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A THEORY OF NON-NOETHERIAN GORENSTEIN RINGS

by

Livia M. Miller

A DISSERTATION

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A THEORY OF NON-NOETHERIAN GORENSTEIN RINGS

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In Noetherian rings there is a hierarchy among regular, Gorenstein and Cohen-Macaulay rings. Regular non-Noetherian rings were originally defined by Bertin in 1971. In 2007, Hamilton and Marley used Čech cohomology to introduce a theory of Cohen-Macaulay for non-Noetherian rings, answering a question posed by Glaz. This dissertation provides a theory of non-Noetherian Gorenstein rings agreeing with the Noetherian definition, and for which regular rings are Gorenstein, and coherent Gorenstein rings are Cohen-Macaulay. The relationship between Gorenstein rings and FP-injective dimension as defined by Stenström is also explored. Finally, an additional characterization of Gorenstein rings involving homological dimensions is examined in the non-Noetherian case.

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Contents

Chapter 1

Introduction

The theories of regular, Gorenstein, and Cohen-Macaulay Noetherian rings form a rich theory within commutative algebra. These three rings enjoy the following cascading relationship in the context of local rings:

$$\text{Regular} \Rightarrow \text{Gorenstein} \Rightarrow \text{Cohen-Macaulay}.$$

All three rings have origins and applications in homological algebra, algebraic geometry, and combinatorics. The goal of this work is to extend the meaning of Gorenstein to the non-Noetherian case while maintaining the relationships shown above.

The definitions of non-Noetherian regular and Cohen-Macaulay rings have been previously explored, primarily in the context of coherent rings. Coherence, like the Noetherian property, is a finiteness condition. Given a ring R, an R-module is *coherent* if it is finitely generated and every finitely generated submodule is finitely presented. A ring is coherent if it is coherent as a module over itself. Noetherian rings are clearly coherent; however there are many examples of non-Noetherian coherent rings. For instance, given a field k, $k[x_1, x_2, \dots]$ is a coherent ring, but obviously not Noetherian.

In 1971, Bertin [4] defined regular rings in the non-Noetherian case. While exploring invariant subrings of polynomial rings over a coherent ring, Glaz raised the question in [13] and [14] of the existence of a definition of non-Noetherian Cohen-Macaulay rings compatible with (coherent) regular rings. Hamilton and Marley [15] provided a positive answer, using Čech cohomology to introduce a theory of non-Noetherian Cohen-Macaulay rings. In addition, several of the usual properties of Noetherian Cohen-Macaulay rings carry over to the non-Noetherian theory. Given the cascading relationship above, a natural question arose as to whether there is a theory of non-Noetherian Gorenstein rings for which coherent regular rings are Gorenstein, and Gorenstein rings are Cohen-Macaulay. Chapter 5 develops a theory of Gorenstein rings which is compatible with the Noetherian definition and provides an affirmative answer to this question.

Gorenstein dimension, or G-dimension, is among the major protagonists in the theory of non-Noetherian Gorenstein rings. In a local Noetherian regular ring every module has finite projective dimension. Auslander and Bridger [2] introduced G-dimension in 1969 to create a similar characterization for Gorenstein rings. A Noetherian ring is Gorenstein if every finitely generated module has finite G-dimension [2]. However the theory of G-dimension was restricted to finitely generated modules over a Noetherian ring. In response to this restriction, Enochs and Jenda [10] introduced Gorenstein projective dimension for arbitrary modules over an arbitrary commutative ring. Avramov, Buchweitz, Martsinkovsky, and Reiten (see the remark following Theorem 4.2.6 in [9]) showed that for finitely generated modules over a Noetherian ring, Gorenstein projective dimension and G-dimension are the same. In the meantime, a theory of Gorenstein projective dimension for Noetherian rings has been developed by Avramov, Christensen, Enochs, Foxby, Holm, Jenda, and Martsinkovsky, among others. More recently some of these results have been extended to the non-Noetherian case by Holm [18] and White [28].

Amongst the results of G-dimension is an extension of the Auslander-Buchsbaum Formula, referred to as the Auslander-Bridger formula. If $\text{G-dim}_R M$ denotes the Gorenstein dimension of an R-module M we have:

> If R is a local Noetherian ring and M is an R-module of finite G-dimension, then $\operatorname{depth}_R M + \text{G-dim}_R M = \operatorname{depth}_R R$.

The generalization of the Auslander-Bridger Formula to the non-Noetherian context is a cornerstone in the theory of non-Noetherian Gorenstein rings. However, the path to the non-Noetherian case is fraught with many hurdles. The first stems from the behavior of grade in the non-Noetherian context. For a Noetherian ring R, $\operatorname{grade}_R(I, M) > 0$ for any finitely generated R-module M if and only if $(0 :_M I) = 0$, yet there are examples of non-Noetherian rings where this is not the case (see [15] or [27]). Hochster, Northcott [24], and Alfonsi [1] played roles in the development of polynomial grade to overcome this inconsistency in the behavior of grade. The definition of polynomial grade is based upon grade over polynomial rings (see Section 2.2):

> Let R be a ring; the *polynomial grade* of an ideal I on an R-module M is $\text{p-grade}_R(I, M) := \lim_{m\to\infty} \operatorname{grade}_{R[t_1,\ldots,t_m]}(IR[t_1, \ldots, t_m], R[t_1, \ldots, t_m] \otimes_R M)$.

If $(R, \mathfrak{m})$ is *quasi-local* (that is, R has a unique maximal ideal $\mathfrak{m}$), define the polynomial depth, or p-depth of a module M to be $\operatorname{grade}_R(\mathfrak{m}, M)$. Some properties of polynomial grade are explored in Section 2.2.

With polynomial grade in hand we can generalize the Auslander-Bridger Formula to the non-Noetherian case, replacing polynomial depth for depth; this is the content of Corollary 4.2.2 (Chapter 4). However, after removing the Noetherian assumption, what assumptions are needed for the Auslander-Bridger Formula to hold for non-Noetherian rings? This is where the theory of non-Noetherian Gorenstein rings comes full circle. The obvious choice is to assume the ring is coherent. While the Auslander-Bridger does hold for coherent rings, our proof of the Auslander-Bridger Formula holds in a more general setting. If R is a coherent ring, $R[x]$ is not coherent in general. In order for the Auslander-Bridger Theorem to hold under the coherence assumption, one must pass to a polynomial ring and maintain coherence. One way this hurdle can be overcome is to assume that R is *stably coherent*, that is, to assume $R[x_1, \ldots, x_n]$ is coherent for every $n \geq 0$. However, this is an unsatisfactory solution; in fact we are able to do much better than coherence.

This is where modules of type $(FP)^R_\infty$ and $\mathcal{BE}(R)$-modules enter the scene. The module class $\mathcal{BE}(R)$ extends Bieri's notion of modules of type $(FP)^R_\infty$, a class of modules admitting degreewise finite projective resolutions. The class $\mathcal{BE}(R)$ consists of modules M of type $(FP)^R_\infty$ such that $\mathrm{Ext}^i_R(M, R)$ are of type $(FP)^R_\infty$ for all $i \geq 0$. Using modules of type $(FP)^R_\infty$ we consider a restricted form of G-dimension, denoted $\tilde{G}$-dimension, which agrees with G-dimension for modules in $\mathcal{BE}(R)$. We then can prove a form of the Auslander-Bridger Formula for modules of finite $\tilde{G}$-dimension (Theorem 4.2.1), with no conditions on the ring.

In Chapter 5, we define non-Noetherian Gorenstein rings; using the Generalized Auslander-Bridger Formula (Corollary 4.2.2) we are able to show the following relationships for quasi-local rings:

$$\text{Coherent Regular} \Rightarrow \text{Coherent Gorenstein} \Rightarrow \text{Cohen-Macaulay.}$$

A local Noetherian Gorenstein ring is characterized by having finite injective dimension. In an attempt to generalize this behavior to non-Noetherian rings, we introduce Stenström's [26] FP-injective dimension in Section 2.4; in Section 5.3 we find a relationship between coherent Gorenstein rings and FP-injective dimension.

This connection lends more credence to our definition of Gorenstein rings.

In the Noetherian case there are numerous other characterizations of Gorenstein rings; Chapter 6 contains preliminary results for an additional characterization. This characterization combines the approaches of FP-injective dimension with $\mathcal{BE}(R)$-modules. While this approach yields some results, it does not form as strong a theory as the one given in Chapter 5.

Chapter 2

Background

Throughout this work, rings are commutative and contain a multiplicative unit. The presence of Noetherian rings will always be explicitly stated. If a Noetherian ring has a unique maximal ideal the ring is called *local*; if the ring is not necessarily Noetherian, the terminology *quasi-local* will be used.

2.1 Coherent Rings and Modules

The notion of coherence plays an important part in the theory of non-Noetherian Cohen-Macaulay and Gorenstein rings. This section will give an overview of coherence and introduce the properties that help link Gorenstein and Cohen-Macaulay rings.

The definition of coherent modules relies on the notion of finitely presented modules. Given a ring R, an R-module M is *finitely presented* if there is an exact sequence $R^n \to R^m \to M \to 0$ for positive integers n and m.

Definition 2.1.1. Let R be a ring. An R-module M is *coherent* if M is finitely generated and every finitely generated submodule is finitely presented.

Subsequently a ring R is *coherent* if it is coherent as an R-module; thus every finitely generated ideal of a coherent ring is finitely presented. Noetherian rings themselves are coherent.

This definition of coherence can be, at times, unwieldy to work with. The following result of Chase [8] provides alternate characterizations of coherent rings that will become useful in Chapter 5.

Before we begin, recall the *annihilator* of an ideal I of a ring R is defined by $(0 :_R I) = \{r \in R | rI = 0\}$.

Theorem 2.1.2. *The following conditions are equivalent for a ring R.*

(i) *R is a coherent ring.*

(ii) *Every finitely presented R-module is a coherent module.*

(iii) *$(0 :_R r)$ is a finitely generated ideal for every element $r \in R$. In addition, the intersection of any two finitely generated ideals of R is finitely generated.*

Proof. See the proof of Theorem 2.3.2 of [12]. □

With these characterizations of coherence in hand, we next consider change of ring results for coherent rings and modules.

Theorem 2.1.3. *[16], [17] If R is a coherent ring and S is an R-algebra that is finitely presented as an R-module, then S is a coherent ring.*

Proof. Similar to the proof of Theorem 2.4.1 in [12]. □

Theorem 2.1.4. *[16], [17] Given a ring R and an ideal I of R,*

(i) *If M is a finitely presented R-module, then M/IM is a finitely presented R/I-module.*

(ii) If I is finitely generated and M is an R/I-module, then M is a finitely presented R-module if and only if M is a finitely presented R/I-module.

Proof. See the proof of Theorem 2.1.8 in [12]. □

Theorem 2.1.5. *[12, Theorem 2.2.6] Let R be a ring and U a multiplicatively closed subset of R. If M is a coherent R-module, then M_U is a coherent R_U-module.*

From Theorem 2.1.5 it follows immediately that if R is a coherent ring, then R_U is a coherent ring.

Starting in Chapter 3 we will consider homological dimensions over coherent rings. The following results are very useful in this context.

Theorem 2.1.6. *[12, Theorem 2.5.1] Let R be a coherent ring and $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. If any two of the modules are finitely presented, then so is the third.*

This leads to a characterization of finitely presented modules in a coherent ring closely related to the $\mathcal{BE}(R)$-modules introduced later in this chapter.

Corollary 2.1.7. *[12, Corollary 2.5.2] If R is a coherent ring and M a finitely presented R-module, then M admits a resolution of finitely generated free modules*

$$\cdots \to F_1 \to F_0 \to M \to 0.$$

Corollary 2.1.8. *[12, Corollaries 2.2.5 and 2.5.3] If R is coherent ring, and M and N are coherent R-modules, then the following modules are coherent:*

(i) $\mathrm{Tor}_n^R(M,N)$ *for* $n \geq 0$, *and*

(ii) $\mathrm{Ext}_R^n(M,N)$ *for* $n \geq 0$.

The final result of this section does not require R to be coherent, but will be used frequently while working with finitely presented modules over coherent rings.

Theorem 2.1.9. *[7] Let R be a ring and S a flat R-algebra. Given R-modules M and N where M admits a resolution of finitely generated free modules,*

$$\operatorname{Ext}_R^n(M, N) \otimes_R S \cong \operatorname{Ext}_S^n(M \otimes_R S, N \otimes_R S)$$

for all $n \geq 0$. In particular, this implies that localization commutes with Ext.

2.2 Polynomial Grade

The classical notion of grade has been extended to the non-Noetherian setting through the work of Hochster, Northcott [24] and Alfonsi [1]. This extension relies upon the addition of indeterminates to a ring to force the existence of non-zero-divisors in situations where they must exist if the ring were Noetherian. In the Noetherian case $\operatorname{grade}_R(I, M) > 0$ for any finitely generated module M if and only if $(0 :_M I) = 0$. However, there are examples of non-Noetherian rings where this is not the case (see [15] or [27]). Extending to polynomial rings fixes this incongruity; the following result forms the basis of polynomial grade.

Theorem 2.2.1. *[24, Chapter 5, Theorem 7] Let R be a ring, $f = a_n x^n + \dots a_1 x + a_0 \in R[x]$ and set $I = (a_0, \dots, a_n)R$. Then $(0 :_R I) = 0$ if and only if f is a non-zero-divisor on $R[x]$.*

The *polynomial grade* of an ideal I on an R-module M is defined by

$$\operatorname{p-grade}_R(I, M) := \lim_{m \to \infty} \operatorname{grade}_{R[t_1, \dots, t_m]}(IR[t_1, \dots, t_m], R[t_1, \dots, t_m] \otimes_R M).$$

If $(R, \mathfrak{m})$ is a quasi-local ring, the *polynomial depth* of M is

$\text{p-depth}_R\, M := \text{p-grade}_R(\mathfrak{m}, M)$.

Remark 2.2.2. Given a finitely generated ideal $I = (x_1, \ldots, x_n)$ and an R-module M, $\text{grade}_R(I, M) \leq n$. Hence $\text{p-grade}_R(I, M) \leq n < \infty$ via the definition given above.

The following proposition summarizes some results about polynomial grade that will be used throughout this work.

Proposition 2.2.3. *Let R be a ring, I an ideal, and M an R-module.*

(i) *If there exists an exact sequence $F_n \to F_{n-1} \to \cdots \to F_0 \to R/I \to 0$ with the F_i finitely generated free modules, then* $\text{p-grade}_R(I, M) \geq n$ *if and only if* $\text{Ext}^i_R(R/I, M) = 0$ *for* $0 \leq i < n$.

(ii) $\text{p-grade}_R(I, M) = \text{p-grade}_R(p, M)$ *for some prime ideal p containing I. In particular,* $\text{p-grade}_R(I, M) = \text{p-grade}_R(\sqrt{I}, M)$.

(iii) $\text{p-grade}_R(I, M) = \sup\{\text{p-grade}_R(J, M) | J \subseteq I, J$ *a finitely generated ideal*$\}$.

(iv) *If* $M = \bigoplus_{i=1}^{n} M_n$*, where the M_j are R-modules, then* $\text{p-grade}_R(I, M) = \min_{1 \leq i \leq n} \{\text{p-grade}_R(I, M_i)\}$.

(v) *If I is finitely presented and S is a faithfully flat extension of R, then* $\text{p-grade}_R(I, M) = \text{p-grade}_S(IS, M \otimes_R S)$.

(vi) *Given an indeterminate x over R,* $\text{p-grade}_R(I, M) > 0$ *if and only if* $\text{grade}_{R[x]}(IR[x], M[x]) > 0$.

(vii) If $\mathbf{x} = x_1, \ldots, x_n \in I$ *form an* M*-regular sequence, then*

$$\begin{aligned}\text{p-grade}_R(I, M) &= \text{p-grade}_R(I, M/(x_1, \ldots, x_n)M) + n \\ &= \text{p-grade}_{R/(\mathbf{x})}((I + (\mathbf{x}))/(\mathbf{x}), M/(\mathbf{x})M) + n.\end{aligned}$$

Proof. The proof of (i) can be found in [12], Theorem 7.1.2 and is related to the results in [1]. The proofs of parts (iii) and (iv) can be found in Chapter 5 of [24]; (v) follows from the remark following Theorem 7.18 in [12]. Part (vi) is a partial restatement of Chapter 5, Theorem 7 in [24]. The proof of the first equality in (vii) can be found in Chapter 5, Theorem 15 of [24]. The second equality holds since for $J = (x_1, \ldots, x_n)$ an ideal of R, in terms of classical grade the following holds for all n since $J[y_1, \ldots, y_n] \subset \operatorname{Ann}_{R[y_1,\ldots,y_n]} R[y_1, \ldots, y_n] \otimes_R M/JM$.

$$\begin{aligned}&\text{grade}_{R[y_1,\ldots,y_n]}(IR[y_1, \ldots, y_n], R[y_1, \ldots, y_n] \otimes_R M/JM) \\ &= \text{grade}_{R[y_1,\ldots,y_n]/J[y_1,\ldots,y_n]}\left(\frac{(I+J)R[y_1, \ldots, y_n]}{J[y_1, \ldots, y_n]}, R[y_1, \ldots, y_n] \otimes_R M/JM\right) \\ &= \text{grade}_{(R/J)[y_1,\ldots,y_n]}\left(\left(\frac{I+J}{J}\right)(R/J)[y_1, \ldots, y_n], (R/J)[y_1, \ldots, y_n] \otimes_{R/J} M/JM\right).\end{aligned}$$

Taking limits over n the second equality on p-grade holds. □

There is also a relationship between the p-depth of modules in a short exact sequence.

Lemma 2.2.4. *[24, Chapter 5, Lemma 13] Let* R *be a quasi-local ring and let* $0 \to M_1 \to M_2 \to M_3 \to 0$ *be a short exact sequence of* R*-modules contained in* $[FP]^R_\infty$*. If* $\text{p-depth}_R M_2 > \text{p-depth}_R M_3$*, then* $\text{p-depth}_R M_1 = \text{p-depth}_R M_3 + 1$.

Proof. Set $\text{p-depth}_R M_2 = m$ and $\text{p-depth}_R M_3 = n$. For a finitely generated ideal I, apply $\text{Hom}_R(R/I, -)$ to the short exact sequence to obtain the long exact sequence

$$\cdots \to \text{Ext}^i_R(R/I, M_2) \to \text{Ext}^i_R(R/I, M_3) \to \text{Ext}^{i+1}_R(R/I, M_1) \\ \to \text{Ext}^{i+1}_R(R/I, M_2) \to \cdots .$$

Since $\text{Ext}^{i+1}_R(R/I, M_2) = 0$ and $\text{Ext}^i_R(M_3, R) = 0$ for all $i < n$ by Proposition 2.2.3(i), we have $0 = \text{Ext}^i_R(R/I, M_3) \cong \text{Ext}^{i+1}_R(R/I, M_1)$ for all $i < n$. Furthermore, $0 \to \text{Ext}^n_R(R/I, M_3) \to \text{Ext}^{n+1}_R(R/I, M_1)$ is exact, so $\text{Ext}^{n+1}_R(R/I, M_1) \neq 0$. Hence $\text{p-depth}_R\, M_1 = n + 1$. □

2.3 Non-Noetherian Regular and Cohen-Macaulay Rings

A Noetherian local ring is *regular* if every ideal of R has finite projective dimension. A Noetherian ring is regular if each of its localizations at a prime ideal is regular. This characterization was extended by Bertin [4] to the non-Noetherian case.

Definition 2.3.1. A quasi-local ring R is *regular* if every finitely generated ideal of R has finite projective dimension. In general, a ring R is regular if R_p is regular for all $p \in \text{Spec } R$.

Hamilton and Marley based their definition of non-Noetherian Cohen-Macaulay rings on the following characterization of Cohen-Macaulay:

Theorem 2.3.2. *A Noetherian ring R is Cohen-Macaulay if every sequence $\mathbf{x} = x_1, \ldots, x_n$ such that $\text{ht}(\mathbf{x})R = n$ is regular.*

Hamilton and Marley looked to a generalization of (partial) system of parameters, called *parameter sequences*, as a substitute for the role of height in Theorem 2.3.2.

