A})$, must ϕ also be linear extreme?

1.3 The Connection to Quantum Computing

Completely positive maps play an important part in the area of quantum information theory. The (very brief) introductory material given here is taken largely from [16]. In quantum computing, a closed quantum system is associated to a Hilbert space $\mathcal{H}$, and the state of the system at a given time is a unit vector $v \in \mathcal{H}$, or the rank one projection $vv^* \in \mathcal{B}(\mathcal{H})$. If we do not know the state of the system completely, but only its probability distribution, then the operator

$$P = \sum p(v)vv^*,$$

where $p(v)$ is the probability the system is in state v, and the sum ranges over all states $v \in \mathcal{H}$, represents the state of the system. The operator P is positive with trace one; i.e, P is a density operator. A map

$$\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$$

given by

$$\pi(A) = U^*AU,$$

where $U \in \mathcal{B}(\mathcal{H})$ is a unitary operator, describes the evolution of a closed quantum system. The map π is positive and trace-preserving, so that density operators will evolve to density operators.

An open quantum system, associated with a Hilbert space $\mathcal{H}$ is in interaction with an environment, associated with a larger Hilbert space $\mathcal{K} \supseteq \mathcal{H}$. Evolution in the open system can be viewed as a compression of the unitary evolution on the closed system containing it. Thus, evolution on the open system is described by a completely positive trace-preserving map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, where

$$\phi = V^* \pi V$$

for a unitary transformation $\pi : \mathcal{B}(\mathcal{K}) \to \mathcal{B}(\mathcal{K})$ and a bounded linear map $V : \mathcal{H} \to \mathcal{K}$.

Although current attention is focused on finite dimensional Hilbert spaces, future work is expected to include infinite dimensional spaces. It is not clear what role C^*-extreme maps might play in quantum computing. The natural topology on $S_{\mathcal{H}}(\mathcal{A})$ is the bounded weak topology. A bounded net $\phi_\lambda \in S_{\mathcal{H}}(\mathcal{A})$ converges to ϕ in the bounded weak topology if and only if the corresponding net $\phi_\lambda(a)$ converges in the weak operator topology to $\phi(a)$, for every $a \in \mathcal{A}$. When $\mathcal{H}$ is finite dimensional, Farenick and Morenz have proved a Krein-Milman type theorem [6, Theorem 3.5], showing that the C^*-convex hull of the C^*-extreme points of $S_{\mathcal{H}}(\mathcal{A})$ is dense, with respect to the bounded-weak or BW-topology, in $S_{\mathcal{H}}(\mathcal{A})$. So any $\phi \in S_{\mathcal{H}}(\mathcal{A})$ can be written as a C^*-convex combination of C^*-extreme maps. Thus C^*-extreme maps in $S_{\mathcal{H}}(\mathcal{A})$ may reasonably be viewed as the "building blocks" for unital completely positive maps. It is not known if this result holds in the infinite dimensional case.

The new results presented in this thesis pertain to positive unital maps $\phi : C(X) \to \mathcal{B}(\mathcal{H})$, so it is not immediately apparent if there will be any possibility of application of this work to the field of quantum computing. Given a quantum channel $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, we could consider the restriction of ϕ to any commutative C^*-subalgebra of $\mathcal{B}(\mathcal{H})$. Applying the results of Chapter 2 or 3 to the restricted map might determine whether the restricted map is C^*-extreme, but will not determine whether the original map was C^*-extreme.

1.4 Non-Commutative Convexity

In addition to C^*-convexity, two other types of non-commutative convexity appear in the literature. CP-convexity has been studied by Fujimoto [9, 8, 10, 11, 12]; matrix convexity was introduced by Wittstock [20, 21] and further pursued by Effros and Winkler [4] and

Farenick [5]. This section gives a brief discussion of both of these versions of non-commutative convexity, and discusses their relationship to C^*-convexity.

A map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is *completely contractive* if $\phi^{(n)}$ is contractive for every $n \in \mathbb{N}$. Equivalently, ϕ is completely contractive if

$$\|\phi\|_{cb} := \sup_{n\in\mathbb{N}} \|\phi^{(n)}\| \leq 1.$$

In [12], Fujimoto defines a CP-state on a C^*-algebra $\mathcal{A}$ to be any completely contractive map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and denotes the collection

$$\{\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H}) \,|\, \phi \text{ is completely positive and } \|\phi\|_{cb} \leq 1\}$$

by $Q_{\mathcal{H}}(\mathcal{A})$. Note that maps in $Q_{\mathcal{H}}(\mathcal{A})$ need not be unital. A CP-convex combination for a map $\phi \in Q_{\mathcal{H}}(\mathcal{A})$ is given by

$$\phi = \sum t_i^* \psi_i t_i,$$

where $\psi_i \in Q_{\mathcal{H}}(\mathcal{A})$, each t_i is an operator (which need not be invertible) in $\mathcal{B}(\mathcal{H})$ and $\sum t_i^* t_i \leq I$. The sum, which may be infinite, will converge in the bounded-strong or BS-topology. (In the BS-topology, a bounded net (ϕ_α) converges if and only if the net $(\phi_\alpha(a))$ converges strongly in $\mathcal{B}(\mathcal{H})$ for each $a \in \mathcal{A}$.) Clearly, a C^*-convex combination of maps $\psi_i \in S_{\mathcal{H}}(\mathcal{A})$ will also be a CP-convex combination in $Q_{\mathcal{H}}(\mathcal{A})$. Fujimoto denotes the CP-extreme maps in $Q_{\mathcal{H}}(\mathcal{A})$ by $D_{\mathcal{H}}(\mathcal{A})$ and has demonstrated that any CP-extreme map of $Q_{\mathcal{H}}(\mathcal{A})$ is either a pure approximately unital completely positive map (i.e., a map ϕ with minimal Stinespring decomposition $\phi = V^*\pi V$ where $V^*V = I$) or an irreducible representation of $\mathcal{A}$ on $\mathcal{H}$; note that when $\mathcal{H}$ has infinite dimension, only the latter alternative is possible. Since there are potentially many more CP-convex combinations representing a given map ϕ than there are C^*-extreme combinations, a C^*-extreme map $\phi \in S_{\mathcal{H}}(\mathcal{A})$ need not be CP-extreme in $Q_{\mathcal{H}}(\mathcal{A})$. However, Farenick and Morenz have shown that all pure and multiplicative maps in $S_{\mathcal{H}}(\mathcal{A})$ are C^*-extreme [6, Proposition 1.2], all CP-extreme maps will be C^*-extreme.

Another non-commutative analog of convexity is matrix convexity, introduced by Wittstock [20, 21] and further developed by Effros and Winkler [4] and Webster and Winkler [19].

Definition 1.4.1. A *matrix convex set* [21, 1.1] $K = (K_n)_{n\in\mathbb{N}}$ in a vector space V is a collection of non-empty convex sets where each $K_n \subseteq M_n(V)$ satisfies the following two conditions:

1. for any matrix $\alpha \in M_{r,n}$ with $\alpha^*\alpha = 1$, we have $\alpha^* K_r \alpha \subseteq K_n$
2. for any $m, n \in \mathbb{N}$, $K_m \oplus K_n \subseteq K_{m+n}$.

Wittstock uses matrix convexity to define *matrix sublinear functionals* [20, 2.1.2], analogous to sublinear (real-valued) functionals, and prove a matricial Hahn-Banach theorem [20, Theorem 2.3.1]. Effros and Winkler prove a second Hahn-Banach type theorem [4, Theorem 6.1] which uses *matrix gauges*, rather than the matrix sublinear functionals employed by Wittstock. Subsequent work by Webster and Winkler [19] includes the following definition of extreme points appropriate to this type of convexity.

Definition 1.4.2. [19, Definition 2.1] Let V be a vector space and $\mathbf{K} = (K_n)_{n\in\mathbb{N}}$ a matrix convex set. Then $v \in K_n$ is a *matrix extreme point* if whenever

$$v = \sum_{i=1}^{k} \gamma_i^* v_i \gamma_i$$

with $\gamma_i \in M_{n_i,n}$ right invertible (i.e., there exists $\alpha \in M_{n,n_i}$ with $\gamma\alpha = I_{n_i}$), $v_i \in K_{n_i}$, and $\sum \gamma_i^* \gamma_i = I_n$, then each $n_i = n$ and $v_i \sim v$.

Webster and Winkler observe that if n is fixed in this definition (i.e., when $n_i = n$ and $K_i = K_n$ for each i), we have exactly the definition of a C^*-extreme point in K_n; thus matrix extreme points are always C^*-extreme. If $v \in K_1$ then v is matrix extreme if and only if v is matrix extreme, so that, if K_1 is compact, there are always matrix extreme points in $\mathbf{K}$. However, Webster and Winkler give an example [19, Example 2.2] in which there are no matrix extreme points in K_n when $n > 1$. They then go on to prove a Krein-Milman Theorem for compact matrix convex sets [19, Theorem 4.3].

