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The Game of Nim

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Expository Paper

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The game of Nim is possibly one of the most frustrating games I have ever played. Just when I started to feel that I had figured the strategy out, my brother, who is a computer programmer, blew me out of the water. I should have known better than to take on a computer wizard.

Many sources claim that Nim possibly originated in China. It was played with pieces of paper, coins or whatever objects they could find. Charles Bouton, a professor at Harvard University, gave Nim its' name around 1901. He named it after an archaic English word meaning to steal or to take away. Some people have noticed that if you turn the Nim upside-down and backwards it results in Win. In 1901 Bouton published a proof containing a winning strategy for Nim. His strategy was based on the Binary (Base 2) number system.

The system of numbers that we use in the United States is called the Hindu-Arabic system. This system is a base ten system and uses the 10 digits 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Whole numbers in this system have a value equal to the sums of the products of the digits and the place values. The place values in this system are as shown here:

Etc.;	10,000;	1000;	100;	10;	1.
	10^4	10^3	10^2	10^1	10^0

For instance the numeral 1302 has the value of:

Etc.;	10,000;	1000;	100;	10;	1
		1	3	0	2

which is 1 times 1000 plus 3 times 100 plus 0 times 10 plus 2 times 1.

All systems do not use 10 digits. Some systems use fewer than 10 digits, and some use more than 10 digits. Each system uses the same number of digits as the base of the system.

The modern binary number system was fully documented in 1705 by Gottfried Lebniz in his article 'Expliction de l'Arithmetique Binaire. In the Binary (Base 2) number system we use the two digits 0 and 1. The values of the places in the base 2 system are (shown as base 10 numbers)

Etc.;	128;	64;	32;	16;	8;	4;	2;	1.
	2^7	2^6	2^5	2^4	2^3	2^2	2^1	2^0

The first place to the left of the decimal point has a value of 1. To get the next value, we multiply 1 by 2 and get 2. To get the next value, we multiply 2 by 2 and get 4. To get the next value, we multiply 4 by 2 to get 8, etc. Each place has a value twice the value of the place to its right. As an example the base 2 numeral of 10111 has a base 10 value of:

Etc.;	128;	64;	32,	16,	8,	4,	2,	1.
				1	0	1	1	1

which is 1 times 16 plus 0 times 8 plus 1 times 4 plus 1times 2 plus 1 times 1 that all equals 23 (base 10).

Because the native language of a computer was based on the binary system, the creation of a Nim playing computer was only a matter of time. Westinghouse Electrical Corporation built the first Nim playing computer in 1940 and exhibited it at the New York Worlds Fair. In 1951, at the Festival of Britian, a Nim-playing robot, called Nimrod, was exhibited. So the next time someone calls you a nimrod you can thank him or her for comparing you to a computer ran robot.

The game of Nim is a two-player mathematical game of strategy. Players take turns removing objects from distinct piles. A player must remove at least one object on their turn but may remove any number of objects provided they all come from the same pile. Nim is a game of strategy where the outcome is influenced through the interaction with the environment and other players. The crucial factor that separates this type of game from others is that there is no element of chance involved. Both players have equal and complete knowledge of all elements of the game. There is also no physical skill required.

Nim was originally played as a misere game, which means that the player to take the last object loses. Nim can also be played as a normal game which means that the person who removes the last object wins. This is called normal play because most games follow this convention, even though the true game of Nim usually does not. The game of Nim can be played either way and the strategy for winning is just reversed. On to the game.

For this paper I am going to discuss the particular game of Nim given to me and the winning strategies. My version of Nim is as follows:

Imagine you have 3 piles of beans where the first pile has 3 beans, the second pile has 5 beans, and the third pile has 8 beans. The game of Nim is a 2-person game in which a player removes as many beans on their turn as they want, provided the beans are all from the same pile. The winner is the player who takes the last bean. As was stated above the strategy to winning Nim lies in the binary numbers. Whether you are playing with one pile on any number of piles here are the winning secrets.

Before we start with my 3 pile game, let’s consider a somewhat easier one pile version. This one pile game does not rely on binary numbers but on multiples on some other fixed number.

