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Spherical Geometry

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By

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Spherical geometry was studied in ancient times as a subset of Euclidean three-dimensional space. It was a logical outcome as the earth is a sphere. The word geometry literally means the measure of the earth. However, the undefined terms, axioms and postulates of Euclidean geometry take on a new meaning when studied on a sphere.

The purpose of this paper was to give a brief explanation of historical information on non-Euclidean geometry, specifically, spherical geometry. Comparisons between geometry on a spherical surface and geometry on a plane surface are also discussed.

The Thesis contains two questions from Math in the Middle coursework, a narration on reasoning and proof and an expository paper on spherical geometry. This paper seeks to explain the concept of a straight line on a sphere.

Master of Arts in Teaching Masters Exam
Part IB.
Spherical Geometry

When one speaks of geometry, it is often assumed one is speaking of Euclidean geometry. There are geometries besides Euclidean geometry. One way to discover other geometries is to see what Euclidean geometry is not. Euclid used a set of axioms (a statement which is assumed to be true without proof) and then proved results based on those assumptions. Axioms are sometimes referred to as postulates. In the 1800's it was discovered Euclid had not explicitly stated all of his assumptions, so several axioms are assumed, in some form, when studying Euclidean geometry. Let's look at five of his axioms and postulates:

1. A straight-line segment can be drawn joining any two points.
2. Any straight-line segment can be extended indefinitely in a straight line.
3. Given any straight-line segment, a circle can be drawn having the segment as radius and one endpoint as center.
4. All right angles are congruent.
5. If two lines are drawn which intersect a third in such a way that the sum of the inner angles on one side is less than two right angles, then the two lines inevitably must intersect each other on that side if extended far enough. This postulate is equivalent to what is known as the parallel postulate.

This is a small group intended to be so obvious they cannot be refuted. And for a long time the fifth postulate was thought to be unnecessary. It was in the nineteenth century that Lobachevski (1793-1856), Bolyai (1777-1855) and Gauss (1802-1860) put an end to this thought. Lobachevski developed theorems using Euclid's first four postulates and the negation of the Parallel Postulate. He expected to eventually develop two theorems that contradicted each other. This would prove that negating the

Parallel Postulate is inconsistent with the first four postulates - thereby proving the Parallel Postulate (and making it the Parallel Theorem). To his surprise, however, he never obtained a contradiction. Instead, he developed a complete and consistent geometry the first non-Euclidean Geometry. This alternative Euclidean geometry is called Lobachevskian geometry in the former Soviet Union and called hyperbolic geometry in the rest of the world.

This was of great mathematical and philosophical interest. “From the time of the Greeks, it was believed that geometric theorems were such pure and perfect Truth that they did not need to be scrutinized by observations of the real world. Now, those statements are only true for some geometries.”(Castellanos). What is truth now depends on what type of geometry one is talking about.

“Historically, the study of geometry proceeds from a small number of accepted truths (axioms or postulates), then builds up true statements using a systematic and rigorous step-by-step proof.” (Weisstein) To explain spherical geometry it would be beneficial to define some terms commonly used when discussing concepts. An axiom is a statement accepted as true without proof. In this writing a mathematical proof is used as a demonstration that a mathematical proposition is true based on axioms, definitions, and proven theorems. A definition will be used as the ability to make another person understand by words or what the term defined stands for. A theorem will be a proposition that has been or is to be proven on the basis of certain assumptions.

Basic plane geometry is concerned with points and lines and their interactions, a collection of rules for computing lengths, areas, and volumes. Plane geometry was used to construct, navigate and survey the world in which people lived.

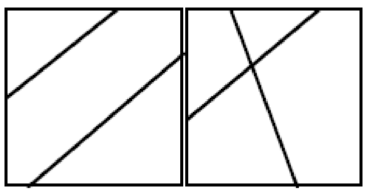
Spherical geometry, the study of objects on the surface of a sphere gives cause to come to an agreement on what lines and line segments are on a sphere. A great circle on a sphere is the intersection of a sphere with a plane passing through the center, the straight line of spherical geometry. The shortest distance between two points on the sphere is along the segment of the great circle joining them--geodesics. Any two lines on a sphere meet in two points because two lines on a sphere are two great circles. There are no parallel lines in spherical geometry. The angle between two lines in spherical geometry is the angle between the planes of the corresponding great circles, and a spherical triangle is defined by three angles. A spherical triangle is defined just like a planar triangle. It consists of three points called vertices, the arcs of great circles that join the vertices, called great arcs, and the area that is enclosed therein. Spherical distance will be defined as the length of the shortest arc of the great circle joining two points.