Effros and Winkler include in their examples of matrix convex sets the collection $\mathcal{M}(V,W) = (\mathcal{M}_n(V,W))_{n\in\mathbb{N}}$ where V, W are operator systems and

$$\mathcal{M}_n(V,W) = \{\phi : V \to M_n(W) \,|\, \phi \text{ is completely positive and unital}\}.$$

As pointed out in [7], if we take $V = \mathcal{A}$ to be a C^*-algebra and $W = \mathbb{C}$, we have

$$\mathcal{M}_n(\mathcal{A},\mathbb{C}) = S_{\mathbb{C}^n}(\mathcal{A}).$$

So the collection $S(\mathcal{A}) = (S_{\mathbb{C}^n}(\mathcal{A}))_{n\in\mathbb{N}}$ is a matrix convex set. Webster and Winkler show that the matrix extreme points of $S_{\mathcal{C}^n}(\mathcal{A})$ are exactly the pure maps in $S_{\mathbb{C}^n}(\mathcal{A})$ [19, Example 2.3]. Thus, if $\mathcal{A} = C(X)$ is a commutative C^*-algebra, $S_{\mathbb{C}^n}(C(X))$ will contain no matrix extreme points for $n > 1$. In contrast, Farenick and Morenz have shown that these generalized state spaces contain many C^*-extreme points. In fact, they prove a Krein-Milman type theorem for C^*-extreme points in the generalized state spaces $S_{\mathcal{C}^n}(\mathcal{A})$: $S_{\mathcal{C}^n}(\mathcal{A})$ is the closed C^*-convex hull of its C^*-extreme points (where the closure is taken with respect to the bounded weak topology) [6, Theorem 3.5]. These results of Webster and Winkler and Farenick on matrix extreme states give information about the C^*-extreme maps in the generalized state spaces $S_{\mathbb{C}^n}(\mathcal{A})$, whereas the major results of this thesis concern C^*-extreme maps in $S_{\mathcal{H}}(C(X))$, where the dimension of $\mathcal{H}$ is infinite.

Chapter 2

C^*-Extreme Maps on $C(X)$

2.1 Operator-Valued Measures

We begin with a discussion of $\mathcal{B}(\mathcal{H})$-valued measures, which closely follows the development given in Paulsen [17]. These operator-valued measures play a key role in the proof of Theorem 2.2.1, which is the main result of this chapter. Given a bounded linear map $\phi : C(X) \to \mathcal{B}(\mathcal{H})$ and vectors $x, y \in \mathcal{H}$, the bounded linear functional

$$f \mapsto \langle \phi(f)x, y \rangle$$

corresponds, via the Riesz Representation Theorem, to a unique regular complex Borel measure $\mu_{x,y}$ on X such that

$$\int f d\mu_{x,y} := \langle \phi(f)x, y \rangle \text{ for any } f \in C(X).$$

Denote the σ-algebra of Borel sets of X by $\mathcal{S}$. For a set $B \in \mathcal{S}$, the sesquilinear form

$$(x, y) \mapsto \mu_{x,y}(B)$$

then determines an operator $\mu(B)$. Thus we obtain an *operator-valued measure* $\mu : \mathcal{S} \longrightarrow \mathcal{B}(\mathcal{H})$ which is:

1. *weakly countably additive*, i. e., if $\{B_i\}_{i=1}^{\infty} \subseteq \mathcal{S}$ are pairwise disjoint, and $B = \bigcup_{i=1}^{\infty} B_i$ then
$$\langle \mu(B)x, y\rangle = \sum_{i=1}^{\infty} \langle \mu(B_i)x, y\rangle \quad \text{for every } x, y \in \mathcal{H}.$$

2. *bounded*, i. e., $\|\mu\| := \sup\{\|\mu(B)\| : B \in \mathcal{S}\} < \infty$.

3. *regular*, i. e., for each pair of vectors x and y in $\mathcal{H}$, the complex measure $\mu_{x,y}$ is regular.

Furthermore, this process works in reverse: given a regular, bounded, and weakly countably additive operator-valued measure $\mu : \mathcal{S} \longrightarrow \mathcal{B}(\mathcal{H})$, define Borel measures

$$\mu_{x,y}(B) := \langle \mu(B)x, y\rangle$$

for each $x, y \in \mathcal{H}$. Then the operator $\phi(f)$ is uniquely determined by the equations

$$\langle \phi(f)x, y\rangle := \int f d\mu_{x,y};$$

the map $\phi : C(X) \longrightarrow \mathcal{B}(\mathcal{H})$ is then seen to be bounded and linear. This construction shows that each operator valued-measure gives rise to a unique bounded linear map, and vice-versa. The following proposition summarizes properties shared by operator valued-measures and their associated linear maps. A proof of (2) is given; the proof of (1) is similar, and is omitted. The proof of (3), also omitted, is similar to the well-known proof of (4), which is, of course, the Spectral Theorem.

Proposition 2.1.1. *[17, Proposition 4.5] Given an operator-valued measure μ and its associated linear map ϕ,* *1. ϕ is self-adjoint if and only if μ is self-adjoint,*

2. ϕ is positive if and only if μ is positive,

3. ϕ is a homomorphism if and only if $\mu(B_1 \cap B_2) = \mu(B_1)\mu(B_2)$ for all $B_1, B_2 \in \mathcal{S}$,

4. ϕ is a $*$-homomorphism if and only if μ is spectral (i.e., projection-valued).

Proof. Assume that ϕ is positive linear map. For any positive function $f \in C(X)$,

$$\phi(f) \geq 0.$$

Thus, for any vector $x \in \mathcal{H}$

$$\langle \phi(f)x, x\rangle \geq 0.$$

Equivalently,

$$\int f d\mu_{x,x} \geq 0.$$

As this holds for any positive function $f \in C(X)$, $\mu_{x,x}$ is a positive measure. So for any Borel set $B \subseteq X$,

$$\mu_{x,x}(B) \geq 0.$$

Equivalently,

$$\langle \mu(B)x, x\rangle \geq 0.$$

Hence $\mu(B) \geq 0$ and μ is a positive operator-valued measure.

Conversely, assume that μ is a positive operator-valued measure. Fix $x \in \mathcal{H}$; for any $B \in \mathcal{S}$

$$\langle \mu(B)x, x\rangle = \int_X \chi_{_B} d\mu_{x,x} \geq 0.$$

Thus $\mu_{x,x}$ is a positive (real-valued) measure. If f is a positive non-zero function in $C(X)$, then there is some open set $G \subseteq \text{supp } f$ such that $f|_G \geq \frac{1}{2}\|f\|$. Then

$$\int_X f d\mu_{x,x} \geq \frac{1}{2}\|f\| \int_X \chi_G d\mu_{x,x}.$$

Equivalently

$$\langle \phi(f)x, x\rangle \geq \frac{1}{2}\|f\|\langle \mu_\phi(G)x, x\rangle,$$

so that

$$\phi(f) \geq \frac{1}{2} \parallel f \parallel \mu(G) \geq 0.$$

Thus ϕ is a positive linear map. $\square$

Remark 2.1.2. Let $\mathfrak{B}_X = \{f : X \to \mathbb{C} \mid f \text{ is a bounded Borel measurable function}\}$. If $\phi : C(X) \to \mathcal{B}(\mathcal{H})$ is a positive map, we may use the corresponding positive operator-valued measure to extend ϕ to a map $\tilde{\phi} : \mathfrak{B}_X \to \mathcal{B}(\mathcal{H})$ by defining

$$\tilde{\phi}(f) = \int_X f d\mu_\phi,$$

for every $f \in \mathfrak{B}_X$. The measure μ_ϕ may then be viewed as the restriction of $\tilde{\phi}$ to the characteristic functions of Borel sets. For simplicity, we will simply write ϕ, rather than $\tilde{\phi}$, and use the notations $\mu_\phi(F)$ and $\phi(\chi_F)$ interchangeably.

Proposition 2.1.3. *Let $\phi \in S_{\mathcal{H}}(C(X))$, μ_ϕ the associated operator valued map, and F a Borel set of X. If $\mu_\phi(F)$ is a projection, $\mu_\phi(F) \in \phi(C(X))'$.*

Proof. Suppose that $\mu_\phi(F)$ is a projection, and choose $f \in C(X)$ with $0 \leq f \leq 1$. Write

$$f = \chi_F f + (1 - \chi_F)f.$$

Then $\phi(\chi_F f) \leq \mu_\phi(F)$, so these operators commute. Similarly, $\phi((1-\chi_F)f) \leq \mu_\phi(X \setminus F) =$

$I - \mu_\phi(F)$, so that $\phi((1-\chi_F)f)$ also commutes with $\mu_\phi(F)$. Therefore $\phi(f)$ commutes with $\mu_\phi(F)$. If f is any continuous function, we can express f as a linear combination of positive functions with ranges in $[0,1]$. Thus $\phi(f)$ will commute with $\mu_\phi(F)$. $\square$

Lemma 2.1.4. *Positive maps $\phi, \psi : C(X) \to \mathcal{B}(\mathcal{H})$ are unitarily equivalent if and only if the corresponding positive operator-valued measures μ_ϕ and μ_ψ are also unitarily equivalent.*

