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REGULARITY FOR SOLUTIONS TO PARABOLIC SYSTEMS AND NONLOCAL MINIMIZATION PROBLEMS

by

Joe Geisbauer

A DISSERTATION

Presented to the Faculty of

The Graduate College at the University of Nebraska-Lincoln

In Partial Fulfilment of Requirements

For the Degree of Doctor of Philosophy

Major: Mathematics

Under the Supervision of Professor Mikil Foss

Lincoln, Nebraska

May, 2013

REGULARITY FOR SOLUTIONS TO PARABOLIC SYSTEMS AND NONLOCAL MINIMIZATION PROBLEMS

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University of Nebraska-Lincoln, 2013

Advisor: Mikil Foss

The goal of this dissertation is to contribute to both the nonlocal and local settings of regularity theory within the calculus of variations. In the nonlocal theory, we first establish the existence of minimizers for two classes of functionals. However, the main result of Chapter 2 states an analogue for higher differentiability of minimizers in the setting of nonlocal functionals, which is established through an application of the difference quotient method. This nonlocal analogue is stated in terms of the fractional order difference quotient, which corresponds to the order of the Besov space to which the solution belongs.

In the third chapter, we investigate the regularity of solutions to the parabolic system

$$u_t - \operatorname{div}(a(x,t,u,Du)) = 0.$$

In particular, we show that, under subquadratic growth and ellipticity conditions, solutions of the above system will be Hölder continuous with exponent $\alpha \in (0,1)$ when the coefficients are continuous. In other words, it is shown that there is an open subset of full measure, when compared to the domain for the problem, on which the solution is Hölder continuous. In order to prove the result, we appeal to the A-caloric Approximation Method.

DEDICATION

To my wife, Kayde,

without whose support in all aspects of life

and sacrifices over the last 6 years

this manuscript would have never been possible.

ACKNOWLEDGMENTS

I would like to begin by thanking my advisor, Mikil Foss, for his constant guidance professionally, academically, and personally. His concern for me as a person went above and beyond the professional relationship required of an advisor. For this and for the vast training he provided me in mathematics, in particular the calculus of variations, I will be forever grateful. I would also like to thank my committee, Petronela Radu, Mohammad Rammaha, Allan Peterson, and Kevin Cole, for their constant support throughout the process of completing this dissertation. I would especially like to thank Petronela for her concern and advice throughout the last two years as well as the opportunities she was able to provide for me, such as Math in the City and invitations to speak at conferences.

During my career as a graduate student, I have been afforded many great opportunities. These opportunities were made possible through the hard work of various faculty members throughout the Department of Mathematics. Know that I am extremely grateful for the hard work (grants, paperwork, recommendation letters, etc.) that made these great teaching and professional endeavors possible. This truly is a great department that fosters the educational, professional, and personal development of its graduate students. Thank you for allowing me to be a part of this special community.

A huge part of what makes this educational community so special are my current and former graduate student colleagues. While there are several individuals that have made my graduate career fun, exciting, and at times managed to keep it bearable, I would like to take a moment and mention a few by name. While here at UNL, I was lucky enough to claim the best officemates available. I owe a lot to each and every one of them! Zahava, I will never forget the gyros lunch we had around 3 years ago and the encouragement you provided. Know that I will always be grateful for the advice and support you provided. By the way, I am still pretty sure that Kayde called you to set up that lunch! As for Derrick Stolee, he

was always there to be a sounding board. Of course, there were those times that he missed my GSS talks, but the friendship he has provided me more than makes up for those petty things. I was able to finish my graduate career sharing an office with James Carraher. There has been a lot of laughter between James and I in the past two years, and I can definitely say that sometimes laughter is the best medicine. While diversity is a great thing that serves to challenge and enhance ones ideas and beliefs, it is also important to have a friend in a similar situation who also has similar goals, beliefs, priorities, and ideas. Throughout my graduate career, but even more so recently, this person has been Amanda Croll. Thank you, Amanda, for the many insightful conversations we have had together as well as the support you have provided. Of course, part of this special friendship is the friendship that I gained with your husband, Nick. I owe him many thanks as well!

I would also like to take a moment and thank the faculty who were in the Department of Mathematics at the University of Arkansas-Fort Smith during my tenure there as a young, very naive, mathematician. In particular, I will always be grateful to Jill Guerra, Kathy Pinzon, Dan Pinzon, Myron Rigsby, and Jack Jackson for the support they provided throughout my undergraduate career as well as the continued support they show to this day. I cannot thank them enough for all that they have done for me both as a mathematician and a person. I would especially like to thank Jill Guerra for mentoring me throughout my undergraduate career and the tidbits of advice she offered throughout graduate school as well. Yes Jill, believe it or not, I still listen!

There are also many people who despite the fact that I am a mathematician have supported me in all endeavors of my life including graduate school. These people are the many family and friends that I have been blessed with throughout life. Each of them has provided me a characteristic which has given me the ability to reach this point in my mathematical career, and I cannot thank them enough. Mom and Dad, I owe you too much to tell all here, but know that your sacrifices, love, character-building, and the strength of your marriage

have enabled me to be who I am today. Thank you! To my in-laws, thank you for the wonderful gift of your daughter. She has been my rock over the past six years. While I owe you many more thanks as well, I cannot thank you enough for this gift. I would also like to thank my brothers and their families and my brother-in-law for their tremendous friendship and support throughout these six years! Sometimes a little brotherhood is needed to take your mind off of the challenges faced in graduate school.

Behind every successful person, there is a mentor, someone who teaches, guides, challenges, and supports that person. While several people have taken me under their wings throughout my life, one person has always been there from the very beginning. Thank you, Godfather, for everything! Know that you hold a very special place in my heart! While you shouldn't let your wife Linda take all of the credit, she should know that I am grateful to her for the constant support and wisdom she imparted as well!

Lastly, I would like to thank my wife again. Kayde, few people know what it is like to be married to a graduate student while having already started your career. The sacrifices, love, support, and patience you have shown over the last six years have been tremendous. I will never be able to repay you for these, but know that I will always be grateful for them. Thank you for everything!

GRANT INFORMATION

This work was partially supported by the NSF grant DMS-0838463, Mentoring through Critical Transition Points.

Contents

Chapter 1

Introduction

1.1 GENERAL THEORY AND METHODS

Within the calculus of variations, there are two overarching topics one often studies. The first pertains to questions about the quantitative nature of solutions to variational problems, such as existence and uniqueness, while the second searches for qualitative properties of solutions. The qualitative study of solutions seeks to answer questions about the asymptotic behavior and regularity of solutions. By regularity theory, we mean the investigation of questions pertaining to increased smoothness or integrability of solutions.

Interest in regularity theory increased after the proposal of Hilbert's 19^{th} problem which questioned whether solutions to regular variational problems must be analytic. The question was resolved in the positive by both Ennio de Giorgi and John Nash independently in [20] and [42], respectively. Their results showed solutions to linear elliptic equations with measurable coefficients were Hölder continuous. This was the key component that allowed one to establish continuity of higher order derivatives through the method of bootstrapping. Attesting to the significance of these results, continuity results of the same nature as those contained in [20] and [42] are now commonly referred to as DeGiorgi-Nash-Moser results.

Before considering regularity theory in more detail, we introduce the various methods from the calculus of variations that are used in this work. The application of variational principles to partial differential equations or minimization problems involving functionals begins by confirming the existence of solutions to such problems. The primary method one employs in order to obtain such a result is the direct method. One begins the method by selecting an arbitrary minimizing sequence. The coercivity of the functional and the reflexivity of the underlying spaces to which the admissible class belongs can then be used to deduce the existence of a limit for this minimizing sequence. Lastly, the method concludes with showing that the limit is contained in the admissible class and deducing that the limit minimizes the functional by means of the lower semicontinuity of the functional.

Once the existence of solutions has been established, one transitions to studying the regularity properties of these solutions. As mentioned before, regularity of solutions can take many forms, but we will only discuss higher differentiability results and continuity results in this work. The two methods used to achieve these results are the difference quotient method and the harmonic approximation method, respectively. We note that the last method is referred to as the harmonic approximation method when studying elliptic equations and the caloric approximation method when studying solutions to parabolic equations. The reasons for this will be discussed in the explanation of the method.

1.1.1 THE DIFFERENCE QUOTIENT METHOD

As previously mentioned, the difference quotient method is used to establish higher order derivatives for solutions to partial differential equations or minimizations problems involving functionals. For example, consider a minimizer of the functional

$$\int_\Omega F(Du)\,dx,$$

where F satsifies certain coercivity, growth, and uniform convexity conditions. These properties will be stated more eplicitly later, and we mention them here only to provide an understanding of the overall method. For simplicity, we take $\Omega \subset \mathbb{R}^n$ and $u : \mathbb{R}^n \to R$. We also note that Du represents the gradient of the solution u. The method begins by considering variations $\varphi \in C_c^\infty(\Omega)$, or smooth functions with compact support in Ω. By appealing to classical methods in variational calculus, one is able to establish the following equation involving the first variation of the functional

$$\int_\Omega F_\xi(Du(x)) \cdot D\varphi(x)\, dx = 0. \tag{1.1}$$

As the variation φ has compact support, one is then able to translate this variation by substituting $\varphi(x - he_j)$ into the above equation and changing variables. Taking the difference of these two objects then gives

$$\int_\Omega \Big(F_\xi(Du(x + he_j)) - F_\xi(Du(x)) \Big) \cdot D\varphi(x)\, dx.$$

Then using Leibniz's rule for integrals, one can rewrite this difference as an integral of the secord order derivative of the integrand and obtain

$$\int_\Omega \int_0^1 F_{\xi\xi}\Big(\mathscr{L}(Du) \Big)(Du(x + he_j) - Du(x)) \cdot D\varphi(x)\, ds\, dx = 0,$$

where $\mathscr{L}(Du) := Du(x) + s(Du(x + he_j) - Du(x))$.

One then makes a particular choice for φ which gives rise to multiple terms in the above functional. Using the uniform convexity of the integrand F and the compact support of φ, these terms can be bounded below by

$$\int_V \left| \frac{Du(x + h) - Du(x)}{h} \right|^2 dx.$$

Here, V is an arbitrary compact subset of Ω, so that one is able to deduce the local existence of a second derivative in $L^2(\Omega)$ at the end of the argument. All of the other terms arising from the particular choice of φ are then bounded above by the functional evaluated at the minimizer or the norm of $u \in W^{1,2}(\Omega)$, which are both finite by our assumptions. This is achieved by using the growth assumptions imposed on the integrand F and then appealing to the coercivity of the functional. As these directional derivatives are uniformly bounded, one can then use an embedding result to deduce the local existence of a second order derivative, which is the desired result.

1.1.2 THE HARMONIC/CALORIC APPROXIMATION METHOD

Having finished the discussion on the application of the difference quotient method, we now proceed to discuss the harmonic approximation method which is used to establish the partial Hölder continuity for a solution to a system of elliptic or parabolic partial differential equations or its gradient. The harmonic approximation lemma facilitates the approximation of an elliptic or parabolic system of partial differential equations by a linear elliptic or parabolic system with constant coefficients. Solutions to such linear elliptic systems are referred to as harmonic functions, and so the name harmonic approximation was given to the method. As solutions to linear parabolic systems with constant coefficients are referred to as caloric functions, the method is referred to as the caloric approximation method in this setting. Often the technical notation and and complexity of the approximation method detract from the understanding behind the method. In an effort to provide an understanding of the ideas behind the method, we make a concerted effort to avoid any technical notation in this discussion.

One begins applying the method by establishing a linearization lemma that captures the error in approximating a quasilinear system of elliptic equations,

$$\text{div}(a(x, u, Du)) = 0,$$

with the following system with constant coefficients,

$$\text{div}(a(x_0, \ell(x_0), Dw)) = 0.$$

Here $\ell : \Omega \to \mathbb{R}^N$, where $\Omega \subset \mathbb{R}^n$, is a fixed affine map and $x_0 \in \Omega$ is fixed. Hence the latter system is in fact linear and has constant coefficients. Ultimately this approximation can only be expected to be strong enough in small neighborhoods of Lebesgue points of Du. This, however, is sufficient to at least establish the desired partial continuity result.

After the linearization of the problem has occurred, the harmonic approximation lemma is then established and applied to the system. This lemma is the cornerstone of the argument and states that, as long as the error of the above approximation is small enough, solutions to the quasilinear system can be compared to solutions of the linear system with constant coefficients. In order to establish the desired continuity result one needs to show the solution satisfies a particular decay estimate, namely one needs to show that

$$\fint_{B_\rho(x_0)} |Du - (Du)_\rho|^2 \, dx \tag{1.2}$$

decays fast enough as $\rho \to 0$. Here $\rho > 0$ and $(Du)_\rho$ represents the integral mean of Du over the ball $B_\rho(x_0)$. The utility of the approximation lemma comes from the fact that it allows one to compare solutions to harmonic functions which satisfy very nice decay properties. Using these decay properties and the approximation lemma followed by an iteration lemma, one can then deduce a decay estimate for (1.2). We should note though that this is a very strong estimate in that it would allow one to deduce the partial Hölder continuity of Du following an embedding result.

However, the method can be modified if one is not able to obtain the rapid decay of the gradient Du, as is the case in our particular problem in Chapter 3. Instead, we are able to use an excess functional that allows us to obtain a bound on how quickly Du blows-up in regions around Lebesgue points and establish a partial decay estimate. For now, we forego the discussion of the excess functional used to measure the oscillations in u due to the extensive notation required to define it; however, we will provide a detailed discussion of the particular excess functional used to obtain our result in the introduction to Chapter 3. Once this partial decay estimate is established for the first iteration, an iteration lemma is then used to show the solution itself is in fact decaying quickly enough, at least locally; in particular

$$\fint_{B_\rho(x_0)} |u - (u)_\rho|^2 \, dx$$

decreases quickly enough as $\rho \to 0$. Here $(u)_\rho$ is the intergral mean of u on the ball $B_\rho(x_0)$. From this, we deduce that the solution belongs to a Campanato space which can be embedded into the set of partially Hölder continuous functions using a result by Da Prato. As mentioned earlier, we will provide a more detailed outline of the caloric method in Chapter 3; however, a basic understanding of the above outline will suffice for the rest of this chapter.

1.2 REGULARITY AND NONLOCAL MINIMIZATION PROBLEMS

As the results pertaining to the nonlocal and local problems in this dissertation are notationally and mathematically very different, we wait to introduce the necessary notation and more specific mathematical background for each problem within each chapter. In doing so, we force the more intricate discussions of the methods to occur within each chapter as well. As such, the discussion within this section of the dissertation will aspire provide an understanding of where the results of this work fit within the mathematical literature and provide a more general historical background for each of the problems we consider.

