1918

A Theorem on Semi-Continuous Functions

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A THEOREM ON SEMI-CONTINUOUS FUNCTIONS.

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(Read before the American Mathematical Society December 1, 1917.)

RECENTLY G. C. Young* and A. Denjoy† have communicated theorems—those in Denjoy's memoir are of an especially comprehensive character—dealing, in particular, with point sets where the four derivatives of a given continuous function are identical. It is the purpose of this note to treat an analogous problem that arises when "derivative" is replaced by "saltus."‡ However, instead of confining ourselves to "saltus," we prove a more general theorem that applies essentially to all semi-continuous functions.§ We preface the proof of this theorem with the following

LEMMA. *Let $f_1(x)$ and $f_2(x)$ be two real, single-valued functions, defined in the linear continuum, such that everywhere $f_1(x) \geq f_2(x)$, and moreover, for every fixed real number k, the set S_k of points x where $f_1(x) \geq k$ and $f_2(x) < k$ is countable. Then $f_1(x)$ and $f_2(x)$ are identical except at most in a countable set.*

Proof. Let $\{k_n\}$, $n = 1, 2, \cdots \infty$, be a set of k's everywhere dense in the linear continuum. The set‖ $S = \mathfrak{S}(S_{k_n})$, which consists of all of the elements of every S_{k_n}, is also countable. We show that $f_1(x) = f_2(x)$ for every given x not in S. For let $\{k_{i_n}\}$ be a monotone decreasing sequence of k_n's having $f_2(x)$ as limit. Since x is given as not belonging to S, it must be that $f_1(x) < k_{i_n}$ for every n; for from $f_1(x) \geq k_{i_n}$ and $f_2(x) < k_{i_n}$, we would conclude that x belonged to $S_{k_{i_n}}$ and

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* *Acta Mathematica*, vol. 37 (1914), p. 141.

† *Journal de Mathématiques*, ser. 7, vol. 1 (1915), p. 105.

‡ By the saltus (= oscillation) of a given function $f(x)$ at the point x we understand the greatest lower bound of the saltus of $f(x)$ in the interval (ξ, η), for all intervals (ξ, η) enclosing x as interior point. Cf. Hobson, The Theory of Functions of a Real Variable (1907), art. 180, and the author's paper, "Certain general properties of functions," *Annals of Mathematics*, vol. 18 (1917), p. 147. For the comprehensiveness of our result see the remark at the close, in conjunction with the theorem of this note and the corollaries.

§ "Essentially" in the sense that every semi-continuous function is exhibitable as the "associated" function $\phi(x)$ of a "monotone decreasing interval-function ϕ_{ab}" (see theorem below).

‖ In regard to the notation, compare Hausdorff, Grundzüge der Mengenlehre (1914), p. 5.

hence to S. It follows that $f_1(x) \leq f_2(x)$, and hence, since $f_1(x) \geq f_2(x)$ by hypothesis, that $f_1(x) = f_2(x)$.

Definitions. ϕ_{ab} is said to be a "monotone, decreasing, real interval function,"* if ϕ_{ab} is a real number for every (closed) interval (a, b) and, in addition, $\phi_{\alpha\beta} \leq \phi_{ab}$ if (α, β) is contained in (a, b). With every such interval function, we associate three point functions: (1) $\phi(x)$, the greatest lower bound of all ϕ_{ab} such that the fixed point x is in the interior of (a, b); (2) $\phi^{(r)}(x)$, the greatest lower bound of ϕ_{xb}, for all b's greater than x; and (3) $\phi^{(l)}(x)$, the greatest lower bound of ϕ_{ax} for all a's less than x.