Proof. Suppose that

$$\psi = u^*\phi u,$$

for a unitary operator $u \in \mathcal{B}(\mathcal{H})$. Fix vectors $x, y \in \mathcal{H}$; for any $f \in C(X)$

$$\langle \psi(f)x, y\rangle = \langle \phi(f)ux, uy\rangle.$$

Equivalently,

$$\int f d\mu_{\psi_{x,y}} = \int f d\mu_{\phi_{ux,uy}}.$$

Thus

$$\mu_{\psi_{x,y}} = \mu_{\phi_{ux,uy}}.$$

It follows that for any Borel set $B \subseteq X$

$$\langle \mu_\psi(B)x, y\rangle = \langle \mu_\phi(B)ux, uy\rangle.$$

As this holds for any choice of $x, y \in \mathcal{H}$,

$$\mu_\psi \sim \mu_\phi.$$

Conversely, let μ_ψ, μ_ϕ be two positive operator-valued measures $\mathcal{S} \to \mathcal{B}(\mathcal{H})$, and ϕ and ψ the corresponding positive maps. Suppose that

$$\mu_\psi = u^* \mu_\phi u$$

for a unitary operator $u \in \mathcal{B}(\mathcal{H})$. Then for any Borel set $B \in \mathcal{S}$

$$\langle \mu_\psi(B)x, y \rangle = \langle \mu_\phi(B)ux, uy \rangle,$$

or equivalently,

$$\int \chi_B d\mu_{\psi_{x,y}} = \int \chi_B d\mu_{\phi_{ux,uy}}.$$

If $f \in C(X)$, f is the pointwise limit of a bounded sequence of simple functions f_n, so

$$\begin{aligned}
\langle \psi(f)x, y \rangle &= \int f d\mu_{\psi_{x,y}} \\
&= \lim_{n\to\infty} \int f_n d\mu_{\psi_{x,y}} \\
&= \lim_{n\to\infty} \int f_n d\mu_{\phi_{ux,uy}} \\
&= \int f d\mu_{\phi_{ux,uy}} \\
&= \langle u^* \phi(f)ux, y \rangle.
\end{aligned}$$

This is true for any choice of $x, y \in \mathcal{H}$; thus

$$\psi = u^* \phi u.$$

□

Let us say that a unital positive operator-valued measure $\mu_\phi : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ is C^*-extreme in the set of unital positive $\mathcal{B}(\mathcal{H})$-valued measures on X if, whenever μ_ϕ is written

$$\mu_\phi = t_1^*\mu_1 t_1 + ... + t_n^*\mu_n t_n,$$

where each $t_j \in \mathcal{B}(\mathcal{H})$ is invertible, $\sum_{j=1}^n t_j^* t_j = I$ and each μ_j is a unital positive $\mathcal{B}(\mathcal{H})$-valued measure, then $\mu_j \sim \mu_\phi$ for each $j = 1, ...n$. Lemma 2.1.4 now yields the following proposition, which is vital in the proofs of Theorems 2.2.1 and 3.1.2.

Proposition 2.1.5. *A map $\phi \in S_H(C(X))$ is C^*-extreme if and only if the associated operator-valued measure μ_ϕ is C^*-extreme.*

Proof. Assume that $\phi \in S_{\mathcal{H}}(\mathcal{A})$ is C^*-extreme, let μ_ϕ be the associated positive operator-valued measure, and suppose that

$$\mu_\phi = t_1^*\mu_1 t_1 + \ldots + t_n^*\mu_n t_n$$

expresses μ_ϕ as a proper C^*-convex combination of unital positive $\mathcal{B}(\mathcal{H})$-valued measures. For each j, let ψ_j be the positive map given by μ_j. Then

$$\phi = t_1^*\psi_1 t_1 + \ldots + t_n^*\psi_n t_n$$

expresses ϕ as a proper C^*-convex combination of $\psi_1, \ldots, \psi_n$. Therefore, for each j there is a unitary operator $u_j \in \mathcal{B}(\mathcal{H})$ such that

$$\phi = u_j^*\psi_j u_j.$$

By Lemma 2.1.4, $\mu_\phi = u^*\mu_j u$. Thus μ_ϕ is seen to be C^*-extreme. Conversely, if μ_ϕ is C^*-extreme, and

$$\phi = t_1^*\psi_1 t_1 + \ldots + t_n^*\psi_n t_n$$

is a proper C^*-convex combination, we can reverse the above process, letting μ_j be the unital positive operator-valued measure determined by ψ_j. In this case, there are unitary operators u_j such that

$$\mu_\phi = u_j^* \mu_j u_j.$$

Once again, Lemma 2.1.4 implies that the maps ϕ and ψ_j will also be unitarily equivalent. $\square$

A final result about positive operator-valued measures relates the ranges of a unital positive map ϕ the corresponding positive operator-valued measure: the range of μ_ϕ is contained in the weak operator topology closure of $\phi(C(X))$, which will be denoted by WOT-cl $\phi(C(X))$. The proof of this fact requires some care, because while $\phi(C(X))$ is an operator space, it is not generally an algebra.

Proposition 2.1.6. *Let $\phi : C(X) \to \mathcal{B}(\mathcal{H})$ be a positive bounded linear map, and μ_ϕ the associated operator-valued measure. Then for each Borel set $F \subseteq X$, $\mu_\phi(F) \in$ WOT-cl $\phi(C(X))$.*

Proof. Let $G \subseteq X$ be an open set. Then a basic WOT-open set in $\mathcal{B}(\mathcal{H})$ centered at $\phi(\chi_G)$ has the form:

$$\begin{aligned} \mathcal{O} &= \mathcal{O}_{\varepsilon, x_1, \ldots, x_n, y_1, \ldots, y_n}(\phi(\chi_G)) \\ &= \{T \in \mathcal{B}(\mathcal{H}) \,:\, |\langle (T - \phi(\chi_G)) x_j, y_j \rangle| < \varepsilon \text{ for } j = 1 \ldots n\}, \end{aligned}$$

where $x_j, y_j \in \mathcal{H}$ and $\varepsilon > 0$. We wish to show that for any such open set there is a function $f \in C(X)$ with $\phi(f) \in \mathcal{O}$. Recall that a complex measure μ is regular if its total variation, $|\mu|$ is regular. For each $j = 1, \ldots, n$, choose a compact set $K_j \subseteq G$ with

$$|\mu_{x_j, y_j}|(G \setminus K_j) < \frac{\varepsilon}{2}.$$

Then, setting

$$K = \bigcup_{j=1}^{n} K_j$$

we have, for each $j = 1, \dots, n$,

$$|\mu_{x_j,y_j}|(G \setminus K) < \frac{\varepsilon}{2}.$$

Urysohn's Lemma now guarantees the existence of a continuous function $f : X \to [0,1]$ with $f|_K = 1$ and $f|_{G^C} = 0$. Then, since $\chi_K \le f \le \chi_G$, we have

$$0 \le f - \chi_K \le \chi_{G \setminus K}.$$

Hence, for each $j = 1, \dots, n$,

$$\begin{aligned}
|\langle(\mu_\phi(G) - \phi(f))x_j, y_j\rangle| &\le |\langle(\phi(\chi_G) - \phi(\chi_K))x_j, y_j\rangle| + |\langle(\phi(f) - \phi(\chi_K))x_j, y_j\rangle| \\
&\le |\mu_{x_j,y_j}(G \setminus K)| + \left| \int (f - \chi_K) d\mu_{x_j,y_j} \right| \\
&\le |\mu_{x_j,y_j}|(G \setminus K) + \int (f - \chi_K) d|\mu_{x_j,y_j}| \\
&< \varepsilon.
\end{aligned}$$

Therefore $\phi(f) \in \mathcal{O}$, as required.

Now let

$$\mathcal{F} = \{F \subseteq X \,:\, \text{F is a Borel set and } \phi(\chi_F) \subseteq \textsc{wot}\text{-cl } \phi(C(X))\}.$$

We have just shown that $\mathcal{F}$ contains every open set of X. If $B_1, \dots, B_m \in \mathcal{F}$ are disjoint sets, then

$$\bigcup_{i=1}^{m} B_i \in \mathcal{F}$$

also. Next suppose that $\{B_i\}$ is a countable family of sets in $\mathcal{F}$; we may assume without loss of generality that $\{B_i\}$ are a disjoint family. Set $B = \bigcup_{i=1}^{\infty} B_i$. Then, since μ_ϕ is weakly countably additive,

$$\langle \mu_\phi(B)x, y\rangle = \sum_{i=1}^{\infty} \langle \mu_\phi(B_i)x, y\rangle$$

for any $x, y \in \mathcal{H}$. That is,

$$\mu_\phi(B) = \text{WOT-}\lim_N \mu_\phi\Big(\bigcup_{i=1}^{N} B_i\Big);$$

thus, $B \in \mathcal{F}$. This shows that $\mathcal{F}$ is a σ-algebra containing the open sets of X. Therefore, $\mathcal{F}$ is the σ-algebra of Borel sets of X. □

Notice that Proposition 2.1.6 shows that if the range of ϕ is contained in a C^*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, then the range of μ_ϕ is contained in the weak operator topology closure of $\mathcal{A}$, i.e, $\mathcal{A}''$.

2.2 A Necessary Condition for C^*-Extreme Maps

We can now prove the major result of this chapter, a theorem which gives a necessary condition for a positive map ϕ on a commutative C^*-algebra (or equivalently its associated positive operator-valued measure) to be C^*-extreme.

Theorem 2.2.1. *Let X be a compact Hausdorff space, and $\phi : C(X) \longrightarrow \mathcal{B}(\mathcal{H})$ a unital, positive map. Denote by μ_ϕ the unique positive operator-valued measure associated to ϕ. If ϕ is C^*-extreme, then for every Borel set $F \subset X$, either*

(1) *$\mu_\phi(F)$ is a projection, so that by Proposition 2.1.3 $\mu_\phi(F) \in \phi(C(X))'$, or*

(2) $\sigma(\mu_\phi(F)) = [0, 1]$.

