1975

ELECTRIC FIELDS AND POTENTIALS FROM CONTINUOUS CHARGE DISTRIBUTIONS

STUDY GUIDE

ELECTRIC FIELDS AND POTENTIALS FROM CONTINUOUS CHARGE DISTRIBUTIONS

INTRODUCTION

Too bad! In case you have not realized it, not all charges come packaged as points, spheres, infinite cylinders, or infinite planes. Ah, if only it were so: Life would be much easier from a calculational viewpoint, although somewhat limited in geometrical options. But then, mechanics would be simpler if only constant accelerations were observed in nature...Not to mention centers of mass; moments of inertia, etc.; all would be considerably simpler to calculate in that wonderful world of point masses, constant accelerations, massless strings, and frictionless boards.

Once again calculus is needed to assist us in analyzing and understanding natural phenomena that are often manifested in hunks of mass, variable accelerations, and globs of charge.

This module introduces no new fundamental physics. Instead, you will learn to extend the concepts of electric field and potential to charge distributions that defy solution by superposition of point-charge fields and potentials or application of Gauss' law.

PREREQUISITES

Before you begin this module, you should be able to:	Location of Prerequisite Content
*Integrate polynomial, sine, and cosine functions (needed for Objectives 1 and 2 of this module)	Calculus Review
*Determine the electric field of a point charge (needed for Objectives 1 through 3 of this module)	Coulomb's Law and the Electric Field Module
*Determine the electric potential of a point charge (needed for Objectives 1 through 3 of this module)	Electric Potential Module

LEARNING OBJECTIVES

After you have mastered the content of this module, you will be able to:

1. Line charges - Given a rectilinear charge distribution, set up, and in some cases evaluate, the definite integral for:

(a) the total charge on a specified segment of the line;
(b) the electric potential at a specified point;
(c) the electric field at a specified point.

2. Ring and disk charges - Given a charge distribution on a circular arc, sector, or disk, set up, and in some cases evaluate, the definite integral for:
(a) the total charge on a specified portion of the distribution;
(b) the electric potential and electric field at the center of the circular arc, sector, or disk and on the axis of the disk.

3. Limiting cases - Demonstrate that the integrals of Objectives 1 and 2 reduce in limiting cases to results expected for simpler charge distributions.

GENERAL COMMENTS

Determining the electric potential V or the electric field E from a continuously distributed charge generally requires the use of integral calculus. Unless the charge distribution has sufficient symmetry so as to permit the use of Gauss' law to determine $\vec{E}$, a calculation of either $\vec{E}$ or V requires that you: (a) use physics to set up a definite integral; and (b) use calculus (or numerical techniques) to evaluate this integral. Since this is a physics course, your attention will focus on step a. Step b can range in difficulty from trivial to impossible depending upon the complexity of the charge distribution and upon your facility at evaluating integrals. Although we shall not emphasize the mathematical gymnastics of integral evaluation, you should feel free to try your hand at any that you simply cannot resist!

When calculating $\vec{E}$ or V from a distributed charge, the essential idea is reasonably simple. It goes like this. Select a very small (some would say infinitesimal) charge dq within the distribution. Treating it as a point charge, write the expression for either the potential dV or the field $d\vec{E}$ at the specified field point. Then superimpose the contributions from the total distribution by means of an integral.

Let us look at this procedure in more detail. Consider Figure 1. There are three vectors you must be sure you understand -

Position vector for the field point P:

$\vec{R} = x\hat{i} + y\hat{j} + z\hat{k}$;

Position vector for dq:

$\vec{\underline{R}} = \underline{x}\hat{i} + \underline{y}\hat{j} + \underline{z}\hat{k}$;

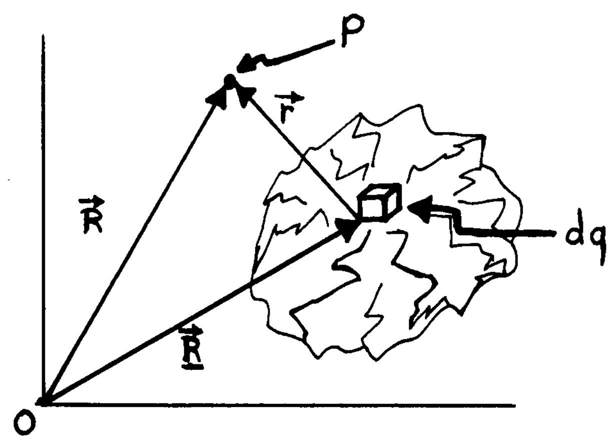


Figure 1

Displacement vector from dq to P:

$$\vec{r} = \vec{R} - \underline{\vec{R}} = (x - \underline{x})\hat{i} + (y - \underline{y})\hat{j} + (z - \underline{z})\hat{k}.$$

As you know from your study of electric potential, the potential at P attributable to the charge dq is

$$dV = (k\ dq)/r \quad (k = 1/4\pi\varepsilon_0 = 9.0 \times 10^9\ \mathrm{N\ m^2/C^2}),$$

where r is the distance from dq to P, i.e.,

$$r = |\vec{r}| = \sqrt{(x - \underline{x})^2 + (y - \underline{y})^2 + (z - \underline{z})^2}\ .$$

Writing out the expression for dV more explicitly gives us

$$dV = (k\ dq)/\sqrt{(x - \underline{x})^2 + (y - \underline{y})^2 + (z - \underline{z})^2}\ .$$

This emphasizes the dependence of dV upon the coordinates of both the field point P and the charge point. The potential at P from the total charge distribution is obtained by summing (integrating) over the charge.

$$V(\vec{R}) = \int_{\text{all charges}} \frac{k\ dq}{r}\ .$$

The dependence of the electric potential on the coordinates of the field point is emphasized by the functional dependence of V on $\vec{R}$.

For a given charge distribution the limits on the integral will be determined by the geometry of the charge distribution. This will be discussed in more detail in the Problem Set.