The functions $\phi(x)$, $\phi^{(r)}(x)$ and $\phi^{(l)}(x)$ associated with a monotone, decreasing, real interval function ϕ_{ab} that has a finite lower bound necessarily exist, and $\phi(x)$ is an upper semi-continuous function. (For the proof, the reader is referred to a former paper by the author.)†

We now come to the

THEOREM. *Let ϕ_{ab} be a monotone, decreasing, real interval function having a finite lower bound. Then the three associated point functions $\phi(x)$, $\phi^{(r)}(x)$ and $\phi^{(l)}(x)$ (exist and) are identical except at most in a countable set.*

Proof. It is evidently sufficient to show that $\phi(x)$ and $\phi^{(r)}(x)$ are identical except in a countable set. From the definition of these functions, it follows that $\phi(x) \geq \phi^{(r)}(x)$ for every x. For if (a, b) contains x as interior point, we have, according to the monotoneity property of ϕ, the inequality $\phi_{xb} \leq \phi_{ab}$. We may thus associate with every number ϕ_{ab} of the set of which $\phi(x)$ is the greatest lower bound a smaller or equal number ϕ_{xb} of the set of which $\phi^{(r)}(x)$ is the greatest lower bound. Let now S_k represent, for a fixed real number k, the set of points x where simultaneously $\phi(x) \geq k$ and $\phi^{(r)}(x) < k$.‡ If x belongs to S_k, there must exist, on account of the relation $\phi^{(r)}(x) < k$, an interval (x, b) such that $\phi_{xb} < k$. From the last inequality, it follows that $\phi(\xi) < k$ for every ξ interior to (x, b); hence ξ does not belong to S_k. We have thus associated with every point x of S_k an interval (x, b) having no interior points belonging to S_k.§ Clearly two such

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* Defined, as is the case with all the functions treated in this note, in the entire linear continuum.

† *Annals of Mathematics,* l. c.

‡ Cf. G. C. Young, l. c., p. 143.

§ This association may be made unique by taking the point b as far to the right as possible, and allowing, if necessary, the "ideal" interval (x, ∞).

associated intervals cannot overlap. Since a set of non-overlapping intervals in the linear continuum is at most countable, it follows that S_k is at most countable. By the use of the lemma we now conclude, identifying $f_1(x)$ with $\phi(x)$ and $f_2(x)$ with $\phi^{(r)}(x)$, that $\phi(x)$ and $\phi^{(r)}(x)$ differ at most in a countable set.

If, in particular, ϕ_{ab} means the saltus of a given real function $f(x)$ in the interval (a, b), we obtain as associated point functions of ϕ the ordinary saltus and two other functions, which may be designated as the "right saltus" and the "left saltus." We thus have the

COROLLARY. *The three functions $\phi(x)$, $\phi^{(r)}(x)$, and $\phi^{(l)}(x)$, representing respectively the saltus, the right saltus and the left saltus of a given function $f(x)$ at the point x, are identical except at most in a countable set.*

The f, d, e, n.d. and z-saltus functions, which are the saltus functions when respectively finite, denumerable, exhaustible, non-dense sets and sets of zero measure may be neglected,* likewise yield particular cases of our theorem, and we have the

COROLLARY. *The f, d, e, n.d. and z-saltus functions are identical with the corresponding right and the left saltus functions except at most in a countable set.*

As a special case of our first corollary we have the

COROLLARY. *The points where a given function $f(x)$ is discontinuous but continuous on the right (left) constitute at most a countable set.*

Remark. The exhaustive character of our theorem may be judged from the fact that if $C = \{x_1, x_2, \cdots\}$ is any given countable set of real numbers, a function $f(x)$ exists whose saltus, f-saltus, e-saltus, etc., coincides everywhere respectively with its right saltus, right f-saltus, right e-saltus, etc., except precisely at the points of C. In fact,

$$f(x) = \sum_{x_n \leqq x} \frac{1}{2^n}$$

—i. e., the sum of all the fractions $1/2^n$ such that $x_n \leqq x$—is such a function. For it may be readily shown that $f(x)$ has its right saltus, right f-saltus, etc., everywhere 0, and its saltus, f-saltus, etc., everywhere 0 except at every point x_n of C, where these saltus functions all take the value $1/2^n$.

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* Cf. *Annals of Mathematics*, l. c.