We begin this discussion by considering the results we have obtained for nonlocal functionals. At this point, we should note that the term nonlocal functional has been used to describe a vast number of different functionals. In many cases, the term has been used to describe functionals in which the integrand depends on the evaluation of its arguments at finitely many different points in the domain. These are not the types of nonlocal functionals we consider; rather, we consider nonlocal functionals that take into account interactions between points within small enough neighborhoods. While variational methods have been well-developed in the setting of local functionals, applications of variational methods to the type of nonlocal functionals we consider are not as prevalent. However, a few results have been established concerning the existence of solutions and regularity for solutions in this setting. In fact, one such existence result is due to Hinds and Radu in [36]. Here the authors use the direct method to establish the existence of minimizers for a nonlocal p-Laplacian system related to peridynamics. They are then able to deduce the well-posedness of the problem from their results.

As for nonlocal regularity, Silvestre, Caffarelli, and Kassmann have made the most contributions thus far. The nonlocal functionals studied by this group mostly involved a Levy process, or jump process, e.g.

$$Tu(x) := \int_{\mathbb{R}^n} (u(x+y) - u(x) - Du \cdot y\chi_{B(y)})k(x,y)dy. \tag{1.3}$$

In [45], Silvestre considers solutions to the problem $Tu(x) = f(x)$, where f is a bounded function and the kernel k satisfies, among other assumptions, the following bounds:

$$\frac{\nu}{|y|^{n+2s(x)}} \leq k(x,y) \leq \frac{M}{|y|^{n+2s(x)}}.$$

Here $0 < \nu \leq M$, n is the space dimension, and $0 < s(x) < 1$. In this work, Silvestre shows that solutions to this equation are Hölder continuous.

Later Kassmann showed that equations of the form

$$\int_\Omega \int_\Omega (u(y) - u(x))(\phi(x) - \phi(y))k(x,y)\,dx\,dy = 0, \text{ for all } \phi \in C_0^\infty(\Omega),$$

where k takes the form

$$\frac{\nu}{|x-y|^{n+2s}} \leq k(x,y) \leq \frac{M}{|x-y|^{n+2s}},$$

have solutions that are Hölder continuous. Again, we have here that $0 < \nu \leq M$, n is the space dimension, and $0 < s < 1$. This work was contained in [37], but was then extended in [38] where the author showed that the Hölder constant and exponent both stay positive as s goes to 1.

The results by Kassmann were then followed by a series of papers by Caffarelli, Silvestre, et. al. in which they studied integro-differential equations involving a functional with a form similar to that in (1.3), a fractional order nonlocal Laplacian, and a min-max problem involving a similar functional to that in (1.3). In this series of papers, namely [11], [5], and [12], the authors presented more results pertaining to the Hölder continuity, or Lipschitz continuity for the problem involving the fractional order Laplacian, of solutions.

The above results differ from the results we present in Chapter 2 in many different ways. Perhaps the most noticebale difference is in the types of results obtained in both cases. All of the above regularity results discuss the continuity of solutions to particular nonlocal problems, whereas the results of Chapter 2 establish a nonlocal analogue of higher differentiability results. We should also note that the functionals considered in the above results have a very different nonlocal structure than the functional we consider. For instance, the above results mostly assume the existence of a full derivative, with the exception of Kassmann. The functionals discussed in Chapter 2 assume only the existence of a fractional order derivative. Moreover, the results we prove are valid in the vectorial setting and not restricted to the scalar setting. It should be noted though that the kernels considered within the above results are often times more general than the kernels of our nonlocal functionals.

The nonlocal functionals we consider ultimately stem from questioning whether the full gradient need be a part of the integrand in order to obtain regularity or if a fractional order derivative will allow one to obtain regularity for solutions as well. This question, along with the increased interest in the use of nonlocal models of late, led us to consider minimizers of the following functionals:

$$J[u] := \int_\Omega \int_0^H F(x, u, \mathscr{G}u(x,h)h^{-1/p})\, dh\, dx$$

and

$$K[u] := \int_\Omega \int_{|h|\leq H} F(x, u, \Lambda u(x,h)h^{-1/p})\, dh\, dx.$$

Here $\Omega \subset \mathbb{R}^n$, and u is a vector-valued map into $\mathbb{R}^N$. Moreover, $\mathscr{G}u(\cdot,\cdot) : \Omega \times (0, H) \to \mathbb{R}^{Nn}$ is defined by $\mathscr{G}^i_j u(x,h) = \frac{u^i(x+he_j)-u^i(x)}{h^s}$, where e_j represents the basis vector with 1 in the j^{th} component and zeros elsewhere and $h \in \mathbb{R}$. Also, $\Lambda u(\cdot,\cdot) : \Omega \times B_H(0) \to R^N$ is defined by $\Lambda^i u(x,h) = \frac{u^i(x+h)-u^i(x)}{|h|^s}$, where $h \in \mathbb{R}^n$. Throughout the rest of this section, $\mathscr{G}u(x,h)$ and $\Lambda u(x,h)$, will be referred to as fractional order difference quotients in order to avoid the otherwise technical notation and nonessential differences between the two difference quotients until Chapter 2.

As previously mentioned, we begin Chapter 2 by showing the existence of minimizers for $J[\cdot]$ and $K[\cdot]$ over an admissible class that lies within a particular Besov space. The need to consider admissible classes contained in Besov spaces is due to the nonlocal nature of $J[\cdot]$ and $K[\cdot]$. However the reflexivity of these spaces, allows for the application of the direct method in order to establish the existence of minimizers. By now, this variational method has become classical for reflexive Banach spaces, and the inclusion of these results serves to provide completeness and motivation for the later results of the chapter.

The main results of this chapter are nonlocal analogues of higher differentiability results. Within the local setting, one considers the higher differentiability of solutions $u \in W^{1,2}(\Omega)$ to minimization problems involving a functional similar to

$$\int_\Omega F(x, u, Du)\, dx,$$

i.e. they are able to deduce that $u \in W^{2,2}_{loc}(\Omega)$. The main results contained in Chapter 2 show that minimizers of

$$J[u] := \int_\Omega \int_0^H F(x, \mathscr{G}u(x,h)h^{-1/2})\, dh\, dx,$$

and

$$K[u] := \int_\Omega \int_{|h|\leq H} F(x, \Lambda u(x,h)|h|^{-1/2})\, dh\, dx$$

have a similar property. We note that the form of $J[\cdot]$ and $K[\cdot]$ only requires minimizers to belong to the Besov space $B^{s,2}(\Omega)$ originally, where 2 pertains to the integrability of the solution and $0 < s < 1$ represents the order of the Besov space. In the latter half of Chapter 2, we are able to show that minimizers of these functionals are contained in the Besov space

$B^{2s,2}_{loc}(\Omega)$ if $0 < s < \frac{1}{2}$ and $B^{\alpha,2}_{loc}(\Omega)$ for any $\alpha \in (0,1)$ if $\frac{1}{2} < s < 1$. As the order, s, corresponds to the fractional order of the difference quotient of the solution, these results are in fact a nonlocal analogue of the local results pertaining to higher differentiability.

1.3 REGULARITY AND PARTIAL DIFFERENTIAL EQUATIONS

We now proceed to discuss the contents of the last chapter in this work. This requires that we move from the nonlocal setting within the calculus of variations to the local setting. The study of regularity theory in the local setting is extensive compared to the nonlocal setting, and so we begin by providing a very brief history of regularity results that led to the study of partial Hölder continuity. We will then introduce our results and comment on the use of the caloric approximation method.

In this brief historical review of regularity theory, we will try to discuss results in both the elliptic and parabolic settings; however, we will only present equations in the elliptic setting in order to maintain clarity. Moreover, we also provide a more in-depth introduction to regularity results for the parabolic setting in Chapter 3. In [20], E. de Giorgi showed that solutions, $u \in W^{1,2}(\Omega)$, to the linear second order partial differential equation

$$\sum_{i,j} \frac{\partial(c^{ij}(x)\frac{\partial u}{\partial x_j})}{\partial x_i} = 0, \tag{1.4}$$

where $c^{i,j}$ are the terms of a symmetric, real valued, uniformly elliptic matrix, are Hölder continuous. Assuming F is smooth enough, taking $u = Dw$ for some $w \in W^{2,2}(\Omega)$, and setting $c^{ij}(x) = F_{\xi\xi}(Dw)$, then (1.4) is the system of Euler-Lagrange equations for the functional

$$\int_\Omega F(Dw)\, dx.$$

Here $F_{\xi\xi}$ is the second derivative of F with respect to the argument Dw. Since (1.4) is the Euler-Lagrange equations for the above functional, one can then use de Giorgi's result to show $w \in C^{1,\alpha}(\Omega)$. The continuity of higher order derivatives could then be established via bootstrapping.

It was later recognized by Ladyzenskaja and Ural′tseva that the above method did not rely on the linearity of the equation. In [39], the two authors proved that weak solutions to quasilinear equations of the form

$$\text{div}(a(x, u, Du)) = b(x, u, Du),$$

where $a(\cdot,\cdot,\cdot), b(\cdot,\cdot,\cdot) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, also belong to $C^{1,\alpha}(\Omega)$, as long as a and b satisfy:

$$\begin{cases} a(x, u, Du) \cdot Du \geq \nu |Du|^p - c(x), \text{ almost everywhere in } \Omega, \\ |a(x, u, Du)| + |b(x, u, Du)| \leq L|Du|^{p-1} + c(x). \end{cases}$$

Here $p > 1$ and $0 < \nu \leq L$ are given, and $c(x) \in L^{\infty}_{loc}(\Omega)$. In general, this is the best result one could hope for as the coefficients in the above equation need only be measurable for the theorem to apply. The parabolic analogue of this result has been established in the series of works [22], [23], and [21] by DiBenedetto and Friedman.

It is important to note that the results in the parabolic setting took longer to establish due to the subtleties that arise in the parabolic setting. For instance, the scaling in the time and spatial directions do not match when $p \neq 2$. This led to the development of the now well-known intrinsic geometry by DiBenedetto. Another such subtlety is the lack of regularity in the time direction. This does not allow one to use the solution itself as a valid test function when extending the methods of de Giorgi from the elliptic setting. In order to overcome this lack of regularity, one must use Steklov averages of the solution when constructing proper test functions. While these averages are defined more precisely later, they are integral averages in the time direction which converge back to the solution as the diameter of the domain goes to zero. While all of these were overcome by DiBenedetto and Friedman, the results in the parabolic setting are only able to be established for $p > \frac{2n}{n+2}$. In order to keep the introduction moderately short, we forego the discussion of this bound and only note that it exists.

After the full regularity of solutions to quasilinear elliptic equations was established, mathematicians began to question the extent to which the results in the scalar setting could be applied to the vectorial setting, i.e. the setting in which the solution $u \in W^{1,p}(\Omega; \mathbb{R}^N)$ with $N > 1$. It was eventually shown in the vectorial setting that one cannot expect everywhere Hölder continuity of the solution when considering systems of quasilinear partial differential equations. Many counterexamples have demonstrated this phenomenon; however, we cite only two examples, one of which can be found in both [34] and [35]. For a more refined counterexample, one may also consult [43]. Similar results were also established in the parabolic setting by the authors of [47], [48], and [49]. In [49], Struwe was able to show that a solution to a nonhomogeneous uniformly parabolic diagonal system developed a blow-up discontinuity in finite time when the right-hand side of this system was assumed to have quadratic growth. The authors of [47], then extended this result by showing blow-up of a solution in finite time for the corresponding homogeneous system. Finally, the authors of [48] were able to improve the results of the previous two papers by showing a solution to a system involving real analytic coefficients blows-up in finite time. The paper is also nice in that the calculations are easier to follow than the previous papers. These examples all show that Hölder continuity does not play the significant role in the regularity of solutions to systems of partial differential equations but rather partial Hölder continuity. Partial Hölder continuity establishes the local Hölder continuity of solutions to systems of partial differential equations on an open set $\Omega_0 \subseteq \Omega$, where Ω is the domain of the system and the Lebesgue measure of $\Omega \backslash \Omega_0$ is zero.

As partial regularity of a solution is unattainable without some continuity assumptions on the coefficients of the system, the research has focused on weakening the continuity assumptions on the coefficients of the system and deducing the regularity that can be obtained for the solution. To this end, Sergio Campanato provided efforts to establish the partial Hölder continuity of solutions to quasilinear elliptic and parabolic systems while only assuming continuity of the coefficients, as opposed to some stronger form of continuity such as Hölder continuity, in [15], [14], and [16], respectively. While his results were valid for some $p > 1$, they were found to be invalid for certain $p > 1$.

In [27], Duzaar and Steffen presented the harmonic approximation method for the first time. The result has become the standard method used to obtain partial ölder continuity results, and in Chapter 3, we use the A-caloric approximation method to prove the partial Hölder continuity of solutions for the following quasilinear parabolic system:

$$u_t - \text{div } a(x, t, u, Du) = 0. \tag{1.5}$$

Here $\Omega \subseteq \mathbb{R}^n$, $\Omega_T := \Omega \times (-T, 0)$, Du is the spatial gradient of u, and $a(\cdot, \cdot, \cdot) : \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is a vector field satisfying subquadratic growth, $\frac{2n}{n+2} < p < 2$, and ellipticity conditions. Moreover we assume that the vector field satisfies a continuity condition slightly weaker than Hölder continuity with respect to its third argument while only assuming the map $a(\cdot, \cdot, \xi) : \Omega_T \times \mathbb{R}^N \to \mathbb{R}^{Nn}$ is continuous for all $\xi \in \mathbb{R}^{Nn}$. Many results concerning the partial Hölder continuity of the spatial gradient have been established previously under stronger continuity assumptions on the coefficients, and these results are discussed thoroughly in the introduction to Chapter 3. However, we note that this is the best result one could, in general, hope to obtain due to the previous examples of systems with measurable coefficients that have solutions with blow-up in finite time. We also note that this result extends the work of Campanato to all $\frac{2n}{n+2} < p < 2$. The extension of Campanato's result to all $p \geq 2$ was previously established by Bögelein, Foss, and Mingione in [6].

Chapter 2

Nonlocal Functionals

2.1 INTRODUCTION

As stated earlier, the aim of this chapter of the dissertation is to contribute to the nonlocal theory within the calculus of variations by establishing the higher differentiability, in the context of Besov spaces, for minimizers for a class of nonlocal functionals. The motivation for the study of these functionals arose from two considerations. The first is from the recent inclusion of more nonlocal features within models in continuum mechanics, math biology, and image processing, for example in [36], [32], and [33], and a lack of literature on nonlocal functionals in regards to variational methods. As mentioned earlier, the second comes from considering whether the full gradient is needed in order to establish regularity of the solution to minimization problems or whether a fractional derivative will suffice. We also note that the results of this chapter were originally published in the manuscript [30] in order to include these results in the mathematical literature in a timely manner.

Due to the nonlocal nature of the functionals we study here, the natural space over which to consider minimizers is a Besov space, which is denoted by $B^{s,p,q}(\Omega;\mathbb{R}^N)$ throughout this chapter. Here $\Omega \subset \mathbb{R}^n$ is open and bounded, $1 < p, q < \infty$, and $s \in (0,1)$. Functions contained in these spaces are said to have a fractional derivative of order s and are characterized by the integrability of their fractional order difference quotient. In this chapter, we are able to use this fractional order difference quotient to show that minimizers of particular nonlocal functionals can be expected to belong to a higher order Besov space. As the order of the Besov space corresponds to the order of the fractional difference quotient, this result can be said to extend higher differentiability results to the nonlocal setting.