The electric field at P attributable to dq is

$$d\vec{E} = (k\vec{r}\ dq)/r^3.$$

Using the integral to superpose the contributions from all the charge gives us

$$\vec{E}(\vec{R}) = \int (k\vec{r}\ dq)/r^3,$$

where, again, the dependence of the field on the coordinates of the field point is emphasized.

You should note that the essential difference between the expressions for V and $\vec{E}$ is that V is a scalar sum, but $\vec{E}$ results from a vector sum. Details of setting up these integrals and seeing how to check them are covered in the Problem Set.

TEXT: Frederick J. Bueche, Introduction to Physics for Scientists and Engineers (McGraw-Hill, New York, 1975), second edition

SUGGESTED STUDY PROCEDURE

Begin by studying the General Comments. Then read Sections 18.6 and 18.7 up to Illustration 18.4 on p. 336 in Chapter 18 of your text, and Section 20.8 in Chapter 20. Next study Problems A through D and work Problems J, L, M, and N. Study Problems E and F and work Problem K. Problem S is challenging and optional. Next study Illustration 18.4 and Problems G and H. Work Problem O - Problem T is challenging, but optional. Study Sections 20.12 and 20.13, Problem I, and work Problems P, Q, and R.

Take the Practice Test, and work some Additional Problems if necessary, before attempting a Mastery Test.

BUECHE

Objective Number	Readings	Problems with Solutions: Study Guide	Assigned Problems: Study Guide	Additional Problems
1	Secs. 18.6, 18.7, 20.8	A, B, C	J, L, M, N	S; Chap. 18, Probs. 13, 14
2	Secs. 18.7, 20.12, 20.13	G, H, I	O, Q	T; Chap. 18, Probs. 15, 16, 17; Chap. 20, Probs. 18, 19
3	Sec. 20.8	D, E, F	K, M, N, P, R	S

TEXT: David Halliday and Robert Resnick, Fundamentals of Physics (Wiley, New York, 1970; revised printing, 1974)

SUGGESTED STUDY PROCEDURE

Begin by studying the General Comments and Problems A through D. Then work Problem J. Next study Problem E and work Problems K and L; study Problem F and work Problems M and N. Problem S is a challenging example that you may work if you so wish. Next study Problems G and H before working Problem O and, if you like, the challenging Problem T. Now go to your text and read Example 5 in Chapter 23 (on pp. 439, 440) and Example 6 in Chapter 25 (p. 473). Then study Problem I and work Problems P, Q, R.

Take the Practice Test, and work some Additional Problems if necessary, before attempting a Mastery Test.

HALLIDAY AND RESNICK

Objective Number	Readings	Problems with Solutions Study Guide	Assigned Problems Study Guide	Additional Problems
1	General Comments	A, B, C, F	J, L, M, N	S; Chap. 23, Probs. 28, 29
2	Chap. 23, Ex.[a] 5; Chap. 25, Ex. 6	G, H, I	O, Q	T; Chap. 23, Prob. 27
3	General Comments	D, E, F	K, M, N, P, R	S

[a]Ex. = Example(s).

TEXT: Francis Weston Sears and Mark W. Zemansky, University Physics (Addison-Wesley, Reading, Mass., 1970), fourth edition

SUGGESTED STUDY PROCEDURE

Begin by studying the General Comments. Then read Section 25-2 in Chapter 25, and study Problems A through D before working Problem J. Next study Problem E and work Problems K and L. Then study Problem F and work Problems M and N, and Problem S if you like (challenging but optional). Study Problems G and H and work Problem O; Problem T is optional. Study Problem I and work Problems P, Q, and R.

Take the Practice Test, and work some Additional Problems if necessary, before attempting a Mastery Test.

SEARS AND ZEMANSKY

Objective Number	Readings	Problems with Solutions	Assigned Problems	Additional Problems
		Study Guide	Study Guide	
1	General Comments, Sec. 25-2	A, B, C, F	J, L, M, N	S
2		G, H, I	O, Q	T
3		D, E, F	K, M, N, P, R	S

TEXT: Richard T. Weidner and Robert L. Sells, Elementary Classical Physics (Allyn and Bacon, Boston, 1973), second edition, Vol. 2

SUGGESTED STUDY PROCEDURE

Begin by studying the General Comments. Then study Problems A through D and work Problem J. Read Section 23-3 in Chapter 23 up to and including Example 23-2. Then study Problem E and work Problems K and L, study Problem F and work Problems M and N. Next study Problems G and H and work Problem O. Problems S and T are challenging optional problems. Study Problem I and work Problems P, Q, and R.

Take the Practice Test before attempting a Mastery Test.

WEIDNER AND SELLS

Objective Number	Readings	Problems with Solutions Study Guide	Assigned Problems Study Guide	Additional Problems
1	General Comments, Sec. 23-3	A, B, C, F	J, L, M, N	S
2		G, H, I	O, Q	T, 23-9
3		D, E, F	K, M, N, P, R	S

PROBLEM SET WITH SOLUTIONS

A(1). A charge Q is uniformly distributed over the interval $0 \leq \underline{x} \leq L$ along the x axis.
(a) Determine the linear charge density in $0 \leq \underline{x} \leq L$.
(b) Determine the charge dq on a segment $d\underline{x}$ in $0 \leq \underline{x} \leq L$.
(c) Set up and evaluate an integral for the electric potential at a point (x, 0, 0) on the x axis to the right (x > L) of the charge.