One Pile

In a normal game of nim where you have one nim pile, and you are allowed to remove from 1 to 4 objects at a time, you can win if you leave the pile with a multiple of 5 objects after your turn. Thereafter, no matter what your opponent does, you can always restore the number of objects to a multiple of 5 when you take your turn. If you cannot leave the pile with a multiple of 5 objects, then your opponent has the advantage, and only an error by your opponent can put you in a position to win the game. So you always want to leave a multiple that is equal to (the maximum removals plus one) to win a normal game of nim.

An example of a normal game of one pile

For example, lets start a game with 13 objects and you can remove 1-4 objects on one turn. I would remove 3 objects to leave 10 objects for my opponent. If my opponent took one object then I would take four leaving 5 objects left. If my opponent to two object I would have then took three objects again only leaving 5 objects left. If he took three objects then I would have taken two and if he had taken four objects then I would have removed one. At the end of each of my second move I have left five objects. With this strategy I can now guarantee myself of being able to remove the last object by following the same pattern. Along this same thought process if the game starts with 10 or 15 objects then the player that goes first can not win the game unless his opponent makes a mistake.

In a misere game you want your opponent to take the last object so think of the game in reverse order, that is instead of taking the last object you want to leave one object at the end of your turn.

An example of a misere game of one pile (in reverse)

For this example you are allowed to remove one, two, or three objects during one turn. If there were two objects at the beginning of your turn, you'd take one and win. If there were three objects at the beginning of your turn, you'd take two and win. If there were four objects at the beginning of your turn, you'd take three and win.
So, if you can get your opponent to leave two, three, or four objects at the end of his turn, you can win easily.

Let's say there are five objects at the end of your next-to-last turn. If your opponent takes one, then there are four left and you can win. If your opponent takes two, then there are three left and you can win. If your opponent takes three, then there are two left and you can win. So, no matter what move your opponent makes when he/she sees five objects, you win.

If the game starts with 15 objects and you can remove 1, 2, or 3 objects in one turn then this is what the first few turns would look like. I go first and remove 2 objects leaving 13. My opponent removes one object to leave me with 12. I would then remove three objects to leave 9. If opponent removed two objects on his first turn (leaving 11) then I would have removed two objects. If opponent had removed three objects on his first turn (leaving 10) then I would have removed one, again leaving 9. With this same strategy I can now get so that I leave 5 on my next move and 1 on my next move after that. So, if you can leave five objects after your turn, you win. By this same reasoning, if you can leave nine objects after your turn, you win, because that allows you to get to five. You can figure out the other critical point. So to win at the misere game of nim you want to leave one plus the multiple that is equal to (the sum of maximum removals plus one).

Normal game with more than one heap

In a game of nim that involves nim piles where you can take as many objects as you want from any one of the piles during your turn, you need to be able to compute a *nim sum*, which characterizes the configuration of the game. Here's how to do it:

- Express the number of objects in each nim pile as a binary number, with the only digits being 0 and 1.
- Fill out the smaller binary numbers with '0's on the left, if necessary, so that all the numbers have the same number of digits.
- Sum the binary numbers (in base 10), but do not carry
- Replace each digit in the sum with the remainder that results when the digit is divided by 2 (same as replacing with a 0 if even and 1 if odd).
- This yields the *nim sum*.
- To win at nim, always make a move, when possible, that leaves a configuration with a nim sum of 0. If you cannot do this, your opponent has the advantage, and you have to depend on his or her committing an error in order to win.
    - Note that if the configuration you are given has a nim sum not equal to 0, there always is a move that creates a new configuration with a nim sum of 0. However, there are usually also moves that will yield configurations that give nim sums not equal to 0, and you need to avoid making these.
    - Also note that if you are given a configuration that has a nim sum of 0, there is no move that will create a configuration that also has a nim sum of 0. You lose!

Example of Computing a Nim Sum

Let's say you have three nim piles, with 3, 7, and 11 matchsticks, respectively.

- As binary numbers, these quantities are 11, 111, and 1011, respectively.
- Filling out with '0's yields 0011, 0111, and 1011.
- Summing (in base 10) without carrying, we get:

- `  0011`
- ` +0111`
- ` +1011`
- `  ----`
- `  1133`

- Taking the remainders after dividing each digit by 2 yields **1111**, which is not equal to 0. Therefore, this configuration is a winning position for the person who is about to take a turn. It follows that you should try *not* to create this configuration when you take your turn. Now, you need to plan a move that creates a configuration with a nim sum of 0. How about removing 7 matchsticks from the pile that now has 11, leaving 4? We then have:

- `  0011`
- ` +0111`
- ` +0100`
- `  ----`
- `  0222`

- Taking the remainders after dividing by 2 gives us **0000**. This shows that there are no ‘leftovers’ in any column. This is a configuration that we want to leave for our opponent. Whatever move your opponent makes must change exactly one of the binary numbers. So in one of the rows at least one 1 or zero will be changed, thus creating a non-zero Nim sum. If it is your turn with this configuration, then you can only win if your opponent commits an error.