Spherical coordinates (ρ,θ,ϕ), ρ represents the positive length coordinate and both θ and ϕ represent angle coordinates. The length coordinate ρ marks the distance form the origin (0,0,0). $\rho = \sqrt{}x^2 + y^2 + z^2$. The angle ϕ lies between $0°$and $180°$ measured in radians is between 0 and Π. The angle θ is projected onto the xy-plane by moving each point of the segment parallel to the z-axis. The resulting segment lies in the xy-plane and has endpoints at the origin and at the point (x,y,0). The length of this segment is given by $r = \sqrt{}\ x^2 + y^2$ and the angle θ is an angle measured in the xy-plane between this segment and the positive x-axis. In degrees, θ is between $0°$ and $360°$. In radians it is greater than, or equal to 0 and less than 2Π. In effect, r and θ are polar coordinates in the xy-plane. $r = \rho \sin(\phi)$ and $z = \rho \cos(\phi)$.

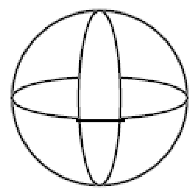
For polar coordinates in the xy-plane, $x = r\cos(\theta)$ and $y=\sin(\theta)$. Substituting in for r; we have: $x = \rho\sin(\phi)\cos(\theta)$, $y = \rho\sin(\phi)\sin(\theta)$ and $z =\rho\cos(\phi)$. Where ρ = radius of the sphere and ϕ = the angle made by the intersection of the great circles.

Looking at some comparisons between plane and spherical geometry:

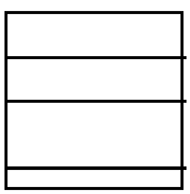
Two lines can never have more than one point of intersection:



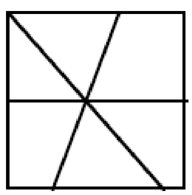
Two distinct great circles have two points of intersection.



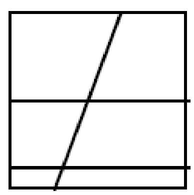
Three lines on a plane can divide a plane into 4,6 or 7 regions. Those lines create 0,1,2 or 3 vertices. Three great circles divide the sphere into 6 or 8 regions with 2 or 6 vertices.



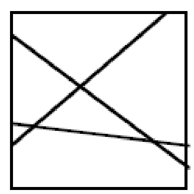
0 vertices,
4 regions



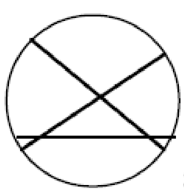
1 vertex
6 regions



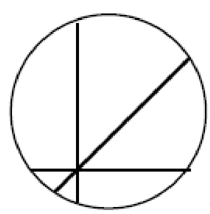
2 vertices
6 regions



3 vertices
7 regions



8 regions, 6vertices



6 regions, 2 vertices

Number of lines	1	2	3	4	N
Maximum number of regions created with lines.	2	4	7	11	$\frac{1+N(N+1)}{2}$
Maximum number of regions created with great circles.	2	4	8	14	2 +(N-1)N

The more lines you draw on the plane or on the sphere, the more regions you create as noted in the chart.

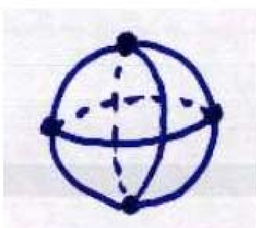
Perpendicular and parallel lines take on different meanings in plane and spherical geometry. Two lines on a plane can be drawn parallel or intersecting, but two lines on a sphere can only be drawn as intersecting. When two lines intersect perpendicularly, a pair of perpendicular lines intersects only once on a plane and creates four right angles with four infinite regions. Whereas, a pair of perpendicular great circles intersects twice on a sphere and creates eight right angles, and four finite congruent regions. Two intersecting lines in plane geometry do not have a common perpendicular but on a sphere, two intersecting great circles can have two common perpendiculars. And, two lines can be parallel on a plane; however, there are no parallel great circles on a sphere.