The definition of parameter sequences incorporates the notion of weakly proregular sequences, as defined by Schenzel [25]. Let $\mathbf{x} = x_1, \ldots, x_t$ be a sequence of elements of R and define $\mathbf{x^n} = x_1^n, \ldots, x_t^n$. The Koszul complex $K(\mathbf{x})$ is defined to be the chain complex $K(x_1) \otimes_R \cdots \otimes_R K(x_t)$ where for each i, $K(x_i)$ is the Koszul chain complex $0 \to R \xrightarrow{x_i} R \to 0$ (where the first R sits in degree one). Denote $H(\mathbf{x})$ to be the homology of $K(\mathbf{x})$. Given $m \geq n$ there exist chain maps $\phi_n^m : K(\mathbf{x}^m) \to K(\mathbf{x}^n)$ given by $\phi_n^m(\mathbf{x}) = \phi_n^m(x_1) \otimes_R \cdots \otimes_R \phi_n^m(x_t)$ where for each i, $\phi_n^m(x_i)$ is the chain map between Koszul complexes

$$\begin{array}{ccccccc} 0 & \longrightarrow & R & \xrightarrow{x_i^m} & R & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle x_i^{m-n}} & & \Big\| & & \\ 0 & \longrightarrow & R & \xrightarrow{x_i^n} & R & \longrightarrow & 0. \end{array}$$

A sequence $\mathbf{x} = x_1, \ldots, x_t$ is *weakly proregular* if for each n there is an $m \geq n$ such that the canonical map $\phi : H_i(\mathbf{x}^m) \to H_i(\mathbf{x}^n)$ is zero for all $i \geq 1$. An element x is weakly proregular if and only if there exists $k > 0$ for which $(0 :_R x^k) = (0 :_R x^{k+1})$. In a Noetherian ring, every sequence is a weakly proregular sequence.

If $I = (x_1, \ldots, x_n)R$, the *ith local cohomology of a module* M is defined to be:

$$H_I^i(M) := \varinjlim \operatorname{Ext}_R^i(R/I^n, M).$$

For a sequence of elements $\mathbf{x} = x_1, \ldots, x_n$, let $C(\mathbf{x})$ denote the Čech complex with respect to $\mathbf{x}$, and set $C(\mathbf{x}; M) := C(\mathbf{x}) \otimes_R M$. The *ith Čech cohomology* $H_{\mathbf{x}}^i(M)$ *of* M with respect to $\mathbf{x}$ is the ith cohomology of $C(\mathbf{x}; M)$. By [25], $\mathbf{x}$ is a weakly proregular sequence if and only if $H_I^i(M) \cong H_{\mathbf{x}}^i(M)$ for all i and R-modules M, where $I = (\mathbf{x})$.

Definition 2.3.3. [15] A sequence $\mathbf{x} = x_1, \ldots, x_n$ is a *parameter sequence* if the following hold:

(i) $\mathbf{x}$ is weakly proregular.

(ii) $(\mathbf{x})R \neq R$.

(iii) $H^n_{\mathbf{x}}(R)_p \neq 0$ for all prime ideals p containing $(\mathbf{x})R$.

Furthermore, $\mathbf{x}$ is a *strong parameter sequence* if $x_1, \ldots, x_i$ is a parameter sequence for each $i = 1, \ldots, n$.

This construction leads to a definition of Cohen-Macaulay rings.

Definition 2.3.4. [15] A ring R is *Cohen-Macaulay* if every strong parameter sequence is regular. We say R is *locally Cohen-Macaulay* if R_p is Cohen-Macaulay for all $p \in \operatorname{Spec} R$.

This definition is equivalent to Theorem 2.3.2 if R is a Noetherian ring. Using coherence, Hamilton and Marley obtained the following result.

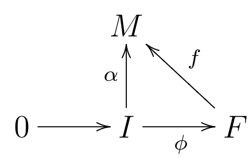
Theorem 2.3.5. *Coherent regular rings are locally Cohen-Macaulay.*

In [15] it is shown that under this definition of Cohen-Macaulay some, but not all, of the standard properties of Noetherian Cohen-Macaulay rings hold in the coherent case. For instance, given a faithfully flat ring homomorphism $f : R \to S$, if S is Cohen-Macaulay, then so is R. Also if all localizations of a ring at its maximal ideals are Cohen-Macaulay, then the ring itself is Cohen-Macaulay. However it is unknown whether the converse holds. In addition, there are examples of Cohen-Macaulay rings R such that $R/(x)$ is not Cohen-Macaulay for some regular element x (see [15]).

2.4 FP-Injective Dimension

FP-injective modules were introduced by Stenström [26] and comprise a class of which the class of injective modules is a subclass. While in the Noetherian case

FP-injective and injective modules coincide, this does not hold in general for non-Noetherian rings. To define FP-injective modules, consider a diagram similar to that of injective modules. An R-module M is *FP-injective* if for any free R-module F and a finitely generated F-submodule I, and given the inclusion $\phi : I \to F$ and a homomorphism $\alpha : I \to M$,



there is a homomorphism f such that $\alpha = f\phi$. In some works FP-injective modules are also referred to as *absolutely pure* modules.

Bass [3] proved that a ring R is Noetherian if and only if an arbitrary direct sum of injective modules is injective. It turns out that over any ring an arbitrary direct sum of FP-injective modules is FP-injective:

Proposition 2.4.1. *[22] Let R be a ring, and $\{M_i\}$ be a finite or infinite family of FP-injective R-modules. Then the following are equivalent.*

(i) M_i is FP-injective for all i.

(ii) $\bigoplus_i M_i$ is FP-injective for all i.

(iii) $\prod_i M_i$ is FP-injective for all i.

Using Bass' result and Proposition 2.4.1 one sees that in any non-Noetherian ring the classes of injective and FP-injective modules are not the same.

The definition of FP-injective modules given above can naturally be rephrased in terms of the vanishing of Ext modules. This characterization leads to a natural definition of FP-injective dimension.

Remark 2.4.2. Stenström [26] defines FP-injective modules in the following manner:

An R-module M is *FP-injective* if $\operatorname{Ext}_R^1(F, M) = 0$ for every finitely presented R-module F.

This definition is equivalent to the one presented at the beginning of this section. Throughout this work we will use Stenström's definition when talking about FP-injective modules.

Definition 2.4.3. [26] An R-module M has *FP-injective dimension at most n*, denoted $\text{FP-id}_R\, M \leq n$, if $\operatorname{Ext}_R^{n+1}(F, M) = 0$ for all finitely presented R-modules F.

We have the following characterizations of FP-injective dimension in the coherent case.

Lemma 2.4.4. *[26, Lemma 1.3] Let R be a coherent ring. For an R-module M, the following are equivalent:*

(i) $\text{FP-id}_R\, M \leq n$.

(ii) $\operatorname{Ext}_R^{n+1}(F, M) = 0$ *for all finitely presented modules F.*

(iii) $\operatorname{Ext}_R^{n+1}(R/I, M) = 0$ *for all finitely generated ideals I.*

(iv) If the sequence $0 \to M \to E_0 \to \cdots \to E_n \to 0$ is exact with $E_0, \ldots, E_{n-1}$ FP-injective, then E_n is also FP-injective.

A ring is *FP-injective* if it is FP-injective as a module over itself. Given an R-module M, let $M^* := \operatorname{Hom}_R(M, R)$. The next result shows that in a coherent FP-injective ring every finitely presented module is reflexive.

Proposition 2.4.5. *[26, Theorem 4.8] If R is a FP-injective coherent ring, for every finitely presented R-module M, $M \cong M^{**}$ via the canonical map.*

In the following we explore some additional properties of FP-injective rings. The next lemma, found in [19], holds for any ring.

Lemma 2.4.6. *[19, Theorem 1] Let R be a ring. The following conditions are equivalent:*

(i) Every homomorphism from a principal ideal of R into R is given by multiplication by an element in R.

(ii) If (a) is a principal ideal of R, then $(0 :_R (0 :_R (a))) = (a)$.

Lemma 2.4.7 and Proposition 2.4.8 are both attributed to [19] by [26]. The proofs are provided as they are not explicitly found in [19].

Lemma 2.4.7. *Let R be a ring. R is FP-injective if and only if for every finitely generated ideal I, every R-homomorphism $I \to R$ is multiplication by an element $r \in R$.*

Proof. Let I be a finitely generated ideal, and consider the short exact sequence

$$0 \to I \xrightarrow{\phi} R \to R/I \to 0.$$

Applying $\mathrm{Hom}_R(-, R)$ we have the exact sequence

$$0 \to \mathrm{Hom}_R(R/I, R) \to \mathrm{Hom}_R(R, R) \xrightarrow{\phi^*} \mathrm{Hom}_R(I, R) \to \mathrm{Ext}^1_R(R/I, R) \to 0.$$

If R is FP-injective, then $\mathrm{Ext}^1_R(R/I, R) = 0$ and ϕ^* is onto. As every map in $\mathrm{Hom}_R(R, R) \cong R$ is given by multiplication by an element in R, and ϕ^* is a restriction map, every map $\phi : I \to R$ is given by multiplication by an element in R.

Conversely, assume every map in $\mathrm{Hom}_R(I, R)$ is given by multiplication in R. Then ϕ^* is onto, so by exactness $\mathrm{Ext}^1_R(R/I, R) = 0$ and R is FP-injective. □

Proposition 2.4.8. *[19] A ring R is FP-injective if and only if* *(i)* $(0 :_R (0 :_R (a))) = (a)$ *for every* $a \in R$, *and*

(ii) $(0 :_R I) + (0 :_R J) = (0 :_R I \cap J)$ *for all finitely generated ideals* I *and* J.

Proof. If R is FP-injective, (i) holds by Lemmas 2.4.6 and 2.4.7. Consider the short exact sequence

$$0 \to R/(I \cap J) \xrightarrow{\phi} R/I \oplus R/J \to R/(I + J) \to 0$$

where $\phi(r + (I \cap J)) = (r + I, -r + J)$, $r \in R$. Then we have the exact sequence

$$\mathrm{Hom}_R(R/I, R) \oplus \mathrm{Hom}_R(R/J, R) \xrightarrow{\phi^*} \mathrm{Hom}_R(R/(I \cap J), R) \to \mathrm{Ext}^1_R(R/(I + J), R) \tag{2.4.1}$$

where given the projections $\pi_1 : R/I \oplus R/J \to R/I$ and $\pi_2 : R/I \oplus R/J \to R/J$, we have $\phi^*(f, g) = f\pi_1\phi + g\pi_2\phi$. Note $\mathrm{Ext}^1_R(R/(I + J), R) = 0$ since R is FP-injective and $I + J$ is finitely generated. However (2.4.1) is chain isomorphic to

$$(0 :_R I) \oplus (0 :_R J) \xrightarrow{\alpha} (0 :_R I \cap J) \to 0$$

where $\alpha(r, s) = r + s$. Note $(0 :_R I) + (0 :_R J) \subseteq (0 :_R I \cap J)$; since α is a surjection, $(0 :_R I) + (0 :_R J) = (0 :_R I \cap J)$. Hence (ii) holds.

Conversely, assume (i) and (ii) hold. Let I be a finitely generated ideal of R; we proceed by induction on the number of generators of I to show R is FP-injective. Consider the short exact sequence $0 \to I \to R \to R/I \to 0$. Assume first that $I = (a)$ is a principal ideal. Applying $\mathrm{Hom}_R(-, R)$ we get the exact sequence

$$R \xrightarrow{\phi} \mathrm{Hom}_R(I, R) \to \mathrm{Ext}^1_R(R/I, R) \to 0.$$

By Lemma 2.4.6, every element of $\mathrm{Hom}_R(I, R)$ is given by multiplication by an element in R. Hence ϕ is surjective and by exactness $\mathrm{Ext}^1_R(R/I, R) = 0$ for I principal.

Assume that $I = (x_1, \ldots x_n) = (x_1, \ldots, x_{n-1}) + (x_n)$. Consider the exact sequence

$$0 \to R/((x_1, \ldots, x_{n-1}) \cap (x_n)) \to R/(x_1, \ldots, x_{n-1}) \oplus R/(x_n) \to R/I \to 0. \quad (2.4.2)$$

By induction $\mathrm{Ext}^1_R(R/(x_1, \ldots, x_{n-1}), R) = \mathrm{Ext}^1_R(R/(x_n), R) = 0$; thus applying $\mathrm{Hom}_R(-, R)$ to (2.4.2) yields the exact sequence

$$\begin{aligned} &\mathrm{Hom}_R(R/(x_1, \ldots, x_{n-1}), R) \oplus \mathrm{Hom}_R(R/(x_n), R) \\ &\qquad \xrightarrow{\theta} \mathrm{Hom}(R/((x_1, \ldots, x_n) \cap (x_n)), R) \to \mathrm{Ext}^1_R(R/I, R) \to 0. \end{aligned} \quad (2.4.3)$$

However (2.4.3) is chain isomorphic to

$$\begin{aligned} 0 \to (0 :_R (x_1, \ldots, x_{n-1})) \oplus (0 :_R x_n) &\xrightarrow{\theta} (0 :_R (x_1, \ldots, x_{n-1}) \cap (x_n)) \\ &\to \mathrm{Ext}^1_R(R/I, R) \to 0, \end{aligned}$$

so by (ii), θ is surjective. By exactness $\mathrm{Ext}^1_R(R/I, R) = 0$ for all finitely generated I. Thus by definition R is FP-injective.

□

Given an ideal I of a ring R, we say I is *irreducible* if it cannot be written as $I = J \cap K$ with $J \neq I$ and $K \neq I$. The previous results reveal the following property of quasi-local FP-injective rings.

Proposition 2.4.9. *If* $(R, \mathfrak{m})$ *is a quasi-local* FP*-injective ring, then* (0) *is irreducible.*

Proof. By Proposition 2.4.8, if R is FP-injective, then $(0 :_R I) + (0 :_R J) = (0 :_R I \cap J)$ for all finitely generated ideals I and J.

Let I and J be non-zero finitely generated ideals of R, and assume that $I \cap J = (0)$. But then

$$(0 :_R I) + (0 :_R J) = (0 :_R I \cap J) = R. \tag{2.4.4}$$

Note that $(0 :_R I)$ and $(0 :_R J) \neq R$ as $I, J \neq 0$. Hence $(0 :_R I)$ and $(0 :_R J) \subset \mathfrak{m}$. From (2.4.4), there exist $x \in (0 :_R I)$ and $y \in (0 :_R J)$ such that $x + y = 1$. But this implies $x = 1 - y$ is a unit, contradicting $x \in \mathfrak{m}$ and $I \neq 0$. Hence $I \cap J \neq (0)$.

In general, let I and J be arbitrary non-zero ideals of R. Since $I \cap J = (0)$, then there exist elements $x \in I \setminus J$ and $y \in J \setminus I$. By the previous argument, $(x) \cap (y) \neq (0)$ implying $I \cap J \neq (0)$. Hence (0) is irreducible in R. □

2.5 $[FP]^R_\infty$- and $\mathcal{BE}(R)$-Modules

Using the notation of Bieri [5], for possibly infinite n, an R-module M *is of type* $(FP)^R_n$ if there is a projective resolution $\mathbf{P}$ of M of length n such that each P_i is finitely generated. Equivalently, M has a free resolution $\mathbf{F}$ of length n such that each F_i is finitely generated. Define $[FP]^R_n$ to be the class of modules of type $(FP)^R_n$.

The focus of this section lies primarily with the properties of $[FP]^R_\infty$-modules, however a few results will appear in terms of $[FP]^R_n$-modules. Since modules of type $(FP)^R_0$ are the finitely generated modules, modules of type $(FP)^R_\infty$ are clearly both finitely generated and finitely presented; in addition, we have the following characteristics of $[FP]^R_\infty$-modules.

Theorem 2.5.1. *[5, Corollary 1.6] The following are equivalent for an R-module M:*

(i) M is of type $(FP)^R_\infty$.

(ii) $\operatorname{Ext}^i_R(M, -)$ commutes with direct limits for all $i \geq 0$.

(iii) $\varinjlim \operatorname{Ext}^i_R(M, N_t) = 0$ *for all* $i \geq 0$ *and all directed systems* $\{N_t\}$ *of* R*-modules with* $\varinjlim N_t = 0$.

The following corollary extends a result of Bieri [5].

Corollary 2.5.2. *Suppose* $M \in [FP]^R_\infty$. *For* $i \geq 0$, *the following are equivalent.*

(i) $\operatorname{Ext}^i_R(M, R) = 0$.

(ii) $\operatorname{Ext}^i_R(M, Q) = 0$ *for all projective modules* Q.

(iii) $\operatorname{Ext}^i_R(M, T) = 0$ *for all flat modules* T.

Proof. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are clear; it remains to show (i) $\Rightarrow$ (iii). Note that if $F \cong R^n$, then $\operatorname{Ext}^i_R(M, F) \cong \bigoplus_{i=1}^{n} \operatorname{Ext}^i_R(M, R) = 0$. By [21], $T = \varinjlim F_i$ for F_i finitely generated free. Thus by Theorem 2.5.1(iii),

$$\operatorname{Ext}^i_R(M, T) = \varinjlim \operatorname{Ext}^i_R(M, F_i) = 0.$$

□

Proposition 2.5.3. *[5, Proposition 1.4] Let* $0 \to M' \to M \to M'' \to 0$ *be a short exact sequence of* R*-modules. Then the following hold:*

(i) If $M' \in [FP]^R_{n-1}$ *and* $M \in [FP]^R_n$, *then* $M'' \in [FP]^R_n$.

(ii) If M *and* $M'' \in [FP]^R_n$, *then* $M' \in [FP]^R_{n-1}$.

(iii) If M' *and* $M'' \in [FP]^R_n$, *then* $M \in [FP]^R_n$.

Corollary 2.5.4. *[5] Given an exact sequence* $0 \to M \to M' \to M'' \to 0$ *of* R*-modules, if any two modules are* $[FP]^R_\infty$*-modules, then so is the third.*

Corollary 2.5.5. *Let R be a ring. If $M \in [FP]_n^R$ for some possibly infinite $n \geq 1$, then given a presentation*

$$F_{n-1} \to \cdots \to F_0 \to M \to 0,$$

where F_i is a finitely generated free R-module for each i, $K = \ker(F_{n-1} \to F_{n-2})$ is finitely generated.

Proof. Proposition 2.5.3 provides the proof in the case where $M \in [FP]_1^R$. If $M \in [FP]_n^R$ for $n > 1$, let

$$F_{n-1} \to \cdots \to F_0 \to M \to 0$$

be a presentation of M by finitely generated free modules F_i. Setting $K_i = \ker(F_i \to F_{i-1})$, consider the short exact sequences

$$\begin{gathered} 0 \to K_{n-1} \to F_{n-1} \to K_{n-2} \to 0 \\ \vdots \\ 0 \to K_0 \to F_0 \to M \to 0. \end{gathered}$$

By Proposition 2.5.3, $K_i \in [FP]_{n-1-i}^R$ for all $0 \leq i \leq n-1$; hence K_{n-1} is finitely generated. $\square$

Definition 2.5.6. Given a ring R, define $\mathcal{BE}(R)$ to be the class of R-modules M such that M and $\operatorname{Ext}_R^i(M,R) \in [FP]_\infty^R$ for all $i \geq 0$.

All finitely presented modules M in a coherent ring are contained in $\mathcal{BE}(R)$ by Corollaries 2.1.7 and 2.1.8. This fact will play an important role in the development of the theory of non-Noetherian Gorenstein rings.

Chapter 3

Gorenstein Dimension

3.1 $\tilde{G}(R)$ and $\tilde{G}$-dimension

The following class of modules plays an important role in the proof of the generalized Auslander-Bridger Formula in Chapter 4. In the following, given a ring R and an R-module M, define $M^* := \mathrm{Hom}_R(M, R)$.

Definition 3.1.1. Given a ring R, $\tilde{G}(R)$ denotes the class of R-modules M such that

(i) M and $M^* \in [FP]^R_\infty$.

(ii) $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > 0$.

(iii) $\mathrm{Ext}^i_R(M^*, R) = 0$ for all $i > 0$.

(iv) The canonical map $M \to M^{**}$ is an isomorphism.

Remark 3.1.2. From the definition, we have the following results concerning $\tilde{G}(R)$-modules.

(i) Finitely generated free R-modules are contained in $\tilde{G}(R)$.