Solution

(a) See Figure 2. Since Q is uniformly distributed the linear charge density λ is constant in $0 \leq \underline{x} \leq L$ and given by

$$\lambda = Q/L.$$

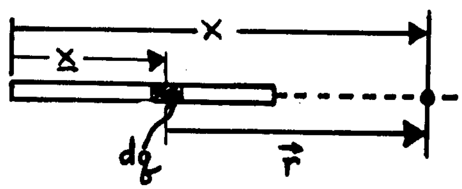


Figure 2

(b) The charge dq in a segment $d\underline{x}$ in the charge region is equal to the charge density multiplied by the length of the interval,

$$dq = \lambda\, d\underline{x} = (Q\, d\underline{x})/L.$$

(c) The vector from dq to the field point is $\vec{r} = (x - \underline{x})\hat{i}$, and its magnitude is

$$r = |\vec{r}| = x - \underline{x} \quad \text{for } x > \underline{x}.$$

Then the potential dV at x attributable to dq is

$$dV = \frac{k\, dq}{r} = \left(\frac{kQ}{L}\right)\left(\frac{d\underline{x}}{x - \underline{x}}\right),$$

and the potential at x is

$$V(x) = \frac{kQ}{L}\int_0^L \frac{d\underline{x}}{x - \underline{x}}.$$

The limits on the integral are determined by the charge boundaries. In a one-dimensional distribution such as this, the smaller boundary coordinate ($\underline{x} = 0$) is the lower limit and the larger ($\underline{x} = L$) is the upper limit. The integral for V(x) is evaluated by the substitution $u \equiv x - \underline{x}$ to get

$$V(x) = \frac{kQ}{L} \int_{x - L}^{x} \frac{du}{u} = \frac{kQ}{L} \ln\left(\frac{x}{x - L}\right) .$$

B(1). Determine $\vec{E}$ at (x, 0, 0) (x > L) for the charge distribution of Problem A.

Solution

The field $d\vec{E}$ at (x, 0, 0) from dq is

$$d\vec{E} = \frac{k\vec{r}\, dq}{r^3} = \left(\frac{kQ}{L}\right)\frac{(x - \underline{x})\hat{i}\, d\underline{x}}{(x - \underline{x})^3} = \left(\frac{kQ}{L}\right)\hat{i}\left(\frac{d\underline{x}}{(x - \underline{x})^2}\right)$$

and

$$\vec{E}(x) = \left(\frac{kQ}{L}\right)\hat{i} \int_0^L \frac{d\underline{x}}{(x - \underline{x})^2} .$$

Using the substitution $u = x - \underline{x}$ gives us

$$\vec{E}(x) = \left(\frac{kQ}{L}\right)\hat{i} \int_{x - L}^{x} \frac{du}{u^2} = \left(\frac{kQ}{L}\right)\hat{i}\left(\frac{1}{x - L} - \frac{1}{x}\right) = \frac{kQ}{x(x - L)}\hat{i}.$$

Comment: Recall from your study of potential that if the potential is a function of one variable only (in this case, x), then

$$E(x) = -dV/dx.$$

Let us check this in this case.

$$-\frac{dV}{dx} = -\left(\frac{kQ}{L}\right)\left(\frac{d}{dx}\right)[\ln x - \ln(x - L)] = -\frac{kQ}{L}\left(\frac{1}{x} - \frac{1}{x - L}\right) = \frac{kQ}{x(x - L)} = E(x),$$

as expected.

C(1). Charge is distributed along the x axis as given by the linear charge density

$$\lambda(\underline{x}) = \alpha \underline{x}^2 \quad \text{for } 0 \le \underline{x} \le L.$$

(a) Determine the total charge Q in this distribution.
(b) Express the constant α in terms of Q and L.

Solution

(a) $\lambda(\underline{x})$ is the linear density in coulombs per meter (C/m). The charge dq on an infinitesimal segment $d\underline{x}$ at position $\underline{x}$ is $dq = \lambda(\underline{x})\, d\underline{x}$. Since the total charge contained in a region $a \le \underline{x} \le b$ is given by $\int_a^b \lambda(\underline{x})\, d\underline{x}$, the charge Q is

$$Q = \int_0^L \alpha \underline{x}^2 \, d\underline{x} = \tfrac{1}{3}\alpha L^3.$$

(b) Solving for α gives us

$$\alpha = 3Q/L^3.$$

Thus

$$\lambda(\underline{x}) = 3Q\underline{x}^2/L^3.$$

D(3). Show that $\vec{E}(x)$ from Problem B reduces as expected for $x >> L$.

Solution

First, how do we expect $\vec{E}$ to behave for $x >> L$? In this case the field point is so distant from the charge that $\vec{E}$ should be very close to that from a point charge Q at the origin, i.e.,

$$\vec{E}(x >> L) \simeq (kQ/x^2)\hat{i}.$$

Now let us see if this is the case. The Problem B result can be written

$$\vec{E}(x) = \frac{kQ}{x^2(1 - L/x)}\hat{i}.$$

For $L << x$, $L/x << 1$, and $1 - L/x \simeq 1$. Therefore

$$\vec{E}(x >> L) \simeq (kQ/x^2)\hat{i}.$$

E(3). Show that the integral for V(x) in Problem J reduces as expected for $x >> L$.

Solution

For $x >> L \geq \underline{x}$, $x - \underline{x} \simeq x$. Thus

$$V(x >> L) \simeq \frac{3kQ}{L^3}\int_0^L \frac{\underline{x}^2 \, d\underline{x}}{x} = \frac{3kQ}{L^3 x}\left(\frac{\underline{x}^3}{3}\Big|_0^L\right) = \frac{kQ}{x},$$

which is the potential for a point charge Q at the origin.

F(1, 3). For the charged rod of Problem A (charge Q uniformly distributed over $0 \leq \underline{x} \leq L$):
(a) Set up an integral for the electric field at $\vec{R} = y\hat{j}$ (a point in the plane perpendicular to the $x = 0$ end of the rod).

(b) Show that this integral reduces as expected for $y \gg L$.