Moreover, if (2) occurs and $\mu_\phi(F)$ has an eigenvalue in $(0, 1)$, then the point spectrum of $\mu_\phi(F)$ must contain $(0, 1)$.

Proof. Suppose there is a Borel set $F \subseteq X$ so that $\mu_\phi(F)$ is not a projection and $\sigma(\mu_\phi(F)) \neq [0,1]$. We will show that ϕ is not C^*-extreme by constructing a proper C^*-convex combination

$$t_1^* \psi_1 t_1 + t_2^* \psi_2 t_2 = \phi$$

in which ψ_1 and ψ_2 are not unitarily equivalent to ϕ. Choose $x \in (0,1) \setminus \sigma(\mu_\phi(F))$ and let (a,b) be the largest open subinterval of $(0,1)$ which contains x but does not intersect $\sigma(\mu_\phi(F))$. To be precise, let

$$(a,b) = \bigcup \{(\alpha,\beta) \subseteq (0,1) : x \in (\alpha,\beta),\ (\alpha,\beta) \cap \mu_\phi(F) = \varnothing\}$$

Note that this choice of the interval (a,b) insures that at least one of the pair $\{a,b\}$ is in $\sigma(\mu_\phi(F))$. In particular, if $a > 0$ then $a \in \sigma(\mu_\phi(F))$ and if $b < 1$ then $b \in \sigma(\mu_\phi(F))$. Choose $s_1 \in \left(\frac{1}{2}\left(\frac{a-ab}{b-ab}\right), \frac{1}{2}\right)$, and set $s_2 = 1 - s_1$. For $k = 1,2$, define

$$Q_k = \frac{1}{2}\mu_\phi(F) + s_k \mu_\phi(F^C) = s_k I + (\frac{1}{2} - s_k)\mu_\phi(F).$$

Note that $0 \notin \sigma(Q_k) = s_k + (\frac{1}{2} - s_k)\sigma(\mu_\phi(F))$, so that both Q_k's are invertible. Now define new positive operator-valued measures μ_1 and μ_2 by

$$\mu_k(B) = Q_k^{-\frac{1}{2}} \left(\frac{1}{2}\mu_\phi(B \cap F) + s_k \mu_\phi(B \cap F^C)\right) Q_k^{-\frac{1}{2}},$$

where B is any Borel set of X. Observe that each of the μ_k's is a positive operator-valued measure with $\mu_k(X) = I$. Next, define $t_k = Q_k^{\frac{1}{2}}$, for $k = 1,2$. Then, for any Borel set B of X,

$$
\begin{aligned}
t_1^*\mu_1(B)t_1 + t_2^*\mu_2(B)t_2 &= Q_1^{\frac{1}{2}}Q_1^{-\frac{1}{2}}\left(\frac{1}{2}\mu_\phi(B\cap F) + s_1\mu_\phi(B\cap F^C)\right)Q_1^{-\frac{1}{2}}Q_1^{\frac{1}{2}} \\
&\quad + Q_2^{\frac{1}{2}}Q_2^{-\frac{1}{2}}\left(\frac{1}{2}\mu_\phi(B\cap F) + s_2\mu_\phi(B\cap F^C)\right)Q_2^{-\frac{1}{2}}Q_2^{\frac{1}{2}} \\
&= \mu_\phi(B\cap F) + \mu_\phi(B\cap F^C) \\
&= \mu_\phi(B).
\end{aligned}
$$

Each t_k is invertible and

$$t_1^*t_1 + t_2^*t_2 = Q_1 + Q_2 = \mu_\phi(F) + \mu_\phi(F^C) = I.$$

Thus $t_1^*\mu_1t_1 + t_2^*\mu_2t_2$ is a proper C^*-convex combination of μ_1 and μ_2.

It is still necessary to show that μ_ϕ is not unitarily equivalent to at least one of μ_1 or μ_2. Set $g(t) = [s_k + (s_k - \frac{1}{2})t]^{-\frac{1}{2}}$. As g is continuous on $[0,1]$, and $Q_k^{-\frac{1}{2}} = g(\mu_\phi(F))$, $Q_k^{-\frac{1}{2}}$ commutes with $\mu_\phi(F)$. Thus, for $k = 1, 2$, we have

$$
\begin{aligned}
\mu_k(F) &= Q_k^{-1/2}\left(\frac{1}{2}\mu_\phi(F)\right)Q_k^{-1/2} \\
&= \frac{1}{2}\mu_\phi(F)\left(s_kI + \left(\frac{1}{2} - s_k\right)\mu_\phi(F)\right)^{-1}.
\end{aligned}
$$

Let $f_k(t) = \frac{1}{2}t\left(s_k + \left(\frac{1}{2} - s_k\right)t\right)^{-1}$. Observe that each f_k is continuous on $[0,1]$, and that $\mu_k(F) = f_k(\mu_\phi(F))$. Therefore, by the spectral mapping theorem, $\sigma(\mu_k(F)) = f_k(\sigma(\mu_\phi(F)))$. It is easy to check that for $t \in (0,1)$, $t < f_1(t) < 1$, while $0 < f_2(t) < t$, and that both f_k's are strictly increasing. In addition, since $s_1 > \frac{1}{2}\left(\frac{a-ab}{b-ab}\right)$, if $a > 0$,

$$a < f_1(a) = \frac{1}{2}a\left(\frac{1}{s_1 + (\frac{1}{2} - s_1)a}\right) < \frac{a}{\frac{a(1-b)}{b(1-a)}(1-a) + a} = b \leq f_1(b).$$

A similar computation shows that when $b < 1$, $f_2(a) \leq a < f_2(b) < b$. Consider the following two cases (these are not mutually exclusive, but at least one must occur):

Case (i) $a \neq 0$. In this case $a \in \sigma(\mu_\phi(F))$. Thus $f_1(a) \in \sigma(\mu_1(F))$, but since $f_1(a) \in (a, b)$, $f_1(a) \notin \sigma(\mu_\phi(F))$. Also $a \in \sigma(\mu_\phi(F)) \setminus \sigma(\mu_2(F))$. This shows that $\sigma(\mu_\phi(F)) \neq \sigma(\mu_k(F))$; therefore μ_ϕ and μ_k are not unitarily equivalent.

Case (ii) $b \neq 1$. In this case, $b \in \sigma(\mu_\phi(F))$. As above, it follows that $f_2(b) \in \sigma(\mu_2(F)) \setminus \sigma(\mu_\phi(F))$ and $b \in \sigma(\mu_\phi(F)) \setminus \sigma(\mu_1(f))$. It follows that neither μ_1 nor μ_2 is unitarily equivalent to μ_ϕ.

Let ψ_k be the unital positive maps associated with the positive operator-valued measures μ_k. Then $\phi = t_1^*\psi_1 t_1 + t_2^*\psi_2 t_2$; this is a proper C^*-convex combination of ψ_1 and ψ_2, where ϕ is not unitarily equivalent to either ψ_k. Therefore, ϕ is not C^*-extreme.

Now suppose that $\sigma(\mu_\phi(F)) = [0, 1]$ and that there exist $a, b \in (0, 1)$ with $a \in \sigma_{pt}(\mu_\phi(F))$ and $b \in \sigma(\mu_\phi(F)) \setminus \sigma_{pt}(\mu_\phi(F))$. Assume that $a < b$ (if not, simply exchange the roles of a and b). Using essentially the same construction as above, we define

$$Q_k = \frac{1}{2}\mu_\phi(F) + s_k\mu_\phi(F^C) = s_k I + (\frac{1}{2} - s_k)\mu_\phi(F)$$

where $s_1 = \frac{1}{2}\left(\frac{a-ab}{b-ab}\right)$, and $s_2 = 1 - s_1$. Then setting

$$\mu_k(B) = Q_k^{-\frac{1}{2}}\left(\frac{1}{2}\mu_\phi(B \cap F) + s_k\mu_\phi(B \cap F^C)\right)Q_k^{-\frac{1}{2}},$$

and $t_k = Q_k^{-\frac{1}{2}}$, we have μ_ϕ expressed as a proper C^*-convex combination of ψ_1 and ψ_2:

$$\mu_\phi = t_1^*\psi_1 t_1 + t_2^*\psi_2 t_2.$$

As above, $\mu_1(F) = f_1(\mu_\phi(F))$, and $f_1(a) = b$. Thus, b is an eigenvalue of $\mu_1(F)$. Since the point spectrum is also a unitary invariant, this shows that ϕ is not C^*-extreme. □

Remark 2.2.2. Notice that the result concerning eigenvalues in Theorem 2.2.1 shows that if a map $\phi \in S_{\mathcal{H}}(C(X))$ is C^*-extreme, then the operators in the range of μ_ϕ may have eigenvalues only when $\mathcal{H}$ is a non-separable Hilbert space.

In their paper of 1997 [6], Farenick and Morenz show that a positive map from a commutative C^*-algebra into a matrix algebra M_n is C^*-extreme if and only if it is a $*$-homomorphism. In view of the spectral condition given by Theorem 2.2.1, a shorter proof is possible.