(ii) $A \oplus B \in \tilde{G}(R)$ if and only if $A, B \in \tilde{G}(R)$.

(iii) Finitely generated projective modules are contained in $\tilde{G}(R)$.

(iv) If $M \in \tilde{G}(R)$, then $M^* \in \tilde{G}(R)$.

Proof. (i) follows directly from Definition 3.1.1. Notice that (ii) holds via the isomorphism

$$\mathrm{Ext}^i_R(A \oplus B, R) \cong \mathrm{Ext}^i_R(A, R) \oplus \mathrm{Ext}^i_R(B, R)$$

for all $i \geq 0$. Since any finitely generated projective module P can be written as the direct summand of a finitely generated free module, (i) and (ii) imply (iii). Finally, if $M \in \tilde{G}(R)$, then $M \cong M^{**}$ and (iv) follows immediately. □

Proposition 3.1.3. *Let R be a ring and $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules with $N \in \tilde{G}(R)$. Then $M \in \tilde{G}(R)$ if and only if $L \in \tilde{G}(R)$.*

Proof. This is similar to the proof of [9] Lemma 1.1.10(a).

Applying $\mathrm{Hom}_R(-, R)$ we obtain the long exact sequence

$$0 \to N^* \to M^* \to L^* \to \mathrm{Ext}^1_R(N, R) \to \cdots$$
$$\cdots \to \mathrm{Ext}^i_R(N, R) \to \mathrm{Ext}^i_R(M, R) \to \mathrm{Ext}^i_R(L, R) \to \cdots .$$

Since $N \in \tilde{G}(R)$, $\mathrm{Ext}^i_R(N, R) = 0$ for all $i > 0$, yielding the short exact sequence

$$0 \to N^* \to M^* \to L^* \to 0 \tag{3.1.1}$$

and $\mathrm{Ext}^i_R(M, R) \cong \mathrm{Ext}^i_R(L, R)$ for all $i \geq 1$. Applying $\mathrm{Hom}_R(-, R)$ to (3.1.1) yields the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \delta_L} & & \downarrow{\scriptstyle \delta_M} & & \downarrow{\scriptstyle \delta_N}\,{\scriptstyle \cong} & & \\ 0 & \longrightarrow & L^{**} & \longrightarrow & M^{**} & \longrightarrow & N^{**}. & & \end{array}$$

If either L or $M \in \tilde{G}(R)$, then $\operatorname{Ext}^1_R(M,R) \cong \operatorname{Ext}^1_R(L,R) = 0$, and thus

$$0 \to L^{**} \to M^{**} \to N^{**} \to 0 \tag{3.1.2}$$

is exact. By the Snake Lemma, δ_L is an isomorphism if and only if δ_M is an isomorphism. Applying $\operatorname{Hom}_R(-,R)$ to (3.1.1), along with the exactness of (3.1.2), yields a long exact sequence that also shows that $\operatorname{Ext}^i_R(L^*,R) \cong \operatorname{Ext}^i_R(M^*,R)$ for all $i > 0$.

Finally, Corollary 2.5.4 combined with (3.1.1) and our original sequence shows that $L \in [FP]^R_\infty$ if and only if $M \in [FP]^R_\infty$, and $L^* \in [FP]^R_\infty$ if and only if $M^* \in [FP]^R_\infty$. $\square$

Remark 3.1.4. Let R be a ring and M be an R-module. Then $M \in [FP]^R_\infty$ if and only if there is an exact sequence

$$\cdots \to G_i \to G_{i-1} \to \cdots \to G_0 \to M \to 0$$

where $G_i \in \tilde{G}(R)$ for all i.

Proof. If $M \in [FP]^R_\infty$, then F has an infinite resolution by finitely generated free modules, which themselves are also in $\tilde{G}(R)$. Conversely, assume M has an infinite resolution

$$\cdots \to G_i \to G_{i-1} \to \cdots \to G_0 \to M \to 0$$

by modules $G_i \in \tilde{G}(R)$. Then M is clearly finitely generated, that is, $M \in [FP]^R_0$. Assume that any such M is contained in $[FP]^R_{n-1}$ for some n. Set $K = \ker(G_0 \to M)$.

Then the sequences

$$0 \to K \to G_0 \to M \to 0$$

and

$$\cdots \to G_2 \to G_1 \to K \to 0$$

are exact.

By induction, $K \in [FP]^R_{n-1}$. Since $G_0 \in [FP]^R_\infty$, then $M \in [FP]^R_n$ by Proposition 2.5.3. □

Definition 3.1.5. A *$\tilde{G}$-resolution* of an R-module M is a complex $\tilde{\mathbf{G}}$

$$\cdots \to G_i \to G_{i-1} \to \cdots \to G_1 \to G_0 \to 0$$

such that each $G_i \in \tilde{G}(R)$, $H_i(\tilde{\mathbf{G}}) = 0$ for $i > 0$ and $H_0(\tilde{\mathbf{G}}) = M$.

Definition 3.1.6. Let $M \in [FP]^R_\infty$ be a nonzero R-module. Define the $\tilde{G}$-dimension of M, denoted $\tilde{\mathrm{G}}\text{-}\dim_R M$, to be

$$\tilde{\mathrm{G}}\text{-}\dim_R M = \inf\{n | 0 \to G_n \to \cdots \to G_0 \to M \to 0 \text{ is a } \tilde{G}\text{-resolution of } M\}.$$

If M has no finite $\tilde{G}$-resolution, $\tilde{\mathrm{G}}\text{-}\dim_R M = \infty$.

Lemma 3.1.7. *Let R be a ring and M be an R-module with $\tilde{\mathrm{G}}\text{-}\dim_R M < \infty$. If $\mathrm{Ext}^i_R(M, R) = 0$ for all $i > 0$, then $M \in \tilde{G}(R)$.*

Proof. The proof of this result is analogous to Lemma 1.2.6 in [9].

Assume $\tilde{\mathrm{G}}\text{-}\dim_R M \leq 1$; then there is an exact sequence

$$0 \to G_1 \to G_0 \to M \to 0,$$

where $G_0, G_1 \in \tilde{G}(R)$. Applying $\mathrm{Hom}_R(-, R)$ we obtain the exact sequence

$$0 \to M^* \to G_0^* \to G_1^* \to 0$$

since $\mathrm{Ext}_R^1(M, R) = 0$. Note that $M^* \in [FP]_\infty^R$ by Corollary 2.5.4. Since G_1^* and $G_0^* \in \tilde{G}(R)$, the sequence

$$0 \to G_1^{**} \to G_0^{**} \to M^{**} \to 0$$

is exact, and shows $\mathrm{Ext}_R^m(M^*, R) = 0$ for all $m > 0$. Hence we have the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_1 & \longrightarrow & G_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \downarrow & & \\ 0 & \longrightarrow & G_1^{**} & \longrightarrow & G_0^{**} & \longrightarrow & M^{**} & \longrightarrow & 0, \end{array}$$

and thus $M \cong M^{**}$ via the Five Lemma. Hence $M \in \tilde{G}(R)$.

For $n > 1$ assume $\tilde{\mathrm{G}}\text{-}\dim_R M \leq n$, and consider the $\tilde{G}$-resolution

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0.$$

Setting $K_{n-1} = \ker(G_{n-2} \to G_{n-3})$ we see that $K_{n-1} \in [FP]_\infty^R$ and $\tilde{\mathrm{G}}\text{-}\dim_R K_{n-1} \leq 1$ as we have the short exact sequence

$$0 \to G_n \to G_{n-1} \to K_{n-1} \to 0$$

Using the fact that $\mathrm{Ext}_R^m(K_{n-1}, R) \cong \mathrm{Ext}_R^m(K_{n-2}, R)$ and iterating along the K_i, we see that $\mathrm{Ext}_R^m(K_{n-1}, R) \cong \mathrm{Ext}_R^{n+m-1}(M, R) = 0$ for all $m > 0$, so $K_{n-1} \in \tilde{G}(R)$. Thus the exact sequence

$$0 \to K_{n-1} \to G_{n-2} \to \cdots \to G_1 \to G_0 \to M \to 0$$

yields $\tilde{\mathrm{G}}\text{-}\dim_R M \leq n-1$, and thus by induction $M \in \tilde{G}(R)$. □

The following result provides additional characterizations of $\tilde{G}$-dimension, and is an adaptation of Theorem 1.2.7 in [9].

Theorem 3.1.8. *Let R be a ring and $M \in [FP]^R_\infty$. The following are equivalent*

(i) $\tilde{\mathrm{G}}\text{-}\dim_R M \leq n$.

(ii) $\tilde{\mathrm{G}}\text{-}\dim_R M < \infty$ *and* $\mathrm{Ext}^i_R(M, R) = 0$ *for all* $i > n$.

(iii) $\tilde{\mathrm{G}}\text{-}\dim_R M < \infty$ *and* $\mathrm{Ext}^i_R(M, Q) = 0$ *for* $m > n$ *and any flat module* Q.

(iv) In any $\tilde{G}$-resolution of M

$$\cdots \to G_n \to G_{n-1} \to \cdots \to G_0 \to 0,$$

the kernel $K = \ker(G_{n-1} \to G_{n-2})$ *is in* $\tilde{G}(R)$.

Thus if $\tilde{\mathrm{G}}\text{-}\dim_R M < \infty$*, then*

$$\tilde{\mathrm{G}}\text{-}\dim_R M = \sup\{n \in \mathbb{N} | \mathrm{Ext}^n_R(M, R) \neq 0\}.$$

Proof. The following is an adaptation of Theorem 1.2.7 in [9].

Notice that once (i) ⇔ (ii) is established the final equality holds. Also notice that (iii) is equivalent to (ii) via an application of Corollary 2.5.2.

If $n = 0$, the four conditions are equivalent by definition and Lemma 3.1.7.

Assume $n > 0$.

(i) $\Rightarrow$ (ii): If $\tilde{\text{G}}\text{-dim}_R\, M \leq n$ then M has a $\tilde{G}$-resolution

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0.$$

By applying $\text{Hom}_R(-, R)$ to the short exact sequences formed by the kernels, we have $\text{Ext}_R^{m+n}(M, R) \cong \text{Ext}_R^m(G_n, R) = 0$ for $m > 0$, that is, $\text{Ext}_R^n(M, R) = 0$ for all $n > m$.

(ii) $\Rightarrow$ (i): M has a finite length $\tilde{G}$-resolution:

$$0 \to G_\ell \to \cdots \to G_1 \to G_0 \to M \to 0.$$

If $\ell \leq n$, we are done; assume $\ell > n$. Set $K = \ker(G_{n-1} \to G_{n-2})$ and consider the exact sequence

$$0 \to K \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

where $K \in [FP]_\infty^R$ and $\tilde{\text{G}}\text{-dim}_R\, K \leq \ell - n$. As above, we also have $\text{Ext}_R^m(K, R) \cong \text{Ext}_R^{m+n}(M, R) = 0$ for $m > 0$. Hence by Lemma 3.1.7, $K \in \tilde{G}(R)$ and M has a $\tilde{G}$-resolution of length n.

(i) $\Leftrightarrow$ (iv): As (iv) $\Rightarrow$ (i) is obvious, assume $\tilde{\text{G}}\text{-dim}_R\, M \leq n$, so there is a $\tilde{G}$-resolution of length n:

$$0 \to G_n \to \cdots G_1 \to G_0 \to 0.$$

It is sufficient to show that if

$$0 \to H_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

$$0 \to K_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0$$

are exact sequences with the P_i finitely generated projective and the $G_i \in \tilde{G}(R)$, then

$H_n \in \tilde{G}(R)$ if and only if $K_n \in \tilde{G}(R)$ is as well. As the P_i are projective there exist $f_n, \ldots, f_0$

$$\begin{array}{cccccccccccc} \mathbf{P}: 0 & \longrightarrow & H_n & \longrightarrow & P_{n-1} & \longrightarrow & \cdots & \longrightarrow & P_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & & & \downarrow{\scriptstyle f_0} & & \\ \mathbf{G}: 0 & \longrightarrow & K_n & \longrightarrow & G_{n-1} & \longrightarrow & \cdots & \longrightarrow & G_0 & \longrightarrow & 0 \end{array}$$

such that f is a chain map between the complexes $\mathbf{P}$ and $\mathbf{G}$ lifting the identity map $H_0(P) \to H_0(G)$. Let $\mathbf{C}$ be the mapping cone of f. From the short exact sequence of complexes

$$0 \to \mathbf{G} \to \mathbf{C} \to \mathbf{P}[-1] \to 0$$

we get the exact sequence

$$\cdots \to H_i(\mathbf{G}) \to H_i(\mathbf{C}) \to H_{i-1}(\mathbf{P}) \to H_{i-1}(\mathbf{G}) \to \cdots .$$

The map from $H_i(\mathbf{P})$ to $H_i(\mathbf{G})$ is the map on homology induced by f. In particular, the map $H_0(\mathbf{P}) = M \to H_0(\mathbf{G}) = M$ is the identity map, as f_0 is a lifting of the identity map on M. Placing this in the long exact sequence above and using that $H_i(\mathbf{P}) = H_i(\mathbf{G}) = 0$ for all $i > 0$, one gets that $H_i(C) = 0$ for all i. Hence the mapping cone is exact.

Recall the mapping cone is as follows:

$$0 \to H_n \to K_n \oplus P_{n-1} \to G_{n-1} \oplus P_{n-2} \to \cdots \to G_1 \oplus P_0 \to G_0 \to 0.$$

By Remark 3.1.2 each $G_i \oplus P_{i-1} \in \tilde{G}(R)$. Repeated applications of Proposition 3.1.3 yields that $H_n \in \tilde{G}(R)$ if and only if $K_n \oplus P_{n-1} \in \tilde{G}(R)$. So again by Remark 3.1.2, $H_n \in \tilde{G}(R)$ if and only if $K_n \in \tilde{G}(R)$. □

Corollary 3.1.9. *Let R be a ring and $0 \to M \to G \to N \to 0$ be a short exact* *sequence of R-modules such that* $G \in \tilde{G}(R)$ *and* $N, M \in [FP]_\infty^R$.

(i) If $N \in \tilde{G}(R)$*, then so is* M.

(ii) If $N \notin \tilde{G}(R)$*, then* $\tilde{\text{G}}\text{-dim}_R\, M = \tilde{\text{G}}\text{-dim}_R\, N - 1$.

Proof. (i) follows immediately from Proposition 3.1.3.

To prove (ii), notice first that any $\tilde{G}$-resolution of M of length n results in a $\tilde{G}$-resolution of N of length $n + 1$. Hence if $\tilde{\text{G}}\text{-dim}_R\, M < \infty$, then $\tilde{\text{G}}\text{-dim}_R\, N < \infty$. On the other hand, if $\tilde{\text{G}}\text{-dim}_R\, N = n < \infty$, consider the resolution

$$\cdots \to G_{n-1} \to G_{n-2} \to \cdots \to G_0 \to M \to 0$$

of M, where $G_i \in \tilde{G}(R)$ for all i. Then

$$\cdots \to G_{n-1} \to G_{n-2} \to \cdots \to G_0 \to G \to N \to 0$$

is a $\tilde{G}$-resolution of N. By Theorem 3.1.8(iv), $K = \ker(G_{n-2} \to G_{n-3})$ is contained in $\tilde{G}(R)$. Thus the sequence

$$0 \to K \to G_{n-2} \to \cdots \to G_0 \to M \to 0$$

is exact and $\tilde{\text{G}}\text{-dim}_R\, M < \infty$. Hence $\tilde{\text{G}}\text{-dim}_R\, M$ and $\tilde{\text{G}}\text{-dim}_R\, N$ must be simultaneously finite; in particular if one is infinite, the equality holds.

Assume $\tilde{\text{G}}\text{-dim}_R\, N < \infty$ (and hence $\tilde{\text{G}}\text{-dim}_R\, M < \infty$). Apply $\text{Hom}_R(-, R)$ to the short exact sequence to get

$$0 \to N^* \to G^* \to M^* \to \text{Ext}_R^1(N, R) \to 0 \to \text{Ext}_R^1(M, R) \to \text{Ext}_R^2(N, R) \to 0 \to \cdots.$$

Thus $\operatorname{Ext}_R^i(M,R) \cong \operatorname{Ext}_R^{i+1}(N,R)$ for all $i > 0$, and by Theorem 3.1.8 $\tilde{\mathrm{G}}\text{-}\dim_R M = \tilde{\mathrm{G}}\text{-}\dim_R N - 1$. □

Proposition 3.1.10. *Let R be a ring and $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules contained in $[FP]_\infty^R$.*

(i) If $\tilde{\mathrm{G}}\text{-}\dim_R L \leq n$ and $\tilde{\mathrm{G}}\text{-}\dim_R N \leq n$, then $\tilde{\mathrm{G}}\text{-}\dim_R M \leq n$.

(ii) If $\tilde{\mathrm{G}}\text{-}\dim_R M \leq n$ and $\tilde{\mathrm{G}}\text{-}\dim_R N \leq n$, then $\tilde{\mathrm{G}}\text{-}\dim_R M \leq n$.

(iii) If $\tilde{\mathrm{G}}\text{-}\dim_R L \leq n$ and $\tilde{\mathrm{G}}\text{-}\dim_R M \leq n$, then $\tilde{\mathrm{G}}\text{-}\dim_R N \leq n+1$.

In particular, if any two of the modules has finite $\tilde{G}$-dimension, then so does the third.

Proof. To prove (i), let $(\mathbf{F}, \phi)$ and $(\mathbf{F}', \phi')$ be free-resolutions of L and N, respectively, consisting of finitely generated free R-modules. By the Horseshoe Lemma, there exists a free resolution $(\mathbf{F}'', \phi'')$ of M consisting of finitely generated free R-modules such that the sequence

$$0 \to \mathbf{F} \to \mathbf{F}'' \to \mathbf{F}' \to 0$$

is exact.

Let K_n, K_n', and K_n'' denote the kernels of ϕ_{n-1}, ϕ_{n-1}', and ϕ_{n-1}'', respectively. Then the sequence

$$0 \to K_n \to K_n'' \to K_n' \to 0$$

is exact. Since $\tilde{\mathrm{G}}\text{-}\dim_R L \leq n$ and $\tilde{\mathrm{G}}\text{-}\dim_R N \leq n$, K_n and K_n' are contained in $\tilde{G}(R)$ by Theorem 3.1.8. Hence $K_n'' \in \tilde{G}(R)$ by Proposition 3.1.3, and $\tilde{\mathrm{G}}\text{-}\dim_R M \leq n$.

The proof of (ii) proceeds in a similar way. While the proof of (iii) also begins in the same manner, once arriving at the short exact sequence

$$0 \to K_n \to K_n'' \to K_n' \to 0 \tag{3.1.3}$$

we see instead that $\tilde{\text{G}}\text{-dim}_R\, K_n' \leq 1$ as $K_n, K_n'' \in \tilde{G}(R)$. Pasting together $\mathbf{F}'$ and (3.1.3) yields the exact sequence

$$0 \to K_n \to K_n'' \to F_{n-1}' \to \cdots \to F_0' \to N \to 0$$

Hence $\tilde{\text{G}}\text{-dim}_R\, N \leq n+1$. □

3.2 Gorenstein Dimension

Removing the restrictions placed on $\tilde{G}(R)$-modules M that M and M^* be in $[FP]_\infty^R$ results in a resolving class that forms the basis of Gorenstein dimension. Gorenstein dimension was originally defined by Auslander and Bridger [2] to characterize Gorenstein rings in a manner similar to the characterization of regular rings. Instead of using projective modules to resolve a given module, one uses totally reflexive modules, which are defined below.

Definition 3.2.1. A finitely generated module M is *totally reflexive* if and only if the following conditions hold:

- $\text{Ext}_R^i(M, R) = 0$ for all $i > 0$.
- $\text{Ext}_R^i(M^*, R) = 0$ for all $i > 0$.
- The canonical map $M \to M^{**}$ is an isomorphism.

Note any finitely generated projective module, as well as any module in $\tilde{G}(R)$, is totally reflexive.

Remark 3.2.2. Note that a totally reflexive R-module M is contained in $\tilde{G}(R)$ if and only if M and $M^* \in [FP]_\infty^R$.

Using totally reflexive modules, one can build a theory of Gorenstein dimension where totally reflexive modules are the modules of Gorenstein dimension zero.