We begin the present chapter by considering the following two functionals

$$J[u] := \int_\Omega \int_0^H F(x, u, \mathscr{G}u(x,h)h^{-1/p})\, dh\, dx \tag{2.1}$$

and

$$K[u] := \int_\Omega \int_{|h|\leq H} F(x, u, \Lambda u(x,h)|h|^{-1/p})\, dh\, dx. \tag{2.2}$$

Here $p \in (1, \infty)$, and $\mathscr{G}u(x,h)$ and $\Lambda u(x,h)$ represent the difference quotients of fractional order s as mentioned previously in Chapter 1. While the functionals will be fully described in Section 2.2, the two functionals can be understood in the following sense. The J-functional can be described as a functional that relies upon changes in coordinate directions of the domain, whereas the K-functional takes into account changes in u over all radial directions. We note that a similar functional to $K[\cdot]$ was used in [32], where the authors utilized the related functional to denoise images. There is also discussion that nonlocal functionals will denoise images that possess some internal periodicity more accurately than previous models.

To the authors' knowledge, the consideration of variational methods within the theory of nonlocal functionals has been confined to the scalar setting. So we begin by showing the existence of minimizers for $J[\cdot]$ and $K[\cdot]$ by means of the direct method. The direct method uses the coercivity and convexity of the integrand F and weak sequential lower semicontinuity of the functional to prove the existence of minimizers. The method is considered classical within the calculus of variations, and more thorough introductions to this strategy can be found in [19] and [28]. The existence and uniqueness results mentioned here comprise Section 2.3 of this work. As these methods are well-known and the existence of minimizers for the K-functional is proved in a similar manner, we provide detailed proofs for the J-functional only and outline the arguments for the K-functional.

In the last two sections of this chapter, we investigate the regularity for minimizers of

$$J[u] := \int_\Omega \int_0^H F(x, \mathscr{G}u(x,h)h^{-1/2})\, dh\, dx \tag{2.3}$$

and

$$K[u] := \int_\Omega \int_{|h|\le H} F(x, \Lambda u(x,h)|h|^{-1/2})\, dh\, dx. \tag{2.4}$$

Note that here we have taken $p = 2$ and dropped the explicit dependence of the functional on the minimizer u. Taking $p = 2$ in the above integrands corresponds to the assumption of quadratic growth for the integrand with respect to the fractional order difference quotient. As mentioned previously, the regularity result we are able to show says that minimizers of $J[\cdot]$ and $K[\cdot]$ belong to a higher order Besov space than originally assumed. In particular, if u is a minimizer of the given functional and is assumed to belong to $B^{s,2,2}(\Omega;\mathbb{R}^N)$, then $u \in B^{t,2,2}_{loc}(\Omega;\mathbb{R}^N)$, for some $t > s$.

The regularity thoerems mentioned in the preceding paragraph are obtained through the difference quotient method, which is discussed for example in [28] and [34]. As mentioned in Chapter 1, this method uses the convexity and coercivity of the integrand F to elicit a bound on an iterated difference quotient. We then employ an embedding theorem in order to bound the higher order Besov norm and establish the result. The major obstacle in these proofs is adapting the difference quotient method to account for the iterated difference quotients that occur. In the local setting, one does not need to worry about this as the method gives rise to a single difference quotient involving the gradient of the solution. However, the problem becomes unavoidable in the nonlocal setting due to the appearance of the fractional order difference quotient in the third argument of the integrand.

2.2 BACKGROUND

Let $\Omega \subset \mathbb{R}^n$ be open and bounded and define

$$\Omega_\alpha := \begin{cases} \displaystyle\bigcup_{x\in\Omega} B_\alpha(x), & \text{if } \alpha > 0, \\ \{x \in \Omega : B_{|\alpha|}(x) \subset \Omega\}, & \text{if } \alpha < 0. \end{cases}$$

If $\alpha > 0$, we will often refer to the set $\Omega_\alpha \backslash \Omega$ as the collar of size α around Ω. Consider a map $u : \Omega_H \to \mathbb{R}^N$ and define $\mathscr{G}u(\cdot,\cdot) : \Omega \times (0,H) \to \mathbb{R}^{Nn}$ by $\mathscr{G}^i_j u(x,h) = \frac{u^i(x+he_j)-u^i(x)}{h^s}$, where e_j represents the basis vector with 1 in the j^{th} component and zeros elsewhere. We say u is in the Besov Space $B^{s,p,q}(\Omega;\mathbb{R}^N)$, where $1 < p, q < \infty$ and $0 < s < 1$, if

$$\|u\|_{B^{s,p,q}(\Omega;\mathbb{R}^N)} := \|u\|_{L^p(\Omega;\mathbb{R}^N)} + \sum_{i,j} \left(\int_0^\infty \left\| \mathscr{G}^i_j u(x,h) \right\|^q_{L^p(\Omega_{-h};\mathbb{R}^N)} \frac{dh}{h} \right)^{1/q} < \infty. \quad (2.5)$$

Note that once h reaches one-half of the diameter of Ω, Ω_{-h} is the empty set and the integrand in the previous seminorm becomes zero. Thus we are allowed to write the integral in this form. In fact, we can replace infinity in the above seminorm with any $0 < H \leq \infty$ and all of the norms are equivalent. We will however take H corresponding to the upper limit of integration in the functional for our applications.

We can also define an equivalent norm to that above for $B^{s,p,q}(\Omega;\mathbb{R}^N)$ by measuring changes in every radial direction from the point x. We do this as follows. Let $B_H(0)$ be the ball of radius $H > 0$ in $\mathbb{R}^n$, and define $\Lambda u(\cdot,\cdot) : \Omega \times B_H(0) \to R^N$ by $\Lambda^i u(x,h) = \frac{u^i(x+h)-u^i(x)}{|h|^s}$. Then u is in $B^{s,p,q}(\Omega;\mathbb{R}^N)$ if and only if

$$\|u\|_{B^{s,p,q}(\Omega;\mathbb{R}^N)} := \|u\|_{L^p(\Omega;\mathbb{R}^N)} + \left(\int_{|h|<\infty} \|\Lambda u(x,h)\|^q_{L^p(\Omega_{-|h|};\mathbb{R}^N)} \frac{dh}{|h|^n}\right)^{1/q}. \tag{2.6}$$

Note once again that the integrand becomes zero once h is large enough, and again the norms are all equivalent when we replace infinity in the definition of the seminorm with $0 < H \leq \infty$. We denote the set of all $u \in B^{s,p,q}(\Omega;\mathbb{R}^N)$ such that $u \equiv g$ on $\Omega_H \backslash \Omega$ by $Bd_g^{s,p,H}(\Omega;;\mathbb{R}^N)$, and $B^{s,p,p}(\Omega;\mathbb{R}^N)$ by $B^{s,p}(\Omega;\mathbb{R}^N)$. We further use $B^{s,p,q}_{loc}(\Omega;\mathbb{R}^N)$ to denote the space of all u such that for each $V \subset\subset \Omega$ we have $u \in B^{s,p,q}(V;\mathbb{R}^N)$.

The two norms above are equivalent when $q = p$, which can be established through the the equivalence of their respective seminorms on all of $\mathbb{R}^n$. The equivalence of the following seminorms is established by Proposition 14.40 in [40],

$$\sum_{i,j}\left(\int_0^\infty \|\mathscr{G}^i_j u(x,h)\|^p_{L^p(\mathbb{R}^n;\mathbb{R}^N)} \frac{dh}{h}\right)^{1/p} \quad \text{and} \quad \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n-sp}}\,dx\,dy\right)^{1/p};$$

whereas, the equivalence of the following

$$\left(\int_{|h|<\infty} \|\Lambda u(x,h)\|^p_{L^p(\mathbb{R}^n;\mathbb{R}^N)} \frac{dh}{|h|^n}\right)^{1/p} \quad \text{and} \quad \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^p}{|x-y|^{n-sp}}\,dx\,dy\right)^{1/p}$$

can then be established using a simple change of variables. Hence all three of the seminorms are equivalent, and we can represent them all by $|u|_{B^{s,p}(\mathbb{R}^n;\mathbb{R}^N)}$.

Recall the following definition from the Direct Method in the Calculus of Variations:

Defintion 2.2.1 (Caratheodory Function)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open set and $f : \Omega \times \mathbb{R}^K \to \overline{\mathbb{R}}$. Then f is said to be a Carathéodory function if*

1. $x \mapsto f(x,\gamma)$ *is measurable for every* $\gamma \in \mathbb{R}^K$*.*

2. $\gamma \mapsto f(x,\gamma)$ *continuous for almost every* $x \in \Omega$*.*

With this definition and the previous notation in mind, we consider the following functionals throughout the paper:

$$J[u] = \int_\Omega \int_0^H F(x, u, \mathscr{G}u(x+h)h^{-1/p})\, dh\, dx,$$

and

$$K[u] = \int_\Omega \int_{|h|\le H} F(x, u, \Lambda u(x+h)|h|^{-n/p})\, dh\, dx,$$

where the function F is a bounded, uniformly convex, and coercive Catheodory function.

The first step to obtaining an existence result for the functionals $J[\cdot]$ and $K[\cdot]$ by means of the direct method is to show the functionals $J[\cdot]$ and $K[\cdot]$ are weakly lower semicontinous. In order to obtain the weak lower semicontinuity of these functionals, we will first prove the result for a sequence which converges strongly, and then extend the result to weakly convergent sequences by means Mazur's Theorem. We will then proceed to prove the existence of minimizers of $J[\cdot]$ and $K[\cdot]$, for which it is necessary to use the following extension of the Sobolev-Gagliardo-Nirenberg inequality to Besov Spaces:

Theorem 2.2.1. *Let* $u \in L^1_{loc}(\mathbb{R}^n)$ *be a function vanishing at infinity such that* $|u|_{B^{s,p,q}(\mathbb{R}^n)}$ *is finite for some* $0 < s < 1, 1 \le p < \frac{n}{s}$*, and* $1 \le q \le \frac{np}{n-sp}$*. Then there exists* $C^* = C^*(n,p,s,q) > 0$ *such that*

$$\left(\int_{\mathbb{R}^n} |u(x)|^{\frac{np}{n-sp}}\, dx\right)^{\frac{n-sp}{np}} \le C^* |u|_{B^{s,p,q}(\mathbb{R}^n)}.$$

In particular, $B^{s,p,q}(\mathbb{R}^n)$ *is continuously embedded in* $L^\theta(\mathbb{R}^n)$ *for all* $p \le \theta \le \frac{np}{n-sp}$*.*

This result is proved in [40] for the semi-norm associated with the J-functional on all of $\mathbb{R}^n$. However, we know from the above discussion that the two seminorms associated with the J-functional and K-functional are equivalent on $\mathbb{R}^n$ when $q = p$. Hence, the embedding also holds for the semi-norm associated with the K-functional on all of $\mathbb{R}^n$ when $q = p$.

In order to apply the above extension of the Sobolev-Gagliardo-Nirenberg inequality on bounded domains, we will will need H to be large enough to approximate the seminorm on all of $\mathbb{R}^n$ by our seminorm on Ω. The following lemmas state specifically when this is possible.

Lemma 2.2.1. *Let $u \in B^{s,p}(\Omega;\mathbb{R}^N)$ be such that $u \equiv 0$ on $\mathbb{R}^n \backslash \Omega$, where $0 < s < 1$ and $1 < p < \frac{np}{n-sp}$, and assume $H > H^* := \left(\dfrac{c(n,N,p)C^* 2^{p-1}}{sp}\right)^{1/sp}$. Then,*

$$\|u\|_{L^p(\Omega;\mathbb{R}^N)} \leq C^{**} |u|_{B^{s,p}(\Omega_H;\mathbb{R}^N)}$$

where $C^{**} = \left(\dfrac{C(n,N,p)C^* spH^{sp}}{spH^{sp} - c(n,N,P)C^* 2^{p-1}}\right)^{1/p} > 0$. Here C^* is the constant from Theorem 2.2.1, and c(n,N,p) is the constant given for a change from the L^p-norm to the p^{th} power of the norm.

Proof. Begin by noting that since $u \equiv 0$ on $\mathbb{R}^n \backslash \Omega$

$$\begin{aligned}
\int_0^\infty \|\mathscr{G}u(x,h)\|_{L^p(\mathbb{R}^n;\mathbb{R}^N)} \frac{dh}{h} &\leq \int_0^H \|\mathscr{G}u(x,h)\|^p_{L^p(\mathbb{R}^n;\mathbb{R}^N)} \frac{dh}{h} \\
&\quad + 2^{p-1} \|u\|^p_{L^p(\Omega,\mathbb{R}^N)} \int_H^\infty \frac{1}{h^{sp-1}}\, dh \\
&= \int_0^H \|\mathscr{G}u(x,h)\|^p_{L^p(\Omega_{H-h};\mathbb{R}^N)} \frac{dh}{h} \\
&\quad + 2^{p-1} \|u\|^p_{L^p(\Omega,\mathbb{R}^N)} \frac{1}{spH^{sp}}
\end{aligned}$$

Then, we must have

$$
\begin{aligned}
|u|^p_{B^{s,p}(\mathbb{R}^n;\mathbb{R}^N)} =& \left(\sum_{n,N} \left(\int_0^\infty \|\mathscr{G}u\|^p_{L^p(\mathbb{R}^n;\mathbb{R}^N)} \frac{dh}{h} \right)^{1/p} \right)^p \\
\leq& c(n,N,p) \sum_{n,N} \int_0^\infty \|\mathscr{G}u\|^p_{L^p(\mathbb{R}^n;\mathbb{R}^N)} \frac{dh}{h} \\
\leq& c(n,N,p) \bigg(\int_0^H \|\mathscr{G}u(x,h)\|^p_{L^p(\Omega_{H-h};\mathbb{R}^N)} \frac{dh}{h} \\
&+ 2^{p-1} \|u\|^p_{L^p(\Omega,\mathbb{R}^N)} \frac{1}{spH^{sp}} \bigg).
\end{aligned}
$$

By applying Theorem (2.2.1) to u followed by the above estimate, we have

$$
\begin{aligned}
\|u\|^p_{L^p(\Omega;\mathbb{R}^N)} = \|u\|^p_{L^p(\mathbb{R}^n;\mathbb{R}^N)} \leq C^* |u|^p_{B^{s,p}(\mathbb{R}^n;\mathbb{R}^N)} \\
\leq c(n,N,p)C^* \left(\int_0^H \|\mathscr{G}u(x,h)\|^p_{L^p(\Omega_{H-h};\mathbb{R}^N)} \frac{dh}{h} \quad + 2^{p-1} \|u\|^p_{L^p(\Omega,\mathbb{R}^N)} \frac{1}{spH^{sp}} \right)
\end{aligned}
$$

So as long as $H > \left(\dfrac{c(n,N,p)C^* 2^{p-1}}{sp} \right)^{1/sp}$, we can subtract the L^p-norm on the right-hand side from both sides of the inequality to obtain

$$
\|u\|^p_{L^p(\Omega;\mathbb{R}^N)} \leq \left(\frac{C(n,N,p)C^* spH^{sp}}{spH^{sp} - c(n,N,P)C^* 2^{p-1}} \right) |u|^p_{B^{s,p}(\Omega_H;\mathbb{R}^N)}.
$$

□

Similarly, one can show the same inequality holds for the K-functional.