The following table compares similarities and differences of a plane versus a sphere:

Plane:	Sphere:
A straight line is the shortest distance between two points. It is infinite; you will never return to the beginning point.	A straight line begins and ends at the same point. It is a great circle on the sphere.
Two points determine one line. The two points divide the line into one finite and two infinite sections.	Two points determine two finite areas.
The smallest n-gon is a triangle.	The smallest n-gon is a lune. Which is a two-sided polygon.
There are many units of measure for a line such as mm, cm, in,yd,ft.	The unit of measure is the degree.
Two points can be any distance apart, there is no greatest distance.	The greatest distance on a sphere is 180 degrees.
A point and a circle can be drawn exactly alike on a plane and a sphere.	A plane that intersects a sphere and includes the center point of a sphere is called a great circle.
The sum of the interior of a polygon is: 180(n-2)/n degrees, where "n" is the number of sides of the polygon.	The sum of the interior angles on a sphere is between 180 and 540 degrees.
The sum of the exterior angles on a plane is 360 degrees.	The sum of the exterior angles on a sphere is between 0 and 360 degrees.
Two lines on a plane can be drawn parallel or intersecting.	Two lines on a sphere can only be drawn as intersecting.
Two distinct stratght lines wiith no points of intersection are called parallel lines.	There are no parallel lines on a sphere.
Two distinct straight lines with exactly one point in common are called intersecting lines.	Two distinct great circles with exactly two points in common are called intersecting great circles.
Two distinct lines can never have more than one point of intersection.	Two distinct great circles have two points of intersection.

When two great circles intersect on a sphere, you have two lunes, or bi-angles, having opposite angles that are congruent. They can be right, acute or obtuse. Four bi-angles are always formed by two intersecting great circles. In some references bi-angles are called lunes.

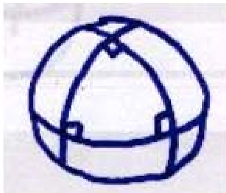
It is possible to argue that a polygon is formed on a sphere because lunes, or bi-angles enclose a definite area, however, the edges of the bi-angles are arcs and a polygon can only be constructed of line segments. The lune creates a two-sided polygon; in plane geometry there are no two-sided polygons. In plane geometry two rays with a common endpoint will never meet, so that a two-sided figure is impossible whereas in spherical geometry two great circles will always meet to form at least two finite regions.



Bi-angles, or lunes that form four congruent regions.

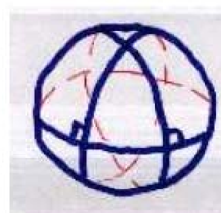
Bi-angles, or lunes that form two congruent regions.

Three great circles can be placed so that they are perpendicular to each other, or one great circle can be perpendicular to two non-perpendicular great circles on a sphere.

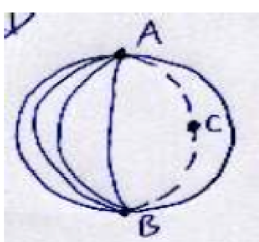


A great circle that is perpendicular
to two perpendicular great circles.

A great circle that is perpendicular to two non-perpendicular great circles:



Three points on a sphere determine eight different possible spherical triangles. This is because two points can be connected with two different great circles. But if two

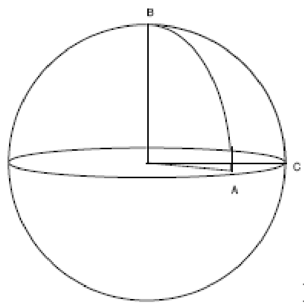


of the points are on the poles, there are an infinite number of triangles. If we define a straight line in plane geometry as the shortest distance between two points, there is only one unique triangle determined by three non-collinear points.

In spherical geometry, all three sides of the right triangle bounding an octant of the unit sphere have length equal to $\pi/2$; this violates the Euclidean Pythagorean theorem because $(\pi/2)^2 + (\pi/2)^2 \neq (\pi/2)^2$. In Euclidean geometry $a^2 + b^2 = c^2$, or the sum of the square of the legs of a right triangle is equal to the square of the hypotenuse holds true.

The area of a triangle in Euclidean geometry is ½ bh. How would we find the area of a lune? Given that the surface area of a sphere is $4\Pi r^2$, if you use a unit sphere where the radius is 1, the area of a lune becomes 4Π times the degrees of the lune/360°. If you have a triangle with a third side on the equator of the sphere, then the area of that triangle is the area of the lune divided by 2. Taking the sum of the angles measured in radians and subtracting pi can compute the area of a triangle with end points anywhere on a sphere. The amount by which the sum of the angles exceeds 180° is called the spherical excess E: $E = \alpha + \beta + \gamma - 180°$. Where α , β and γ are the angles of the triangle, measured in degrees. This surplus determines the surface area of any spherical triangle. To determine this, the spherical excess must be expressed in radians; the surface area A is then given in terms of the sphere's radius R by the expression: $A = R^2 \cdot E$. Unlike Euclidean geometry where all triangles have the same sum of 180°, in spherical geometry, the angle sum of a triangle is related to its angle sum. A triangle on a sphere has the interesting property that the sum of the angles is greater than 180 degrees!