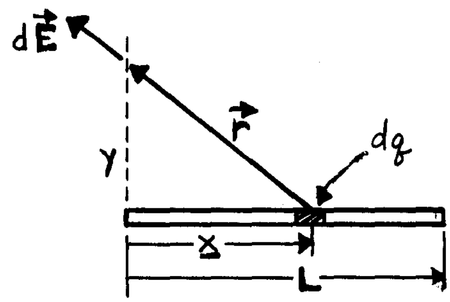


Figure 3

Solution

(a) See Figure 3. The electric field at $y\hat{j}$ resulting from the presence of $dq = (Q\,d\underline{x})/L$ at $\underline{x}\hat{i}$ is

$$d\vec{E} = (kQ/L)(\vec{r}\,d\underline{x}/r^3),$$

where $\vec{r} = -\underline{x}\hat{i} + y\hat{j}$ (Why the negative sign?), and

$$r = |\vec{r}| = (\underline{x}^2 + y^2)^{1/2}.$$

Thus

$$d\vec{E} = (kQ/L)[(-\underline{x}\hat{i} + y\hat{j})\,d\underline{x}]/(\underline{x}^2 + y^2)^{3/2}.$$

Integrating to get $\vec{E}$ at (y, 0, 0) gives us

$$\vec{E}(y) = \frac{kQ}{L}\int_0^L \frac{(-\underline{x}\hat{i} + y\hat{j})\,d\underline{x}}{(\underline{x}^2 + y^2)^{3/2}},$$

which may be rewritten

$$\vec{E}(y) = \left(-\frac{kQ}{L}\int_0^L \frac{\underline{x}\,d\underline{x}}{(\underline{x}^2 + y^2)^{3/2}}\right)\hat{i} + \left(\frac{kQy}{L}\int_0^L \frac{d\underline{x}}{(\underline{x}^2 + y^2)^{3/2}}\right)\hat{j}.$$

Thus the components of $\vec{E}$ are

$$E_x(y) = -\frac{kQ}{L}\int_0^L \frac{\underline{x}\,d\underline{x}}{(\underline{x}^2 + y^2)^{3/2}}, \qquad E_y(y) = \frac{kQy}{L}\int_0^L \frac{d\underline{x}}{(\underline{x}^2 + y^2)^{3/2}}.$$

(b) Before seeing what happens to these results for $y >> L$, what do we expect? If the rod length L is small compared to the distance from the rod to the field point, the field should be very nearly the same as that of a point charge Q at the origin. That is,

$$\vec{E}(y >> L) \simeq (kQ/y^2)\hat{j}.$$

Let us see. Consider the y component first. Since $y >> L$ and $0 \leq x \leq L$,

$$x^2 + y^2 \simeq y^2,$$

so that

$$E_y(y >> L) \simeq \frac{kQy}{L}\int_0^L \frac{dx}{y^3} = \frac{kQ}{y^2}.$$

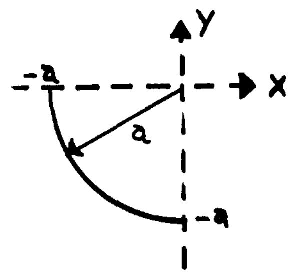


Figure 4

Just what we expected! What about E_x?

$$E_x(y >> L)| \simeq -\frac{kQ}{L}\int_0^L \frac{x\,dx}{y^3} = -\frac{kQL}{2y^3}.$$

What's this? We expected E_x to be zero, and it apparently isn't. But things are not as bad as they might appear. Watch. Rewrite the result for E_x:

$$|E_x(y >> L)| \simeq \frac{kQ}{y^2}\left(\frac{L}{2y}\right) \simeq \frac{L}{2y} E_y(y >> L) << E_y(y >> L).$$

Thus although E_x is not identically zero, it is negligible compared to E_y. The integrals for E_x and E_y can be evaluated by standard substitution techniques. The appropriate substitutions and the results are given. Have a try at it if you are so inclined.

E_x: $u = x^2 + y^2$, $E_x(y) = -(kQ/Ly)[1 - y/(y^2 + L^2)^{1/2}]$,

E_y: $\tan\theta = \frac{x}{y}$, $E_y(y) = kQ/y(y^2 + L^2)^{1/2}$.

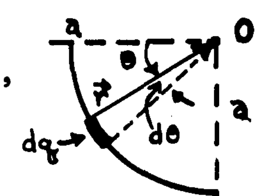


Figure 5

G(2). A charge is uniformly distributed along the circular arc shown in Figure 4. Determine the electric potential and field at the origin.

Solution

Let dq be the charge on a segment of arc as in Figure 5. Since each dq is the same distance from 0, the electric potential is particularly easy to determine:

$$V_0 = \int\frac{k\,dq}{a} = \frac{k}{a}\int dq = \frac{kQ}{a}.$$

The displacement vector from dq to the origin is

$$\vec{r} = (a \cos \theta)\hat{i} + (a \sin \theta)\hat{j}.$$

The electric field at 0 attributable to dq is then

$$d\vec{E} = (k\vec{r}\ dq)/r^3 = k[(\cos \theta)\hat{i} + (\sin \theta)\hat{j}]/a^2\ dq.$$

Since dq subtends an arc segment of length a dθ and since Q is uniformly distributed along the length $\pi a/2$,

$$dq = \frac{Q}{\pi a/2} a\ d\theta = \frac{2Q}{\pi} d\theta.$$

Thus

$$\vec{E}_0 = \frac{k}{a^2} \int_0^{\pi/2} [(\cos \theta)\hat{i} + (\sin \theta)\hat{j}]\frac{2Q}{\pi} d\theta = \frac{2kQ}{\pi a^2}[\int_0^{\pi/2} (\cos \theta\ d\theta)\hat{i} + \int_0^{\pi/2}(\sin \theta\ d\theta)\hat{j}]$$

$$= (2kQ/\pi a^2)(\hat{i} + \hat{j}).$$

Thus

$$E_x = E_y = 2kQ/\pi a^2.$$

Since $E_x = E_y$, $\vec{E}_0$ makes an angle of $\pi/4$ with the positive x axis and has a magnitude

$$E_0 = (E_x^2 + E_y^2)^{1/2} = 2\sqrt{2}kQ/\pi a^2.$$

H(2). A charge Q is uniformly distributed over one-quarter of the circle shown in Figure 6 as a shaded region. Determine the electric potential at the origin 0.