Definition 3.2.3. A *G-resolution* of an R-module M is a complex $\mathbf{G}$

$$\cdots \to G_t \to G_{t-1} \to \cdots \to G_1 \to G_0 \to 0$$

such that each G_i is totally reflexive, $H_i(\mathbf{G}) = 0$ for $i > 0$, and $H_0(\mathbf{G}) \cong M$.

Definition 3.2.4. Given a ring R, suppose the R-module M has a G-resolution. The *Gorenstein dimension*, or *G-dimension*, of an R-module M is defined as follows:

$$\text{G-dim}_R\, M = \inf\{n | 0 \to G_n \to \cdots \to G_0 \to M \to 0 \text{ is a } G\text{-resolution of } M\}.$$

If M has no finite G-resolution, $\text{G-dim}_R\, M = \infty$.

Gorenstein projective dimension, an extension of G-dimension developed by Enochs and Jenda, appears more widely in the literature. Let's take a look at its construction.

Definition 3.2.5. An exact resolution of projective modules

$$\mathbf{P} : \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

is a *complete projective resolution* if $\text{Hom}_R(\mathbf{P}, Q)$ is exact for every projective R-module Q.

Definition 3.2.6. An R-module M is *Gorenstein projective* if there exists a complete projective resolution $\mathbf{P}$ with $M \cong \text{Im}(P_0 \to P^0)$.

It is clear from the definition of Gorenstein projective modules that projective modules are also Gorenstein projective. Notice that Gorenstein projective modules, unlike totally reflexive modules, need not be finitely generated.

A chain complex $\mathbf{G}$

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to 0$$

is a *Gorenstein projective resolution of* M *of length* n if each G_i is a Gorenstein projective R-module, $G_n \neq 0$, $H_i(\mathbf{G}) = 0$ for all $i \neq 0$, and $H_0(\mathbf{G}) \cong M$.

Definition 3.2.7. The *Gorenstein projective dimension* of an R module M is given by

$$\begin{aligned} \operatorname{Gpd}_R M = \inf\{n | 0 \to G_n \to \cdots \to G_0 \to M \to 0 \\ \text{is a Gorenstein projective resolution of } M\}. \end{aligned}$$

If M has no finite Gorenstein projective resolution, $\operatorname{Gpd}_R M = \infty$.

For finitely generated modules over a Noetherian ring, G-dimension and Gorenstein projective dimension are equal (Avramov, Buchweitz, Martsinkovsky, and Reiten; see remark following Theorem 4.2.6 in [9]). It was shown in [28] that in the case of $\mathcal{BE}(R)$-modules there is no distinction between these two classes of modules.

Proposition 3.2.8. *Let* $M \in \mathcal{BE}(R)$. *Then the following are equivalent:*

(i) M *is Gorenstein projective.*

(ii) M *is totally reflexive.*

(iii) M *has a complete projective resolution consisting of finitely generated free modules.*

Proof. See [28] Lemma 5.3 and Theorem 5.4 □

Corollary 3.2.9. *[28, Corollary 5.5] Let R be a ring. If $M \in \mathcal{BE}(R)$, then* $\text{G-dim}_R\, M = \text{Gpd}_R\, M$.

For the remainder of this work, we prefer to use the notation of G-dimension since we primarily use the properties of Definition 3.2.1 in our proofs.

We now explore the properties of G-dimension that will be used in subsequent chapters. Some of these results have already appeared in the context of $\tilde{G}$-dimension. We will frequently use [9] as a reference for the the basic properties of totally reflexive modules and G-dimension. Throughout [9] the underlying assumption is that R is Noetherian; however many of these results hold over arbitrary rings. The proof of these results will appear only if substantial changes are needed for them to hold in the present context, otherwise the corresponding result in [9] will be cited.

Analogous to Proposition 3.1.3, we have:

Proposition 3.2.10. *Let R be a ring and $0 \to L \to M \to N \to 0$ be an exact sequence of finitely generated R-modules with N totally reflexive. Then M is totally reflexive if and only if L is totally reflexive.*

Proof. The proof of this result is similar to the proofs of [9] Lemma 1.1.10(a) and Proposition 3.1.3. □

Corollary 3.2.11. *Let R be a ring and assume M is a totally reflexive R-module with $\tilde{\text{G}}\text{-dim}_R\, M < \infty$. Then $M \in \tilde{G}(R)$.*

Proof. Assume $\tilde{\text{G}}\text{-dim}_R\, M = n$. Then M has a $\tilde{G}$-resolution

$$0 \to G_n \to \cdots \to G_0 \to M \to 0. \tag{3.2.1}$$

We may break up (3.2.1) into the short exact sequences

$$0 \to G_n \to G_{n-1} \to K_{n-1} \to 0$$

$$0 \to K_i \to G_{i-1} \to K_{i-1} \to 0,\ 1 \le i \le n-1$$

$$0 \to K_0 \to G_0 \to M \to 0.$$

Iterated applications of Corollary 2.5.4 shows $M \in [FP]^R_\infty$. Also by repeated applications of Corollary 3.1.9, K_i is totally reflexive for each $0 \le i \le n$. Applying $\mathrm{Hom}_R(-, R)$ to the sequences above yields the exact sequences

$$0 \to K^*_{n-1} \to G^*_{n-1} \to G^*_n \to 0$$

$$0 \to K^*_{i-1} \to G^*_{i-1} \to K^*_i \to 0,\ 1 \le i \le n-1$$

$$0 \to M^* \to G^*_0 \to K^*_0 \to 0,$$

and applying Corollary 2.5.4 gives $M^* \in [FP]^R_\infty$. □

A means of measuring G-dimension that will become particularly useful is the vanishing of Ext-modules.

Lemma 3.2.12. *Let R be a ring and M an R-module of finite G-dimension. If $\mathrm{Ext}^m_R(M, R) = 0$ for all $m > 0$, then M is totally reflexive.*

Proof. The proof of this result is analogous to Lemma 1.2.6 in [9], as well as Lemma 3.1.7.

□

Theorem 3.2.13. *Let R be a ring and $M \in [FP]^R_\infty$ be an R-module The following are equivalent:*

(i) $\text{G-dim}_R\, M \le n$.

(ii) $\text{G-dim}_R M < \infty$ *and* $\operatorname{Ext}^i_R(M, R) = 0$ *for* $i > n$.

(iii) $\text{G-dim}_R M < \infty$ *and* $\operatorname{Ext}^i_R(M, Q) = 0$ *for* $m > n$ *and any flat module* Q.

(iv) In any G-resolution of M

$$\cdots \to G_n \to G_{n-1} \to \cdots \to G_0 \to 0,$$

the kernel $K = \ker(G_{n-1} \to G_{n-2})$ *is totally reflexive.*

In addition, if $\text{G-dim}_R M < \infty$ *then*

$$\text{G-dim}_R M = \sup\{i \in \mathbb{N}_0 | \operatorname{Ext}^i_R(M, R) \neq 0\}.$$

Proof. The proof of the equivalence of (i), (ii), and (iv), as well as the last statement, is similar to that of Theorem 1.2.7 in [9]. The proof is also similar to that of Theorem 3.1.8.

Condition (iii) is equivalent to (ii) via an application of Corollary 2.5.2.

□

Lemma 3.2.14. *Let* R *be a ring and* $0 \to K \to G \to M \to 0$ *be a short exact sequence of* R*-modules where* $G \in \tilde{G}(R)$*. If* $M \in \mathcal{BE}(R)$*, then* $K \in \mathcal{BE}(R)$*.*

Proof. By Corollary 2.5.4, $K \in [FP]^R_\infty$. Applying $\operatorname{Hom}_R(-, R)$ to the given sequence we obtain:

$$0 \to M^* \to G^* \xrightarrow{\phi} K^* \to \operatorname{Ext}^1_R(M, R) \to 0 \to \operatorname{Ext}^1_R(K, R) \to \operatorname{Ext}^2_R(M, R) \to 0 \to \cdots.$$

Then $\operatorname{Ext}^i_R(K, R) \cong \operatorname{Ext}^{i+1}_R(M, R) \in [FP]^R_\infty$ for all $i > 0$. Hence it remains to show that $K^* \in [FP]^R_\infty$. Let $L = \operatorname{im}\phi$; consider the short exact sequences

$$0 \to M^* \to G^* \to L \to 0 \tag{3.2.2}$$

$$0 \to L \to K^* \to \operatorname{Ext}^1_R(M,R) \to 0. \tag{3.2.3}$$

Sequence (3.2.2) implies $L \in [FP]^R_\infty$, which then implies $K^* \in [FP]^R_\infty$ by sequence (3.2.3). □

We next show that if $M \in \mathcal{BE}(R)$ has finite G-dimension, then M has finite $\tilde{G}$-dimension.

Proposition 3.2.15. *Let R be a ring and $M \in \mathcal{BE}(R)$ such that* $\operatorname{G-dim}_R M = n < \infty$. *Then M has a G-resolution $0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to 0$ such that $G_i \in \tilde{G}(R)$ for $i = 0, \ldots, n$.*

Proof. Let

$$\cdots \to F_k \to F_{k-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

be a degreewise finite free resolution of M. For $i \geq 0$ set $K_i = \ker(F_i \to F_{i-1})$. By Theorem 3.2.13,

$$0 \to K_{n-1} \to F_{n-1} \to \cdots \to F_1 \to M \to 0 \tag{3.2.4}$$

is a resolution such that K_{n-1} and each F_i are totally reflexive. Since each F_i is finitely generated and free, $F_i \in \tilde{G}(R)$. Set $K_0 = \ker(F_0 \to M)$. We now have the exact sequence

$$0 \to K_0 \to F_0 \to M \to 0,$$

and for $1 \leq i \leq n$, exact sequences

$$0 \to K_i \to F_i \to K_{i-1} \to 0.$$

Repeated applications of Lemma 3.2.14 yields $K_i \in \mathcal{BE}(R)$ for all $1 \leq i \leq n$. In particular $K_n \in \tilde{G}(R)$ by Theorem 3.2.13, and (3.2.4) is the desired resolution. □

3.3 Equality of $\tilde{G}$-dimension and Gorenstein Dimension

In the previous two sections we have seen close parallels between the results for $\tilde{G}$-dimension and G-dimension. In this section we see that for R-modules in $\mathcal{BE}(R)$, G-dimension and $\tilde{G}$-dimension are equivalent notions.

Proposition 3.3.1. *Let R be a ring and $M \in [FP]^R_\infty$. Then* $\text{G-dim}_R\, M \leq \tilde{\text{G}}\text{-dim}_R\, M$*. Moreover, if* $\tilde{\text{G}}\text{-dim}_R\, M < \infty$*, then* $\tilde{\text{G}}\text{-dim}_R\, M = \text{G-dim}_R\, M$*.*

Proof. If $\tilde{\text{G}}\text{-dim}_R\, M = \infty$, there is nothing to prove for the inequality. So assume $\tilde{\text{G}}\text{-dim}_R\, M = n < \infty$. Then there is a $\tilde{G}$-resolution

$$0 \to G_n \to \cdots \to G_1 \to G_0 \to M \to 0.$$

Since each G_i is also totally reflexive, $\text{G-dim}_R\, M = m \leq n$.

Consider a $\tilde{G}$-resolution of M

$$0 \to G_n \to \cdots \to G_0 \to M \to 0.$$

Set $K_{m-1} = \ker(G_{m-1} \to G_{m-2})$; K_{m-1} is totally reflexive by Theorem 3.2.13. Since

$$0 \to G_n \to G_{n-1} \to \cdots \to G_m \to K_{m-1} \to 0 \tag{3.3.1}$$

is exact, the $G_i \in \tilde{G}(R)$ implies $K_{m-1} \in \tilde{G}(R)$ by Corollary 3.2.11. Therefore

$$0 \to K_{m-1} \to G_{m-1} \to \cdots \to G_0 \to M \to 0$$

is a $\tilde{G}$-resolution of length m. Thus the equality holds. □

Proposition 3.3.2. *Let R be a ring and $M \in \mathcal{BE}(R)$. Then* $\text{G-dim}_R M = \tilde{\text{G}}\text{-dim}_R M$.

Proof. By Proposition 3.3.1, it suffices to show $\tilde{\text{G}}\text{-dim}_R M \leq \text{G-dim}_R M$.

Assume $\text{G-dim}_R M = n < \infty$, by Proposition 3.2.15 M has a G-resolution

$$0 \to G_n \to \cdots \to G_0 \to M \to 0$$

where each $G_i \in \mathcal{BE}(R)$. But then $G_i \in \tilde{G}(R)$ for all i. Hence
$\tilde{\text{G}}\text{-dim}_R M \leq \text{G-dim}_R M$. □

Chapter 4

The Auslander-Bridger Formula

The Auslander-Buchsbaum Formula relates the depth of a ring and a module to the projective dimension of the module.

Theorem 4.0.3. *If R is a local Noetherian ring, and M is a finitely generated R-module of finite projective dimension, then* $\operatorname{depth}_R M + \operatorname{pd}_R M = \operatorname{depth}_R R$.

In the Noetherian case, the Auslander-Bridger Formula [2] is an extension of the Auslander-Buchsbaum Formula relating depth and G-dimension:

Theorem 4.0.4. *If R is a local Noetherian ring and M is an R-module of finite G-dimension, then* $\operatorname{depth}_R M + \text{G-dim}_R M = \operatorname{depth}_R R$.

The Auslander-Buchsbaum Formula itself has been generalized to the non-Noetherian case:

Theorem 4.0.5. *[24, Ch 6, Theorem 2] If R is a quasi-local ring, and M an R-module with a degreewise finite free resolution of finite length, then* $\operatorname{pd}_R M + \text{p-depth}_R M = \text{p-depth}_R R$.

In this section we will prove a version of the Auslander-Bridger Formula for coherent rings and finitely presented modules with finite G-dimension, replacing depth with p-depth. However our main tools will not be G-dimension and the class $\mathcal{BE}(R)$, but rather results from $\tilde{G}$-dimension. Thus given the equivalence in Proposition 3.3.2, the Auslander-Bridger Formula will appear in several different forms, first in the $\tilde{G}$-dimension case, and then for G-dimension.

4.1 Additional Properties of $\tilde{G}$-dimension

Before proving the Auslander-Bridger Formula for modules of finite $\tilde{G}$-dimension, several results are needed. The first results investigate the behavior of $\tilde{G}$-dimension under flat liftings. The proof of the Auslander-Bridger Formula (Theorem 4.2.1) requires the existence of regular elements, necessitating the passage from the ring R to a polynomial ring over R. However in general, if R is coherent, $R[x]$ is not necessarily coherent. Unlike coherence, $\tilde{G}$-dimension remains stable across faithfully flat liftings as shown by Lemma 4.1.1 and Proposition 4.1.2. This fact plays a crucial role in the proof of a generalized Auslander-Bridger Formula.

Lemma 4.1.1. *Let R be a ring and $R \to S$ be a flat ring homomorphism. Let M be an R-module.*

(i) If $M \in [FP]^R_\infty$, then $M \otimes_R S \in [FP]^S_\infty$.

(ii) If $M \in \mathcal{BE}(R)$, then $M \otimes_R S \in \mathcal{BE}(S)$.

(iii) If $M \in \tilde{G}(R)$, then $M \otimes_R S \in \tilde{G}(S)$.

If S is faithfully flat, then the converses to (i), (ii), *and* (iii) *hold.*

Proof. If $M \in [FP]^R_\infty$ there is an exact sequence

$$\cdots \to F_2 \to F_1 \to F_0 \to M \to 0$$

where each F_i is a finitely generated free R-module. The resolution

$$\cdots \to F_2 \otimes_R S \to F_1 \otimes_R S \to F_0 \otimes_R S \to M \otimes_R S \to 0$$

is a degreewise finite free S-resolution of $M \otimes_R S$, and hence $M \otimes_R S \in [FP]^S_\infty$.

Conversely, assume S is faithfully flat and $M \otimes_R S \in [FP]^S_\infty$. First, we show by induction that if $M \otimes_R S \in [FP]^S_n$, then $M \in [FP]^R_n$. If $n = 0$, then $M \otimes_R S$ is finitely generated. Let $x_1, \ldots, x_t \in M$ be such that $x_1 \otimes 1, \ldots, x_t \otimes 1$ generate $M \otimes_R S$ as an S-module, and set $N = x_1 R + \cdots + x_t R$. Consider the exact sequence

$$0 \to N \to M \to M/N \to 0. \tag{4.1.1}$$

Applying $- \otimes_R S$, S flat yields

$$0 \to N \otimes_R S \to M \otimes_R S \to (M/N) \otimes_R S \to 0.$$

Notice that the map $N \otimes_R S \to M \otimes_R S$ is an isomorphism, thus $(M/N) \otimes_R S = 0$. By faithfulness of S, $M/N = 0$, and hence $M = N$ and M is a finitely generated R-module.

Assume $n > 0$. Since M is finitely generated, there is an exact sequence

$$0 \to K \to F \to M \to 0$$

where F is a finitely generated free R-module. By Proposition 2.5.3, it suffices to show that $K \in [FP]^R_{n-1}$. Applying $- \otimes_R S$, we have the exact sequence

$$0 \to K \otimes_R S \to F \otimes_R S \to M \otimes_R S \to 0.$$

Since $F \otimes_R S \in [FP]^S_\infty$,and $M \otimes_R S \in [FP]^S_n$, $K \otimes_R S \in [FP]^S_{n-1}$ by Proposition 2.5.3. By induction, this implies that $K \in [FP]^R_{n-1}$.

Since $M \otimes_R S \in [FP]^S_\infty$ if and only if $M \in [FP]^R_n$ for all $n \geq 0$, then this argument implies $M \in [FP]^R_\infty$.

Assume that $M \in \mathcal{BE}(R)$; by (i), $M \otimes_R S$ and $\operatorname{Ext}^m_R(M, R) \otimes_R S \in [FP]^S_\infty$ for all $m \geq 0$. Since S is flat and $M \in [FP]^R_\infty$, for all $m \geq 0$, by Theorem 2.1.9 we have the isomorphism

$$\operatorname{Ext}^m_S(M \otimes_R S, S) \cong \operatorname{Ext}^m_R(M, R) \otimes_R S \in [FP]^S_\infty. \tag{4.1.2}$$

Hence $M \otimes_R S \in \mathcal{BE}(S)$.

Conversely, assume that $M \otimes_R S \in \mathcal{BE}(S)$ and that S is faithful; the converse to (i) shows $M \in [FP]^R_\infty$. By Theorem 2.1.9 $\operatorname{Ext}^i_S(M \otimes_R S, S) \cong \operatorname{Ext}^i_R(M, R) \otimes_R S$ for all $i \geq 0$, thus the argument from the $[FP]^R_\infty$ case can be used to show $\operatorname{Ext}^i_R(M, R) \in [FP]^R_\infty$ for each $i \geq 0$. Thus $M \in \mathcal{BE}(R)$.

Assume that $M \in \tilde{G}(R)$. As S is flat and $M, M^* \in [FP]^R_\infty$, we have the natural isomorphisms:

$$\operatorname{Ext}^i_R(M, R) \otimes_R S \cong \operatorname{Ext}^i_S(M \otimes_R S, S),$$

and

$$\operatorname{Ext}^i_R(\operatorname{Hom}_R(M, R), R) \otimes_R S \cong \operatorname{Ext}^i_S(\operatorname{Hom}_S(M \otimes_R S, S), S)$$

for all $i \geq 0$. In particular, $M^{**} \otimes_R S \cong (M \otimes_R S)^{**}$ where the latter double dual is with respect to S-modules. Let $\phi : M \to M^{**}$ and $\psi : M \otimes_R S \to (M \otimes_R S)^{**}$ be the canonical maps. Let $K = \ker \phi$, $K' = \ker \psi$, $C = \operatorname{coker} \phi$, and $C' = \operatorname{coker} \psi$. Then we have the following commutative diagram with exact rows.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K \otimes_R S & \longrightarrow & M \otimes_R S & \xrightarrow{\phi \otimes 1_S} & M^{**} \otimes_R S & \longrightarrow & C \otimes_R S & \longrightarrow & 0 \\
 & & \downarrow & & \| & & \downarrow \cong & & \downarrow & & \\
0 & \longrightarrow & K' & \longrightarrow & M \otimes_R S & \xrightarrow{\psi \otimes 1_S} & (M \otimes_R S)^{**} & \longrightarrow & C' & \longrightarrow & 0
\end{array}$$

By the Five Lemma, $K \otimes_R S \cong K'$ and $C \otimes_R S \cong C'$. Hence if $M \in \tilde{G}(R)$, then ϕ is an isomorphism and $K = C = 0$. Thus $\psi \otimes 1_S$ is an isomorphism, and $M \otimes_R S \in \tilde{G}(S)$. Similarly, if S is faithfully flat, the converse holds. $\square$

Proposition 4.1.2. *Let R be a ring and $R \to S$ be a flat ring extension. If $M \in [FP]^R_\infty$, then $\tilde{\text{G}}\text{-dim}_R(M \otimes_R S) \leq \tilde{\text{G}}\text{-dim}_R M$. If S is faithfully flat, then equality holds.*

Proof. If $\tilde{\text{G}}\text{-dim}_R M = 0$ the result holds by Lemma 4.1.1.