Lemma 2.2.2. *Let $u \in B^{s,p}(\Omega;\mathbb{R}^N)$ be such that $u \equiv 0$ on $\mathbb{R}^n \backslash \Omega$, where $0 < s < 1$ and $1 < p < \frac{np}{n-sp}$, and assume $H > H^* := \left(\dfrac{C^* 2^{p-1}}{sp} \right)^{1/sp}$. Then,*

$$
\|u\|_{L^p(\Omega;\mathbb{R}^N)} \leq C^{**} |u|_{B^{s,p}(\Omega_H;\mathbb{R}^N)}
$$

where $C^{**} = \left(\dfrac{C^* spH^{sp}}{spH^{sp} - C^* 2^{p-1}}\right)^{1/p} > 0$*. Here* C^* *is the constant from Theorem 2.2.1.*

Lastly, we mention a few results that will be used to obtain the higher fractional order differentiability in Section 2.4. In order to show such a result via the difference quotient method, it is necessary to apply an embedding result that relates higher order Besov spaces and iterated difference quotients in Besov spaces. Let $V \subset\subset \Omega$. We say that $u \in \mathscr{B}^{s_1;p;q}(\mathscr{B}^{s_2,p,q}(V;\mathbb{R}^N))$ if

$$\begin{aligned}&\|u\|_{\mathscr{B}^{s_2;p;q}(\mathscr{B}^{s_1,p,q}(V;\mathbb{R}^N))}\\ &:= \|u\|_{L^p(V;\mathbb{R}^N)} + \sum_{i,j}\left(\int_0^L\int_0^L \left\|\mathscr{G}\left(\mathscr{G}_j^i u(x,h)\right)(x,\ell)\right\|^q_{L^p(V_{-(h+\ell)};\mathbb{R}^N)} \frac{dh}{h}\frac{d\ell}{\ell}\right)^{1/q} \quad (2.7)\end{aligned}$$

is finite when considering Besov spaces in the context of the J functional, and if

$$\begin{aligned}&\|u\|_{\mathscr{B}^{s_2;p;q}(\mathscr{B}^{s_1,p,q}(V;\mathbb{R}^N))}\\ &:= \|u\|_{L^p(V;\mathbb{R}^N)} + \left(\int_{|\ell|\le L}\int_{|h|\le L} \left\|\Lambda\left(\Lambda u(x,h)\right)(x,\ell)\right\|^q_{L^p(V_{-(h+\ell)};\mathbb{R}^N)} \frac{dh}{|h|^n}\frac{d\ell}{|\ell|^n}\right)^{1/q}\end{aligned} \quad (2.8)$$

is finite in the context of the K functional. The relationship between the iterated norms above and higher order Besov spaces is the main topic of [2] and [9]. In Lemma 3 of [9], the authors show a more general version of the following lemma; however, we will only need the result as stated here.

Lemma 2.2.3. *Let* $V \subset \mathbb{R}^n$ *be open and bounded,* $1 \le p, q \le \infty$*, and* $0 < H \le \infty$*. Then,*

$$\|u\|_{B^{s_1+s_2,p,q}(V)} \le \frac{c(n,,q)}{\nu_n^{1/q}} \|u\|_{\mathscr{B}^{s_2,p,q}(\mathscr{B}^{s_1,p,q}(L^p(V)))} \quad (2.9)$$

for $s_1 + s_2 < 1$*. Here* ν_n *represents the measure of the unit ball in* $\mathbb{R}^n$*.*

Following a similar argument to the authors in [9], we will prove that such a result also holds for the norm in (2.6). In order to do so, it is necessary to employ the following inequality which is stated and proved in more generality in Section 5.3 of [10]:

Lemma 2.2.4. *Let $1 \leq p \leq \infty$ and $V \subset \mathbb{R}^n$ be open and bounded. Then for all functions measurable on V and for all $h \in R^n$ we have*

$$\|\Lambda u(x,h)\|_{L^p(\Omega)} \leq \frac{c(n)}{\nu_n |h|^n} \int_{\eta \leq |h|} \|\Lambda u(x,\eta)\|_{L^p(\Omega_{-h})} \, d\eta,$$

where ν_n represents the measure of the unit ball in $\mathbb{R}^n$.

Lastly, we provide the following definition which will be referred to in the last two sections of this chapter. It allows us to state more general assumptions under which the theorems of these sections are valid.

Defintion 2.2.2. *We say that $\Omega \subset \mathbb{R}^n$ is an extension domain, in the setting of Besov spaces, if there exists a bounded linear operator $L : B^{s,p,q}(\Omega;\mathbb{R}^N) \to B^{s,p,q}(\mathbb{R}^n;\mathbb{R}^N)$.*

2.3 EXISTENCE AND UNIQUENESS OF MINIMIZERS

In the following section, we present the proofs for the lower semicontinuity and existence results in terms of the J-functional. As the analogous proofs for the K-functional only require notational changes and the direct method is well-known, we will not present these proofs for the K-functional. However, we state the two theorems separately in order to keep the notation consistent and precise.

2.3.1 LOWER SEMICONTINUITY

We begin by showing the lower semicontinuity of the functional, which follows from the coercivity and convexity of the integrand. We will then use the lower semicontinuity to show the existence of minimizers for $J[\cdot]$ and $K[\cdot]$. The lower semicontinuity result for $J[\cdot]$ is stated as follows:

Theorem 2.3.1. *Let $\Omega \subseteq \mathbb{R}^n$ and $\{u_k\}_{k=1}^{\infty} \subseteq Bd_g^{s,p,H}(\Omega;\mathbb{R}^N)$ be such that $u_k \rightharpoonup u$ in $B_g^{s,p}(\Omega;\mathbb{R}^N)$, where $g \in B^{s,p}(\Omega_H;\mathbb{R}^N)$. Assume $F : \Omega\times\mathbb{R}^N\times\mathbb{R}^{Nn} \to \mathbb{R}$ is a Caratheodory function which is convex with respect to $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$ and satisfies*

$$F(x,u,\xi) \geq a|\xi|^p + b|u|^q + c(x) \tag{2.10}$$

for almost every $x \in \Omega$ and for every $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$. Here we assume $a > 0$ and $c \in L^1(\Omega;\mathbb{R}^N)$.

I. If $b \geq 0$ and $p, q \geq 1$, then

$$J[u] \leq \liminf_{k\to\infty} J[u_k].$$

II. Assume $H > H^*$, $\frac{-a}{2HC^{**}} < b < 0$, and $1 < q = p \leq \frac{np}{n-sp}$. Then $J[\cdot]$ is again weakly lower semicontinuous.

Proof. First assume that $u_k \to u$ strongly in $Bd_0^{s,p,H}(\Omega;\mathbb{R}^N)$, and we will weaken the notion of convergence on $\{u_k\}_{k=0}^{\infty}$ later. We can also assume that $c(x) = 0$ almost everywhere in Ω without loss of generality; otherwise consider the functional with integrand $F(\xi) - c(x)$ which is still convex for $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$. First note that by convexity we have

$$J[u_k] - J[u] \geq 2\int_\Omega \int_0^H F\left(x, \frac{1}{2}u_k - \frac{1}{2}u, \left[\frac{1}{2}\mathscr{G}u_k - \frac{1}{2}\mathscr{G}u\right] h^{-1/p}\right) dh\, dx.$$

The coercivity of F, Fubini-Tonelli, and the equivalence of norms on finite dimensional spaces, allows us to show

$$\begin{aligned}J[u_k]-J[u] \geq & \frac{1}{2^{p-1}}\int_\Omega\int_0^H a|\mathscr{G}u_k-\mathscr{G}u|^p\frac{1}{h}+b|u_k-u|^q\,dh\,dx\\ =&\frac{1}{2^{p-1}}\int_0^H\int_\Omega a|\mathscr{G}u_k-\mathscr{G}u|^p\frac{1}{h}+b|u_k-u|^q\,dx\,dh\\ =&\frac{a}{2^{p-1}}\left(\frac{1}{2}|u_k-u|^p_{B^{s,p}(\Omega_H;\mathbb{R}^N)}+\frac{bH}{a}\,\|u_k-u\|^q_{L^q(\Omega;\mathbb{R}^N)}\right).\end{aligned}$$

If at this point $b \geq 0$, then we certainly have that the right-hand side of the above inequality is greater than or equal to zero. If however, $H > H^*$ and $0 \geq b \geq \frac{-a}{2HC^{**}}$, then we can apply Lemma 2.2.1 to the right-hand side of the above inequality to find $C_0 > 0$ such that

$$J[u_k]-J[u] \geq C_0|u_k-u|^p_{B^{s,p}(\Omega;\mathbb{R}^N)} \geq 0.$$

In both cases, we have

$$\liminf_{k\to\infty} J[u_k]-J[u] \geq 0.$$

Thus J is sequentially lower semicontinuous and we only need to extend the result to weakly convergent subsequences.

Suppose now that $\{u_k\}_{k=1}^\infty \subseteq Bd_0^{s,p,H}(\Omega;\mathbb{R}^N)$ is such that $u_k \rightharpoonup u$ in $Bd_0^{s,p,H}(\Omega;\mathbb{R}^N)$. Define $L := \liminf_{k\to\infty} J[u_k]$ which is finite or we are done. So possibly taking a subsequence, for each $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $J[u_k] \leq L+\epsilon$. By Mazur's Theorem and the boundary condition placed on the $\Omega_H\backslash\Omega$ there exists $\{v_\ell\}_{\ell=1}^\infty \subseteq$ co $\{u_k\}_{k=K}^\infty$ such that

1. $v_\ell \to u$ strongly in $Bd_0^{s,p,H}(\Omega;\mathbb{R}^N)$,

2. $v_\ell = \sum_{r=K}^{m_\ell} \alpha_{\ell_r} u_r$ with $\alpha_{\ell_r} \geq 0$ and $\sum_{r=K}^{m_\ell} \alpha_{\ell_r} = 1$.

By the convexity of F, we have

$$J[v_\ell] = J\left[\sum_{r=K}^{m_\ell} \alpha_{\ell_r} u_r\right] \leq \sum_{r=K}^{m_\ell} \alpha_{\ell_r} J[u_r] \leq \sum_{r=K}^{m_\ell} \alpha_{\ell_r}(L+\epsilon) \leq L+\epsilon.$$

The lower semicontinuity of $J[\cdot]$ and the strong convergence of $\{v_k\}$ to u in $Bd_0^{s,p,H}(\Omega;\mathbb{R}^n)$ gives

$$J[u] \leq \liminf_{j\to\infty} J[v_j] \leq L+\epsilon.$$

As $\epsilon > 0$ was arbitrary, we have $J[u] \leq \liminf_{k\to\infty} J[u_k]$.

In order to establish the result for weakly convergent sequences in $Bd_g^{s,p,H}(\Omega;\mathbb{R}^N)$, we note that with the substitution the following functionals are equivalent:

$$\int_\Omega \int_0^H F(x, w+g, \mathscr{G}(w+g))\, dh\, dx \tag{2.11}$$

and

$$\int_\Omega \int_0^H F(x, u, \mathscr{G}(u))\, dh\, dx. \tag{2.12}$$

Since $g \in B^{s,p}(\Omega_H;\mathbb{R}^N)$ is fixed, we can establish the lower semicontinuity with respect to $w \in Bd_0^{s,p,H}(\Omega;\mathbb{R}^N)$ for (2.11) if we show the following functional is lower semicontinuous with respect to w:

$$\int_\Omega \int_0^H F(x, w+g, \mathscr{G}(w+g)) + a|\mathscr{G}g|^p + b|g|^q\, dh\, dx. \tag{2.13}$$

As (2.13) satisfies the coercivity condition and $w \in Bd_0^{s,p,H}(\Omega;\mathbb{R}^N)$, we can appeal to the lower semicontinuity result previously established for zero boundary conditions to obtain the lower semicontinuity of (2.13). Hence we have established the lower semicontinuity of (2.11) with respect to w and, through the above discussion of equivalence, the lower semicontinuity of (2.12) with respect to u. □

One can also show a similar result for the K-functional. In particular, we establish the following:

Theorem 2.3.2. *Let $\Omega \subseteq \mathbb{R}^n$ and $\{u_k\}_{k=1}^{\infty} \subseteq Bd_g^{s,p,H}(\Omega;\mathbb{R}^N)$ be such that $u_k \rightharpoonup u$ in $B_g^{s,p}(\Omega;\mathbb{R}^N)$, where $g \in B^{s,p}(\Omega_H;\mathbb{R}^N)$. Assume $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function which is convex with respect to $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$ and satisfies*

$$F(x,u,\xi) \geq a|\xi|^p + b|u|^q + c(x)$$

for almost every $x \in \Omega$ and for every $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$. Here we assume $a > 0$ and $c \in L^1(\Omega;\mathbb{R}^N)$.

I. If $b \geq 0$ and $p,q \geq 1$, then

$$K[u] \leq \liminf_{k\to\infty} K[u_k].$$

II. Assume $H > H^*$, $\frac{-a}{2HC^{**}} < b < 0$, and $1 < q = p \leq \frac{np}{n-sp}$. Then $K[\cdot]$ is again weakly lower semicontinuous.

Proof. We note that once again we can assume without loss of generality that $c(x) = 0$ almost everywhere in Ω. Furthermore, we assume $u_k \to u$ strongly in $Bd_0^{s,p,H}(\Omega;\mathbb{R}^N)$ as previously done. Note that by the convexity of the integrand, we have

$$K[u_k] - K[u] \geq 2\int_\Omega \int_{|h|\leq H} F\left(x, \frac{1}{2}u_k - \frac{1}{2}u, \left[\frac{1}{2}\Lambda u_k - \frac{1}{2}\Lambda u\right] h^{-1/p}\right) dh\, dx.$$

As in the proof for the J-functional, we are able to apply the Fubini-Tonelli theorem to obtain

$$K[u_k] - K[u] \geq \frac{a}{2^{p-1}}\left(\frac{1}{2}|u_k - u|^p_{B^{s,p}(\Omega;\mathbb{R}^N)} + \frac{bH}{a}\,\|u_k - u\|^p_{L^p(\Omega;\mathbb{R}^N)}\right).$$

If at this point $b \geq 0$, then we can discard the second term on the right-hand side of the above inequality. If however, $H > H^*$ and $\frac{-a}{2HC^{**}} < b < 0$, we can use Lemma 2.2.2 to show that for some $C_0 > 0$, we have

$$K[u_k] - K[u] \geq C_0|u_k - u|^p_{B^{s,p}(\Omega;\mathbb{R}^N)}.$$

So in either case, we have

$$\lim_{k\to\infty} K[u_k] - K[u] \geq 0.$$

Thus $K[\cdot]$ is also sequentially lower semicontinuous.