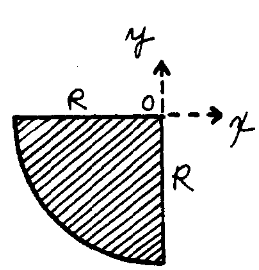


Figure 6

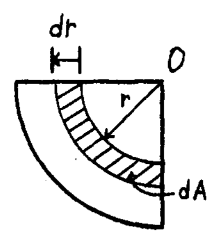


Figure 7

Solution

See Figure 7. The essential idea here is: (a) to determine the potential at 0 from the charge on the ring of radius r and width dr; and (b) to add these contributions to get V. In Problem G the contribution from the ring was determined to be

$$dV = (k\ dq)/r,$$

where dq is the charge on the ring. To get dq we use the fact that Q is uniformly distributed over the area $(1/4)\pi R^2$, and thus the density on this surface is uniform and given by

$$\sigma = \frac{Q}{(1/4)\pi R^2} = \frac{4Q}{\pi R^2}.$$

The area dA of the ring under consideration is

$$dA = (\text{length}) \times (\text{width}) = (1/2)\pi r\ dr.$$

Therefore the charge on the ring is

$$dq = \sigma\ dA = (2Qr\ dr)/R^2.$$

The potential at 0 attributable to this ring is then

$$dV = 2kQ/R^2\ dr.$$

Summing over all rings, i.e., integrating over r from 0 to R, gives us

$$V_0 = \int_0^R \frac{2kQ}{R^2}\,dr = \frac{2kQ}{R}.$$

Comment: By rewriting this as

$$V_0 = \frac{kQ}{R/2},$$

we see that the potential at 0 is the same as if all the charge were placed a distance (1/2)R away.

I(2). A charge Q is uniformly distributed along a ring of radius a as in Figure 8. Determine the electric potential and field at the point P on the axis of the ring.

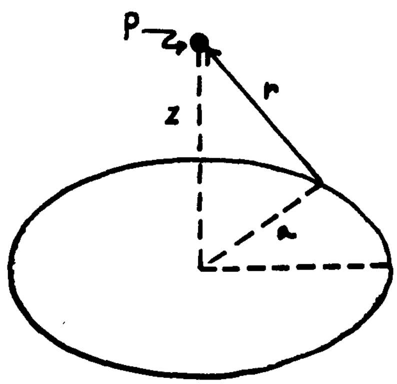


Figure 8

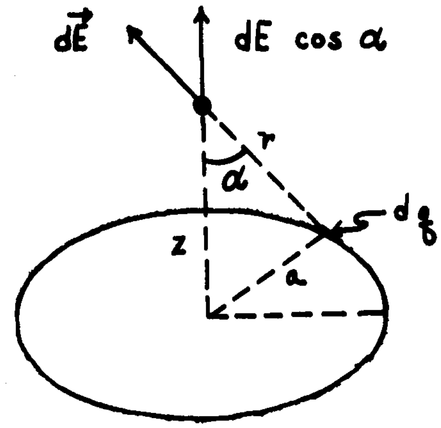


Figure 9

Solution

See Figure 9. The distance from each elementary charge dq on the ring to the field point is constant,

$$r = (z^2 + a^2)^{1/2}.$$

Hence, the potential at P is

$$V(z) = \int\frac{k\ dq}{r} = \frac{kQ}{(z^2 + a^2)^{1/2}} .$$

The components of E perpendicular to the axis sum to zero. This is ensured by the fact that charges on opposite sides of the circle contribute fields with equal components along the axis but oppositely directed perpendicular components. From the figure the magnitude of the field at P by dq is

$$dE = (k\ dq)/r^2,$$

and the axial component is

$$dE_z = dE \cos\alpha = dE(z/r) = (kz\ dq)/(z^2 + a^2)^{3/2}.$$

Therefore,

$$E_z(z) = \int kz/(z^2 + a^2)^{3/2}\, dq,$$

but since a and z do not change for different dq's,

$$E_z(z) = [kz/(z^2 + a^2)^{3/2}]\int dq = kzQ/(z^2 + a^2)^{3/2}.$$

Problems

J(1). Set up (but you need not evaluate) integrals for V and $\vec{E}$ at the point (x, 0, 0), x > L, for the linear charge distribution of Problem C, $\lambda(\underline{x}) = 3Q\underline{x}^2/L$.

K(3). Show that the integral for $\vec{E}(x)$ in Problem J reduces as expected for x >> L.

L(1). For the charge distribution of Problem C,

$\lambda(\underline{x}) = 3Q\underline{x}^2/L^3$ for $0 \leq \underline{x} \leq L$,

determine the electric potential and field at the origin.

M(1, 3). (a) Set up the integral for the electric potential at the point $y\hat{j}$ as in Problem F.
(b) Show that this integral reduces appropriately for y >> L.

N(1, 3). (a) For the uniformly charged rod in Figure 3, set up integrals for the electric potential and field at an arbitrary field point $x\hat{i} + y\hat{j} + z\hat{k}$.
(b) Show that your integrals of part (a) are identical to earlier integrals for the following cases:
(i) x > L, y = z = 0 from Problems A and B.
(ii) y > 0, x = z = 0 from Problems F and M.

O(2). A charge Q is uniformly distributed along the circular arc shown in Figure 10. Determine the electric potential and field at the center.

P(3). Show that the results obtained in Problem I behave as expected for z = 0 and z >> a.

Q(2). A charge Q is uniformly distributed over a disk of radius R as in Figure 11. Set up, but do not evaluate, integrals for the electric potential and field at a point on the axis of the disk.

R(3). Show that the integrals of Problem I reduce appropriately for z >> R. Q

S(1, 3). (<u>Optional - challenging</u>). (a) Use the integrals of Problem N to evaluate V and $\vec{E}$ at $(1/2)L\hat{i} + y\hat{j}$, a point in the midplane of the charged rod. <u>Hint</u>: Use the substitution

$$\tan\theta = \frac{\underline{x} - L/2}{y}.$$

(b) Suppose $y \ll L$ for part (a). In this case the field point is very close to a long charged rod. Show that your result for $\vec{E}$ reduces to that obtained for a long charged rod using Gauss' law, namely, $(2kQ/Ly)\hat{j}$.