Assume $\tilde{\text{G}}\text{-dim}_R M = n > 0$. Consider the exact sequence

$$0 \to K \to G \to M \to 0$$

where $G \in \tilde{G}(R)$. Then $\tilde{\text{G}}\text{-dim}_R K = n - 1$ by Corollary 2.5.4 and Corollary 3.1.9. By induction, $\tilde{\text{G}}\text{-dim}_S(K \otimes_R S) \leq n - 1$. Since $G \otimes_R S \in \tilde{G}(S)$ by Lemma 4.1.1, the sequence

$$0 \to K \otimes_R S \to G \otimes_R S \to M \otimes_R S \to 0$$

shows $\tilde{\text{G}}\text{-dim}_S(M \otimes_R S) \leq n$.

If in addition S is faithful, then by induction
$\tilde{\text{G}}\text{-dim}_S(K \otimes_R S) = \tilde{\text{G}}\text{-dim}_R K = n - 1$. Notice $M \otimes_R S \notin \tilde{G}(S)$, for otherwise

$M \in \tilde{G}(R)$. Therefore, by Corollary 3.1.9,

$$\tilde{\text{G}}\text{-dim}_S(M \otimes_R S) = \tilde{\text{G}}\text{-dim}_S(K \otimes_R S) + 1 = n = \tilde{\text{G}}\text{-dim}_R M.$$

□

Corollary 4.1.3. *Let R be a ring and and $M \in [FP]^R_\infty$.*

(i) If $p \in \operatorname{Spec} R$, then $\tilde{\text{G}}\text{-dim}_{R_p} M_p \leq \tilde{\text{G}}\text{-dim}_R M$.

(ii) If $(R, \mathfrak{m})$ is quasi-local, then $\tilde{\text{G}}\text{-dim}_{R[x]_{\mathfrak{m}R[x]}} M \otimes_R R[x]_{\mathfrak{m}R[x]} = \tilde{\text{G}}\text{-dim}_R M$.

Remark 4.1.4. Let R be a ring and M an R-module. Assume $f \in M^* = \operatorname{Hom}_R(M, R)$ and that x is a non-zero-divisor on R. If $xf = 0$, then $xf(M) = 0$. However, since $f(M) \subset R$ and x is a non-zero-divisor on R then $f(M) = 0$. Hence x is M^*-regular and M^* is torsion-free. If $M \cong M^{**}$, then M is torsion-free. In particular, if M is reflexive, then M is torsion-free. It follows that if M is totally reflexive or $M \in \tilde{G}(R)$, any R-regular element is also M-regular. The existence of such an R- and M-regular element is necessary in many of the results to come.

The following result is stated in terms of Noetherian rings in Lemma 1.3.4 of [9], however its proof requires no modification to hold for non-Noetherian rings.

Lemma 4.1.5. *Let R be a ring and M be an R-module. If $x \in R$ is M- and R-regular, then the following hold*

(i) $\operatorname{Tor}^R_m(M, R/(x)) = 0$ for $m > 0$.

(ii) If $\operatorname{Ext}^1_R(M, R) = 0$, then $\operatorname{Hom}_{R/(x)}(M/xM, R/(x)) \cong M^*/xM^*$.

(iii) *If* $\operatorname{Ext}^1_R(M,R) = 0 = \operatorname{Ext}^1_R(M^*,R)$, *then*

$$\operatorname{Hom}_{R/(x)}(\operatorname{Hom}_{R/(x)}(M/xM, R/(x)), R/(x)) \cong M^{**}/xM^{**}.$$

The next lemma is a result attributed to Rees.

Lemma 4.1.6. *[23, p. 140, Lemma 2] Let R be a ring, M and N be R-modules, and $x \in R$ be an R- and M-regular element. If $xN = 0$ then*

(i) $\operatorname{Hom}_R(N,M) = 0$ *and* $\operatorname{Ext}^{i+1}_R(N,M) \cong \operatorname{Ext}^i_{R/(x)}(N, M/xM)$ *for all* $i \geq 0$,

(ii) $\operatorname{Ext}^i_R(M,N) \cong \operatorname{Ext}^i_{R/(x)}(M/xM, N)$ *for all* $i \geq 0$, *and*

(iii) $\operatorname{Tor}^R_i(M,N) \cong \operatorname{Tor}^{R/(x)}_i(M/xM, N)$ *for all* $i \geq 0$.

Using these results we explore the relationship between $\tilde{\text{G}}\text{-dim}_R\, M$ and $\tilde{\text{G}}\text{-dim}_{R/(x)}\, M/xM$ for an R-module M and an M- and R-regular element x.

Lemma 4.1.7. *Let R be a ring, $M \in [FP]^R_\infty$, and x be an M- and R-regular element. Then $M/xM \in [FP]^{R/(x)}_\infty$.*

Proof. By Lemma 4.1.5, $\operatorname{Tor}^R_i(M, R/(x)) = 0$ for all $i \geq 1$. Therefore, if $\mathbf{F}$ is a free resolution of M consisting of finitely generated free R-modules, $\mathbf{F} \otimes_R R/(x)$ is a free resolution of M/xM consisting of finitely generated free $R/(x)$-modules. Hence $M/xM \in [FP]^{R/(x)}_\infty$. □

Proposition 4.1.8. *Let R be a ring, $M \in \tilde{G}(R)$, and x be an M- and R-regular element. Then $M/xM \in \tilde{G}(R/(x))$.*

Proof. By Remark 4.1.4 x is M-regular, and by Lemma 4.1.7 $M/xM \in [FP]^{R/(x)}_\infty$. Applying $\operatorname{Hom}_R(M,-)$ to the short exact sequence

$$0 \to R \xrightarrow{x} R \to R/(x) \to 0$$

we get

$$0 \to M^* \xrightarrow{x} M^* \to \operatorname{Hom}_R(M, R/(x)) \to 0. \tag{4.1.3}$$

Notice by Lemmas 4.1.6 and 4.1.5

$$\operatorname{Hom}_{R/(x)}(M/xM, R/(x)) \cong \operatorname{Hom}_R(M, R/(x)) \cong M^*/xM^*.$$

Since x is M^*-regular and $M^* \in [FP]_\infty^R$, by Lemma 4.1.7,
$\operatorname{Hom}_{R/(x)}(M/xM, R/(x)) \cong M^*/xM^* \in [FP]_\infty^{R/(x)}$.

Since $M \in \tilde{G}(R)$, (4.1.3) shows $\operatorname{Ext}_R^i(M, R/(x)) = 0$ for all $i > 0$, and Lemma 4.1.6 gives $\operatorname{Ext}_{R/(x)}^i(M/xM, R/(x)) \cong \operatorname{Ext}_R^i(M, R/(x)) = 0$ for all $i > 0$.

As x is also M^*-regular, and $M^* \in \tilde{G}(R)$, the same argument shows $\operatorname{Ext}_{R/(x)}^i((M/xM)^*, R/(x)) = 0$ for all $i > 0$.

As the biduality map $\delta_M : M \to M^{**}$ is an isomorphism, so is $\delta_M \otimes_R R/(x)$. By Lemma 4.1.5(iii) we have the commutative diagram:

$$\begin{array}{ccc} M \otimes_R R/(x) & \xrightarrow[\cong]{\delta_M \otimes_R R/(x)} & M^{**} \otimes_R R/(x) \\ \cong \downarrow & & \cong \downarrow \\ M/xM & \xrightarrow[\delta_{M/xM}]{} & \operatorname{Hom}_{R/(x)}(\operatorname{Hom}_{R/(x)}(M/xM, R/(x)), R/(x)) \end{array}$$

showing $\delta_{M/xM} : M/xM \to \operatorname{Hom}_{R/(x)}(\operatorname{Hom}_{R/(x)}(M/xM, R/(x)), R/(x))$ is an isomorphism. Hence $M/xM \in \tilde{G}(R/(x))$.

□

Lemma 4.1.9. *Let R be a ring and M an R-module with* $\tilde{\text{G}}\text{-dim}_R\, M = n < \infty$*. Then* $\operatorname{Ext}_R^n(M, R)$ *is a finitely generated R-module.*

Proof. We proceed by induction on $\tilde{\text{G}}\text{-dim}_R\, M = n$. If $n = 0$, $M \in \tilde{G}(R)$ and $M^* \in [FP]_\infty^R$ is finitely generated.

If $n > 0$, consider the exact sequence

$$0 \to K \to G \to M \to 0$$

where $G \in \tilde{G}(R)$. Since $K \in [FP]^R_\infty$, by Corollary 3.1.9 $\tilde{\text{G}}\text{-dim}_R K = n - 1$. By induction $\text{Ext}^{n-1}_R(K, R)$ is finitely generated. Applying $\text{Hom}_R(-, R)$ yields the long exact sequence

$$\cdots \to \text{Ext}^{n-1}_R(G, R) \to \text{Ext}^{n-1}_R(K, R) \to \text{Ext}^n_R(M, R) \to \text{Ext}^n_R(G, R) \to \cdots .$$

However $\text{Ext}^i_R(G, R) = 0$ for all $i > 0$ and thus if $n > 1$, $\text{Ext}^{n-1}_R(K, R) \cong \text{Ext}^n_R(M, R)$. Hence $\text{Ext}^n_R(M, R)$ is finitely generated if $n > 1$. If $n = 1$, then there is a surjection $\text{Hom}_R(K, R) \to \text{Ext}^1_R(M, R) \to 0$. By induction $\text{Hom}_R(K, R)$ is finitely generated, and hence $\text{Ext}^1_R(M, R)$ is finitely generated. □

Proposition 4.1.10. *Let $(R, \mathfrak{m})$ be a quasi-local ring and M an R-module with $\tilde{\text{G}}\text{-dim}_R M < \infty$. Let $x \in \mathfrak{m}$ be an M- and R-regular element. If $M/xM \in \tilde{G}(R/(x))$, then $M \in \tilde{G}(R)$.*

Proof. By Lemma 4.1.6 $\text{Ext}^i_R(M, R/(x)) \cong \text{Ext}^i_{R/(x)}(M/xM, R/(x)) = 0$ for all $i \geq 0$. Suppose $\tilde{\text{G}}\text{-dim}_R M = n > 0$. By Theorem 3.1.8 $\text{Ext}^n_R(M, R) \neq 0$ and is finitely generated by Lemma 4.1.9. Apply $\text{Hom}_R(M, -)$ to the exact sequence

$$0 \to R \xrightarrow{x} R \to R/(x) \to 0$$

to obtain the exact sequence

$$\text{Ext}^n_R(M, R) \xrightarrow{x} \text{Ext}^n_R(M, R) \to 0.$$

By Nakayama's Lemma $\mathrm{Ext}_R^n(M,R) = 0$ a contradiction to $\tilde{\mathrm{G}}\text{-}\dim_R M = n$. Thus $\tilde{\mathrm{G}}\text{-}\dim_R M = 0$. □

Combining Propositions 4.1.8 and 4.1.10, we have

Corollary 4.1.11. *Let $(R,\mathfrak{m})$ be a quasi-local ring and M an R-module with $\tilde{\mathrm{G}}\text{-}\dim_R M < \infty$. If $x \in \mathfrak{m}$ is an R- and M-regular element, then $M \in \tilde{G}(R)$ if and only if $M/xM \in \tilde{G}(R/(x))$.*

Lemma 4.1.12. *Let $(R,\mathfrak{m})$ be a quasi-local ring, M an R-module, and $x \in \mathfrak{m}$ be an R- and M-regular element. If $\tilde{\mathrm{G}}\text{-}\dim_R M < \infty$, then $\tilde{\mathrm{G}}\text{-}\dim_{R/(x)} M/xM \leq \tilde{\mathrm{G}}\text{-}\dim_R M$.*

Proof. Proposition 4.1.8 proves the result when $\tilde{\mathrm{G}}\text{-}\dim_R M = 0$. Assume $\tilde{\mathrm{G}}\text{-}\dim_R M = n > 0$ and consider the $\tilde{G}$-resolution

$$0 \to G_n \to G_{n-1} \to \cdots \to G_0 \to M \to 0.$$

Set $K_i = \ker(G_i \to G_{i-1})$ for $0 \leq i \leq n-1$. Then we have the exact sequences

$$0 \to G_n \to G_{n-1} \to K_{n-2} \to 0$$
$$0 \to K_i \to G_i \to K_{i-1} \to 0, \text{ for } 1 \leq i \leq n-2$$
$$0 \to K_0 \to G_0 \to M \to 0.$$

Since x is R-regular, and $G_i \in \tilde{G}(R)$ for all i, x is G_i regular for all i. Thus x is K_i-regular for all i. Hence by Lemma 4.1.5(i) the sequences

$$0 \to G_n/xG_n \to G_{n-1}/xG_{n-1} \to K_{n-2}/xK_{n-2} \to 0$$
$$0 \to K_i/xK_i \to G_i/xG_i \to K_{i-1}/xK_{i-1} \to 0, \text{ for } 1 \leq i \leq n-2$$
$$0 \to K_0/xK_0 \to G_0/xG_0 \to M/xM \to 0$$

are exact, and thus

$$0 \to G_n/xG_n \to \cdots \to G_0/xG_0 \to M/xM \to 0$$

is exact where $G_i/xG_i \in \tilde{G}(R/(x))$ for each i by Proposition 4.1.8. Therefore $\tilde{\mathrm{G}}\text{-}\dim_{R/(x)} M/xM \leq \tilde{\mathrm{G}}\text{-}\dim_R M = n$. □

Proposition 4.1.13. *Let $(R, \mathfrak{m})$ be a quasi-local ring, M be an R-module, and $x \in \mathfrak{m}$ be an M and R-regular element. If $\tilde{\mathrm{G}}\text{-}\dim_R M < \infty$, then*

$$\tilde{\mathrm{G}}\text{-}\dim_{R/(x)} M/xM = \tilde{\mathrm{G}}\text{-}\dim_R M.$$

Proof. In light of Lemma 4.1.12, it suffices to prove $\tilde{\mathrm{G}}\text{-}\dim_{R/(x)} M/xM \geq \tilde{\mathrm{G}}\text{-}\dim_R M$. Proposition 4.1.10 proves the result when $\tilde{\mathrm{G}}\text{-}\dim_{R/(x)} M/xM = 0$.

Suppose $\tilde{\mathrm{G}}\text{-}\dim_{R/(x)} M/xM = t > 0$ and consider the short exact sequence

$$0 \to K \to G \to M \to 0$$

with $G \in \tilde{G}(R)$. Then by Corollary 3.1.9 $\tilde{\mathrm{G}}\text{-}\dim_R K = \tilde{\mathrm{G}}\text{-}\dim_R M - 1$. As x is M- and R-regular we have the short exact sequence

$$0 \to K/xK \to G/xG \to M/xM \to 0,$$

with $G/xG \in \tilde{G}(R/(x))$. Since $\tilde{\mathrm{G}}\text{-}\dim_{R/(x)} M/xM > 0$,

$$\tilde{\mathrm{G}}\text{-}\dim_{R/(x)} K/xK = \tilde{\mathrm{G}}\text{-}\dim_{R/(x)} M/xM - 1.$$

By induction, $\tilde{\text{G}}\text{-dim}_{R/(x)} K/xK = \tilde{\text{G}}\text{-dim}_R K$. Hence by Corollary 3.1.9, $\tilde{\text{G}}\text{-dim}_R M = \tilde{\text{G}}\text{-dim}_{R/(x)} M/xM$. □

Lemma 4.1.14. *[9, Lemma 1.4.4] Let $(R, \mathfrak{m})$ be a coherent ring, M a finitely presented R-module, and $x \in \mathfrak{m}$ an R- and M-regular element. Then $M \in \tilde{G}(R)$ if and only if $M/xM \in \tilde{G}(R/(x))$.*

Proof. The proof is similar to that of Lemma 1.4.4 [9]; coherence allows us to apply Nakayama's Lemma within this proof. □

Proposition 4.1.15. *Let $(R, \mathfrak{m})$ be a coherent ring, M a finitely presented R-module, and $x \in \mathfrak{m}$ be a M- and R-regular element. Then $\tilde{\text{G}}\text{-dim}_R M = \tilde{\text{G}}\text{-dim}_{R/(x)} M/xM$ if either $\tilde{\text{G}}\text{-dim}_R M < \infty$ or $\tilde{\text{G}}\text{-dim}_{R/(x)} M/xM < \infty$.*

Proof. If $\tilde{\text{G}}\text{-dim}_R M < \infty$, the result follows by Proposition 4.1.13

Suppose $\tilde{\text{G}}\text{-dim}_R M/xM = t < \infty$. If $t = 0$, this is Lemma 4.1.14.

Suppose $t > 0$. Let

$$0 \to K \to F \to M \to 0$$

be an exact sequence where F is a finitely generated free R-module. As x is R- and M-regular, by Lemma 4.1.5 $\text{Tor}_1^R(M, R/(x)) = 0$. Thus the sequence

$$0 \to K/xK \to F/xF \to M/xM \to 0$$

is exact. Since $F/xF \in \tilde{G}(R/(x))$, $\tilde{\text{G}}\text{-dim}_{R/(x)} K/xK = t - 1$ by Corollary 3.1.9. As x is K-regular, by induction $\tilde{\text{G}}\text{-dim}_R K = t - 1$; hence $\tilde{\text{G}}\text{-dim}_R M = t$. □

The following result provides an important tool for finding R- and M-regular elements in a ring.

Lemma 4.1.16. *Let* $(R, \mathfrak{m})$ *be a quasi-local ring with* $\operatorname{p-depth}_R R > 0$ *and* M *an* R*-module such that* $\operatorname{p-depth}_R M > 0$. *Then there exists* $y \in \mathfrak{m}R[t]_{\mathfrak{m}R[t]}$ *such that* y *is* $R[t]_{\mathfrak{m}R[t]}$*- and* $M \otimes_R R[t]_{\mathfrak{m}R[t]}$*-regular.*

Proof. By Proposition 2.2.3(iv), $\operatorname{p-depth}_R(M \oplus R) > 0$. Then by Proposition 2.2.3(vi) $\operatorname{depth}_{R[t]_{\mathfrak{m}R[t]}}(M \otimes_R R[t]_{\mathfrak{m}R[t]} \oplus_R R[t]_{\mathfrak{m}R[t]}) > 0$, so there exists $y \in \mathfrak{m}R[t]_{\mathfrak{m}R[t]}$ that is $M \otimes_R R[t]_{\mathfrak{m}R[t]} \oplus R[t]_{\mathfrak{m}R[t]}$-regular. □

Remark 4.1.17. In light of Lemma 4.1.1 and Corollary 4.1.3, given a quasi-local ring $(R, \mathfrak{m})$ we may pass to the ring $R[t]_{\mathfrak{m}R[t]}$ while retaining assumptions involving $\tilde{G}$-dimension as well as the class $[FP]^R_\infty$. In addition, by the definition of polynomial grade, $\operatorname{p-depth}_R R = \operatorname{p-depth}_{R[t]_{\mathfrak{m}R[t]}} R[t]_{\mathfrak{m}R[t]}$. By Proposition 2.2.3(v), $\operatorname{p-depth}_R M = \operatorname{p-depth}_{R[t]_{\mathfrak{m}R[t]}}(M \otimes_R R[t]_{\mathfrak{m}R[t]})$. Given these properties, passing to $R[t]_{\mathfrak{m}R[t]}$ will prove to be a highly useful tool in future results.