So, let us now try this on the normal game given to us where we have piles of 8 beans, 5 beans, and 3 beans.

We have three nim piles, with 8, 5, and 3 beans, respectively.

- As binary numbers, these quantities are 1000, 101, and 11, respectively.
- Filling out with '0's yields 1000, 0101, and 0011.
- Summing with carrying, we get:
- `  1000`
- ` +0101`
- ` +0011`
- `  ----`
- `  1112`

- Taking the remainders after dividing each digit by 2 yields **1110**, which is not equal to 0. Therefore, this configuration is a winning position for the person who is about to take a turn. It follows that you should try *not* to create this configuration when you take your turn. Now, you need to plan a move that creates a configuration with a nim sum of 0. How about removing 2 beans from the pile that now has 8, leaving 6? We then have:
- `  0110`
- ` +0101`
- ` +0011`
- `  ----`
- `  0222`

  Taking the remainders after dividing by 2 gives us **0000**. This is a configuration that we want to leave for our opponent.

- Try the Nim Sum Calculator Applet. You will need to have a Java 2 plug-in to do this.

If the nim-sum is even to start the game, this is known as a zero game for there is no way, short of an error by your opponent that you can win. If your opponent plays perfectly there are no options for you. If you see this situation at the beginning of the game, offer to let your opponent go first. Of course if he/she also knows the strategy they will probably refuse.

There are many different variations to the game of nim. All you must do is change the number of objects in each pile or change the number of piles. The strategy stays the same. Plainim is the most challenging variation of nim that deviates the first step of finding binary representations. Plainim is played on a 5 by 5 checkerboard by removing and/or adding objects. You therefore have 5 piles of 0, 1, 2, 3, 4, or 5 objects depending on the game being played. The simple rules are:

*On a single move, you may only remove/add objects in a single row.

*Only one or zero objects are allowed per square.

*You may only add chips to the right of a chip being removed on the same move.

*The one to remove the last chip wins.

So you are not only playing the takeaway game but now you can also add objects to a certain pile to get you nim-sum. It sounds so easy doesn't it? But even when you know the strategy and have the game figured out your opponent can add an object to the board and force a whole different game at you.

Fibonacci nim is another variation with just one pile but different takeaway rules. In Fibonacci nim the rules are:

- There is one pile of counters. The first player may remove any positive number of counters, but not the whole pile.
- Thereafter, each player may remove at most twice the number of counters his opponent took on the previous move.
- The player who removes the last counter wins.
- You move first: you can win from the initial position, but don't make any mistakes!

The mathematical strategy to winning Fibonacci nim is very complicated and has to do something with Grundy numbers and what you can do with them. In short you need to take the number of counters you have and rewrite it as a sum of smaller Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, 21, 34, 55, etc.). You will then remove the number of counters equal to the smallest Fibonacci number within this sum expression. As an example let's play a normal game with 26 objects. On my first move I will rewrite 26 as 13 + 8 + 5 and then remove 5 objects leaving 21. If my opponent removes, say 4 objects, which leaves 17. I can write 17 as 13 + 3 + 1 and therefore remove one object to leave 16. With this strategy you have a great chance of winning. You must be careful as you reach the lower numbers and remember that you opponent has the option of removing 2x's what you last removed so make sure that your sum expression is with the smallest Fibonacci numbers. If you start a game and do not start with a Fibonacci number then the person that starts has the advantage. If you start a game with a Fibonacci number of objects and your opponent knows the strategy, then she/he will win unless you can convince she/he to go first.

Other variations limit you on the number of objects you may remove from a pile. Still another variation, called Wolf nim, allows you to remove multiple objects from different rows. Wolf nim is usually played as a misere game in which the last to remove an object loses.

The game of Nim, and all its' variations, can be made simple enough for upper elementary students to be successful at or make extremely difficult to challenge the most experienced Mathematician. They are fun, frustrating, and frighteningly habit forming. That has a nim-sum of one, so I win, game over.

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