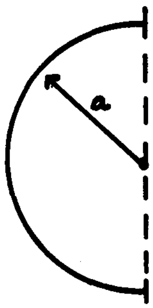


Figure 10

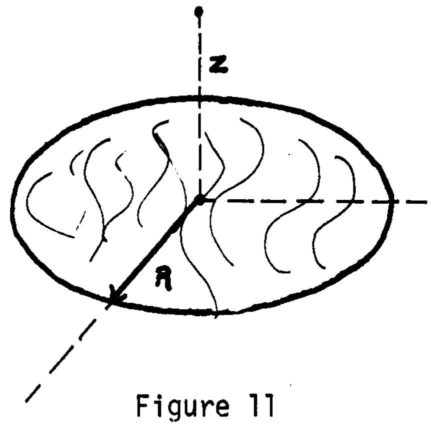


Figure 11

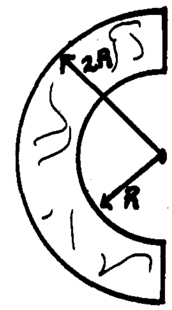


Figure 12

T(2). (Optional - challenging). A charge Q is uniformly distributed over the area shown in Figure 12. Determine the electric potential and field at the center.

Solutions

J(1). $V(x) = \frac{3kQ}{L^3}\int_0^L \frac{\underline{x}^2\, d\underline{x}}{x - \underline{x}}$, $\quad \vec{E}(x) = \frac{3kQ}{L^3}\int_0^L \frac{\underline{x}^2\, d\underline{x}}{(x - \underline{x})^2}\hat{i}$.

K(3). $\vec{E}(x \gg L) \simeq (kQ/x^2)\hat{i}$.

L(1). $V(0) = 3kQ/2L$, $\vec{E}(0) = -(3kQ/L^2)\hat{i}$.

M(1, 3). $V(y) = \frac{kQ}{L}\int_0^L \frac{d\underline{x}}{(\underline{x}^2 + y^2)^{1/2}} \simeq \frac{kQ}{y}$ for $y \gg L$.

Comment: The substitution $\tan\theta = \underline{x}/y$ leads to

$$V(y) = (kQ/L)\ \ln\ [(y^2 + L^2)^{1/2} + L]/y.$$

Remember, you are not required to do this integral.

N(1, 3). $V(x, y, z) = \frac{kQ}{L}\int_0^L \frac{d\underline{x}}{[(x - \underline{x})^2 + y^2 + z^2]^{1/2}}$,

$$\vec{E}(x, y, z) = \frac{kQ}{L}\int_0^L \frac{[(x - \underline{x})\hat{i} + y\hat{j} + z\hat{k}]}{[(x - \underline{x})^2 + y^2 + z^2]^{3/2}}\, d\underline{x}.$$

O(2). $V_0 = kQ/a$, $\vec{E}_0 = (2kQ/\pi a^2)\hat{i}$.

P(3). $z = 0$: $V(a) = kQ/a$, $E_z(0) = 0$.

$z \gg a$: $V \simeq kQ/z$, $E_z \simeq kQ/z^2$.

Q(2). $$V(z) = \frac{2kQ}{R^2}\int_0^R \frac{r\,dr}{(r^2 + z^2)^{1/2}},$$

$$\vec{E}(z) = \frac{2zkQ}{R^2}\int_0^R \frac{r\,dr}{(r^2 + z^2)^{3/2}}\hat{k}.$$

R(3). $V(z \gg R) \simeq kQ/z^2$, $E(z \gg R) \simeq (kQ/z^2)\hat{k}$.

S(1, 3). $V(L/2, y, 0) = (2kQ/L)\ln\{[(4y^2 + L^2)^{1/2} + L]/2y\}$,

$\vec{E}(L/2, y, 0) = [2kQ/y(4y^2 + L^2)^{1/2}]\hat{j}$.

T(2). $V_0 = 2kQ/3R$, $\vec{E}_0 = (4kQ/3\pi R^2)(\ln 2)\hat{i}$.

PRACTICE TEST

1. A linear charge is distributed along the x axis with the density

   $\lambda(x) = \alpha x^3$ for $0 \le x \le L$.

   (a) Determine the total charge Q in terms of α and L.
   (b) Determine the electric potential at the origin. Express your answer in terms of Q, L, and other constants (not including α).

2. A charge Q is uniformly distributed along the arc shown in Figure 13.
   (a) Determine $\vec{E}$ at the origin.
   (b) Use

   $$\lim_{\alpha \to 0} \frac{\sin \alpha}{\alpha} = 1$$

   to show that your result reduces as expected as $\alpha \to 0$.

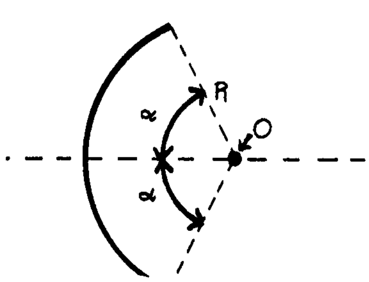


Figure 13

Practice Test Answers

1. (a) $Q = (1/4)\alpha L^4$. (b) $V = 4kQ/3L$.
2. (a) $\vec{E}_0 = \{(kQ/R^2)[(\sin \alpha)/\alpha]\}\hat{i}$.

ELECTRIC FIELDS AND POTENTIALS
FROM CONTINUOUS CHARGE DISTRIBUTIONS

Date ________________

Mastery Test Form A

pass recycle

1 2 3

Name ______________________ Tutor ______________________

1. Charge is distributed along the x axis according to the linear charge density

   $\lambda(\underline{x}) = \beta \sin(\pi\underline{x}/L), \qquad 0 \leq \underline{x} \leq L.$

   (a) Determine the total charge Q.

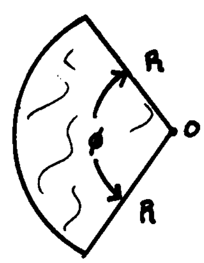
   (b) Set up, but do not evaluate, an integral for the electric potential at the field point $x\hat{i} + y\hat{j}$.

   (c) Set up, but do not evaluate, an integral for the electric field at the field point $x\hat{i} + y\hat{j}$.