Corollary 2.2.3. *[6, Proposition 2.2] Let X be a compact Hausdorff space and $\phi : C(X) \longrightarrow M_n$ a positive map. Then ϕ is C^*-extreme if and only if it is a $*$-homomorphism.*

Proof. It is already known that if ϕ is a representation (i.e.,$*$-homomorphism), then ϕ is C^*-extreme [6, Proposition 1.2]. On the other hand, if ϕ is not a representation, then the associated positive operator-valued measure μ_ϕ is not a spectral measure. In this case, there is a Borel set $F \subset X$ for which $\mu_\phi(F)$ is not a projection. As $\mu_\phi(F)$ is an $n \times n$ matrix, $\sigma(\mu_\phi(F))$ consists of at most n isolated points. We may therefore apply the theorem to conclude that ϕ is not C^*-extreme. □

Note that in the proof of Theorem 2.2.1, Q_k, $Q_k^{-\frac{1}{2}}$ and $t_k = Q_k^{\frac{1}{2}}$ are elements of the C^*-algebra generated by $\mu_\phi(F)$. As noted in Proposition 2.1.6, the range of μ_ϕ is contained in the WOT-closure of the range of ϕ. Thus we have the following corollary to the proof of Theorem 2.2.1:

Corollary 2.2.4. *Let $\mathcal{M} \subseteq \mathcal{B}(\mathcal{H})$ be a von Neumann algebra, $\phi : C(X) \longrightarrow \mathcal{M}$ a unital positive map, and μ_ϕ the positive operator-valued measure associated to ϕ. If ϕ fails to meet the spectral condition described in Theorem 2.2.1, then ϕ can be written as a proper C^*-convex combination*

$$\phi = t_1^* \psi_1 t_1 + t_2^* \psi_2 t_2,$$

where each $t_k \in \mathcal{M}$, each $\psi_k : C(X) \longrightarrow \mathcal{M}$, and neither ψ_k is unitarily equivalent to ϕ in $\mathcal{B}(\mathcal{H})$.

2.3 Examples

Theorem 2.2.1 gives a necessary condition for a positive unital map ϕ to be C^*-extreme. Example 2.3.4 demonstrates that this condition is not sufficient. Before presenting the example, we will need to prove Proposition 2.3.2; a result of Choi, stated here as Lemma 2.3.1, will be of use in its proof.

Lemma 2.3.1. *[2, Theorem 3.1] If ϕ is a 2-positive map between C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, then the set $\{a \in \mathcal{A} : \phi(a^*a) = \phi(a^*)\phi(a)\}$ is a closed subalgebra of $\mathcal{A}$. In fact, it is just the multiplicative domain,*

$$\mathcal{A}_\phi := \{a \in \mathcal{A} : \phi(xa) = \phi(x)\phi(a) \textit{ for all } x \in \mathcal{A}\}.$$

Proposition 2.3.2. *If $X = \{x_1, x_2\}$, then a positive unital map $\phi : C(X) \to \mathcal{B}(\mathcal{H})$ is C^*-extreme if and only if $\phi(z)$ is an isometry or co-isometry, where $z \in C(X)$ is given by $z(x_k) = (-1)^k, k = 1, 2$. In this case, ϕ is also multiplicative.*

Proof. If $\phi(z)$ is an isometry (resp. co-isometry), then

$$\phi(z^*z) = I = \phi(z)^*\phi(z), \text{ (resp., } \phi(zz^*) = I = \phi(z)\phi(z)^*).$$

In this case, $z \in C(X)_\phi$ (resp., $z^* \in C(X)_\phi$), the multiplicative domain of ϕ. However, z (resp. z^*) generates all of $C(X)$ as an algebra. Thus ϕ is multiplicative, and therefore C^*-extreme.

On the other hand, if $\phi(z)$ is neither an isometry nor a co-isometry, then $\phi(z)$ can be written

$$\phi(z) = \frac{1}{2}(S_1 + S_2),$$

where S_1 and S_2 are either isometries or co-isometries [14, Cor 1.2]. Since $\phi(z)$ is not an isometry or co-isometry, it is not unitarily equivalent to either S_j. Note that any function $f \in C(X)$ can be written

$$f(x_j) = \frac{1}{2}[f(x_2) + f(x_1)] + \frac{1}{2}[f(x_2) - f(x_1)]z(x_j).$$

We can now define positive maps $\psi_j, j = 1, 2$ by

$$\psi_j(f) = \frac{1}{2}(f(x_1) + f(x_2))I + \frac{1}{2}(f(x_2) - f(x_1))S_j.$$

Since $\psi_1(z) = S_1$ and $\psi_2(z) = S_2$ are isometries or co-isometries, the maps ψ_1 and ψ_2 are multiplicative, and therefore positive and unital. Thus

$$\phi(f) = \frac{1}{2}(\psi_1(f) + \psi_2(f)),$$

expresses ϕ as a proper C^*-convex combination of maps ψ_1 and ψ_2 which are not unitarily equivalent to ϕ, so ϕ is not C^*-extreme. $\square$

Remark 2.3.3. Note that the proof of Proposition 2.3.2 demonstrates that for $X = \{x_1, .., x_n\}$ and $z(x_n) = e^{2\pi ik/n}$, if $\phi(z)$ is an isometry, then ϕ is multiplicative.

Write $|X|$ for the cardinality of the set X. It may seem natural to try to extend Proposition 2.3.2 to maps in $S_{\mathcal{H}}(C(X))$ where $2 < |X| < \infty$. However this problem turns out to be quite difficult. We will discuss this issue further in Section 2.5.

Consider the Hilbert spaces $L^2(\mathbb{T}, m)$, where m is normalized Lebesgue measure on the unit circle $\mathbb{T}$, and H^2, the classical Hardy space. Let $T_f = PM_fP$ be the Toeplitz operator for f. We will make use of a result of Hartman and Wintner [13, p.868], which shows that for a real-valued function $f \in L^\infty(\mathbb{T}, m)$, the spectrum of T_f is the closed convex hull of the essential range of f. In particular, for a set $S \subsetneq \mathbb{T}$ with $0 < m(S) < 1$, we have

$$\sigma(T_{\chi_S}) = [0, 1].$$

Example 2.3.4. Let $X = \{x_1, x_2\}$ and set $U = \{e^{i\theta} : 0 \leq \theta < \pi\}$, so that U is the top half of the unit circle, and $L = \mathbb{T} \setminus U$. Then we may define an operator-valued measure μ_ϕ by $\mu_\phi(\{x_1\}) = T_{\chi_U}$ and $\mu_\phi(\{x_2\}) = T_{\chi_L}$. The map ϕ corresponding to μ_ϕ will be unital and positive, and by [13, p. 868] will meet the spectral condition set out in Theorem 2.2.1. However, ϕ is not multiplicative, and therefore not C^*-extreme.

We now consider an example of a C^*-extreme map which is not multiplicative. The positive map ϕ defined below was considered by Arveson [1, p. 164] as an example of an extreme point in the generalized state space. Farenick and Morenz [6, Example 2] subsequently showed that ϕ is also a C^*-extreme point, although not a homomorphism.

Example 2.3.5. [1], [6] Consider the representation $\pi : C(\mathbb{T}) \longrightarrow \mathcal{B}(L^2(\mathbb{T}, m))$ given by $\pi(f) = M_f$. The spectral measure associated to π is given by $\mu_\pi(B) = M_{\chi_B}$, where $B \subseteq X$

is a Borel set. Define a unital positive map

$$\phi : C(\mathbb{T}) \longrightarrow \mathcal{B}(H^2)$$

by

$$\phi(f) = PM_f P.$$

Since $\mu_\pi(B) = M_{\chi_B}$, we have $\mu_\phi(B) = PM_{\chi_B}P = T_{\chi_B}$, a Toeplitz operator. Thus $\sigma(\mu_\phi(B)) = \sigma(T_{\chi_B})$ and, if $\mu_\phi(B) \notin \{0, I\}$, then $\sigma(\mu_\phi(B)) = [0,1]$ [13, p.868]. Thus, for any Borel set $B \subseteq X$, either $\mu_\phi(B) = [0,1]$ or $\mu_\phi(B)$ is a trivial projection; that is, ϕ satisfies the conditions of Theorem 2.2.1.

In Example 2.3.5, the image of z under ϕ is T_z, which is unitarily equivalent to the forward shift operator, and is thus, an isometry. The proof given by Farenick and Morenz that the map ϕ of Example 2.3.5 is C^*-extreme uses explicitly the fact that $\phi(z)$ is an isometry, and therefore a C^*-extreme point of the unit ball of $\mathcal{B}(\mathcal{H})$ [14, Corollary 1.2], and also relies implicitly on the fact that the multiplicative domain of ϕ is the non-selfadjoint subalgebra $\mathfrak{A}$ of $C(\mathbb{T})$ generated by z. This algebra is Dirichlet dense in $C(\mathbb{T})$; that is, the norm closure of $\mathfrak{A} + \mathfrak{A}^*$ is $C(\mathbb{T})$.

As the following example shows, it is possible to use the map ϕ of Example 2.3.5 to construct additional non-multiplicative C^*-extreme maps.