For the proof of Lemma 4.1.19 we need the following result

Proposition 4.1.18. *Let* R *be a ring and* $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ *be a short exact sequence of* R*-modules. If* L *is finitely generated and* M *is finitely presented, then* N *is finitely presented.*

Proof. Since M is finitely presented, there is an exact sequence

$$0 \to K \to R^m \xrightarrow{h} M \to 0$$

with K finitely generated. Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & K & \longrightarrow & R^m & \xrightarrow{h} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \| & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & A & \longrightarrow & R^m & \xrightarrow{gh} & N & \longrightarrow & 0 \end{array}$$

where $A = \ker gh$. It suffices to show A is finitely generated. By the Snake Lemma, $\operatorname{coker} \alpha \cong \ker g \cong L$ is finitely generated, and $\ker \alpha = 0$. Since K is finitely generated and $A/\alpha(K)$ is finitely generated, A is finitely generated. □

The following results provide a relationship between $\tilde{G}$-dimension and p-depth.

Lemma 4.1.19. *Let* $(R, \mathfrak{m})$ *be a quasi-local ring with* $\operatorname{p-depth}_R R = 0$*. If* M *is a finitely presented* R*-module, then* $M^* = 0$ *if and only if* $M = 0$*.*

Proof. Clearly, if $M = 0$, then $M^* = 0$.

Conversely, assume $M^* = 0$. If $M \neq 0$, we proceed by induction on the number of generators, $\mu_R(M)$, of M. If $\mu_R(M) = 1$, then $M \cong R/I$ for some finitely generated ideal I of R. Since $\operatorname{p-depth}_R R = 0$, $\operatorname{Hom}_R(R/I, R) \neq 0$ for every finitely generated ideal $I \subset R$, a contradiction.

Assume $\mu_R(M) = n$, and the claim holds for all finitely presented modules N such that $\mu_R(N) < n$. Let $M = x_1R + \cdots + x_nR$ and set $M' = x_1R + \cdots + x_{n-1}R$. Consider the short exact sequence $0 \to M' \to M \to M/M' \to 0$. Since M is finitely presented and M' is finitely generated, M/M' is finitely presented by Proposition 4.1.18. Thus M/M' is finitely presented and $\operatorname{Hom}_R(M/M', R) \neq 0$. But $0 \to \operatorname{Hom}_R(M/M', R) \to \operatorname{Hom}_R(M, R)$ is exact; thus $\operatorname{Hom}_R(M, R) \neq 0$, a contradiction. □

Proposition 4.1.20. *Let* $(R, \mathfrak{m})$ *be a quasi-local ring with* $\operatorname{p-depth}_R R = 0$*. If* M *is an* R*-module with* $\tilde{\text{G}}\text{-dim}_R M < \infty$*, then* $M \in \tilde{G}(R)$*.*

Proof. Notice that by induction it is sufficient to consider the case for $\tilde{\text{G}}\text{-dim}_R M \leq 1$. For, if $\tilde{\text{G}}\text{-dim}_R M \leq n$ there is an exact sequence

$$0 \to K \to G \to M \to 0$$

with $G \in \tilde{G}(R)$ and $\tilde{\text{G}}\text{-dim}_R\, K = \tilde{\text{G}}\text{-dim}_R\, M - 1$. The inductive step will show $K \in \tilde{G}(R)$, and hence $\tilde{\text{G}}\text{-dim}_R\, M \leq 1$.

Thus assume $\tilde{\text{G}}\text{-dim}_R\, M \leq 1$ and consider the short exact sequence

$$0 \to G_1 \to G_0 \to M \to 0$$

with $G_0, G_1 \in \tilde{G}(R)$. Applying $\text{Hom}_R(-, R)$ we have $\text{Ext}^i_R(M, R) = 0$ for all $i \geq 2$ and the exact sequence

$$0 \to M^* \to G_0^* \to G_1^* \to \text{Ext}^1_R(M, R) \to 0. \tag{4.1.4}$$

Again applying $\text{Hom}_R(-, R)$ we obtain the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \text{Ext}^1_R(M,R)^* & \longrightarrow & G_1^{**} & \longrightarrow & G_0^{**} \\ & & \downarrow & & \cong\downarrow & & \cong\downarrow \\ & & 0 & \longrightarrow & G_1 & \longrightarrow & G_0, \end{array}$$

and $\text{Ext}^1_R(M, R)^* = 0$. Since $\text{Ext}^1_R(M, R)$ is finitely presented by (4.1.4) and Proposition 4.1.18, by Lemma 4.1.19 $\text{Ext}^1_R(M, R) = 0$. Thus by Theorem 3.1.8,
$\tilde{\text{G}}\text{-dim}_R\, M = 0$. □

Proposition 4.1.21. *Let $(R, \mathfrak{m})$ be a quasi-local ring with* $\text{p-depth}_R\, R = 0$ *and $M \in \tilde{G}(R)$ a non-zero R-module. Then* $\text{p-depth}_R\, M = 0$.

Proof. Assume $\text{p-depth}_R\, M > 0$. By Remark 4.1.17 and Lemma 4.1.16 we pass to $R[t]_{\mathfrak{m}R[t]}$ and may assume there exists $x \in \mathfrak{m}$ that is M-regular. Applying $\text{Hom}_R(-, R)$ to the short exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

yields the exact sequence

$$0 \to \operatorname{Hom}_R(M/xM, R) \to M^* \xrightarrow{x} M^* \to \operatorname{Ext}^1_R(M/xM, R) \to 0. \tag{4.1.5}$$

Applying $\operatorname{Hom}_R(-, R)$ again, consider the diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \operatorname{Ext}^1_R(M/xM, R)^* & \longrightarrow & M^{**} & \xrightarrow{x} & M^{**} \\ & & \downarrow & & \downarrow \cong & & \downarrow \cong \\ & & 0 & \longrightarrow & M & \xrightarrow{x} & M. \end{array}$$

Hence $\operatorname{Ext}^1_R(M/xM, R)^* = 0$. Since $\operatorname{Ext}^1_R(M/xM, R)$ is finitely presented from (4.1.5), $\operatorname{Ext}^1_R(M/xM, R) = 0$ by Lemma 4.1.19. From (4.1.5) $M^* = xM^*$. Since M^* is finitely generated (and finitely presented), $M^* = 0$ by Nakayama's Lemma, and $M = 0$ by Lemma 4.1.19, a contradiction. □

Proposition 4.1.22. *Let $(R, \mathfrak{m})$ be a quasi-local ring and $M \in \tilde{G}(R)$ a non-zero R-module. Then* $\operatorname{p-depth}_R M = \operatorname{p-depth}_R R$.

Proof. Suppose first that $\operatorname{p-depth}_R R = n < \infty$. If $n = 0$, then $\operatorname{p-depth}_R M = 0$ by Proposition 4.1.21. If $\operatorname{p-depth}_R R > 0$, then by Remark 4.1.17 we may pass to $R[t]_{\mathfrak{m}R[t]}$ to find an $x \in \mathfrak{m}$ that is R-regular. Since $M \in \tilde{G}(R)$, x is also M-regular. Thus $M/xM \in \tilde{G}(R/(x))$ by Proposition 4.1.8. By induction on $\operatorname{p-depth}_R R$, $\operatorname{p-depth}_{R/(x)} M/xM = \operatorname{p-depth}_{R/(x)} R/(x)$, and thus by Proposition 2.2.3(vii), $\operatorname{p-depth}_R R = \operatorname{p-depth}_R M$.

Suppose now that $\operatorname{p-depth}_R R = \infty$ and $\operatorname{p-depth}_R M = m < \infty$. Suppose $m = 0$. Passing to $R[t]_{\mathfrak{m}R[t]}$ we have $x \in \mathfrak{m}$ such that x is R-regular. As M is torsion-free, x is M-regular as well, contradicting that $\operatorname{p-depth}_R M = 0$. If $m > 0$, then passing to $R[t]_{\mathfrak{m}R[t]}$ there is an $x \in \mathfrak{m}$ such that x is R- and M-regular. Then $M/xM \in \tilde{G}(R/(x))$,

$\text{p-depth}_{R/(x)} M/xM = m-1$, and $\text{p-depth}_{R/(x)} R/(x) = \infty$, a contradiction by induction. □

Given the equality in Proposition 3.3.2 between G-dimension and $\tilde{G}$-dimension in the case of modules in $\mathcal{BE}(R)$, all the results in this section can be restated in terms of G-dimension. Hence from this point, any references involving G-dimension will often be made back to these $\tilde{G}$-dimension results whenever the assumptions are such that the equality holds.

4.2 Generalized Auslander-Bridger Formulas

We now have the tools to prove generalized versions of the Auslander-Bridger Formula. The first version is stated in terms of $\tilde{G}$-dimension.

Theorem 4.2.1. *Let* $(R, \mathfrak{m})$ *be a quasi-local ring and* $\tilde{\text{G}}\text{-dim}_R M < \infty$. *Then*

$$\text{p-depth}_R M + \tilde{\text{G}}\text{-dim}_R M = \text{p-depth}_R R.$$

Proof. We first consider the case when $\text{p-depth}_R R = \infty$.

If $\tilde{\text{G}}\text{-dim}_R M = 0$, Proposition 4.1.22 shows $\text{p-depth}_R M = \infty$. Suppose $\tilde{\text{G}}\text{-dim}_R M > 0$ and $\text{p-depth}_R M < \infty$ and consider the exact sequence

$$0 \to K \to G \to M \to 0$$

where $G \in \tilde{G}(R)$. By Proposition 4.1.22, $\text{p-depth}_R G = \infty > \text{p-depth}_R M$. Therefore, by Lemma 2.2.4, $\text{p-depth}_R K = \text{p-depth}_R M + 1 < \infty$ and $\tilde{\text{G}}\text{-dim}_R K = \tilde{\text{G}}\text{-dim}_R M - 1 < \infty$. By induction on $\tilde{\text{G}}\text{-dim}_R K$, $\text{p-depth}_R K = \infty$ which implies $\text{p-depth}_R M = \infty$, a contradiction.

Now assume $\operatorname{p-depth}_R R < \infty$. If $\operatorname{p-depth}_R R = 0$ the theorem holds by Propositions 4.1.21 and 4.1.20. Assume $\operatorname{p-depth}_R R > 0$ and assume first that $\operatorname{p-depth}_R M > 0$. Passing to $R[t]_{\mathfrak{m}R[t]}$ there is an $x \in \mathfrak{m}$ that is M- and R-regular by Lemma 4.1.16. Then $\tilde{\mathrm{G}}\text{-}\dim_{R/(x)} M/xM = \tilde{\mathrm{G}}\text{-}\dim_R M$ by Proposition 4.1.13 and $\operatorname{p-depth}_{R/(x)} R/(x) = \operatorname{p-depth}_R R - 1$. By induction on $\operatorname{p-depth}_R R$,

$$\operatorname{p-depth}_{R/(x)} M/xM + \tilde{\mathrm{G}}\text{-}\dim_{R/(x)} M/xM = \operatorname{p-depth}_{R/(x)} R/(x).$$

Since by Proposition 2.2.3(vii) $\operatorname{p-depth}_{R/(x)} M/xM = \operatorname{p-depth}_R M - 1$, the formula holds.

Now assume that $\operatorname{p-depth}_R M = 0$ and consider the short exact sequence

$$0 \to K \to G \to M \to 0$$

where $G \in \tilde{G}(R)$. Since $\operatorname{p-depth}_R G = \operatorname{p-depth}_R R > \operatorname{p-depth}_R M$, by Lemma 2.2.4 $\operatorname{p-depth}_R K = 1$ and $\tilde{\mathrm{G}}\text{-}\dim_R K = \tilde{\mathrm{G}}\text{-}\dim_R M - 1$. By the $\operatorname{p-depth}_R M > 0$ case,

$$\begin{aligned} \operatorname{p-depth}_R R &= \operatorname{p-depth}_R K + \tilde{\mathrm{G}}\text{-}\dim_R K \\ &= 1 + (\tilde{\mathrm{G}}\text{-}\dim_R M - 1) \\ &= \tilde{\mathrm{G}}\text{-}\dim_R M. \end{aligned}$$

□

Proposition 3.3.2 allows Theorem 4.2.1 to be rewritten in terms of $\mathcal{BE}(R)$-modules instead.

Corollary 4.2.2. *Let $(R, \mathfrak{m})$ be a quasi-local ring and $M \in \mathcal{BE}(R)$. If*

$\text{G-dim}_R\, M < \infty$, *then*

$$\text{p-depth}_R\, M + \text{G-dim}_R\, M = \text{p-depth}_R\, R.$$

The statement of the Auslander-Bridger Formula for coherent rings follows easily.

Corollary 4.2.3. *Let* $(R, \mathfrak{m})$ *be a quasi-local coherent ring and* M *a finitely presented* R*-module. If* $\text{G-dim}_R\, M < \infty$, *then*

$$\text{p-depth}_R\, M + \text{G-dim}_R\, M = \text{p-depth}_R\, R.$$

Chapter 5

Gorenstein Rings

5.1 Gorenstein Rings Defined

G-dimension characterizes Noetherian Gorenstein rings in the following manner:

Theorem 5.1.1. *[2] Let (R, m, k) be a local Noetherian ring. The following are equivalent*

(i) *R is Gorenstein.*

(ii) $\text{G-dim}_R\, M < \infty$ *for all finitely generated R-modules M.*

(iii) $\text{G-dim}_R\, k < \infty$.

This characterization motivates the definition of (non-Noetherian) quasi-local Gorenstein rings.

Definition 5.1.2. A quasi-local ring R is *Gorenstein* if $\text{G-dim}_R\, R/I < \infty$ for every finitely generated ideal I. An arbitrary ring R is Gorenstein if $R_{\mathfrak{m}}$ is Gorenstein for every maximal ideal $\mathfrak{m}$.

If $(R, \mathfrak{m}, k)$ is a local Noetherian ring such that $\text{G-dim}_R R/I < \infty$ for all ideals I, $\text{G-dim}_R R/\mathfrak{m} = \text{G-dim}_R k < \infty$. Thus when R is Noetherian, Theorem 5.1.1 and Definition 5.1.2 agree.

As in the Noetherian case, when R is a coherent ring the Gorenstein property localizes.

Proposition 5.1.3. *Let R be a coherent quasi-local Gorenstein ring and S a multiplicatively closed set. Then R_S is Gorenstein.*

Proof. Consider a finitely generated ideal J of R_S. Then $J = I_S$ for some finitely generated ideal I of R. Since R is coherent Gorenstein, $\text{G-dim}_R R/I = n < \infty$ and Proposition 4.1.2 implies $\text{G-dim}_{R_S}(R/I)_S = \text{G-dim}_{R_S} R_S/J \leq n$. Thus R_S is Gorenstein. □

In the context of quasi-local coherent rings, we may characterize the Gorenstein property via finitely presented modules.

Proposition 5.1.4. *Suppose $(R, \mathfrak{m})$ is a quasi-local coherent ring. Then $\text{G-dim}_R R/I < \infty$ for all finitely generated ideals I if and only if $\text{G-dim}_R M < \infty$ for all finitely presented modules M.*

Proof. Assume $\text{G-dim}_R M < \infty$ for all finitely presented modules M. Let $I = (x_1, \ldots, x_n)$ be a finitely generated ideal of R. Then $R^n \to R \to R/I \to 0$ is exact, and R/I is finitely presented. Thus the conclusion follows.

Assume $\text{G-dim}_R R/I < \infty$ for all finitely generated ideals I. Let M be a finitely presented R-module; we proceed by induction on the number of generators of M, $\mu_R(M)$.

If $\mu_R(M) = 1$, let $M = xR$ for some $x \in M$. The map $R \xrightarrow{x} M \to 0$, shows $M \cong R/(0 :_R x)$. By Theorem 2.1.2, $(0 :_R x)$ is finitely generated so $\text{G-dim}_R M = \text{G-dim}_R R/(0 :_R x) < \infty$.

If $\mu_R(M) > 1$, let $M = x_1R + \cdots + x_nR$ for $x_1, \ldots, x_n \in M$, and set $N = x_1R + \cdots + x_{n-1}R$. Consider the exact sequence $0 \to N \to M \to M/N \to 0$. As M is a finitely presented R-module and R is coherent, M is coherent. Therefore N is finitely presented. Via Proposition 4.1.18 M/N is finitely presented, and by induction $\text{G-dim}_R N < \infty$. Hence $\text{G-dim}_R M/N < \infty$, and $\text{G-dim}_R M < \infty$ by Proposition 3.1.10. □

5.2 Relation to Regular and Cohen-Macaulay Rings

This definition of Gorenstein is very closely related to regular rings.

Proposition 5.2.1. *Coherent regular rings are Gorenstein.*

Proof. Since regular rings remain regular under localization, it suffices to assume R is quasi-local. As R is regular, $\text{pd}_R R/I < \infty$ for every finitely generated ideal I. Since R is coherent, R/I has a finite resolution consisting of finitely generated free modules. As finitely generated free modules are contained in $\tilde{G}(R)$, $\text{G-dim}_R R/I \leq \text{pd}_R R/I < \infty$ for every finitely generated ideal I. Hence R is Gorenstein. □

Thus regular rings such as $k[x_1, x_2, \ldots]$ for any field k, valuation domains, and Prüfer domains are Gorenstein.

A prime ideal p is a *weak Bourbaki prime* (or *weak associated prime*) of an R-module M if p is minimal over $(0 :_R x)$ for some $x \in M$. Denote the set of weak Bourbaki primes by $\text{wAss}_R M$. The following result was shown in [15].

Lemma 5.2.2. *[15] Let R be a ring, M be an R-module, and $p \in \text{wAss}_R M$. Then* $p\text{-depth}_{R_p} M_p = 0$.

This result plays a role in the connection between Gorenstein and Cohen-Macaulay rings.

Theorem 5.2.3. *Let R be a coherent Gorenstein ring. Then R is locally Cohen-Macaulay.*

Proof. Let $q \in \operatorname{Spec} R$. Then R_q is a coherent Gorenstein ring, so we may assume R is quasi-local. Let $\mathbf{x} = x_1, \ldots, x_n$ be a strong parameter sequence of R of length n, and $\mathbf{x}' = x_1, \ldots, x_{n-1}$. By induction, we may assume $\mathbf{x}'$ is a regular sequence. It must be shown that x_n is regular on $R/(\mathbf{x}')$. Assume not; then $x_n \in p \in \text{wAss } R/(\mathbf{x}')$. Localizing at p, $\text{p-depth}_{R_p}(R/(\mathbf{x}'))_p = 0$. Since $\mathbf{x}'$ remains regular on R_p, $\text{p-depth}_{R_p} R_p = n-1$ by Proposition 2.2.3(vii). Replacing R_p with R, we may assume $\text{p-depth}_R R = n-1$. Since $(\mathbf{x})$ is finitely generated, $R/(\mathbf{x})$ is clearly finitely presented. Corollary 4.2.2 gives $\text{G-dim}_R R/(\mathbf{x}) \leq \text{p-depth} R = n - 1$. Thus, $\operatorname{Ext}^n_R(R/(\mathbf{x}), R) = 0$ by Theorem 3.2.13. Similarly, as $\mathbf{x}'^k = x_1^k, \ldots, x_{n-1}^k$ is regular it follows that $\operatorname{Ext}^n_R(R/(\mathbf{x}^k), R) = 0$ for all $k > 0$. Thus $\varinjlim_{k\to\infty} \operatorname{Ext}^n_R(R/(\mathbf{x}^k), R) = H^n_{\mathbf{x}}(R) = 0$, contradicting that $\mathbf{x}$ is a strong parameter sequence. Hence x_n is regular on $R/(\mathbf{x})$, and $\mathbf{x}$ is regular on R. Thus R is Cohen-Macaulay, and hence the original ring is locally Cohen-Macaulay. □

5.3 Gorenstein Rings and FP-Injectivity

In the Noetherian context, a local ring is Gorenstein if and only if it has finite injective dimension. One might ask if a similar relationship exists in the non-Noetherian context. There are several indications that FP-injective rings may be Gorenstein. The first is the compatibility of FP-injective dimension with the following characterization of Artinian Gorenstein rings.

Proposition 5.3.1. *[6, Exercise 3.2.15] Let $(R, \mathfrak{m})$ be an Artinian local ring. The following are equivalent:*

(i) R is Gorenstein.