As in the proof for the J-functional, we now need to extend this result to weakly convergent sequences. Given a weakly convergent sequence in $Bd_0^{s,p,H}(\Omega;\mathbb{R}^N)$, we can again find a strongly convergent sequence $\{v_\ell\}$ in the convex hull of the tail of $\{u_k\}$ such that $v_k \to u$ strongly and $K[v_k] \leq L + \epsilon$ for all $\epsilon > 0$, where $L := \liminf_{k\to\infty} K[u_k]$. So by the lower semicontinuity of J and the strong convergence of $\{v_k\}$, we have $K[u] \leq L + \epsilon$. As $\epsilon > 0$ was arbitrary, we have established the lower semicontinuity of $K[\cdot]$.

Finally, one can affirm the lower semicontinuity of the K-functional with respect to sequences in $Bd_g^{s,p,H}(\Omega;\mathbb{R}^N)$ using the same substitution, $w := u - g$, as in the proof of the J-functional. Of course, one must also change the fractional order difference quotient to Λg instead of $\mathscr{G}$ when modifying the functional to obtain the lower semicontinuity of the modified functional so that the modification makes sense in this setting.

□

2.3.2 EXISTENCE

Having established the lower semicontinuity results for the J-functional and K-functional, we are now in a position to apply the direct method in order to deduce the existence of minimizers for both functionals. We will, again in this section, provide detailed proofs for the J-functional while outlining the proofs for the K-functional only to establish the necessary notational changes that must occur.

Theorem 2.3.3. *Let $\Omega \subseteq \mathbb{R}^n, g \in B^{s,p}(\Omega_H;\mathbb{R}^N)$, and $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}$ is a Caratheodory function which is convex with respect to $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$ and satisfies*

$$F(x,u,\xi) \geq a|\xi|^p + b|u|^q + c(x) \tag{2.14}$$

for almost every $x \in \Omega$ and for every $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$. Here we assume $a > 0$ and $c \in L^1(\Omega;\mathbb{R}^N)$.

I. If $b \geq 0$ and $p, q > 1$, then $J[\cdot]$ has a minimizer in $\mathscr{A}_s := Bd_g^{s,p,H}(\Omega;\mathbb{R}^N)$.

II. Assume $H > H^*$, $\frac{-a}{2HC^{**}} < b < 0$, and $1 < q = p \leq \frac{np}{n-sp}$. Then $J[\cdot]$ has a minimizer in $\mathscr{A}_s := Bd_g^{s,p,H}(\Omega;\mathbb{R}^N)$.

III. Furthermore, the minimizer is unique in both cases provided F is strictly convex with respect to $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^{Nn}$.

In what follows, we will establish the existence of minimizers over the admissible class in which $g \equiv 0$ on $\Omega_H \backslash \Omega$. We will then establish the existence of minimizers in the case of nonzero boundary values by modifying the functional as in the result for the lower semicontinuity argument and appealing to the result for zero boundary values.

Proof. If $\inf\limits_{u\in\mathscr{A}_s} J[u] = \infty$, then any $u \in \mathscr{A}_s$ will be an acceptable minimizer. So we suppose that $\inf\limits_{u\in\mathscr{A}_s} J[u] = \ell < \infty$. The coercivity condition then shows that $\ell > -\infty$. Let $\{u_k\}_{k=1}^{\infty} \subset \mathscr{A}_s$ be a minimizing sequence of J. Without loss of generality, we can take $c(x) = 0$ in the hypotheses for the same reason stated in the proof for lower semicontinuity. Then by the coercivity condition, we have

$$\begin{aligned} J[u_k] &\geq \int_\Omega \int_0^H a|\mathscr{G}u_k(x,h)|^p \frac{1}{h} + b|u_k|^q \, dx \, dh \\ &= \int_0^H \int_\Omega a|\mathscr{G}u_k(x,h)|^p \frac{1}{h} + b|u_k|^q \, dh \, dx \\ &= \frac{a}{2^{p-1}} \left(\frac{1}{2} |u_k|^p_{B^{s,p}(\Omega_H;\mathbb{R}^N)} + \frac{bH}{a} \|u_k\|^q_{L^q(\Omega;\mathbb{R}^N)} \right). \end{aligned} \tag{2.15}$$

If at this point $b > 0$, we can use Hölders inequality to obtain

$$\begin{aligned} J[u_k] &\geq \frac{a}{2^{p-1}} \left(\frac{1}{2} |u_k|^p_{B^{s,p}(\Omega_H;\mathbb{R}^N)} + \frac{bH}{a} \|u_k\|^q_{L^q(\Omega;\mathbb{R}^N)} \right) \\ &\geq \frac{a}{2^{p-1}} \left(|u_k|^p_{B^{s,p}(\Omega_H;\mathbb{R}^N)} + \frac{bH}{a} \|u_k\|^p_{L^p(\Omega:\mathbb{R}^N)} \right). \end{aligned}$$

As $J[u_k] \to \ell < \infty$, $\sup_{k\in\mathbb{N}} \|u_k\|_{B^{s,p}(\Omega_H;\mathbb{R}^N)} < \infty$. In the other case, namely when $H > H^*$ and $0 \geq b > \frac{-a}{2HC^{**}}$, we apply Lemma 2.2.1 to the L^q-norm on right-hand side of (2.15), which is actually an L^p-norm in this setting, to obtain

$$J[u_k] \geq C_0 |u_k|^p_{B^{s,p}(\Omega_H;\mathbb{R}^N)},$$

where $C_0 > 0$. Hence the Besov seminorms of the sequence $\{u_k\}$ are uniformly bounded. We then use Lemma 2.2.1 to show $\{u_k\}$ are uniformly bounded in $L^p(\Omega_H;\mathbb{R}^N)$. In either case, we have shown $\{u_k\}$ is a uniformly bounded sequence in $B^{s,p}(\Omega_H;\mathbb{R}^N)$. Since $B^{s,p}(\Omega_H;\mathbb{R}^N)$ is reflexive, there exists $u \in B^{s,p}(\Omega_H;\mathbb{R}^N)$ such that, possibly taking a subsequence, which we will not relabel, $u_k \rightharpoonup u$ in $B^{s,p}(\Omega_H;\mathbb{R}^N)$. We now need to show that $u \in \mathscr{A}_s$. Note that $Bd_0^{s,p,H}(\Omega;\mathbb{R}^N)$ is a closed linear subspace of $B^{s,p}(\Omega_H;\mathbb{R}^N)$, and thus $\mathscr{A}_s$ is weakly closed by Mazur's Theorem. Hence, $u \in \mathscr{A}_s$. So by the weak lower semicontinuity of J, we have

$$J[u] \leq \liminf_{k\to\infty} J[u_k] \leq J[u].$$

Therefore $u \in \mathscr{A}_s$ is a minimizer of $J[\cdot]$. One can then show uniqueness in the usual way when given that F is stricly convex.

Finally, one can show the prove the existence of minimizers when nonzero boundary conditions are present by modifying the functional in a similar manner to the lower semicontinuity proof and modifying the minimizing class. We begin by taking $w = u - g$ and modifying the minimization problem as follows:

$$\begin{aligned} &\text{minimize } \int_\Omega \int_0^H F(x, w+g, \Lambda(w+g)) + \frac{a}{2}|\Lambda g|^p + \frac{b}{2}|g|^q \, dh \, dx, \\ &\text{subject to } w \in Bd_0^{s,p,H}(\Omega;\mathbb{R}^N). \end{aligned} \tag{2.16}$$

Since $g \in Bd^{s,p}(\Omega_H;\mathbb{R}^N)$ is fixed, the solutions of the above problem also minimize the functional

$$\int_\Omega \int_0^H F(x, w+g, \Lambda(w+g)) \, dh \, dx$$

over the same admissible class. We only subtracted the last two terms in order to meet the coercivity condition so that we are able to apply the result for zero boudary conditions. However, the above functional and its corresponding minimization problem over the admissible class with zero boundary conditions is equivalent to the orginal functional and its minimization problem over the admissible class with $u = g$ on $\Omega_H \backslash \Omega$. Therefore, appealing to the existence result for zero boundary conditions to solve (2.16) in order to deduce the existence o minimizers provides the existence of minimizers for $J[\cdot]$ over the admissible class with nonzero boundary conditions. $\square$

Similarly, we have the following result for the K-functional.

Theorem 2.3.4. *Let $\Omega \subseteq \mathbb{R}^n, g \in B^{s,p}(\Omega_H;\mathbb{R}^N)$, and $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a Caratheodory function which is convex with respect to $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$ and satisfies*

$$F(x,u,\xi) \geq a|\xi|^p + b|u|^q + c(x)$$

for almost every $x \in \Omega$ and for every $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$. Here we assume $a > 0$ and $c \in L^1(\Omega;\mathbb{R}^N)$.

I. If $b \geq 0$ and $p,q > 1$, then $K[\cdot]$ has a minimizer in $\mathscr{A}_s := Bd_g^{s,p,H}(\Omega;\mathbb{R}^N)$.

II. Assume $H > H^*$, $\frac{-a}{2HC^{**}} < b < 0$, and $1 < q = p \leq \frac{np}{n-sp}$. Then $K[\cdot]$ has a minimizer in $\mathscr{A}_s := Bd_g^{s,p,H}(\Omega;\mathbb{R}^N)$.

III. Furthermore, the minimizer is unique in both cases provided F is strictly convex with respect to $(u,\xi) \in \mathbb{R}^N \times \mathbb{R}^N$.

Proof. As in the proof of the J-functional, we can assume without loss of generality that $\inf_{u\in\mathscr{A}} K[u] = \ell < \infty$ and $c(x) = 0$. So by the coercivity condition, one can obtain

$$K[u_k] \geq \frac{ca}{2^{p-1}}\left(\frac{1}{2}|u_k|^p_{B^{s,p}(\Omega_H;\mathbb{R}^N)} + \frac{bH}{a}\left\|u_k\right\|^q_{L^q(\Omega;\mathbb{R}^N)}\right).$$

We can then apply Hölders inequality, if $b > 0$, or Lemma 2.2.2, if $-\frac{-a}{2HC^{**}} < b \leq 0$ and $H > H^*$, to the second term on the right-hand side in order to obtain the lower bound

$$K[u_k] \geq C_0|u_k|^p_{B^{s,p}(\Omega_H;\mathbb{R}^N)}.$$

Thus, we can deduce the existence of the minimizer in $\mathscr{A}_s$ just as we did in the proof of the J-functional by using the lower semicontinuity of the K-functional instead. Again, uniqueness is established using the same proof by contradiction that is used in the local setting.

Finally, one can show the prove the existence of minimizers when nonzero boundary conditions are present by taking $w = u - g$ on Ω_H and appealing to the result for problems with zero boundary conditions to deduce minimizers of the equivalent minimization problem :

$$\text{minimize } \int_\Omega \int_{|h|\leq H} F(x, w+g, \Lambda(w+g)) + \frac{a}{2}|\Lambda g|^p + \frac{b}{2}|g|^q \, dh \, dx,$$
$$\text{subject to } w = 0 \quad \text{on } \Omega_H \backslash \Omega.$$

□

2.4 HIGHER FRACTIONAL DIFFERENTIABILITY

In this section, we consider the functionals

$$J[u] = \int_\Omega \int_0^H F(x, \mathscr{G}u(x,h)h^{-1/2}) \, dh \, dx,$$

and

$$K[u] = \int_\Omega \int_{|h|\leq|H|} F(x, \Lambda u(x,h)|h|^{-1/2}) \, dh \, dx,$$

where we have taken $p = 2$. Recall from the introduction of this chapter that taking $p = 2$ corresponds to the quadratic growth assumptions we assume for the integrand. In what follows we investigate the regularity of minimizers for these two functionals, which is provided in the form of an increase on the order of the Besov space to which the minimizer belongs. We will first show the results for the J-functional as the necessary lemmas are already established in this setting. We will then show that similar results hold for the K-functional.

2.4.1 THE J-FUNCTIONAL

We now proceed to prove the higher differentiability result for the J-functional which can be stated as follows:

Theorem 2.4.1. *Let $F : \Omega \times \mathbb{R}^{Nn} \to \mathbb{R}$ be a coercive function that is $C^2(\Omega \times \mathbb{R}^{Nn})$, uniformly convex with respect to $\xi \in \mathbb{R}^{Nn}$, and quadratic in growth with respect to $\xi \in R^{Nn}$, i.e.*

$$F(x,\xi) \geq a|\xi|^2 + c(x), \quad \textit{for some } a > 0, c(x) \in L^1(\Omega;\mathbb{R}^N),$$

$$F_{\xi\xi}(x,\xi)\theta \cdot \theta \geq \nu|\theta|^2, \quad \textit{for all } \theta > 0 \textit{ and almost every } x \in \Omega,$$

$$|F_{\xi\xi}(x,\xi)| \leq M, \quad \textit{for all } \xi \in \mathbb{R}^{Nn} \textit{ and almost every } x \in \Omega,$$

$$|F_{\xi x}(x,\xi)| \leq M(1+|\xi|), \quad \textit{for all } \xi \in \mathbb{R}^{Nn} \textit{ and almost every } x \in \Omega.$$

Further let $\mathscr{A}_s = Bd_g^{s,2,H}(\Omega;\mathbb{R}^N)$, where $g \in C^\infty(\Omega_H;\mathbb{R}^N)$ for some $0 < s < 1$, and assume $u \in \mathscr{A}_s$ satisfies

$$J[u] = \inf_{f \in B^{s,2}(\Omega;\mathbb{R}^N)} J[f] < \infty.$$

Then $u \in B^{2s,2}_{loc}(\Omega;\mathbb{R}^N)$ if $s < 1/2$, and $u \in B^{\alpha,2}_{loc}(\Omega;\mathbb{R}^N)$ for any $0 < \alpha < 1$ if $1/2 \leq s < 1$. Furthermore, we need only assume $g \in B^{2s,2}(\Omega_H;\mathbb{R}^N)$ if Ω is an extension domain in $B^{2s,2}_{loc}(\Omega;\mathbb{R}^N)$ when $s < 1/2$ and in $B^{\alpha,2}_{loc}(\Omega;\mathbb{R}^N)$ for all $\alpha \in (0,1)$ when $s \geq 1/2$.