Example 2.3.6. Let $\phi : C(\mathbb{T}) \to \mathcal{B}(H^2)$ be the map of Example 2.3.5, which is a C^*-extreme point of $S_{H^2}(C(\mathbb{T}))$. Extend ϕ to a map on the Borel measurable functions of the circle, as in Remark 2.1.2. There is a one-to-one correspondence between Borel measurable functions on $\mathbb{T}$, and Borel measurable functions on $[0, 2\pi]$, given by $f \mapsto g$ where $g(t) = f(e^{it})$. Define a map $\psi : C([0,2\pi]) \to \mathcal{B}(H^2)$ by $\psi(g) = \phi(f)$, where $g(t) = f(e^{it})$. Suppose that

$$\psi = t_1^*\psi_1 t_1 + t_2^*\psi_2 t_2$$

is a proper C^*-extreme convex combination. Define maps $\phi_1, \phi_2 : C(\mathbb{T}) \to \mathcal{B}(\mathcal{H})$ by

$$\phi_j(f) = \psi_j(g)$$

where $f(e^{it}) = g(t)$ Then

$$\phi = t_1^*\phi_1 t_1 + t_2^*\phi_2 t_2.$$

Since ϕ is C^*-extreme, there are unitaries u_j so that

$$\phi(f) = u_j \phi_j(f) u_j.$$

But then

$$\psi(g) = u_j^* \psi_j(g) u_j.$$

Thus ψ is C^*-extreme in $S_{H^2}(C([0,2\pi]))$.

2.4 Zhou's Characterization of C^*-extreme points

In [22, 3.1.2, 3.1.5], Zhou gives two characterizations of C^*-extreme maps; the second of these is stated below as Theorem 2.4.1. The goal of this section is to describe the relationship between Theorem 2.4.1 and Theorem 2.2.1.

Theorem 2.4.1. *[22, 3.1.5] Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a completely positive map. Then ϕ is a*

C^*-extreme point if and only if the following condition is satisfied:

$$(\dagger) \textit{ whenever } \rho \leq \phi, \textit{ we can write } \rho = A^*\phi A \textit{ for some operator } A \in \mathcal{B}(\mathcal{H}).$$

Moreover, if $\rho(1)$ is invertible, then A may be taken to be invertible .

Remark 2.4.2. Note that Theorem 2.4.1 gives an alternate proof of the fact, mentioned in Section 1.2, that every pure map is also C^*-extreme. For, if ϕ is pure and ψ is any completely positive map with $\psi \leq \phi$, then there is some $t \in (0, 1)$ so that $\psi = t\phi$. But then

$$\psi = (\sqrt{t}I)\phi(\sqrt{t}I).$$

Returning to our discussion of the connection between Theorem 2.4.1 and Theorem 2.2.1, we wish to show directly that if a map $\phi \in S_{\mathcal{H}}(C(X))$ satisfies condition (†), then for every Borel set $F \subseteq X$ either $\mu_\phi(F)$ is a projection, or $\sigma(\mu_\phi(F)) = [0, 1]$. As in the proof of Theorem 2.2.1, we will prove the contrapositive. Assume that there is a Borel set $F \subseteq X$ for which $\mu_\phi(F)$ is not a projection and $\sigma(\mu_\phi(F)) \subsetneq [0, 1]$. We will show that ϕ fails to satisfy condition (†); that is, there is a map $\rho \leq \phi$ which cannot be expressed in the form $\rho = A^*\phi A$. Proceeding as we did in the proof of Theorem 2.2.1, choose an interval $(a, b) \subsetneq (0, 1)$ satisfying

1. $(a, b) \cap \sigma(\mu_\phi(F)) = \emptyset$, and
2. any open interval which properly contains (a, b) will intersect $\sigma(\mu_\phi(F))$.

Choose $s_1 \in (\frac{1}{2}\frac{a-ab}{b-ab}, \frac{1}{2})$. Then construct operators Q_k and measures μ_k:

$$Q_k = \frac{1}{2}\mu_\phi(F) + s_k\mu_\phi(F^C) = s_kI + (\frac{1}{2} - s_k)\mu_\phi(F),$$

$$\mu_k(B) = Q_k^{-\frac{1}{2}}\left(\frac{1}{2}\mu_\phi(B \cap F) + s_k\mu_\phi(B \cap F^C)\right)Q_k^{-\frac{1}{2}},$$

for $k = 1, 2$ and for any Borel set B. Letting ψ_k be the maps associated with μ_k, we have

$$\phi = t_1^* \psi_1 t_1 + t_2^* \psi_2 t_2,$$

where $t_k = Q_k^{\frac{1}{2}}$. Then the maps $\rho_1 = t_1^* \psi_1 t_1$ and $\rho_2 = t_2^* \psi_2 t_2$ are both positive maps, and $\rho_k \leq \phi$ for $k = 1, 2$.

Suppose, to seek a contradiction, that $\rho_1 = A^* \phi A$ for some $A \in \mathcal{B}(\mathcal{H})$. Since both ψ_1 and ϕ are unital maps,

$$\rho_1(1) = t_1^* t_1 = A^* A.$$

Set

$$T = (t_1^* t_1)^{\frac{1}{2}} = (A^* A)^{\frac{1}{2}}.$$

Then, the polar decompositions of t_1 and A may be written

$$t_1 = uT, \text{ and } A = wT.$$

Note that, since t_1 is invertible, A must also be invertible, so that both u and w are unitaries. Thus,

$$T = u^* t_1 = w^* A,$$

and,

$$t_1 = uw^* A,$$

where uw^* is a unitary.

It follows that

$$\rho_1 = t_1^* \psi_1 t_1 = (uw^*A)^* \psi_1 (uw^*A)$$
$$= A^*(uw^*)^* \psi_1 (uw^*) A$$

But then,

$$A^* \phi A = A^*(uw^*)^* \psi_1 (uw^*) A.$$

Multiplying on the left by $(A^*)^{-1}$ and on the right by A^{-1}, we obtain

$$\phi = (uw^*)^* \psi_1 (uw^*).$$

But this is a contradiction, as it was shown in the proof of Theorem 2.2.1 that ϕ and ψ are not unitarily equivalent. We conclude that there is no operator A for which

$$\rho_1 = A^* \phi A.$$

That is, ϕ fails to satisfy condition (†).

2.5 When X is Finite

Let X be a finite set, and write $|X|$ for the cardinality of X; denote by z be the function

$$z(x_j) = e^{\frac{2i\pi j}{|X|}}.$$

In Proposition 2.3.2 it was shown that when $X = \{x_1, x_2\}$ a map $\phi : C(X) \to \mathcal{B}(\mathcal{H})$ is C^*-extreme if and only if it is multiplicative. It seems reasonable to hope that this result could be extended to $C(X)$ for any finite set X. However, the proof of Proposition 2.3.2 relied on the fact that the C^*-algebra $C(X)$ is generated as a vector space by the functions 1 and z when $|X| = 2$. When $|X| > 2$, this is no longer the case. Since z does generate C(X) as an algebra when $|X|$ is finite, it follows that whenever $\phi(z)$ is an isometry, ϕ will be multiplicative and therefore C^*-extreme (see Remark 2.3.3). Unfortunately, it is not clear whether $\phi(z)$ must be an isometry (and ϕ therefore be multiplicative) in order for ϕ to be C^*-extreme. The following discussion gives some insight into why this apparently straightforward question has been so resistant to solution.

Let $X = \{x_1, x_2, x_3\}$ and let P_1, P_2, P_3 be positive operators in $\mathcal{B}(\mathcal{H})$ with $I = P_1+P_2+P_3$ and $\sigma(P_j) = [0, 1]$ for $j = 1, 2, 3$. To construct an example of such operators, let

$$S_j = \left\{ e^{i\theta} \in \mathbb{T} : \quad \frac{2\pi(j-1)}{3} \leq \theta < \frac{2\pi j}{3} \right\},$$

for $j = 1, 2, 3$, then set $P_j = T_{\chi_{S_j}}$. Define a positive map $\phi : C(X) \to \mathcal{B}(\mathcal{H})$ by

$$\phi(\delta_j) = P_j,$$

where δ_j is the characteristic function of $\{x\}$. Then a proper C^*-convex combination for ϕ will have the form

$$\phi(\delta_j) = t_1^*\psi_1(\delta_j)t_1 + t_2^*\psi_2(\delta_j)t_2,$$

for $j = 1, 2, 3$. To determine whether ϕ is C^*-extreme, we need to know if such a C^*-convex combination exists in which (without loss of generality) ψ_1 is not unitarily equivalent to ϕ.

Consider the C^*-algebra $\mathcal{A} = C([0, 1]) * C([0, 1])$, the free product of two copies of $C([0, 1])$. Let f_1 and f_2 be generators of the first and second free factors of $\mathcal{A}$, respectively. Given any positive unital map $\phi \in S_{\mathcal{H}}(C(X))$ we can define a representation π_ϕ of $\mathcal{A}$ by setting

$$\pi_\phi(f_j) = P_j,$$

for $j = 1, 2$, and extending π_ϕ. Then two such maps, say ϕ and ψ_1, will be unitarily equivalent if and only if the corresponding representations π_ϕ and π_{ψ_1} are unitarily equivalent.