(ii) All finitely generated R-modules M are reflexive, that is $M \cong M^{**}$.

(iii) $I = (0 :_R (0 :_R I))$ *for all ideals I of R.*

(iv) For all non-zero ideals I and J, $I \cap J \neq 0$.

In light of Propositions 2.4.5 and 2.4.8, as well as the properties of coherence seen in Theorem 2.1.2, this suggests that quasi-local coherent FP-injective rings may be Gorenstein.

In addition, quasi-local FP-injective rings are Cohen-Macaulay, as seen in Proposition 5.3.3. First consider the following lemma from [15].

Lemma 5.3.2. *[15] Let R be a ring and x be an element in the Jacobson radical of R. Then* $H^i_x(R) = 0$ *if and only if x is nilpotent.*

Proposition 5.3.3. *Let $(R, \mathfrak{m})$ be a quasi-local FP-injective ring. Then the only weakly proregular elements in R are units or nilpotents. In particular, R is Cohen-Macaulay.*

Proof. Let $x \in R$ be a weakly proregular element. Then for every n there is an $m \geq n$ such that $(0 :_R x^m) = (0 :_R x^{m-n})$. As R is FP-injective, we have
$(x^m) = (0 :_R (0 :_R x^m)) = (0 :_R (0 :_R x^{m-n})) = (x^{m-n})$. Hence $x^{m-n}\alpha = 0$ for some $\alpha \in R$, and either:

- $x \notin \mathfrak{m}$, and x is a unit, or,

- $x \in \mathfrak{m}$. In this case α is a unit and $x^{m-n} = 0$. Thus x is nilpotent.

Note that units cannot be parameter sequences by definition. Also, by Lemma 5.3.2, nilpotents cannot be parameter sequences. Hence the empty sequence is the only parameter sequence in R, and thus R is Cohen-Macaulay. □

With our definition of Gorenstein in hand, we can make a connection between coherent Gorenstein rings and FP-injective dimension. This connection requires a restriction on p-depth.

Lemma 5.3.4. *Assume R is a coherent ring. Let M be a finitely presented R-module and I be a finitely generated ideal.*

(i) If R is FP-injective, $M = 0$ if and only if $\operatorname{Hom}_R(M, R) = 0$.

(ii) If R is FP-injective, or $\operatorname{G-dim}_R R/I = 0$, *then $I = R$ if and only if* $\operatorname{Hom}_R(R/I, R) = 0$.

Proof. For (i), if $M = 0$, clearly $\operatorname{Hom}_R(M, R) = 0$.

Assume $\operatorname{Hom}_R(M, R) = 0$. By Proposition 2.4.5 M is reflexive, so $M \cong M^{**} = \operatorname{Hom}_R(\operatorname{Hom}_R(M, R), R) = \operatorname{Hom}_R(0, R) = 0$.

For (ii), set $M = R/I$, and apply (i). □

Restricting to $\operatorname{p-depth}_R R = 0$, we obtain an equivalence between quasi-local coherent Gorenstein and FP-injective rings.

Theorem 5.3.5. *Let $(R, \mathfrak{m})$ be a quasi-local coherent ring. R is Gorenstein with* $\operatorname{p-depth}_R R = 0$ *if and only if R is FP-injective.*

Proof. Assume R is Gorenstein. Then $\operatorname{G-dim}_R R/I < \infty$ for every finitely generated ideal I. By Proposition 4.1.20, $\operatorname{G-dim}_R R/I = 0$. Thus by Theorem 3.2.13 $\operatorname{Ext}^i_R(R/I, R) = 0$ for all finitely generated ideals I and $i > 0$, so R is FP-injective.

Conversely, assume R is FP-injective; then given a finitely generated ideal I, $\mathrm{Ext}^i_R(R/I, R) = 0$ for all $i > 0$. By Proposition 2.4.5, $R/I \cong (R/I)^{**}$ for every finitely generated ideal I. Since $\mathrm{Hom}_R(R/I, R)$ is finitely presented by Corollary 2.1.8, for all $i > 0$

$$\mathrm{Ext}^i_R((R/I)^*, R) = \mathrm{Ext}^i_R(\mathrm{Hom}_R(R/I, R), R) = 0.$$

Hence $\text{G-dim}_R\, R/I = 0$ for all finitely generated ideals I.

To show $\text{p-depth}_R\, R = 0$, recall by Proposition 2.2.3(iii) that

$$\text{p-depth}_R\, R = \sup\{\text{p-grade}_R(I, R) | I \subset \mathfrak{m}, I \text{ a finitely generated ideal}\}.$$

By Lemma 5.3.4(ii), for each finitely generated ideal $I \neq R$, $\mathrm{Hom}_R(R/I, R) \neq 0$. So $\text{p-depth}_R\, R = 0$ by Proposition 2.2.3(i). □

Corollary 5.3.6. *If R is a Gorenstein ring with* $\text{p-depth}_R\, R = 0$*, (0) is irreducible.*

Proof. The proof follows from Theorem 5.3.5 and Proposition 2.4.9. □

The following inequality holds without the Gorenstein assumption.

Lemma 5.3.7. *Let $(R, \mathfrak{m})$ be a quasi-local coherent ring. Then*

$$\text{FP-id}_R\, R \geq \text{p-depth}_R\, R.$$

Proof. By Proposition 2.2.3(iii)

$$\text{p-depth}_R\, R = \sup\{\text{p-grade}_R(I, R) | I \subseteq \mathfrak{m}, I \text{ a finitely generated ideal}\}.$$

For each finitely generated ideal $I \subseteq \mathfrak{m}$, denote

$$t_I = \text{p-grade}_R(I, R) = \min\{i \mid \operatorname{Ext}^i_R(R/I, R) \neq 0\}.$$

By Remark 2.2.2, $\text{FP-id}_R\, R \geq t_I$ for each I; hence $\text{FP-id}_R\, R \geq \text{p-depth}_R\, R$. □

With the additional assumption of Gorenstein, equality holds.

Theorem 5.3.8. *If $(R, \mathfrak{m})$ is a quasi-local coherent Gorenstein ring, then*

$$\text{FP-id}_R\, R = \text{p-depth}_R\, R.$$

Proof. If $\text{p-depth}_R\, R = \infty$, equality holds by Lemma 5.3.7.

Assume $\text{p-depth}_R\, R < \infty$. For each finitely generated ideal I, $\text{G-dim}_R\, R/I < \infty$, and in particular, $\text{G-dim}_R(R/I) \leq \text{p-depth}_R\, R$ by Corollary 4.2.2. Theorem 3.2.13 says $\operatorname{Ext}_R^{\text{p-depth}_R R+1}(R/I, R) = 0$ for all finitely generated ideals I. Thus $\text{FP-id}_R\, R \leq \text{p-depth}_R\, R$ by Lemma 2.4.4; Lemma 5.3.7 gives equality. □

5.4 Additional Properties of Gorenstein Rings

Using this connection between FP-injective and Gorenstein rings, the next example provides an example of a Gorenstein ring that remains Gorenstein when reducing by a regular element.

Example 5.4.1. Let V be a d-dimensional valuation domain (and thus Gorenstein). Given a non-zero, non-unit $x \in V$, the ring V/xV is an FP-injective ring, and hence Gorenstein.

Proof. Set $R = V/xV$. As V is a valuation domain, every finitely generated ideal is principal and the ideals of R are totally ordered. Thus every finitely gener-

ated ideal of R is principal as well. Let $y \in V$ such that $y \notin (x)$. We claim that $((x) :_V ((x) :_V y)) = (y)$. As $y \notin (x)$, then $(x) \subseteq (y)$ since V is a valuation domain. Then $x = ry$ for some $r \in V$. As V is a domain and $y \neq 0$, $((x) :_V y) = ((ry) :_V y) = (r)$. As $r \neq 0$ (since $x \neq 0$),
$((x) :_V ((x) :_V y)) = ((ry) :_V r) = (y)$.

Now let I be a non-zero finitely generated ideal of $R = V/xV$. Then $I = yR$ for some $y \in V \setminus xV$. Since V is a valuation domain and $((x) :_V y)$ is finitely generated (since V is coherent), $((x) :_V y) = (z)$ for some $z \in V$. By the claim, $((x) :_V (z)) = (y)$. Hence $((0) :_R y) = zR$ and $((0) :_R z) = yR$. Thus the sequence

$$\cdots \xrightarrow{z} R \xrightarrow{y} R \xrightarrow{z} R \xrightarrow{y} R \to R/(y) \to 0$$

is exact. Truncating the sequence and applying $\mathrm{Hom}_R(-, R)$, yields the complex

$$0 \to R \xrightarrow{y} R \xrightarrow{z} R \xrightarrow{y} R \xrightarrow{z} R \xrightarrow{y} \cdots .$$

This complex is exact except at the 0th spot, hence $\mathrm{Hom}_R(R/(y), R) \neq 0$, and $\mathrm{Ext}^i_R(R/(y), R) = 0$ for all $i > 0$. Thus R is FP-injective, and by Theorem 5.3.5 R is Gorenstein with $\text{p-depth}_R R = 0$. □

In fact, given a coherent Gorenstein ring R, $R/(x)$ is also a Gorenstein ring, as will be shown in Corollary 5.4.4.

Lemma 5.4.2. *Let R be a ring and*

$$0 \to K \to L \to M \xrightarrow{\phi} N \to 0$$

be an exact sequence of R-modules. Suppose *(i)* $\mathrm{Ext}^1_R(L, R) = \mathrm{Ext}^1_R(M, R) = \mathrm{Ext}^1_R(N, R) = 0$, *and*

(ii) $\mathrm{Ext}^2_R(N, R) = 0$.

Then the sequence

$$0 \to N^* \to M^* \to L^* \to K^* \to 0$$

is exact.

Proof. Let $C = \ker \phi$. Then the sequences

$$0 \to K \to L \to C \to 0 \text{ and}$$
$$0 \to C \to M \to N \to 0$$

are exact. Applying $\mathrm{Hom}_R(-, R)$, we have the exact sequences

$$0 \to C^* \to L^* \to K^* \to \mathrm{Ext}^1_R(C, R) \to 0 \tag{5.4.1}$$

and

$$0 \to N^* \to M^* \to C^* \to 0 \to 0 \to \mathrm{Ext}^1_R(C, R) \to 0. \tag{5.4.2}$$

Thus $\mathrm{Ext}^1_R(C, R) = 0$ and the sequence

$$0 \to C^* \to L^* \to K^* \to 0$$

is exact. Pasting together (5.4.1) and (5.4.2) yields the desired exact sequence. □

The following generalizes a result of Peskine and Szpiro found in [2].

Theorem 5.4.3. *Let R be a ring and M a non-zero R-module such that $\tilde{\mathrm{G}}\text{-}\dim_R M = t < \infty$. Suppose $x \in \mathrm{Ann}_R M$ and x is R-regular. Then $\tilde{\mathrm{G}}\text{-}\dim_{R/(x)} M = t - 1$.*

Proof. As x is R-regular and $xM = 0$, it must be that $t \geq 1$ by Remark 4.1.4. We proceed by induction on t.

Suppose $t = 1$; there is an exact sequence

$$0 \to G_1 \to G_0 \to M \to 0 \tag{5.4.3}$$

where $G_0, G_1 \in \tilde{G}(R)$. Since x is R-regular and $xM = 0$, by Lemma 4.1.6 $\operatorname{Hom}_R(M, R) = 0$ and $\operatorname{Ext}^i_R(M, R) = \operatorname{Ext}^{i-1}_{R/(x)}(M, R/(x))$ for all $i \geq 0$. As $\tilde{\text{G}}\text{-dim}_R\, M = 1$, this gives $\operatorname{Ext}^i_{R/(x)}(M, R/(x)) = 0$ for all $i > 0$ and $\operatorname{Hom}_{R/(x)}(M, R/(x)) \cong \operatorname{Ext}^1_R(M, R)$.

Applying $\operatorname{Hom}_R(-, R)$ to (5.4.3) gives

$$0 \to G_0^* \to G_1^* \to \operatorname{Ext}^1_R(M, R) \to 0.$$

Since $G_i^* \in \tilde{G}(R)$ and $x \operatorname{Ext}^1_R(M, R) = 0$, we see that $\tilde{\text{G}}\text{-dim}_R \operatorname{Ext}^1_R(M, R) = 1$. Set $\bar{R} = R/(x)$. Again using Lemma 4.1.6,

$$\operatorname{Ext}^1_R(\operatorname{Ext}^1_R(M, R), R) \cong \operatorname{Hom}_{\bar{R}}(\operatorname{Ext}^1_R(M, R), \bar{R}) \cong \operatorname{Hom}_{\bar{R}}(\operatorname{Hom}_R(M, \bar{R}), \bar{R})$$

and $\operatorname{Ext}^i_{\bar{R}}(\operatorname{Hom}_{\bar{R}}(M, \bar{R}), \bar{R}) = 0$ for all $i > 0$.

For ease of notation define $M^\dagger := \operatorname{Hom}_{\bar{R}}(M, \bar{R})$. Notice that $\operatorname{Tor}^R_1(M, \bar{R}) \cong M$. To see this, apply $M \otimes_R -$ to the exact sequence

$$0 \to R \xrightarrow{x} R \to \bar{R} \to 0$$

to obtain

$$\cdots \to \operatorname{Tor}_1^R(M,R) \to \operatorname{Tor}_1^R(M,\bar{R}) \to M \xrightarrow{x} M \to \bar{M} \to 0,$$

where $\bar{M} = M/xM$. Since R is a free R-module, $\operatorname{Tor}_1^R(M,R) = 0$, hence $\operatorname{Tor}_1^R(M,\bar{R}) = \ker(M \xrightarrow{x} M) = M$ since $x \in \operatorname{Ann}_R M$.

Applying $- \otimes_R \bar{R}$ to (5.4.3) and using both that $\operatorname{Tor}_1^R(M,\bar{R}) = M$ and that by Lemma 4.1.5 $\operatorname{Tor}_1^R(G_0,\bar{R}) = 0$, we have the exact sequence

$$0 \to M \to \bar{G_1} \to \bar{G_0} \to M \to 0,$$

where $\bar{G_i} = G_i/xG_i \in \tilde{G}(\bar{R})$.

Applying Lemma 5.4.2 twice (using $\operatorname{Hom}_{\bar{R}}(-,\bar{R})$), we get the exact sequences

$$0 \to M^\dagger \to \bar{G_0}^\dagger \to \bar{G_1}^\dagger \to M^\dagger \to 0$$

and

$$0 \to M^{\dagger\dagger} \to \bar{G_1}^{\dagger\dagger} \to \bar{G_0}^{\dagger\dagger} \to M^{\dagger\dagger} \to 0.$$

Consider the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \longrightarrow & \bar{G_1} & \longrightarrow & \bar{G_0} & \longrightarrow & M & \longrightarrow & 0 \\ & & \downarrow & & \cong\downarrow & & \cong\downarrow & & \downarrow & & \\ 0 & \longrightarrow & M^{\dagger\dagger} & \longrightarrow & \bar{G_1}^{\dagger\dagger} & \longrightarrow & \bar{G_0}^{\dagger\dagger} & \longrightarrow & M^{\dagger\dagger} & \longrightarrow & 0. \end{array}$$

Since $\bar{G_0}, \bar{G_1} \in \tilde{G}(\bar{R})$, the canonical map $M \to M^{\dagger\dagger}$ is an isomorphism by the Five Lemma. Hence $M \in \tilde{G}(\bar{R})$ and $\tilde{\text{G}}\text{-dim}_{\bar{R}}\, M = 0$.

Suppose $t > 1$; let $\phi : G \to M$ be surjective for some $G \in \tilde{G}(R)$. Then

$\bar{\phi} : G/xG \to M$ is surjective. Let $K = \ker \bar{\phi}$. Then

$$0 \to K \to G/xG \to M \to 0$$

is exact. As

$$0 \to G \xrightarrow{x} G \to G/xG \to 0$$

is exact, we see $\tilde{\text{G}}\text{-dim}_R\, G/xG = 1$. Notice that $\tilde{\text{G}}\text{-dim}_R\, K < \infty$ by Proposition 3.1.10. Then since $\tilde{\text{G}}\text{-dim}_R\, M = t > 1$, the long exact sequence

$$\cdots \to \operatorname{Ext}_R^{t-1}(K, R) \to \operatorname{Ext}_R^t(M, R) \to 0 \to \operatorname{Ext}_R^t(K, M) \to 0 \to \cdots$$

shows $\tilde{\text{G}}\text{-dim}_R\, K = t-1$ as $\operatorname{Ext}_R^t(M, R) \neq 0$. Since $xK = 0$ and $\tilde{\text{G}}\text{-dim}_R\, K = t-1$, by induction, we have $\tilde{\text{G}}\text{-dim}_{R/(x)}\, K = t-2$. As $G/xG \in \tilde{G}(R/(x))$, $\tilde{\text{G}}\text{-dim}_{R/(x)}\, M = t-1$. □

Corollary 5.4.4. *Let R be a coherent Gorenstein ring and $x \in R$ a non-unit regular element. Then $R/(x)$ is Gorenstein.*

Proof. Through localization we may assume that $(R, \mathfrak{m})$ is a quasi-local coherent Gorenstein ring and that $x \in \mathfrak{m}$ is regular. Let $\bar{I} = I/(x)$ be a finitely generated ideal of $\bar{R} := R/(x)$. It suffices to show that $\tilde{\text{G}}\text{-dim}_{\bar{R}}\, \bar{R}/\bar{I} = \tilde{\text{G}}\text{-dim}_{\bar{R}}\, R/I < \infty$. Since R is Gorenstein, $\tilde{\text{G}}\text{-dim}_R\, R/I < \infty$. By Theorem 5.4.3,
$\tilde{\text{G}}\text{-dim}_{\bar{R}}\, R/I = \tilde{\text{G}}\text{-dim}_R\, R/I - 1 < \infty$. □

Given a regular ring R and $x \in R$ an R-regular element, it is not necessarily true that $R/(x)$ is regular. Hence Corollary 5.4.4 provides us with many examples of coherent Gorenstein rings that are not regular.

The converse of Corollary 5.4.4 holds for quasi-local coherent rings.

Theorem 5.4.5. *Let $(R, \mathfrak{m})$ be a coherent quasi-local ring and $x \in \mathfrak{m}$ be an R-regular element. If $R/(x)$ is Gorenstein, then so is R.*

Proof. Let M be a finitely presented R-module. If x is M-regular, then since M/xM is a finitely presented $R/(x)$-module and $R/(x)$ is Gorenstein, $\tilde{\text{G}}\text{-dim}_{R/(x)} M/xM < \infty$. Thus $\tilde{\text{G}}\text{-dim}_R M < \infty$ by Proposition 4.1.15.

If instead x is a zero-divisor on M, let

$$0 \to K \to F \to M \to 0$$

be an exact sequence with F a finitely generated free R-module. Then K is finitely presented by coherence, and x is K-regular. By the previous argument, $\tilde{\text{G}}\text{-dim}_R K < \infty$. Hence $\tilde{\text{G}}\text{-dim}_R M < \infty$ by Corollary 3.1.9. □

Corollary 5.4.6. *Let R be a coherent ring and $x \in R$ an R-regular element. Then R is Gorenstein if $R/(x)$ and R_x are Gorenstein.*

Proof. Let $\mathfrak{m}$ be a maximal ideal. By definition, it suffices to show that $R_\mathfrak{m}$ is Gorenstein. If $x \in \mathfrak{m}$, then $(R/(x))_\mathfrak{m}$ is Gorenstein, and thus $R_\mathfrak{m}$ is Gorenstein by Theorem 5.4.5. If $x \notin \mathfrak{m}$, then $R_\mathfrak{m} \cong ((R_x)_{\mathfrak{m}_x}$ is Gorenstein as R_x is Gorenstein. □

In the next results we explore under what conditions the ring $R[x]$ is Gorenstein if R is a Gorenstein ring.