Proof. Let $u \in \mathscr{A}_s$, for some $0 < s < 1$, be an infimum of $J[\cdot]$. Since $u = g$ on $\Omega_H \backslash \Omega$, we can define $w := u - g$ and note that w must minimize the following functional:

$$J_g[v] := \int_\Omega \int_0^H F(x, \mathscr{G}\{v+g\}(x,h)h^{-1/2})\, dh\, dx,$$

over all $v \in Bd_0^{s,p,H}(\Omega; \mathbb{R}^N)$. Since w is the minimizer of the above functional, w satisfies the following analogue of the weak Euler-Langrange Equations:

$$\int_\Omega \int_0^H h^{-1/2} F_\xi \left(x, \mathscr{G}\{w+g\}(x,h)h^{-1/2}\right) \cdot \left(\frac{\varphi(x+he_j) - \varphi(x)}{h^s} \right) dh\, dx = 0, \quad (2.17)$$

for all $\varphi \in Bd_0^{s,2,H}$. Since φ has compact support, we can test (2.17) with $\varphi(x - \ell e_k, h)$ for ℓ small enough and apply a change of variables to show

$$\int_\Omega \int_0^H h^{-1/2} F_\xi \left(x + \ell e_k, \mathscr{G}\{w+g\}(x+\ell e_k,h)h^{-1/2}\right) \cdot \left(\frac{\varphi(x+he_j) - \varphi(x)}{h^s} \right) dhdz = 0.$$

From the previous two equations, we can add and subtract a term to obtain

$$\begin{aligned}\int_\Omega \int_0^H &\Big\{ F_\xi \left(x + \ell e_k, \mathscr{G}\{w+g\}(x+\ell e_k,h)h^{-1/2}\right) \\ &\quad - F_\xi \left(x, \{\mathscr{G}w(x+\ell e_k) + \mathscr{G}g(x,h)\}h^{-1/2}\right) \Big\} \cdot \mathscr{G}\varphi(x,h)h^{-1/2}\, dh\, dx \\ + \int_\Omega \int_0^H &\Big\{ F_\xi \left(x, \{\mathscr{G}w(x+\ell e_k) + \mathscr{G}g(x,h)\}h^{-1/2}\right) \\ &\quad - F_\xi \left(x, \mathscr{G}\{w+g\}(x,h)h^{-1/2}\right) \Big\} \cdot \mathscr{G}\varphi(x,h)h^{-1/2}\, dh\, dx = 0,\end{aligned}$$

which, by way of Leibniz's Rule, can be rewritten as

$$\int_\Omega \int_0^H \int_0^1 \frac{d}{dt} \big[F_\xi \big(x, \big\{ \mathscr{L}(t, \mathscr{G}w(x,h))$$

$$+\mathscr{G}g(x,h)\big\}h^{-1/2}\big)\big]\cdot\mathscr{G}\varphi(x,h)h^{-1/2}\,dt\,dh\,dx$$

$$=-\int_\Omega\int_0^H\int_0^1\frac{d}{dr}\big[F_\xi\big(x+r\ell e_k,\big\{\mathscr{G}w(w+\ell e_k,h)+\mathscr{G}g(x,h)$$

$$+r\big(\mathscr{G}g(x+\ell e_k,h)-\mathscr{G}g(x,h)\big)\big\}h^{-1/2}\big)\big]\cdot\mathscr{G}\varphi(x,h)h^{-1/2}\,dr\,dh\,dx.$$

Here $\mathscr{L}(s,f(x,h)):=f(x,h)+s(f(x+\ell e_k,h)-f(x,h))$. Then computing the derivatives in the previous equation gives

$$\begin{aligned}
&\int_\Omega\int_0^H\int_0^1 F_{\xi\xi}\big(x,\big\{\mathscr{L}(t,\mathscr{G}w(x,h))+\mathscr{G}g(x,h)\big\}h^{-1/2}\big)\\
&\qquad\times(\mathscr{G}w(x+\ell e_k,h)-\mathscr{G}w(x,h))\cdot\mathscr{G}\varphi(x,h)h^{-1}\,dt\,dh\,dx\\
&=-\int_\Omega\int_0^H\int_0^1 F_{\xi x}\big(x+r\ell e_k,\big\{\mathscr{G}w(x+\ell e_k,h)+\mathscr{L}(r,\mathscr{G}g(x,h))\big\}h^{-1/2}\big)\\
&\qquad\cdot\ell e_k\mathscr{G}\varphi(x,h)h^{-1}\,dr\,dh\,dx \qquad (2.18)\\
&\quad+\int_\Omega\int_0^H\int_0^1 F_{\xi\xi}\big(x+r\ell e_k,\big\{\mathscr{G}w(x,h)+\mathscr{L}(r,\mathscr{G}g(x,h))\big\}h^{-1/2}\big)\\
&\qquad\times(\mathscr{G}g(x+\ell e_k,h)-\mathscr{G}g(x,h))\cdot\mathscr{G}\varphi(x,h)h^{-1}\,dr\,dh\,dx.
\end{aligned}$$

Let $V\subset\subset\Omega$, which means there exists $U\subset\Omega$ such that $V\subset U\subset\Omega$. Define $L:=\min\left\{H,\frac{\operatorname{dist}(V,\Omega_H)}{2}\right\}$. Hence, $V_{H+L}\subset\Omega_H$. Take $\varphi(x,h)=\eta^2(x)\mathscr{G}w(x,\ell)\ell^{s-2\beta}$, where $0<\beta\le s$ and $\eta(x)\in C_0^\infty(\mathbb{R}^n)$ satisfies

$$\eta(x)=\begin{cases}1 & \text{on } V_H,\\ 0 & \text{on } \Omega_H\backslash U_H.\end{cases}$$

Now note that we can rewrite the fractional difference quotient of φ to obtain

$$\begin{aligned}\mathscr{G}\varphi(x,h) =&\eta^2(x+he_j)\left(\frac{\mathscr{G}w(x+\ell e_k,h)-\mathscr{G}w(x,h)}{\ell^{2\beta}}\right)\\&+\mathscr{G}(\eta^2)(x,h)\left(\frac{\mathscr{G}w(x,\ell)}{\ell^{2\beta-s}}\right).\end{aligned}$$

Subsitituting this into (2.18), gives

$$\begin{aligned}&\int_\Omega\int_0^H\int_0^1(\eta^2(x+he_j)F_{\xi\xi}\Big(x,\Big\{\mathscr{L}(t,\mathscr{G}w(x,h))+\mathscr{G}g(x,h)\Big\}h^{-1/2}\Big)\\&\quad\Big(\mathscr{G}w(x+\ell e_k,h)-\mathscr{G}w(x,h))\Big)\cdot\left(\frac{\mathscr{G}w(x+\ell e_k,h)-\mathscr{G}w(x,h)}{\ell^{2\beta}}\right)\,dt\,\frac{dh}{h}\,dx\\&=-\int_\Omega\int_0^H\int_0^1F_{\xi\xi}\big(x,\{\mathscr{L}(t,\mathscr{G}w(x,h))+\mathscr{G}g(x,h)\}h^{-1/2}\big)\\&\quad\Big(\mathscr{G}w(x+\ell e_k,h)-\mathscr{G}w(x,h)\Big)\cdot\mathscr{G}(\eta^2)(x,h)\left(\frac{\mathscr{G}w(x,\ell)}{\ell^{2\beta-s}}\right)\,dt\,\frac{dh}{h}\,dx\qquad(2.19)\\&\quad-\int_\Omega\int_0^H\int_0^1F_{\xi x}\big(x+r\ell e_k,\{\mathscr{G}w(x+\ell e_k,h)+\mathscr{L}(r,\mathscr{G}g(x,h))\}h^{-1/2}\big)\ell e_k\\&\quad\Bigg[\eta^2(x+he_j)\bigg(\frac{\mathscr{G}w(x+\ell e_k,h)-\mathscr{G}w(x,h)}{\ell^{2\beta}}\bigg)\\&\qquad\qquad+\mathscr{G}(\eta^2)(x,h)\bigg(\frac{\mathscr{G}w(x,\ell)}{\ell^{2\beta-s}}\bigg)\Bigg]h^{-1/2}\,dr\,dh\,dx\\&\quad+\int_\Omega\int_0^H\int_0^1F_{\xi\xi}\big(x+r\ell e_k,\{\mathscr{G}w(x,h)+\mathscr{L}(r,\mathscr{G}g(x,h))\}h^{-1/2}\big)\\&\quad(\mathscr{G}g(x+\ell e_k,h)-\mathscr{G}g(x,h))\cdot\Bigg[\eta^2(x+he_j)\bigg(\frac{\mathscr{G}w(x+\ell e_k,h)-\mathscr{G}w(x,h)}{\ell^{2\beta}}\bigg)\\&\qquad+\mathscr{G}(\eta^2)(x,h)\bigg(\frac{\mathscr{G}w(x,\ell)}{\ell^{2\beta-s}}\bigg)\Bigg]\,dr\,\frac{dh}{h}\,dx\\&=:I+II+III.\end{aligned}$$

By the convexity of F, the integral on the left-hand side is bounded below by

$$\nu\sum_{i,j}\int_\Omega\int_0^H\eta^2(x+he_j)\left|\frac{\mathscr{G}^i_ju(x+\ell e_k,h)-\mathscr{G}^i_ju(x,h)}{\ell^\beta}\right|^2\frac{dh}{h}\,dx.$$

By rewriting the the first term on the right-hand side equation 2.19, invoking the boundedness condition on $F_{\xi\xi}$, and using young's inequality, we find

$$
\begin{aligned}
I =& -2\int_\Omega \int_0^H \int_0^1 \eta(x+he_j) F_{\xi\xi}\big(x, \{\mathscr{L}(t,\mathscr{G}w(x,h)) + \mathscr{G}g(x,h)\}h^{-1/2}\big) \\
&\left(\frac{\mathscr{G}w(x+\ell e_k,h)-\mathscr{G}w(x,h)}{\ell^\beta}\right)\cdot \mathscr{G}\eta(x,h)\left(\frac{\mathscr{G}w(x,\ell)}{\ell^{\beta-s}}\right) dt\,\frac{dh}{h}\,dx \\
&+ \int_\Omega \int_0^H \int_0^1 F_{\xi\xi}\big(x, \{\mathscr{L}(t,\mathscr{G}w(x,h)) + \mathscr{G}g(x,h)\}h^{-1/2}\big) \\
&\left(\frac{\mathscr{G}w(x+he_j,\ell)-\mathscr{G}w(x,\ell)}{\ell^{\beta-s}}\right)\cdot (\mathscr{G}\eta(x,h))^2\left(\frac{\mathscr{G}w(x,\ell)}{\ell^{\beta-s}}\right) dt\,\frac{dh}{h}\,dx \\
\le& c(\epsilon,M)\int_\Omega\int_0^H \eta^2(x+he_j)\left|\frac{\mathscr{G}w(x+\ell e_k,h)-\mathscr{G}w(x,h)}{\ell^\beta}\right|^2\frac{dh}{h} \\
&+ c(\epsilon^{-1},M)\int_\Omega\int_0^H \left|\frac{\mathscr{G}w(x,\ell)}{\ell^{\beta-s}}\right|^2 (\mathscr{G}\eta(x,h))^2\,\frac{dh}{h}\,dx \\
&+ M\int_\Omega\int_0^H \left|\frac{\mathscr{G}w(x+he_j,\ell)}{\ell^{\beta-s}}\right|^2 (\mathscr{G}\eta(x,h))^2\,\frac{dh}{h}\,dx.
\end{aligned}
$$

Notice that since $0<\beta<s$ and $0<\ell<1$, we can bound the last two integrals in order to obtain the following, where we have imposed a change of variables on the last integral:

$$
\begin{aligned}
I \le& c(\epsilon,M)\int_\Omega\int_0^H \eta^2(x+he_j)\left|\frac{\mathscr{G}w(x+\ell e_k,h)-\mathscr{G}w(x,h)}{\ell^\beta}\right|^2\frac{dh}{h} \\
&+ c(\epsilon^{-1},M)\,\|D\eta\|^2_{L^\infty(\Omega;\mathbb{R})}\int_\Omega |\mathscr{G}w(x,\ell)|^2\,dx
\end{aligned}
$$

By a similar argument, we can also show

$$
\begin{aligned}
II \le& c(\epsilon,M)\int_\Omega\int_0^H \eta^2(x+he_j)\left|\frac{\mathscr{G}w(x+\ell e_k,h)-\mathscr{G}w(x,h)}{\ell^\beta}\right|^2\frac{dh}{h}\,dx \\
&+ c(\epsilon^{-1},M)\int_\Omega\int_0^H \eta^2(x+he_j)\left|\frac{\mathscr{G}g(x+\ell e_k,h)-\mathscr{G}g(x,h)}{\ell^\beta}\right|^2\frac{dh}{h}\,dx \\
&+ M\,\|D\eta\|^2_{L^\infty(\Omega;\mathbb{R})}\int_\Omega |\mathscr{G}w(x,\ell)|^2\,dx.
\end{aligned}
$$

For III, we use the quadratic growth assumption and then Young's ineqaulity to obtain the following bound:

$$\begin{aligned}
III \leq & c(M) \int_\Omega \int_0^H |\mathscr{G}w(x+\ell e_k,h) + \mathscr{G}g(x,h) + \mathscr{G}g(x+\ell e_k,h)|\ell \\
& \qquad \Bigg[\eta(x+he_j)\bigg(\frac{\mathscr{G}w(x+\ell e_k,h) - \mathscr{G}w(x,h)}{\ell^{2\beta}}\bigg) \\
& \qquad + \bigg(\frac{\mathscr{G}w(x,\ell)}{\ell^{2\beta-s}}\bigg)\mathscr{G}\eta(x,h)|\eta(x+he_j)+\eta(x)|\Bigg] \frac{dh}{h}\,dx \\
\leq & c(\epsilon,M) \int_\Omega \int_0^H \eta^2(x+he_j)\bigg|\frac{\mathscr{G}w(x+\ell e_k,h) - \mathscr{G}w(x,h)}{\ell^{\beta}}\bigg|^2 \frac{dh}{h}\,dx \\
& + c(\epsilon^{-1},M) \int_\Omega \int_0^H \bigg|\frac{\mathscr{G}w(x,\ell)}{\ell^{\beta-s}}\bigg|^2 (\mathscr{G}\eta(x,h))^2 \frac{dh}{h}\,dx \\
& + c(\epsilon^{-1},M) \int_\Omega \int_0^H |\eta(x+he_j)+\eta(x)|^2 |\mathscr{G}w(x+\ell e_k,h)|^2 |\ell^{1-\beta}|^2 \frac{dh}{h}\,dx \\
& + c(\epsilon^{-1},M) \int_\Omega \int_0^H |\eta(x+he_j)+\eta(x)|^2 |\mathscr{G}g(x,h)|^2 |\ell^{1-\beta}|^2 \frac{dh}{h}\,dx \\
& + c(\epsilon^{-1},M) \int_\Omega \int_0^H |\eta(x+he_j)+\eta(x)|^2 |\mathscr{G}g(x+\ell e_k,h)|^2 |\ell^{1-\beta}|^2 \frac{dh}{h}\,dx \\
\leq & III_1 + III_2 + III_3 + III_4 + III_5.
\end{aligned}$$

As before, we have

$$\begin{aligned}
III_1 + III_2 \leq & c(\epsilon,M) \int_\Omega \int_0^H \eta^2(x+he_j)\bigg|\frac{\mathscr{G}w(x+\ell e_k,h) - \mathscr{G}w(x,h)}{\ell^{\beta}}\bigg|^2 \frac{dh}{h} \\
& + c(\epsilon^{-1},M)\, \|D\eta\|^2_{L^\infty(\Omega;\mathbb{R})} \int_\Omega |\mathscr{G}w(x,\ell)|^2\,dx.
\end{aligned}$$