The above construction shows that every $\phi \in S_{\mathcal{H}}(C(X))$ gives rise to a representation of $\mathcal{A}$. However, the reverse is not true. Each representation of $\mathcal{A}$ on $\mathcal{B}(\mathcal{H})$ carries the generators f_1 and f_2 of $\mathcal{A}$ to operators $\pi(f_j)$ in $\mathcal{B}(\mathcal{H})$ with $0 \leq \pi(f_j) \leq I$. In order to define a corresponding unital positive map ϕ, with

$$\phi(\delta_j) = \pi(f_j) \quad \text{for } j = 1, 2, \text{ and}$$

$$\phi(\delta_3) = I - (\phi(f_1) + \phi(f_2)),$$

we need $\pi(f_1) + \pi(f_2) \leq I$. Thus the ideal

$$J = \bigcap_{\phi \in S_{\mathcal{H}}(C(X))} \ker \pi_\phi$$

will be nontrivial. The representations of the algebra $\mathcal{A}/J$ will be in one-to-one correspondence with the maps in $S_{\mathcal{H}}(C(X))$, and the unitary equivalence classes of the representations of $\mathcal{A}/J$ will correspond exactly to those in $S_{\mathcal{H}}(C(X))$. Thus, our attempt to determine whether ϕ is C^*-extreme, leads us to questions about the representation theory of $\mathcal{A} = C([0,1]) * C([0,1])$ and the quotient algebra $\mathcal{A}/J$. The representation theory for $\mathcal{A}$ and $\mathcal{A}/J$ is unlikely to be easily understood.

Chapter 3

C^*-Extreme Maps into $\mathcal{K}^+$

3.1 Maps into $\mathcal{K}^+$

Now let us consider the case of a unital positive map ϕ on a commutative C^*-algebra $C(X)$ whose range is in $\mathcal{K}^+$, the C^*-algebra generated by the compact operators and the identity operator. In [6, Proposition 1.1] Farenick and Morenz show that if such a map ϕ is C^*-extreme, then ϕ is also extreme. It is possible, however, to say more. Theorem 2.2.1 requires the operators in the range of the positive operator-valued measure μ_ϕ either to be projections, or to have spectrum equal to $[0, 1]$. In contrast, the spectrum of a positive operator $K + \alpha I \in \mathcal{K}^+$ must be a sequence of positive numbers with a single limit point at α. This dichotomy suggests that Theorem 2.2.1 may give additional information about these maps. However, Proposition 2.1.6 tells us only that the operators in the range of μ_ϕ must be in WOT-cl $\phi(C(X))$, and WOT-cl $\mathcal{K}^+ = \mathcal{B}(\mathcal{H})$. The desired result is, therefore, not an immediate consequence of Theorem 2.2.1, but will require some additional effort on our part. Both the result of Theorem 2.2.1 (the spectral condition on the operators in the range of μ_ϕ) and the technique used in its proof, will be used below. The result is Theorem 3.1.2, which shows that C^*-extreme maps into $\mathcal{K}^+$ must be multiplicative, and gives their structure.

In the succeeding lemma and theorem, let q be the usual quotient map $q : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, and set $\tau = q \circ \phi$, see Figure 1.

$$\begin{tikzcd} C(X) \arrow[r, "\phi"] \arrow[dr, dashed, "\tau"'] & \mathcal{K}^+ \arrow[d, "q"] \\ & \mathbb{C} \end{tikzcd}$$

Figure 1

Then τ is a positive linear functional, so there is a unique positive real-valued Borel measure μ_τ on X so that

$$\tau(f) = \int_X f d\mu_\tau \text{ for every } f \in C(X).$$

For any function $f \in C(X)$, write $K_f = \phi(f) - \tau()f)I$, so $K_f \in \mathcal{K}$ and

$$\phi(f) = K_f + \tau(f)I.$$

Lemma 3.1.1. *Let $\phi : C(X) \to \mathcal{K}^+$ be unital, positive, and C^*-extreme. Then the map τ is multiplicative.*

Proof. As in the proof of Theorem 2.2.1, we will prove the contrapositive. Assume that τ is not multiplicative; then the support of μ_τ must contain at least two distinct points, which we will call x_1 and x_2. Let N_1 be a neighborhood of x_1 which does not contain x_2. By Urysohn's Lemma, there exists a continuous function $f : X \to [0, 1]$ such that $f(x_1) = 1$ and $f|_{N_1^C} = 0$; note that $\tau(f) \neq 0$.

Choose α and β in $(0, 1)$ with $\alpha > \beta$ and let

$$Q_1 = \alpha\phi(f) + \beta\phi(1 - f) = (\alpha - \beta)\phi(f) + \beta I, \text{ and}$$

$$Q_2 = (1 - \alpha)\phi(f) + (1 - \beta)\phi(1 - f) = (\beta - \alpha)\phi(f) + (1 - \beta)I.$$

Note that since $0 \leq f \leq 1$, the spectrum of $\phi(f)$ is contained in the closed unit interval. Thus,

$$\sigma(Q_1) \subseteq [\beta, \alpha], \text{ and}$$
$$\sigma(Q_2) \subseteq [1-\alpha, 1-\beta].$$

So both Q_j's are invertible positive operators. Define maps ψ_1 and ψ_2 by

$$\psi_1(g) = Q_1^{-\frac{1}{2}} \left[\alpha\phi(fg) + \beta\phi((1-f)g)\right] Q_1^{-\frac{1}{2}}, \text{ and}$$
$$\psi_2(g) = Q_2^{-\frac{1}{2}} \left[(1-\alpha)\phi(fg) + (1-\beta)\phi((1-f)g)\right] Q_2^{-\frac{1}{2}}.$$

Both ψ_j's are positive, unital maps with ranges in $\mathcal{K}^+$. Setting $t_j = Q_j^{\frac{1}{2}}$, we have

$$\begin{aligned} t_1^*\psi_1(g)t_1 + t_2^*\psi_2(g)t_2 &= \alpha\phi(fg) + \beta\phi(g - fg) + (1-\alpha)\phi(fg) + (1-\beta)\phi(g-fg) \\ &= \phi(fg) + \phi(g - fg) \\ &= \phi(g), \quad \text{for every } g \in C(X). \end{aligned}$$

Since $t_1^*t_1 + t_2^*t_2 = I$, the above expression gives ϕ as a proper C^*-convex combination of ψ_1 and ψ_2.

We now wish to show that ψ_1 and ψ_2 are not unitarily equivalent. To this end, let N_2 be a neighborhood of x_2 with $N_1 \cap N_2 = \emptyset$. Then we may choose a continuous function $h : X \to [0,1]$ with $h|_{N_2^C} = 0$ (i.e., supp $h \subseteq N_2$) and $h(x_2) = 1$; thus $fh = 0$,$(1-f)h = h$, and $\tau(h) \neq 0$.

Since $h \in C(X)$, $\phi(h) = K_h + \tau(h)I \in \mathcal{K}^+$. Note that $\tau(h) > 0$, since $h > 0$ on some neighborhood of x_2, and that the essential spectrum of $\phi(h)$ is $\{\tau(h)\}$. Now compute

$$\begin{aligned}\psi_1(h) &= Q_1^{-\frac{1}{2}}\left(\alpha\phi(fh) + \beta\phi((1-f)h)\right)Q_1^{-\frac{1}{2}} \\ &= \beta Q_1^{-\frac{1}{2}}\phi(h)Q_1^{-\frac{1}{2}} \\ &= \beta Q_1^{-\frac{1}{2}}K_h Q_1^{-\frac{1}{2}} + \beta\tau(h)Q_1^{-1}\end{aligned}$$

The first term in this sum is compact, while the second term can be written

$$\beta\tau(h)Q_1^{-1} = \beta\tau(h)[(\alpha-\beta)K_f + ((\alpha-\beta)\tau(f)+\beta)I]^{-1},$$

where $\phi(f) = K_f + \tau(f)I$. Thus

$$(q\circ\psi_1)(h) = \frac{\beta\tau(h)}{(\alpha-\beta)\tau(f)+\beta}I + \mathcal{K}.$$

Similar computations yield

$$\begin{aligned}\psi_2(h) &= (1-\beta)Q_2^{-\frac{1}{2}}K_h Q_2^{-\frac{1}{2}} + (1-\beta)\tau(h)Q_2^{-1}, \text{ and} \\ (q\circ\psi_2)(h) &= \frac{(1-\beta)\tau(h)}{(\beta-\alpha)\tau(f)+(1-\beta)}I + \mathcal{K}.\end{aligned}$$

So the essential spectra of $\psi_1(h)$ and $\psi_2(h)$ are

$$\left\{\frac{\beta\tau(h)}{(\alpha-\beta)\tau(f)+\beta}\right\} \text{ and } \left\{\frac{(1-\beta)\tau(h)}{((\beta-\alpha)\tau(f)+(1-\beta)}\right\},$$

respectively. However, if these are equal, then

$$\beta(\beta-\alpha)\tau(f) + \beta(1-\beta) = (1-\beta)(\alpha-\beta)\tau(f) + \beta(1-\beta),$$

so that,

$$\beta = \beta - 1,$$

which is clearly impossible. This shows that the essential spectra of $\psi_1(h)$ and $\psi_2(h)$ are distinct, so that $\psi_1(h)$ and $\psi_2(h)$ are not unitarily equivalent. So

$$\phi = t_1^* \psi_1 t_1 + t_2^* \psi_2 t_2$$

expresses ϕ as a proper C^*-convex combination of positive unital maps ψ_1 and ψ_2 which are not both unitarily equivalent to ϕ, demonstrating that ϕ is not C^*-extreme. This proves the lemma.