Lemma 5.4.7. *Let R be a ring, x an indeterminate over R, and M an $R[x]$-module. Then there is an exact sequence of $R[x]$-modules*

$$0 \to R[x] \otimes_R M \to R[x] \otimes_R M \to M \to 0.$$

Proof. If t is a second indeterminate over R, then M can be viewed as an $R[t,x]$-module via the ring homomorphism $f : R[t,x] \to R[t,x]/(t-x) \cong R[x]$, where for $f(t,x) \in R[t,x]$ and $m \in M$, $f(t,x)m := f(x,x)m$. Consider the short exact sequence of $R[t,x]$-modules

$$0 \to R[t,x] \overset{t-x}{\to} R[t,x] \to R[x] \to 0.$$

Applying $- \otimes_{R[x]} M$, we get the exact sequence of $R[t,x]$-modules

$$\mathrm{Tor}_1^{R[x]}(R[x], M) \to R[t,x] \otimes_{R[x]} M \to R[t,x] \otimes_{R[x]} M \to R[x] \otimes_{R[x]} M \to 0$$

where $\mathrm{Tor}_1^{R[x]}(R[x], M) = 0$ since $R[x]$ is a free $R[x]$-module. Now $R[x] \otimes_{R[x]} M \cong M$ as $R[t,x]$-modules. Also

$$R[t,x] \otimes_{R[x]} M \cong (R[t] \otimes_R R[x]) \otimes_{R[x]} M \cong R[t] \otimes_R M$$

as $R[t,x]$-modules. Thus we have a short exact sequence of $R[t,x]$-modules

$$0 \to R[t] \otimes_R M \to R[t] \otimes_R M \to M \to 0.$$

This is also a short exact sequence of $R[t]$-modules by restriction of scalars.

Replacing t with x and noting that the $R[t]$-module structure on M is the same as the $R[x]$-module structure, we get the desired short exact sequence. □

Corollary 5.4.8. *Let R be a ring and x an indeterminate. Let R be an $R[x]$-module and suppose* $\tilde{\mathrm{G}}\text{-}\mathrm{dim}_R\, M = 0$. *Then* $\tilde{\mathrm{G}}\text{-}\mathrm{dim}_{R[x]}\, M \leq 1$.

Proof. Since $R[x]$ is a faithfully flat R-module, $\tilde{\mathrm{G}}\text{-}\mathrm{dim}_{R[x]}\, R[x] \otimes_R M = 0$ by Proposition 4.1.2. We may now use the short exact sequence from Lemma 5.4.7 to see that $\tilde{\mathrm{G}}\text{-}\mathrm{dim}_{R[x]}\, M \leq 1$. □

The proof the next result was inspired by a parallel result of [11] for coherent regular rings.

Theorem 5.4.9. *Let R be a ring and x an indeterminate such that $R[x]$ is coherent. Then R is Gorenstein if and only if $R[x]$ is Gorenstein.*

Proof. Since x is a non-unit $R[x]$-regular element, by Corollary 5.4.6 if $R[x]$ is Gorenstein, then so is $R \cong R[x]/(x)$.

Suppose R is Gorenstein and let $\mathfrak{p}$ be a maximal ideal of $R[x]$. Localizing at $\mathfrak{p} \cap R$, we may assume that $(R, \mathfrak{m})$ is quasi-local and $\mathfrak{p} \cap R = \mathfrak{m}$. Since $R[x]$ is coherent, then so is R. Since $\mathfrak{p}/(\mathfrak{m}R[x])$ is a maximal ideal of $R[x]/(\mathfrak{m}R[x])$, $\mathfrak{p} = (\mathfrak{m}, f)R[x]$, where f is a monic polynomial in $\mathfrak{p}$. Then $R[x]/fR[x] \cong R^n$ as R-modules where $n = \deg f$.

Let J be a finitely generated ideal of $R[x]_{\mathfrak{p}}$ Then $J = I_{\mathfrak{p}}$ for some finitely generated ideal I of $R[x]$. As $R[x]$ is coherent, I is finitely presented as an $R[x]$-module. Hence I/fI is finitely presented as an $R[x]/fR[x]$-module by Theorem 2.1.4. Since $R[x]/fR[x] \cong R^n$, I/fI is finitely presented as an R-module. Since R is a quasi-local coherent Gorenstein ring, $\tilde{\text{G}}\text{-dim}_R(I/fI) < \infty$.

Claim 5.4.10. $\tilde{\text{G}}\text{-dim}_{R[x]}\, I/fI < \infty$.

Proof. We proceed by induction on $t = \tilde{\text{G}}\text{-dim}_R\, I/fI$. If $t = 0$, the result follows from Corollary 5.4.8. Suppose $t > 0$. Consider the exact sequence

$$0 \to K \to (R/[x]/fR[x])^n \to I/fI \to 0.$$

Since $R[x]/fR[x]$ is coherent and I/fI is finitely presented, K is a finitely presented $R[x]/fR[x]$-module by Proposition 2.5.3 and coherence. Hence K is finitely presented as an R-module. Moreover, as $R[x]/fR[x]$ is a finitely generated free R-module, $\tilde{\text{G}}\text{-dim}_R(R[x]/fR[x])^n = 0$. Thus $\tilde{\text{G}}\text{-dim}_R\, K = t - 1$.

By induction, $\tilde{\mathrm{G}}\text{-}\dim_{R[x]} K < \infty$ and by Corollary 5.4.8, $\tilde{\mathrm{G}}\text{-}\dim_{R[x]}(R[x]/fR[x])^n < \infty$. Thus $\tilde{\mathrm{G}}\text{-}\dim_{R[x]} I/fI < \infty$ by Proposition 3.1.10. □

Localizing at $\mathfrak{p}$ we get $\tilde{\mathrm{G}}\text{-}\dim_{R[x]_\mathfrak{p}} J/fJ < \infty$. As f is $R[x]_\mathfrak{p}$-regular, and $f \in \mathrm{Ann}_{R[x]_\mathfrak{p}} J/fJ$, $\tilde{\mathrm{G}}\text{-}\dim_{R[x]_\mathfrak{p}/fR[x]_\mathfrak{p}} J/fJ < \infty$ by Theorem 5.4.3. Finally, since f is $R[x]_\mathfrak{p}$-regular and J-regular, and as $R[x]_\mathfrak{p}$ is coherent, $\tilde{\mathrm{G}}\text{-}\dim_{R[x]_\mathfrak{p}} J < \infty$. □

Chapter 6

Other Characterizations of Gorenstein Rings

6.1 $(FP)_\infty$-Injective Dimension and Gorenstein Rings

In the previous chapter we explored the connection between FP-injective dimension and Gorenstein rings. We now consider an injective dimension based on $[FP]_\infty^R$-modules and its connection to the Gorenstein property.

Definition 6.1.1. Given a ring R, an R-module E is $(FP)_\infty$-*injective* if $\text{Ext}_R^1(M, E) = 0$ for all R-modules $M \in [FP]_\infty^R$. Similarly, E is $\mathcal{BE}$-*injective* if $\text{Ext}_R^1(M, E) = 0$ for all R-modules $M \in \mathcal{BE}(R)$.

Definition 6.1.2. An R-module E has $(FP)_\infty$-injective dimension at most n, denoted $(\text{FP})_\infty\text{-id}_R\, E \leq n$, if $\text{Ext}_R^{n+1}(M, E) = 0$ for all $M \in [FP]_\infty^R$.

$\mathcal{BE}$-injective dimension is defined similarly.

We then have the following equivalences.

Proposition 6.1.3. *The following are equivalent for an R-module M:*

(i) $(\mathrm{FP})_\infty\text{-}\mathrm{id}_R\, M \le n$ *(resp.* $\mathcal{BE}\text{-}\mathrm{id}_R\, M \le n$*).*

(ii) $\mathrm{Ext}_R^{n+1}(N, M) = 0$ *for all* N *of type* $(FP)_\infty^R$ *(resp.* $N \in \mathcal{BE}(R)$*).*

(iii) $\mathrm{Ext}_R^i(N, M) = 0$ *for all* $i > n$ *and* N *of type* $(FP)_\infty^R$ *(resp.* $N \in \mathcal{BE}(R)$*).*

(iv) Given an exact sequence $0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to K \to 0$ *with* E_i $(FP)_\infty$*-injective for* $0 \le i \le n-1$*, then* K *is* $(FP)_\infty$*-injective. (The analogous statement holds for* $\mathcal{BE}$*-dimension.)*

Proof. The equivalences (i) ⇔ (ii) and (i) ⇔ (iii) follow by definition.

The equivalence (iii) ⇔ (iv) follows from the isomorphism $\mathrm{Ext}_R^1(N, K) \cong \mathrm{Ext}_R^{n+1}(N, M)$ for any $N \in [FP]_\infty^R$ which is obtained by breaking the sequence

$$0 \to M \to E_0 \to E_1 \to \cdots \to E_{n-1} \to K \to 0$$

into short exact sequences and using the fact that $\mathrm{Ext}_R^i(N, E) = 0$ for any $N \in [FP]_\infty^R$ and all $i > 0$. Note that since every R-module can be embedded in an injective module (which is $(FP)_\infty$-injective), every module M has a resolution of the type shown in (iv) for all n.

The proof for $\mathcal{BE}(R)$-modules is similar. □

Proposition 6.1.4. *Let* R *be a quasi-local coherent ring, and* M *an* R*-module. Then* $\mathrm{FP}\text{-}\mathrm{id}_R\, M = (\mathrm{FP})_\infty\text{-}\mathrm{id}_R\, M$*.*

Proof. Lemma 2.4.4 shows this equality immediately, since any R-module is finitely presented if and only if it is contained in $[FP]_\infty^R$. □

Thus by Theorem 5.3.5,

Corollary 6.1.5. *A quasi-local coherent $(FP)_\infty$-injective ring is Gorenstein.*

Within the context of $\mathcal{BE}(R)$-modules we now consider a possible characterization Gorenstein rings.

Definition 6.1.6. A quasi-local ring R is *$\mathcal{BE}$-Gorenstein* if $\text{G-dim}_R M < \infty$ for all $M \in \mathcal{BE}(R)$.

In light of Proposition 5.1.4, we have the following result.

Proposition 6.1.7. *A quasi-local coherent ring is Gorenstein if and only if it is $\mathcal{BE}$-Gorenstein.*

With this equivalence in hand we next explore whether R being $(FP)_\infty$-injective, is equivalent to R being $\mathcal{BE}$-Gorenstein. An R-module M is *torsionless* if the canonical map $M \to M^{**}$ is injective. Lemma 6.1.8 and Theorem 6.1.9 are similar to Noetherian results found in [26] and Jans [20], but are instead placed in the non-Noetherian context of $(FP)_\infty$- and $\mathcal{BE}$-injective dimensions.

Lemma 6.1.8. *Let R be a ring and $M \in \mathcal{BE}(R)$ a torsionless R-module. Then there exists $N \in [FP]_\infty^R$ such that the sequence $0 \to M \to M^{**} \to \operatorname{Ext}_R^1(N, R) \to 0$ is exact.*

Proof. Since $M \in \mathcal{BE}(R)$, there is a short exact sequence

$$0 \to L \to F \to M \to 0$$

where F is a finitely generated free module and $L \in \mathcal{BE}(R)$. Applying $\operatorname{Hom}_R(-, R)$ and setting $N = \operatorname{coker}(M^* \to F^*)$ yields the exact sequence

$$0 \to M^* \to F^* \to N \to 0.$$

Note that $N \in [FP]^R_\infty$ by Corollary 2.5.4. Apply $\mathrm{Hom}_R(-, R)$ again and consider the following diagram with exact rows and columns.

$$
\begin{array}{ccccccccccc}
 & & K_1 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & L & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow \cong & & \downarrow \delta & & \downarrow & & \\
0 & \longrightarrow & N^* & \longrightarrow & F^{**} & \xrightarrow{\gamma} & M^{**} & \xrightarrow{\alpha} & \mathrm{Ext}^1_R(N, R) & \longrightarrow & \mathrm{Ext}^1_R(F^*, R) \longrightarrow \cdots \\
 & & \downarrow & & \downarrow & & \downarrow \beta & & & & \\
 & & C_1 & & 0 & & C_2 & & & &
\end{array}
$$

Note that $M \to M^{**}$ is injective because M is torsionless, and $F \cong F^{**}$ as F is a finitely generated free module. Since $F^* \cong R^n$, $\mathrm{Ext}^1_R(F^*, R) = 0$. Notice that $\ker \alpha = \operatorname{im} \gamma = \operatorname{im} \delta$ by the commutativity of the diagram. Thus $\mathrm{Ext}^1_R(N, R) \cong C_2$ and we have the short exact sequence

$$0 \to M \xrightarrow{\delta} M^{**} \xrightarrow{\alpha} \mathrm{Ext}^1_R(N, R) \to 0.$$

□

Theorem 6.1.9. *If a ring R is $(FP)_\infty$-injective, then $M \cong M^{**}$ for any $\mathcal{BE}(R)$-module M.*

Proof. Let M be a $\mathcal{BE}(R)$-module; hence we have a short exact sequence

$$0 \to K \to F \to M \to 0 \tag{6.1.1}$$

with F a finitely generated free module. Applying $\mathrm{Hom}_R(-, R)$ twice, consider the following diagram with exact rows and columns.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K & \longrightarrow & F & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \cong\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & K^{**} & \longrightarrow & F^{**} & \longrightarrow & M^{**} & \longrightarrow & \operatorname{Ext}^1_R(K^*, R)
\end{array}$$

By Lemma 3.2.14, the top row gives us that $K \in \mathcal{BE}(R)$; thus $\operatorname{Ext}^1_R(K^*, R) = 0$. Since F is torsionless, by the Snake Lemma so is K. Thus by Lemma 6.1.8 there exists $L \in [FP]^R_\infty$ such that

$$0 \to K \to K^{**} \to \operatorname{Ext}^1_R(L, R) \to 0.$$

By assumption R is $(FP)_\infty$-injective, so $\operatorname{Ext}^1_R(L, R) = 0$ and thus $K \cong K^{**}$. By the Five Lemma, $M \cong M^{**}$. □

Using these results there is a link between $(FP)_\infty$-injective rings and $\mathcal{BE}$-Gorenstein rings.

Theorem 6.1.10. *An $(FP)_\infty$-injective ring is $\mathcal{BE}$-Gorenstein.*

Proof. Let R be an $(FP)_\infty$-injective ring; by Theorem 6.1.9, $M \cong M^{**}$ for any $M \in \mathcal{BE}(R)$. It suffices to show any such M is totally reflexive. This holds since $M, M^* \in [FP]^R_\infty$ so by the $(FP)_\infty$-injectivity of R, $\operatorname{Ext}^i_R(M, R) = \operatorname{Ext}^i_R(M^*, R) = 0$ for all $i > 0$. □

From Theorem 6.1.10 it also holds that all $\mathcal{BE}(R)$-modules in an $(FP)_\infty$-injective ring are totally reflexive.

Corollary 6.1.11. *Let R be a ring. If $(\mathrm{FP})_\infty\text{-id}_R\, R = 0$, then $\operatorname{G-dim}_R M = 0$ for all $M \in \mathcal{BE}(R)$. Conversely, if $\operatorname{G-dim}_R M = 0$ for all $M \in [FP]^R_\infty$, then $(\mathrm{FP})_\infty\text{-id}_R\, R = 0$.*

Proof. Suppose $(\mathrm{FP})_\infty\text{-}\mathrm{id}_R\, R = 0$. Given $M \in \mathcal{BE}(R)$, by definition $\mathrm{Ext}^i_R(M,R) = \mathrm{Ext}^i_R(M^*,R) = 0$ for all $i > 0$. By Theorem 6.1.9, M is totally reflexive.

If $\text{G-dim}_R\, M = 0$ for all $M \in [FP]^R_\infty$, then for all such M, $\mathrm{Ext}^i_R(M,R) = 0$ for all $i > 0$. Hence $(\mathrm{FP})_\infty\text{-}\mathrm{id}_R\, R = 0$. □

This result can easily be partially extended to $\mathcal{BE}$-injective rings.

Corollary 6.1.12. *If* $\text{G-dim}_R\, M = 0$ *for every R-module $M \in \mathcal{BE}(R)$, then R is $\mathcal{BE}$-injective.*

However, to obtain the converse of Corollary 6.1.12, one would need $M \cong M^{**}$ for every $M \in \mathcal{BE}(R)$ in order for M to be totally reflexive.

Thus we have the following inequality for any R-module E

$$\mathcal{BE}\text{-}\mathrm{id}_R\, E \leq (\mathrm{FP})_\infty\text{-}\mathrm{id}_R\, E \leq \text{FP-}\mathrm{id}_R\, E.$$

Bibliography

[1] Bernard Alfonsi. Grade non-noethérien. *Comm. Algebra*, 9(8):811–840, 1981.

[2] Maurice Auslander and Mark Bridger. *Stable module theory.* Memoirs of the American Mathematical Society, No. 94. American Mathematical Society, Providence, R.I., 1969.

[3] H. Bass. *Global dimension of rings.* Ph.D. Thesis. Univ. of Chicago, Chicago, Ill., 1959.

[4] José Bertin. Anneaux cohérents réguliers. *C. R. Acad. Sci. Paris Sér. A-B*, 273:A590–A591, 1971.

[5] Robert Bieri. *Homological dimension of discrete groups.* Mathematics Department, Queen Mary College, London, 1976. Queen Mary College Mathematics Notes.

[6] Winfried Bruns and Jürgen Herzog. *Cohen-Macaulay rings*, volume 39 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1993.

[7] Henri Cartan and Samuel Eilenberg. *Homological algebra.* Princeton University Press, Princeton, N. J., 1956.

[8] Stephen U. Chase. Direct products of modules. *Trans. Amer. Math. Soc.*, 97:457–473, 1960.

[9] Lars Winther Christensen. *Gorenstein dimensions*, volume 1747 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2000.

[10] Edgar E. Enochs and Overtoun M. G. Jenda. Gorenstein injective and projective modules. *Math. Z.*, 220(4):611–633, 1995.

[11] Sarah Glaz. Regular symmetric algebras. *J. Algebra*, 112(1):129–138, 1988.

[12] Sarah Glaz. *Commutative coherent rings*, volume 1371 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989.

[13] Sarah Glaz. Coherence, regularity and homological dimensions of commutative fixed rings. In *Commutative algebra (Trieste, 1992)*, pages 89–106. World Sci. Publ., River Edge, NJ, 1994.

[14] Sarah Glaz. Homological dimensions of localizations of polynomial rings. In *Zero-dimensional commutative rings (Knoxville, TN, 1994)*, volume 171 of *Lecture Notes in Pure and Appl. Math.*, pages 209–222. Dekker, New York, 1995.

[15] Tracy Dawn Hamilton and Thomas Marley. Non-Noetherian Cohen-Macaulay rings. *J. Algebra*, 307(1):343–360, 2007.

[16] Morton E. Harris. Some results on coherent rings. *Proc. Amer. Math. Soc.*, 17:474–479, 1966.

[17] Morton E. Harris. Some results on coherent rings. II. *Glasgow Math. J.*, 8:123–126, 1967.

[18] Henrik Holm. Gorenstein homological dimensions. *J. Pure Appl. Algebra*, 189(1-3):167–193, 2004.

[19] Masatosi Ikeda and Tadasi Nakayama. On some characteristic properties of quasi-Frobenius and regular rings. *Proc. Amer. Math. Soc.*, 5:15–19, 1954.

[20] J. P. Jans. Duality in Noetherian rings. *Proc. Amer. Math. Soc.*, 12:829–835, 1961.

[21] Daniel Lazard. Sur les modules plats. *C. R. Acad. Sci. Paris*, 258:6313–6316, 1964.

[22] B. H. Maddox. Absolutely pure modules. *Proc. Amer. Math. Soc.*, 18:155–158, 1967.

[23] Hideyuki Matsumura. *Commutative ring theory*, volume 8 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1986. Translated from the Japanese by M. Reid.

[24] D. G. Northcott. *Finite free resolutions*. Cambridge University Press, Cambridge, 1976. Cambridge Tracts in Mathematics, No. 71.

[25] Peter Schenzel. Proregular sequences, local cohomology, and completion. *Math. Scand.*, 92(2):161–180, 2003.

[26] Bo Stenström. Coherent rings and FP-injective modules. *J. London Math. Soc. (2)*, 2:323–329, 1970.

[27] Wolmer V. Vasconcelos. Annihilators of modules with a finite free resolution. *Proc. Amer. Math. Soc.*, 29:440–442, 1971.

[28] Diana White. Gorenstein projective dimension with respect to a semidualizing module. *arXiv:math.AC/0611711*, Nov 2006.