2. Show that your integral for $\vec{E}(x, y)$ in Problem 1 reduces appropriately for $\vec{E}(L/2, y >> L)$.

3. A charge Q is uniformly distributed on the flat circular sector shown in Figure 1. Determine the electric potential at the center (point O) in terms of Q, R, ϕ, and other constants as needed.

Figure 1



ELECTRIC FIELDS AND POTENTIALS
FROM CONTINUOUS CHARGE DISTRIBUTIONS

Date ____________________

Mastery Test   Form B

pass		recycle
1	2	3

Name ____________________________   Tutor ______________________________

1. Charge is distributed along the x axis according to the linear charge density

   $\lambda(\underline{x}) = \beta\underline{x}(L - \underline{x}), \qquad 0 \le \underline{x} \le L.$

   (a) Determine the total charge Q.

   (b) Set up, but do not evaluate, an integral for the electric potential at the field point $x\hat{i} + y\hat{j} + z\hat{k}$.

   (c) Set up, but do not evaluate, an integral for the electric field at the field point $x\hat{i} + y\hat{j} + z\hat{k}$.

2. Show that your integral for V(x, y, z) in Problem 1 reduces appropriately for V(x >> L, 0, 0).

3. A charge Q is uniformly distributed on the flat circular section shown in Figure 1. Determine the electric potential at point 0 in terms of Q, R, and other constants as needed.

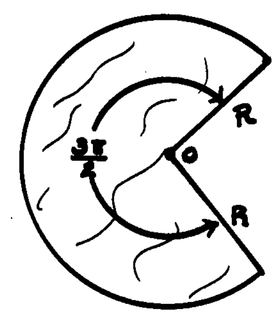


Figure 1

ELECTRIC FIELDS AND POTENTIALS
FROM CONTINUOUS CHARGE DISTRIBUTIONS

Date ____________

Mastery Test Form C

pass recycle

1 2 3

Name ________________ Tutor ________________

1. Charge is distributed along the x axis according to the linear charge density

$$\lambda(\underline{x}) = \beta \cos(\pi\underline{x}/2L), \qquad -L \leq \underline{x} \leq L.$$

   (a) Determine the total charge Q.
   (b) Set up, but do not evaluate, an integral for the electric potential at the field point $x\hat{i} + y\hat{j} + z\hat{k}$.
   (c) Set up, but do not evaluate, an integral for the electric field at the field point $x\hat{i} + y\hat{j} + z\hat{k}$.

2. Show that your integral for E(x, y, z) in Problem 1 reduces appropriately for E(0, 0, z >> L).

3. A charge Q is uniformly distributed on the annular ring shown in Figure 1. Determine the electric potential at a point on the axis of the ring a distance h from the ring's plane.

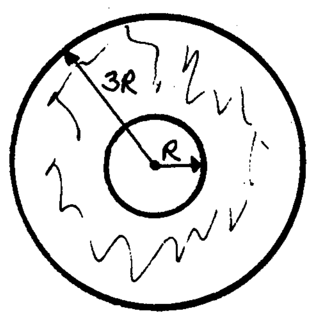


Figure 1

MASTERY TEST GRADING KEY - Form A

1. What To Look For: (a) Correct answer. (b) Be sure denominator of answer is correct. (c) Check for unit vectors in numerator.

Solution: $Q = \int_0^L \lambda(\underline{x})\, d\underline{x} = \int_0^L \beta \sin(\pi\underline{x}/L)\, d\underline{x} = 2\beta L/\pi.$

(b) $V(x, y) = \int \frac{k\, dq}{r} = \int_0^L \frac{k\beta \sin(\pi\underline{x}/L)}{[(x - \underline{x})^2 + y^2]^{1/2}}\, d\underline{x}.$

(c) $\vec{E}(x, y) = \int \frac{k\vec{r}\, dq}{r^3} = \int_0^L \frac{k\beta[(x - \underline{x})\hat{i} + y\hat{j}] \sin(\pi\underline{x}/L)}{[(x - \underline{x})^2 + y^2]^{3/2}}\, d\underline{x}.$

2. What To Look For: (a) No contribution to Ex. (b) Correct result.

Solution:

$$\vec{E}(L/2, y >> L) = \int_0^L \frac{k\beta[(L/2 - \underline{x})\hat{i} + y\hat{j}]}{[(x - \underline{x})^2 + y^2]^{3/2}} \sin(\pi\underline{x}/L)\, d\underline{x}$$

$$\simeq \left(\frac{k\beta}{y^3} \int_0^L \left(\frac{L}{2} - \underline{x}\right) \sin\left(\frac{\pi\underline{x}}{L}\right) d\underline{x}\right)\hat{i} + \left(\frac{k}{y^2} \int_0^L \beta \sin\left(\frac{\pi\underline{x}}{L}\right) d\underline{x}\right)\hat{j}.$$

But $\int_0^L \left(\frac{L}{2} - \underline{x}\right) \sin\left(\frac{\pi\underline{x}}{L}\right) d\underline{x} = 0$ (odd about $\underline{x} = \frac{L}{2}$)

and

$\int_0^L \beta \sin\left(\frac{\pi\underline{x}}{L}\right) d\underline{x} = Q.$

So $\vec{E}(L/2, y >> L) \simeq (kQ/L^2)\hat{j}.$

3. What To Look For: (a) Correct dq. (b) Correct expression for dV. (c) Correct answer.

Solution: See Figure 17. Area of sector $= \phi R^2/2$. Charge density $= \frac{Q}{\text{area}} = \frac{2Q}{\phi R^2}$.

$dA = (r\phi)\, dr$, $dq = (\text{density})dA = (2Q/\phi R^2)r\phi\, dr = (2Qr\, dr)/R^2$.