□

We can now prove the following:

Theorem 3.1.2. *Let $\phi : C(X) \to \mathcal{K}^+$ be unital and positive. Then ϕ is C^*-extreme if and only if ϕ is a homomorphism.*

Proof. If ϕ is multiplicative, then ϕ is C^*-extreme [6, Proposition 1.2]. Conversely, if ϕ is C^*-extreme, Lemma 3.1.1 shows that the map $\tau = q \circ \phi$ is multiplicative, so that $\tau(f) = f(x_0)$ for some point $x_0 \in X$.

Let N be any neighborhood of x_0. Then there exists a continuous function $g_{_N} : X \to [0, 1]$ with $g_{_N}(x_0) = 0$ and $g_{_N}|_{N^C} = 1$.

In this case $\tau(g_{_N}) = 0$, so

$$\phi(g_{_N}) = K_{g_{_N}} \in \mathcal{K}.$$

Note that $\chi_{_{N^C}} \leq g_{_N}$, so that $\phi(\chi_{_{N^C}}) \leq \phi(g_{_N})$. Since $\mathcal{K}$ is hereditary, it follows that $\phi(\chi_{_{N^C}})$ is compact. By Theorem 2.2.1, either $\phi(\chi_{_{N^C}})$ is a projection or $\sigma(\phi(\chi_{_{N^C}})) = [0, 1]$. As a compact operator cannot have the unit interval as its spectrum, $\phi(\chi_{_{N^C}})$ must be a projection of finite rank. So for any closed set $F \not\ni x_0$, $\phi(\chi_F)$ is a finite rank projection. Next we will show that the same is true for any Borel set B which does not contain x_0. Set

$$\Lambda := \{K \subseteq B : K \text{ closed}\},$$

and partially order Λ by inclusion. Then $\mu_\phi(K)$ is an increasing net of projections. Thus the SOT-$\lim\limits_{K} \mu_\phi(K) =: Q$ exists, and is a projection, namely the projection onto $\bigcup\limits_{K\in\Lambda} \operatorname{ran} \mu_\phi(K)$. Since the measures $\mu_{x,x}$ are regular for any choice of $x \in \mathcal{H}$, we have

$$\mu_{x,x}(B) = \sup_{K\in\Lambda} \mu_{x,x}(K)$$

or, equivalently,

$$\begin{aligned} \langle \mu_\phi(B)x, x\rangle &= \sup_{K\in\Lambda} \langle \mu_\phi(K)x, x\rangle \\ &= \langle Qx, x\rangle. \end{aligned}$$

As this holds for any $x \in \mathcal{H}$,

$$Q = \mu_\phi(B).$$

If B is a Borel set in X which does contain x_0, then the preceding argument shows that $\mu_\phi(B^C)$ is a projection. Thus $\mu_\phi(B)$ is also a projection. Hence μ_ϕ is a projection valued measure, and ϕ is a homomorphism.

$\square$

Remark 3.1.3. When $\phi : C(X) \to \mathcal{K}^+$, as in Theorem 3.1.2, we can obtain more information regarding the support of μ_ϕ. We have shown above that for any closed set K with $x_0 \notin K$, $\mu_\phi(K)$ is a finite rank projection, say of rank n. If x_1, x_2 are distinct points of $K\cap$ supp μ_ϕ, let $N_1 \subseteq K$ be a neighborhood of x_1 which does not contain x_2. Then $K \setminus N_1$ is closed and

$x_0 \notin K \setminus N_1$, so $\mu_\phi(K \setminus N_1)$ is a projection of finite rank and

$$0 < \text{rank } \mu_\phi(K \setminus N_1) < \text{rank } \mu_\phi(K) = n.$$

Since

$$\mu_\phi(K) = \mu_\phi(K \setminus N_1) + \mu_\phi(N_1),$$

it follows that $\mu_\phi(N_1)$ is also a projection with

$$0 < \text{rank } \mu_\phi(N_1) < n.$$

Clearly this process can be iterated at most n times; we conclude that any closed set $K \not\ni x_0$ contains at most finitely many points of supp μ_ϕ. Consequently, supp $\mu_\phi \setminus \{x_0\}$ is a discrete set with at most one accumulation point at x_0.

If $\mathcal{H}$ is a separable Hilbert space, then it is clear from the proof of Theorem 3.1.2 and the preceding remark that the support of μ_ϕ must be at most countable with a single limit point at x_0. In this case, ϕ must have the form

$$\phi(f) = \sum_{x \in \text{supp}(\mu_\phi)} f(x) P_x,$$

where $P_x = \mu_\phi(\{x\})$ is a finite rank projection for each $x \neq x_0$. The rank of P_{x_0}, on the other hand, may be finite or infinite. The following example, in which we consider a nonseparable Hilbert space, illustrates the structure of unital positive maps $\phi : C(X) \to \mathcal{K}^+$.

Example 3.1.4. Let $\mathcal{H}$ be a nonseparable Hilbert space with dimension at least as great as the cardinality of $\mathbb{R}$, and let $X = \mathbb{R} \cup \{\omega\}$ be the one point compactification of $(\mathbb{R}, d)$, the reals equipped with the discrete topology. Choose an orthonormal set $\{e_x\}_{x \in \mathbb{R}}$ in $\mathcal{H}$ indexed by the reals, and write P_x for the projection onto the span of e_x. Then, for any function

$f \in C(X)$, the set

$$S(f) := \{x \in X : f(x) \neq f(\omega)\}$$

is at most countable, and

$$\lim_{n\to\infty} f(x_n) = f(\omega),$$

where $\{x_n\}$ is any enumeration of $S(f)$. Define a positive map ϕ on $C(X)$ by

$$\phi(f) = \sum_{x\in S(f)} [f(x) - f(\omega)]P_x + f(\omega)I.$$

Then for each $x \in \mathbb{R}$, the function $\delta_x = \chi_{\{x\}}$ is continuous and $\phi(\delta_x) = \mu_\phi(\{x\}) = P_x$. As in the proof of Theorem 3.1.2, if G is any neighborhood of ω, then G^C is a closed set not containing ω, and $\phi(\chi_G)$ is a projection. In this case the descending net $\phi(\chi_G)$ of projections converges to the trivial projection $\phi(\chi_{\{\omega\}}) = 0$. Thus μ_ϕ is a projection valued measure.

Note that we could define similar maps ϕ_1 and ϕ_2 by

$$\phi_1(f) = \sum_{x\in S(f)\setminus\{0\}} [f(x) - f(\omega)](P_{e^x} + P_{-e^x}) + f(\omega)I,$$

and

$$\phi_2(f) = \sum_{x\in S(f)} [f(x) - f(\omega)]P_{\arctan x} + f(\omega)I.$$

For these two maps, we have $\phi_1(\chi_{\{\omega\}}) = P_0$, while $\phi_2(\chi_{\{\omega\}})$ is the projection onto the closed span $\{\text{ran } P_x : x \in (-\infty, \frac{\pi}{2}] \cup [\frac{\pi}{2}, \infty)\}$. Thus, the measure of $\{\omega\}$ may be a projection of either finite or infinite rank.

3.2 Efforts to Extend Theorem 3.1.2

As we have seen, Theorem 3.1.2 gives a complete characterization of C^*-extreme maps in $S_{\mathcal{H}}(C(X))$ with range in $\mathcal{K}^+$, and leads to a description of the structure of these maps. This section will address the question of whether Theorem 3.1.2 can be extended to a larger class of completely positive maps.

Consider, for example, unital completely positive maps $\phi : \mathcal{A} \to \mathcal{K}^+$, where $\mathcal{A}$ is a CCR algebra. Then there are non-multiplicative C^*-extreme points in $S_{\mathcal{H}}(\mathcal{A})$; Zhou gives the following example of a C^*-extreme map [22, 3.3.3].

Example 3.2.1. Take $\mathcal{A} = M_2 \oplus M_1$ and write elements $a \in \mathcal{A}$ in the form

$$a = \begin{pmatrix} a_{1,1} & a_{1,2} & 0 \\ a_{2,1} & a_{2,2} & 0 \\ 0 & 0 & a_{3,3} \end{pmatrix}.$$

Then the map $\phi : M_3 \to M_2$, given by

$$\phi(a) = \frac{1}{2}\begin{pmatrix} a_{1,1} + a_{3,3} & a_{1,1} - a_{3,3} \\ a_{1,1} - a_{3,3} & a_{1,1} + a_{3,3} \end{pmatrix}$$

is $C*$-extreme. However, ϕ is not multiplicative. For example, taking

$$a = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

we obtain

$$\phi(a)\phi(b) = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

while

$$\phi(ab) = \begin{pmatrix} \frac{5}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{5}{2} \end{pmatrix}.$$

On the other hand, consider maps $\phi : C(X) \to \mathcal{A}$ where $\mathcal{A}$ is an AF algebra other than $\mathcal{K}^+$. The proof of Theorem 3.1.2 depends strongly on the fact that the spectrum of a self-adjoint compact operator is a sequence of real numbers converging to zero. In an AF algebra other than $\mathcal{K}^+$, the spectrum of an operator might very well be the entire unit interval. It seems unlikely, therefore, that the technique used in the proof of Theorem 3.1.2 will yield results about maps with range in other AF algebras.

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