Using a change of variables on III_3 and III_5, we find

$$\begin{aligned}
& III_3 + III_5 \\
\leq & c(\epsilon^{-1},M) \int_\Omega \int_0^H |\eta(x-\ell e_k+he_j)+\eta(x-\ell e_k)|^2 |\mathscr{G}w(x,h)|^2 |\ell^{1-\beta}|^2 \frac{dh}{h}\,dx
\end{aligned}$$

$$+ c(\epsilon^{-1}, M) \int_\Omega \int_0^H |\eta(x - \ell e_k + he_j) + \eta(x - \ell e_k)|^2 |\mathscr{G} g(x,h)|^2 |\ell^{1-\beta}|^2 \,\frac{dh}{h}\,dx.$$

So now

$$\begin{aligned} III_3 + III_4 + III_5 \le & c(\epsilon^{-1}, M)|\ell^{1-\beta}|^2 \int_\Omega \int_0^H |\mathscr{G} w(x,h)|^2 \,\frac{dh}{h}\,dx \\ & + c(\epsilon^{-1}, M)|\ell^{1-\beta}|^2 \int_\Omega \int_0^H |\mathscr{G} g(x,h)|^2 \,\frac{dh}{h}\,dx. \end{aligned}$$

Combining all of our estimates gives

$$\begin{aligned} \nu \sum_{i,j} \int_\Omega \int_0^H & \eta^2(x + he_j) \left| \frac{\mathscr{G}_j^i w(x + \ell e_k, h) - \mathscr{G}_j^i w(x,h)}{\ell^\beta} \right|^2 \frac{dh}{h}\,dx \\ \le & c(\epsilon, M) \int_\Omega \int_0^H \eta^2(x + he_j) \left| \frac{\mathscr{G} w(x + \ell e_k, h) - \mathscr{G} w(x,h)}{\ell^\beta} \right|^2 \frac{dh}{h} \\ & + c(\epsilon^{-1}, M)\, \|D\eta\|^2_{L^\infty(\Omega;\mathbb{R})} \int_\Omega |\mathscr{G} w(x,\ell)|^2 \,dx] \\ & + c(\epsilon^{-1}, M) \int_\Omega \int_0^H \left| \frac{\mathscr{G} g(x + \ell e_k, h) - \mathscr{G} g(x,h)}{\ell^\beta} \right|^2 \frac{dh}{h}\,dx \\ & + c(\epsilon^{-1}, M)\ell^{2-2\beta} \int_\Omega \int_0^H |\mathscr{G} w(x,h)|^2 \,\frac{dh}{h}\,dx \\ & + c(\epsilon^{-1}, M)\ell^{2-2\beta} \int_\Omega \int_0^H |\mathscr{G} g(x,h)|^2 \,\frac{dh}{h}\,dx. \end{aligned}$$

Taking $\epsilon = \frac{\nu}{2M}$ and subtracting the first integral on the right-hand side from both sides of the above equation, we arrive at

$$\begin{aligned} \sum_{i,j} \int_\Omega \int_0^H & \eta^2(x + he_j) \left| \frac{\mathscr{G}_i^j w(x + \ell e_k, h) - \mathscr{G}_i^j w(x,h)}{\ell^\beta} \right|^2 \frac{dh}{h}\,dx \\ \le & c\left(\frac{M}{\nu}\right) \|D\eta\|^2_{L^\infty(\Omega;\mathbb{R})} \int_\Omega |\mathscr{G} w(x,\ell)|^2 \,dx \\ & + c\left(\frac{M}{\nu}\right) \int_\Omega \int_0^H \left| \frac{\mathscr{G} g(x + \ell e_k, h) - \mathscr{G} g(x,h)}{\ell^\beta} \right|^2 \frac{dh}{h}\,dx \end{aligned} \tag{2.20}$$

$$+ c\left(\frac{M}{\nu}\right)\ell^{2-2\beta}\int_\Omega\int_0^H |\mathscr{G}w(x,h)|^2\,\frac{dh}{h}\,dx$$
$$+ c\left(\frac{M}{\nu}\right)\ell^{2-2\beta}\int_\Omega\int_0^H |\mathscr{G}g(x,h)|^2\,\frac{dh}{h}\,dx.$$

Noting that $V_{H+L} \subseteq \Omega_H$, we can now divide by ℓ, integrate from 0 to L over ℓ, and bound the terms involving the smooth function g as we bounded η before to arrive at

$$\begin{aligned}\sum_{i,j,k}\int_\Omega\int_0^L\int_0^H \eta^2(x+he_j)\left|\frac{\mathscr{G}^i_j w(x+\ell e_k,h)-\mathscr{G}^i_j w(x,h)}{\ell^\beta}\right|^2\frac{dh}{h}\,\frac{d\ell}{\ell}\,dx&\\
\leq c\left(\frac{M}{\nu}\right)\|D\eta\|^2_{L^\infty(\Omega;\mathbb{R})}\int_\Omega\int_0^L |\mathscr{G}w(x,\ell)|^2\,\frac{d\ell}{\ell}\,dx&\\
+ c\left(\frac{M}{\nu},L^{2-2\beta}\right)\left[\left\|D^2 g\right\|^2_{L^\infty(\Omega_H;\mathbb{R}^{NnNn})}+\|Dg\|_{L^\infty(\Omega_H;\mathbb{R}^{Nn})}\right]&\\
+ c\left(\frac{M}{\nu},L^{2-2\beta}\right)\int_\Omega\int_0^H |\mathscr{G}w(x,h)|^2\,\frac{dh}{h}\,dx.&\end{aligned}$$

We note that we must take $L < 1$ in order to achieve these bounds from our previous work. Furthermore, the integrands of all the above integrals are positive, so that we can change the order of integration by the Fubini-Tonelli Theorem. Since $0 < L < H$, the integrals in the first and third terms are bounded by the functional evaluated at the minimizer which we assumed was finite. The second term on the right-hand side is finite as g is smooth. Using Fubini-Tonelli once more on the left-hand side, we see

$$\sum_{i,j,k}\int_0^L\int_0^H\int_\Omega \eta^2(x+he_j)\left|\frac{\mathscr{G}^i_j w(x+\ell e_k,h)-\mathscr{G}^i_j w(x,h)}{\ell^\beta}\right|^2\frac{dh}{h}\,dx\,\frac{d\ell}{\ell} < \infty,$$

where the constant in the end depends on s, H, n, N, and M/ν. Recalling the definition of $\eta(\cdot)$, we have

$$\sum_k\left(\int_0^L (\ell^{-2\beta}|\mathscr{G}^i_j w(x,h)|^2_{B^{s,2}(V_H;\mathbb{R}^N)})\,\frac{d\ell}{\ell}\right)^{\frac{1}{2}} < \infty.$$

Since this bounds the norm on $\mathscr{B}^{\beta,2}(\mathscr{B}^{s,2}(V_{H+L};\mathbb{R}^N))$ and $V \subset\subset \Omega$ was arbitrary, it follows from Lemma 2.2.3 that $w \in B^{2s,2}_{loc}(\Omega;\mathbb{R}^N)$ if $s < 1/2$, and $w \in B^{\alpha,2}_{loc}(\Omega;\mathbb{R}^N)$ for any $0 < \alpha < 1$ if $1/2 \leq s < 1$. Now as g is smooth, we further have $u \in B^{2s,2}_{loc}(\Omega;\mathbb{R}^N)$ if $s < 1/2$, and $u \in B^{\alpha,2}_{loc}(\Omega;\mathbb{R}^N)$ for any $0 < \alpha < 1$ if $1/2 \leq s < 1$.

Lastly, we discuss the slight changes needed within the proof in order to establish the regularity result when the assumptions on g are weakened to $g \in B^{2s,2}(\Omega_H;\mathbb{R}^N)$. If Ω is an extension domain, then so is Ω_H. In this case, the term II from (2.20) is less than the iterated norm on the whole of Ω_H. So by Part 2 of Theorem 1 in [9] and the fact that $g \in B^{2s,2}(\Omega_H;\mathbb{R}^N)$, we have

$$\int_\Omega \int_0^H \left| \frac{\mathscr{G}g(x+\ell e_k,h) - \mathscr{G}g(x,h)}{\ell^\beta} \right|^2 \frac{dh}{h}\, dx < \infty. \tag{2.21}$$

Similarly, we can obtain a bound for the last term on the right-hand side of (2.20), although we only need $g \in B^{s,2}(\Omega_H;\mathbb{R}^N)$ for this term. Therefore the result holds for $g \in B^{2s,2}(\Omega;\mathbb{R}^N)$, if Ω is an extension domain. □

2.4.2 THE K-FUNCTIONAL

We will now provide a similar result for the K-functional. However, we first need to establish the analogue of Theorem 2.2.3 for the norm associated with the K-functional. As mentioned before, the following argument follows along the lines of Lemma 2.2.3 in Setion 5.3 of [9].

Lemma 2.4.1. *Let $V \subset \mathbb{R}^n$ be open and bounded, $1 \leq p \leq q \leq \infty$, and $0 < H \leq \infty$. Then,*

$$\|u\|_{B^{s_1+s_2,p,q}(V)} \leq \frac{c}{\nu_n^{1/q}} \|u\|_{\mathscr{B}^{s_2,p,q}(\mathscr{B}^{s_1,p,q}(L^p(V)))} \tag{2.22}$$

for $s_1 + s_2 < 1$. *Here* $c = c(n,q)$.

Proof. We first recall the following inequality from Lemma 4 in [10]

$$\|\Lambda u(x,h)\|_{L^p(V)} \leq \frac{c}{\nu_n |h|^n} \int_{|\eta|\leq|h|} \|\Lambda u(x,\eta)\|_{L^p(V_{-\eta})}\, d\eta. \tag{2.23}$$

Now by using Hölder's inequality on (2.23), we obtain

$$\begin{aligned} \|\Lambda u(x,h)\|_{L^p(V)} &\leq \frac{c}{\nu_n^{\frac{1}{q}}} \left(\frac{1}{|h|^n} \int_{|\eta|\leq|h|} \|\Lambda u(x,\eta)\|^q_{L^p(V_{-\eta})}\, d\eta \right)^{\frac{1}{q}} \\ &\leq \frac{c(n,q)}{\nu_n^{1/q}} \left(\int_{|\eta|\leq|h|} \|\Lambda u(x,\eta)\|^q_{L^p(V_{-\eta})}\, \frac{d\eta}{|\eta|^n} \right)^{\frac{1}{q}}. \end{aligned} \tag{2.24}$$

Now we are ready to show the result. By the previous inequality, we see that

$$\begin{aligned} |u|_{B^{s_1+s_2,p,q}(V)} &\equiv \left(\int_{|h|\leq\infty} \frac{\|\Lambda u(x,h)\|^q_{L^p(V_{-2h})}}{|h|^{q(\ell_1+\ell_2)}}\, \frac{dh}{|h|^n} \right)^{1/q} \\ &\leq c(n,q,\nu_n^{-1/q}) \left(\int_{|h|\leq\infty} \int_{|\eta|\leq|h|} \frac{\left\| \Lambda\Big(\Lambda u(x,h)\Big)(x,\eta) \right\|_{L^p(V_{-\eta-h})}}{|h|^{q(\ell_1+\ell_2)}}\, \frac{d\eta}{|\eta|^n}\, \frac{dh}{|h|^n} \right)^{1/q} \\ &\leq c(n,q,\nu_n^{-1/q}) \left(\int_{|h|\leq\infty} \int_{|\eta|\leq\infty} \frac{\left\| \Lambda\Big(\Lambda u(x,h)\Big)(x,\eta) \right\|_{L^p(V_{-\eta-h})}}{|h|^{q\ell_2}|\eta|^{q\ell_1}}\, \frac{d\eta}{|\eta|^n}\, \frac{dh}{|h|^n} \right)^{1/q} \\ &= c(n,q,\nu_n^{-1/q}) |u|_{\mathscr{B}^{s_2,p,q}(\mathscr{B}^{s_1,p,q}(L^p(V_{2L})))}. \end{aligned}$$

□

We now present the proof of the following higher differentiability result for the K-functional.

Theorem 2.4.2. *Let $F : \Omega \times \mathbb{R}^N \to \mathbb{R}$ be a coercive function that is $C^2(\Omega \times \mathbb{R}^N)$, uniformly convex with respect to $\xi \in \mathbb{R}^N$, and quadratic in growth with respect to $\xi \in \mathbb{R}^N$, i.e.*

$$F(x,\xi) \geq a|\xi|^2 + c(x), \quad \textit{for some } a > 0, c(x) \in L^1(\Omega;\mathbb{R}^N),$$

$$F_{\xi\xi}(x,\xi)\theta \cdot \theta \geq \nu|\theta|^2, \quad \textit{for all } \theta > 0 \textit{ and for almost every } x \in \Omega,$$

$$|F_{\xi\xi}(x,\xi)| \leq M, \quad \textit{for all } \xi \in \mathbb{R}^N \textit{ and almost every } x \in \Omega,$$

$$|F_{\xi x}(x,\xi)| \leq M(1+|\xi|), \quad \textit{for all } \xi \in \mathbb{R}^N \textit{ and almost every } x \in \Omega.$$

Further let $\mathscr{A}_s = Bd_g^{s,2,H}(\Omega;\mathbb{R}^N)$, where $g \in C^\infty(\Omega_H;\mathbb{R}^N)$ for some $0 < s < 1$, and assume $u \in \mathscr{A}_s$ satisfies

$$K[u] = \inf_{f \in B^{s,2}(\Omega)} K[f] < \infty. \tag{2.25}$$

Then $u \in B^{2s,2}_{loc}(\Omega;\mathbb{R}^N)$ if $s < 1/2$, and $u \in B^{\alpha,2}_{loc}(\Omega;\mathbb{R}^N)$ for any $0 < \alpha < 1$ if $1/2 \leq s < 1$. Furthermore, we need only assume $g \in B^{2s,2}(\Omega_H;\mathbb{R}^N)$ if Ω is an extension domain in $B^{2s,2}_{loc}(\Omega;\mathbb{R}^N)$ when $s < 1/2$ and in $B^{\alpha,2}_{loc}(\Omega;\mathbb{R}^N)$ for all $\alpha \in (0,1)$ when $s \geq 1/2$.