$dV = (k\, dq)/r = (2kQ\, dr)/R^2$,

$V = \int_0^R \frac{2kQ}{R^2}\, dr = \frac{2kQ}{R}.$

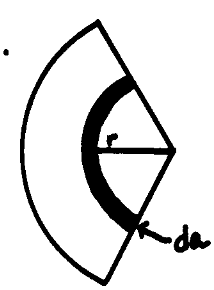


Figure 17

MASTERY TEST GRADING KEY - Form B

1. What To Look For: (a) Correct answer. (b) Correct denominator. (c) Unit vectors in numerator.

   Solution: (a) $Q = \int_0^L \lambda(\underline{x})\, d\underline{x} = \beta \int_0^L \underline{x}(L - \underline{x})\, d\underline{x} = \beta L^3/6.$

   (b) $V(x, y, z) = \int \frac{k\, dq}{r} = \int_0^L \frac{k\beta\underline{x}(L - \underline{x})}{[(x - \underline{x})^2 + y^2 + z^2]^{1/2}}\, d\underline{x}.$

   (c) $\vec{E}(x, y, z) = \int \frac{k\mathbf{r}\, dq}{r^3} = \int_0^L \frac{k\beta[(x - \underline{x})\hat{i} + y\hat{j} + z\hat{k}]\underline{x}(L - \underline{x})}{[(x - \underline{x})^2 + y^2 + z^2]^{3/2}}\, d\underline{x}.$

2. What To Look For: See that integral for Q is correct. Correct answer.

   Solution: $V(x >> L, 0, 0) = \int_0^L \frac{k\beta\underline{x}(L - \underline{x})}{(x - \underline{x})}\, d\underline{x} \approx \frac{k}{x}\int_0^L \beta\underline{x}(L - \underline{x})\, d\underline{x} = \frac{kQ}{x}.$

3. What To Look For: Correct dq. Correct dV. Correct answer.

   Solution: See Figure 18. Area of sector = $(3\pi/2)(R^2/2) = (3\pi R^2/4)$.

   Charge density $\sigma = Q/\text{Area} = 4Q/3\pi R^2$. $dA = (r\phi)\, dr = (3\pi/2)r\, dr$.

   $dq = \sigma\, dA = (3\pi/2)(4Q/3\pi R^2)r\, dr = (2Q/R^2)r\, dr.$

   $dV = \frac{k\, dq}{r} = \frac{2kQ}{R^2}\, dr; \quad V = \int dV = \int_0^R \frac{2kQ}{R^2}\, dr = \frac{2kQ}{R}.$

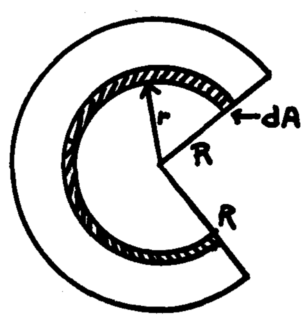


Figure 18

MASTERY TEST GRADING KEY - Form C

1. What To Look For: Correct answer. Correct denominator and limits. Unit vectors in numerator. Limits

Solution: (a) $Q = \int_{-L}^{L} \lambda(\underline{x})\, d\underline{x} = \beta \int_{-L}^{L} \cos(\frac{\pi \underline{x}}{2L})\, d\underline{x} = \frac{4\beta L}{\pi}$.

(b) $V(x, y, z) = \int \frac{k\, dq}{r} = \int_{-L}^{L} \frac{k\beta \cos(\pi \underline{x}/2L)}{[(x - \underline{x})^2 + y^2 + z^2]^{1/2}}\, d\underline{x}$.

(c) $E(x, y, z) = \int \frac{k\vec{r}\, dq}{r^3} = \int_{-L}^{L} \frac{k\beta[(x - \underline{x})\hat{i} + y\hat{j} + z\hat{k}] \cos(\pi \underline{x}/2L)}{[(x - \underline{x})^2 + y^2 + z^2]^{3/2}}\, d\underline{x}$.

2. What To Look For: See that $E_x = E_y = 0$. Correct answer.

Solution: $\vec{E}(0, 0, z >> L) = \int_{-L}^{L} \frac{k\beta(-\underline{x}\hat{i} + z\hat{k}) \cos(\pi \underline{x}/2L)}{(\underline{x}^2 + z^2)^{3/2}}\, d\underline{x}$

$$\simeq -\hat{i}\frac{k\beta}{z^3} \int_{-L}^{L} \underline{x} \cos(\tfrac{\pi \underline{x}}{2L})\, d\underline{x} + \hat{k}\frac{k}{z^2} \int_{-L}^{L} \beta \cos(\tfrac{\pi \underline{x}}{2L})\, d\underline{x}.$$

But $\int_{-L}^{L} \underline{x} \cos(\frac{\pi \underline{x}}{2L})\, d\underline{x} = 0$ (odd about $\underline{x} = 0$), $\int_{-L}^{L} \beta \cos(\pi \underline{x}/2L)\, d\underline{x} = Q$.

So $\vec{E}(0, 0, z >> L) \simeq (kQ/z^2)\hat{k}$.

3. What To Look For: Correct dq. Correct dV. Correct answer.

Solution: See Figure 19. Area of ring $= \pi[(3R)^2 - R^2] = 8\pi R^2$

Density of charge = Q/Area = $Q/8\pi R^2$ $dA = 2\pi r\, dr$.

$dq = (\text{density})dA = (Qr\, dr)/4R^2$. Distance from dq to axial point $= \sqrt{r^2 + h^2}$.

$dV = k\, dq/\sqrt{r^2 + h^2}$.

$V = \frac{kQ}{4R^2} \int_{R}^{3R} \frac{r\, dr}{\sqrt{r^2 + h^2}}$ (substitution: $u = r^2 + h^2$, $du = 2r\, dr$)

$= \frac{kQ}{4R^2} \int_{R^2 + h^2}^{9R^2 + h^2} \frac{1}{2} u^{-1/2}\, du = \frac{kQ}{4R^2}[\sqrt{9R^2 + h^2} - \sqrt{R^2 + h^2}]$.

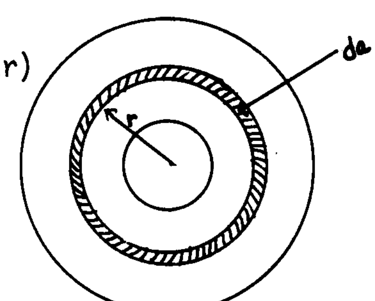


Figure 19