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Relativistic momentum

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Note (2012): The 4th equation in the right-hand column below has been corrected.
It had read: u'_{y1} = ...
It is now corrected to read: u'_{y2} = ...

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Introductory treatments of relativistic dynamics rely on the invariance of momentum conservation (i.e., on the assumption that momentum is conserved in all inertial frames if it is conserved in one) to establish the relationship for the momentum of a particle in terms of its mass and velocity. By contrast, more advanced treatments rely on the transformation properties of the four-velocity and/or proper time to obtain the same result and then show that momentum conservation is invariant. Here, we will outline a derivation of that relationship that, in the spirit of the more advanced treatments, relies on an elemental feature of the transformation of momentum rather than on its conservation but does not have as a prerequisite the introduction of four-vectors and invariants. The steps in the derivation are no more involved than in the usual introductory treatments; indeed, the arithmetic is almost identical.

As usual, we assume that the momentum **p** of a body of mass m and velocity **u** is in the direction of **u** and that **p** approaches $m\mathbf{u}$ when $\mathbf{u}/c$ is small. Namely, we assume that

$$\mathbf{p}=m\mathbf{u}F(\mathbf{u}^2), \tag{1}$$

where F depends only on the magnitude of **u** and that $F(\mathbf{u}^2)\rightarrow 1$ as $\mathbf{u}^2\rightarrow 0$. Our other assumption is based on our expectation that the momentum of a system of particles should transform in the same way as the momentum of a single particle. Consider a particle at rest in frame S so that $\mathbf{u}=0$ and, therefore, $\mathbf{p}=0$ by Eq. (1). In another frame S', moving with velocity **v** along the x axis of S, the particle will have zero velocity component perpendicular to **v** (e.g., $u'_y=0$) and, therefore, zero momentum component perpendicular to **v** (e.g., $p'_y=0$) by Eq. (1). Specifically, our assumption is that if a system of particles also has zero total momentum (i.e., $\mathbf{p}=0$) in frame S, then in frame S' the system will have zero total momentum component perpendicular to **v** (e.g., $p'_y=0$).

Consider now a system of two particles 1 and 2 of equal mass with opposite velocities in frame S: $m_2=m_1=m$ and $-\mathbf{u}_2=\mathbf{u}_1=\mathbf{u}$. Recalling that S' moves with **v** along the x axis of S, the velocity transformation law applied to each particle leads to

$$u'_{x1}=(u_x-v)/(1-u_xv/c^2),$$

$$u'_{y1}=u_y[1-v^2/c^2]^{1/2}/(1-u_xv/c^2),$$

$$u'_{x2}=(-u_x-v)/(1+u_xv/c^2),$$

$$u'_{y2}=-u_y[1-v^2/c^2]^{1/2}/(1+u_xv/c^2).$$

By direct substitution of these expressions one can show that

$$(1-\mathbf{u}_1'^2/c^2)=(1-\mathbf{u}^2/c^2)(1-v^2/c^2)/(1-u_xv/c^2)^2,$$

$$(1-\mathbf{u}_2'^2/c^2)=(1-\mathbf{u}^2/c^2)(1-v^2/c^2)/(1+u_xv/c^2)^2,$$

so that

$$(1-\mathbf{u}_1'^2/c^2)/(1-\mathbf{u}_2'^2/c^2) =((1+u_xv/c^2)/(1-u_xv/c^2))^2. \tag{2}$$

The total momentum of the particles in S is zero, since

$$\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2=m_1\mathbf{u}_1F(\mathbf{u}_1^2)+m_2\mathbf{u}_2F(\mathbf{u}_2^2)$$
$$=m\mathbf{u}F(\mathbf{u}^2)+m(-\mathbf{u})F((-\mathbf{u})^2)=0.$$

In frame S', the total y component of total momentum is

$$p'_y=p'_{y1}+p'_{y2}$$
$$=m_1u'_{y1}F(\mathbf{u}_1'^2)+m_2u'_{y2}F(\mathbf{u}_2'^2)$$
$$=m\{(+u_y)[1-v^2/c^2]^{1/2}/(1-u_xv/c^2)\}F(\mathbf{u}_1'^2)$$
$$+m\{(-u_y)[1-v^2/c^2]^{1/2}/(1+u_xv/c^2)\}F(\mathbf{u}_2'^2)$$
$$=mu_y[1-v^2/c^2]^{1/2}\{F(\mathbf{u}_1'^2)/(1-u_xv/c^2)$$
$$-F(\mathbf{u}_2'^2)/(1+u_xv/c^2)\}.$$

By our assumption, however, $p'_y=0$ so that

$$F(\mathbf{u}_2'^2)/F(\mathbf{u}_1'^2)=(1+u_xv/c^2)/(1-u_xv/c^2)$$
$$=[(1-\mathbf{u}_1'^2/c^2)/(1-\mathbf{u}_2'^2/c^2)]^{1/2},$$

where the last step follows from Eq. (2). From the condition that $F(0)=1$, it follows that F has the form

$$F(\mathbf{u}^2)=1/[1-\mathbf{u}^2/c^2]^{1/2}.$$

Consequently, by Eq. (1), any body of mass m and velocity $\mathbf{u}$ has momentum

$$\mathbf{p}=m\mathbf{u}/[1-\mathbf{u}^2/c^2]^{1/2}.$$

Now that the relativistic expression for momentum has been found, it can be used in the usual way as the basis for the development of the rest of relativistic dynamics, including the invariance of momentum conservation.