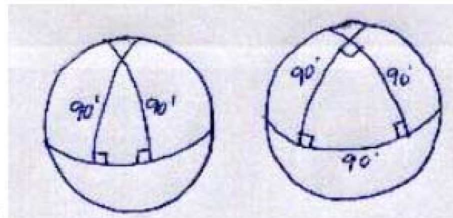
From this formula, it becomes obvious that there are no similar triangles (triangles with equal angles but different side lengths and area) on a sphere. And in fact, two triangles with the same angles are not just similar (as in planar geometry), they are actually congruent! There does not exist any similarity on a sphere because spherical triangles with equal angles must contain the same area, because area of a spherical triangle is calculated by adding the sum of the angles of the triangle measured in radians and subtracting pi from that total. Therefore all triangles with the same measure of angles will contain the same area.



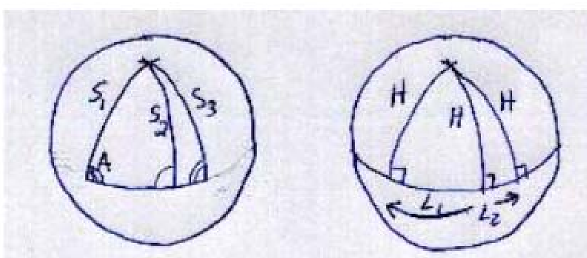
Refer to this diagram in the following paragraph.

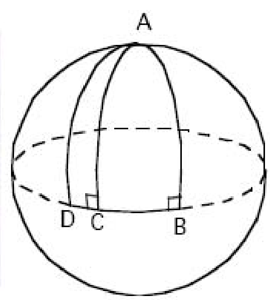
To find the perimeter of a triangle in plane geometry simply add the lengths of all three sides. A spherical triangle is like a planar triangle. It consists of three points called vertices, the arcs of great circles that join the vertices, called the sides, and the area enclosed therein. The perimeter of the spherical triangle can be calculated by using the basic formula for the circumference of a circle: $C = \Pi \cdot D$. For a unit circle that becomes: $\pi \bullet D$ or $3.14 \bullet 2$. Arc BA and BC are 90° angles so they are $90^\circ/360^\circ \approx 1.57$ each or 3.14. Arc AC is an arc with the central angle of AOC Arc AC is the same as that central angle. If in this example that is a 30° angle the total would be $30^\circ/360^\circ = 1/6 \bullet 6.28 \approx 1.05$. So that the perimeter of spherical triangle $ABC = 3.14 + 1.05 \approx 4.19$.

AAS is not true for a sphere. The following sketches show two congruent angles with sides that are congruent on both spheres, yet the third angle is not the same, thus they are not congruent triangles.



The following drawing demonstrates how AAS congruency does not hold for a spherical triangle similarly; the Hypotenuse-Leg Theorem will produce two right triangles that are not congruent.





In the figure on the left, because the sides are pole points they are all congruent. ∠A ≅ ∠A, however, the two triangles are not congruent. Consider the spherical triangles Δ ABD and Δ ACD in the sphere on the right. AD ≅ AD by the reflexive property of congruence, and since A is a pole point of line BC, we know AB ≅ AC. Again since A is a pole point, ∠ABD and ∠ACD are right angles. Yet ΔABC is not congruent to ΔACD.

References:

Beem, John K. "Geometry Connections." Pearson Education/Prentice Hall. Drafted Materials March 7, 2005.

Castellanos, Joel. "Non-Euclid." <http://maths.gla.ac.uk>

Su, Francis E., et al. "Spherical Pythagorean Theorem." Mudd Math Fun Facts. <http://www.math.hmc.edu/funfacts>.

Weeks, Jeffrey R. "Exploring the Shape of Space." Key Curriculum Press. Emeryville, California. 2001.

Weisstein, Eric W. "Euclid's Postulates." From Mathworld—A Wolfram Web Resource. <http:// mathworld.wolfram.com/Euclids-postulates.html>