Proof. Similarly to the proof of Theorem 2.4.1, we can show that if u satisfies the finiteness condition in (2.25), then $w := u - g$ must satisfy

$$\begin{aligned}
&\int_\Omega \int_{|h|\leq H} \int_0^1 F_{\xi\xi}\big(x, \big\{\Lambda w(x,h) + t(\Lambda w(x+\ell,h) - \Lambda w(x,h)) + \Lambda g(x,h)\big\}|h|^{-n/2}\big) \\
&\qquad \cdot (\Lambda w(x+\ell,h) - \Lambda w(x,h)) \cdot \Lambda\varphi(x,h)|h|^{-n}\, dt\, dh\, dx \\
&= -\int_\Omega \int_{|h|\leq H} \int_0^1 F_{\xi x}\big(x + r\ell, \big\{\Lambda w(x+\ell,h) + r(\Lambda g(x+\ell,h) - \Lambda g(x,h)) \\
&\qquad + \Lambda g(x,h)\big\}|h|^{-n/2}\big)\ell \cdot \Lambda\varphi(x,h)|h|^{-n}\, dr\, dh\, dx \\
&\quad + \int_\Omega \int_{|h|\leq H} \int_0^1 F_{\xi\xi}\big(x + r\ell, \big\{\Lambda w(x,h) + r(\Lambda g(x+\ell,h) - \Lambda g(x,h))
\end{aligned}$$

$$+ \Lambda g(x,h)\big\}|h|^{-n/2}\big)(\Lambda g(x+\ell,h) - \Lambda g(x,h)) \cdot \Lambda\varphi(x,h)|h|^{-n}\,dr\,dh\,dx.$$

Let $V \subset\subset \Omega$. So there exists an open set U such that $V \subset U \subset \Omega$, and with L as defined before, $V_{H+L} \subset \Omega_H$. Choose $\varphi(x,h) = \eta^2 \Lambda u(x,\ell)|\ell|^{s-2\beta}$, where $0 < \beta \leq s$ and $\eta(x) \in C_0^\infty(\Omega_H)$ satisfies

$$\eta(x) = \begin{cases} 1 & \text{on } V_H, \\ 0 & \text{on } \Omega_H \backslash U_H. \end{cases}$$

Then, once again, we can show in a similar fashion to the proof of Theorem 2.4.1 that the following bound holds:

$$\begin{aligned}
&\int_\Omega \int_{|h|\leq H} \eta^2(x+h) \left| \frac{\Lambda w(x+\ell,h) - \Lambda w(x,h)}{|\ell|^\beta} \right|^2 \frac{dh}{|h|^n}\,dx \\
&\quad \leq c\left(\frac{M}{\nu}\right) \|D\eta\|^2_{L^\infty(\Omega;\mathbb{R})} \int_\Omega |\Lambda w(x,\ell)|^2\,dx \\
&\quad\quad + c\left(\frac{M}{\nu}\right) \int_\Omega \int_{|h|\leq H} \left| \frac{\Lambda g(x+\ell e_k,h) - \Lambda g(x,h)}{\ell^\beta} \right|^2 \frac{dh}{|h|^n}\,dx \\
&\quad\quad + c\left(\frac{M}{\nu}\right) |\ell|^{2-2\beta} \int_\Omega \int_{|h|\leq H} |\Lambda w(x,h)|^2 \frac{dh}{h}\,dx \\
&\quad\quad + c\left(\frac{M}{\nu}\right) |\ell|^{2-2\beta} \int_\Omega \int_{|h|\leq H} |\Lambda g(x,h)|^2 \frac{dh}{h}\,dx.
\end{aligned}$$

Now dividing by $|\ell|^n$, integrating from zero to L with respect to ℓ, and bounding terms in a similar manner to those in the proof of the J-functional gives

$$\begin{aligned}
&\int_{|\ell|\leq L} \int_{|h|\leq H} \int_\Omega \eta^2(x+h) \left| \frac{\Lambda w(x+\ell,h) - \Lambda w(x,h)}{|\ell|^\beta} \right|^2 \frac{dh}{|h|^n}\,dx\,\frac{d\ell}{\ell} \\
&\quad \leq c\left(\frac{M}{\nu}\right) \|D\eta\|^2_{L^\infty(\Omega;\mathbb{R})} \int_\Omega \int_{|\ell|\leq L} |\Lambda w(x,\ell)|^2 \frac{d\ell}{\ell}\,dx \\
&\quad\quad + c\left(\frac{M}{\nu}, L^{2-2\beta}\right) \left[\left\|D^2 g\right\|_{L^\infty(\Omega_H,\mathbb{R}^{NnNn})} + \|Dg\|_{L^\infty(\Omega_H,\mathbb{R}^{Nn})} \right]
\end{aligned}$$

$$+ c\left(\frac{M}{\nu}, L^{2-2\beta}\right) \int_{\Omega} \int_{|h| \leq H} |\Lambda w(x,h)|^2 \, \frac{dh}{h} \, dx.$$

We note once again that in order obtain the above inequality, we employed the Fubini-Tonelli Theorem which was admissible since all of the integrands are positive. While the second term on the right-hand side of the above inequality is already bounded because g is smooth, the first and third terms are bounded by our functional evaluated at its minimizer u. Hence these terms are finite as well. Thus we have

$$\int_{|\ell| \leq L} \int_{|h| \leq |H|} \int_{\Omega} \eta^2(x+h) \left| \frac{\Lambda w(x+\ell, h) - \Lambda w(x,h)}{\ell^{\beta}} \right|^2 \frac{dh}{|h|^n} \frac{d\ell}{|\ell|^n} \, dx < \infty,$$

or

$$\left(\int_{|\ell| \leq L} (|\ell|^{-2\beta} |\Lambda w(x,h)|^2_{B^{s,2}(V_L)}) \, \frac{dh}{|\ell|^n} \right)^{1/2} < \infty.$$

As this bounds the norm on $\mathscr{B}^{\beta,2}(\mathscr{B}^{s,2}(V;\mathbb{R}^N))$ and $V \subset\subset \Omega$ was arbitrary, it follows from Lemma 2.4.1 that $w \in B^{2s,2}_{loc}(\Omega;\mathbb{R}^N)$ if $s < 1/2$, and $w \in B^{\alpha,2}_{loc}(\Omega;\mathbb{R}^N)$ for any $0 < \alpha < 1$ if $1/2 \leq s < 1$. Since g is smooth, we have $u \in B^{2s,2}_{loc}(\Omega;\mathbb{R}^N)$ if $s < 1/2$, and $u \in B^{\alpha,2}_{loc}(\Omega;\mathbb{R}^N)$ for any $0 < \alpha < 1$ if $1/2 \leq s < 1$. We again need only assume $g \in B^{2s,s}(\Omega_H;\mathbb{R}^N)$ if Ω is an extension domain. This follows from the argument at the end of the proof for the J-functional and the equivalence of the two semi-norms over all of $\mathbb{R}^n$. $\square$

Chapter 3

Local PDE

3.1 INTRODUCTION

We now proceed to present the continuity result discussed in the introduction of the thesis which was originally published and can be found in its final form in the work [29]. We should note here that we are proceeding formally when presenting these results. As mentioned in Chapter 1, solutions to the systems considered in this chapter are not valid for the construction of proper test functions due to their lack of regularity in the time direction. One can overcome this by using the Steklov averages when constructing the proper test function and then taking limits before applying the growth and ellipticity conditions to the integrand. These Steklov averages are defined in the following way:

$$f_h = \begin{cases} \frac{1}{h}\int_t^{t+h} f(\cdot,\tau)\,d\tau, & t \in [-T,-h), \\ 0, & (-h,0), \end{cases}$$

and

$$\overline{f}_h = \begin{cases} \frac{1}{h}\int_{t-h}^{t} f(\cdot,\tau)\,d\tau, & t \in [-T+h, 0), \\ 0, & (-T, -T+h). \end{cases}$$

Lastly, we note that $f_h \to f$ and $\overline{f}_h \to f$ as $h \to 0$.

In what follows, we show the partial Hölder continuity of solutions to the quasilinear homogeneous parabolic system

$$u_t - \text{div}\, a(x, t, u, Du) = 0, \tag{3.1}$$

where $a(\cdot, \cdot, \cdot) : \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ is a vector field satisfying subquadratic growth and ellipticity conditions. Moreover we assume that the vector field satisfies a continuity condition slightly weaker than Hölder continuity with respect to its third argument while only assuming the map $a(\cdot, \cdot, \xi) : \Omega_T \times \mathbb{R}^N \to \mathbb{R}^{Nn}$ is continuous for all $\xi \in \mathbb{R}^{Nn}$. Here $\Omega \subseteq \mathbb{R}^n$, $\Omega_T := \Omega \times (-T, 0)$, and Du is the spatial gradient of u. More precisely, we show a solution u to (3.1) is Hölder continuous on an open set of full measure. This result was conjectured by Campanato several years ago. Unfortunately, his paper was found to contain a flaw as mentioned in [6]. As mentioned in Chapter 1, we establish the result here by means of the A-caloric approximation method.

To facilitate the discussion, we state our hypotheses for the system more explicitly. We assume $a(\cdot, \cdot, \cdot)$ satisfies

$$
\begin{cases}
|a(z,u,w)| \le M\,(1+|w|)^{p-1}\,, \\
\langle \partial_w a(z,u,w)\tilde{w},\tilde{w}\rangle \ge \nu\,(1+|w|)^{p-2}\,|\tilde{w}|^2, \\
|\partial_w a(z,u,w)| \le M\,(1+|w|)^{p-2}\,,
\end{cases}
\tag{3.2}
$$

for all $(z,u) \in \Omega_T \times \mathbb{R}^N$ and $w, \widetilde{w} \in \mathbb{R}^{Nn}$. Here $0 < \nu \le 1 \le M$ and $\frac{2n}{n+2} < p < 2$. Moreover, we assume for all $(z,u),(z_0,u_0) \in \Omega_T \times \mathbb{R}^N$ and $w, w_0 \in \mathbb{R}^{Nn}$ the vector field $a(\cdot,\cdot,\cdot)$ satisfies the following continuity conditions,

$$
\begin{cases}
|a(z,u,w) - a(z_0,u_0,w)| \le M\omega(d_{\text{par}}^2(z,z_0) + |u-u_0|^2)(1+|w|)^{p-1}, \\
|\partial_w a(z,u,w_0) - \partial_w a(z,u,w)| \le M\mu\left(\frac{|w_0-w|}{1+|w_0|+|w|}\right)(1+|w_0|+|w|)^{p-2}\,,
\end{cases}
\tag{3.3}
$$

where parabolic distance $d_{\text{par}}(\cdot,\cdot)$ is given by

$$
d_{\text{par}}(z,z_0) = \max\{|x-x_0|, \sqrt{|t-t_0|}\}
$$

with $z = (x,t)$ and $z_0 = (x_0,t_0)$. Here ω and μ are moduli of continuity, i.e. maps with $\omega(0) = \mu(0) = 0$ which are bounded, nonnegative, concave, and non-decreasing. The assumptions on ω imply the continuity of the map $(z,u) \mapsto a(z,u,w)(1+|w|)^{1-p}$ is uniform for fixed w. We observe that $(3.3)_2$ is satisfied if, for example, we assume the following type of Hölder continuity: there exists $\alpha \in (0,1)$ such that for all $w, w_0 \in \mathbb{R}^{Nn}$,

$$
|\partial_w a(\cdot,\cdot,w_0) - \partial_w a(\cdot,\cdot,w)| \le M|w_0-w|^{\alpha}\,(1+|w_0|+|w|)^{p-2-\alpha}\,.
$$

As ω and μ are bounded, we will assume without loss of generality that $\omega, \mu \le 1$ throughout the paper.

In the scalar setting, i.e. $N = 1$, the above assumptions are sufficient to establish the everywhere regularity of the solution u, see for instance [21]. This paper focuses on the vectorial case, $N \geq 2$. In this setting it has been shown by others that everywhere regularity cannot be expected. For some counterexamples in the parabolic setting, one may consult [47], [48], and [49]. Assuming ω is Hölder continuous, more precisely $\omega(\tau) \leq \tau^{\alpha}$ for some $\alpha \in (0,1)$, Duzaar, Mingione, and Steffen established the partial Hölder continuity for the gradient of the solution Du assuming $p \geq 2$ in hypotheses (3.2) and (3.3) [24]. More recently, Scheven has produced the analogous result for the the subquadratic case [44], and in [3], Baroni was able to show the continuity of the gradient Du while only assuming the Dini continuity of $\omega(\cdot)$. Bögelein, Duzaar, and Mingione were then able to extend the Hölder continuity out to the parabolic boundary in [7] and [8]. These results for parabolic problems are analogues of results that have been established in the elliptic setting. For an extensive survey of the regularity theory for both elliptic and parabolic problems, we refer the interested reader to the manuscript [41].

As indicated above, it is possible to establish the partial continuity of the gradient Du under the assumption of Hölder continuous coefficients. To obtain such a result it is critical to establish uniform bounds on the mean values of Du in neighborhoods of Lebesgue points. We denote the mean value of Du over the parabolic cylinder $Q_\rho(z_0)$ by $(Du)_\rho$. In order to roughly describe the argument for estimating $|(Du)_{\vartheta^j\rho}|$, define $\widetilde{\Psi}(z_0,\rho) := \fint_{Q_\rho(z_0)} |Du - (Du)_\rho|^2\,dx + \omega(\rho)$, where $\omega(\cdot)$ represents the modulus of continuity for the coefficients. Using an iteration argument along with a decay estimate for $\widetilde{\Psi}$, one can show

$$\begin{aligned}
|(Du)_{\vartheta^j\rho}| &\leq |(Du)_\rho| + \sum_{m=1}^{j} |(Du)_{\vartheta^m\rho} - (Du)_{\vartheta^{m-1}\rho}| \\
&\leq L + C\sum_{m=0}^{j-1} \sqrt{\vartheta^{2m\alpha}\widetilde{\Psi}(\rho) + c(M)\omega(\vartheta^m\rho)}.
\end{aligned}$$

Here $\vartheta \leq 1$ and we set $L := |(Du)_\rho|$. Assuming the Hölder continuity of $\omega(\cdot)$, we continue with

$$\begin{aligned} |(Du)_{\vartheta^j \rho}| &\leq L + C \sum_{m=0}^{j-1} \sqrt{\vartheta^{2m\alpha} \widetilde{\Psi}(\rho) + c(L)(\vartheta^m \rho)^{2\beta}} \\ &\leq L + C \sum_{m=0}^{\infty} \sqrt{\vartheta^{2m\alpha} \widetilde{\Psi}(\rho) + c(L)(\vartheta^m \rho)^{2\beta}} \\ &\leq L + C \left(\frac{\sqrt{\widetilde{\Psi}(\rho)}}{1 - \vartheta^\alpha} + \frac{\sqrt{c(L)\rho^{2\beta}}}{1 - \vartheta^\beta} \right). \end{aligned} \tag{3.4}$$

Hence, for each $j \in \mathbb{N}$, this yields a bound on each $|(Du)_{\vartheta^j \rho}|$ that is independent of j. The weakest assumption on $\omega(\cdot)$ that ensures convergence of the series in (3.4) is Dini continuity. For more details on achieving a bound on $|(Du)_\rho|$, one may consult [25], [26], and [44].

Since we are not even assuming Dini continuity of $\omega(\cdot)$, we can expect neither boundedness nor partial continuity of Du. On the other hand, the partial Hölder continuity of a solution u itself has been established by Foss and Mingione in the elliptic setting [31]. Bögelein, Foss, and Mingione then extended the result to parabolic problems with $p \geq 2$ in [6]. Also, the analogue of Foss and Mingione's result for subquadratic elliptic problems was provided in [4] by Beck. In what follows, we establish the parabolic version of Beck's result. More precisely, we have the following:

Theorem 3.1.1. *Let* $u \in C^0(-T, 0; L^2(\Omega, \mathbb{R}^N)) \bigcap L^p(-T, 0; W^{1,p}(\Omega, \mathbb{R}^N))$ *be a solution to* (3.1) *in* Ω_T *under the assumptions* (3.2) *and* (3.3)*. Then for each* $\alpha \in (0, 1)$ *there exists an open subset* $\Omega_0 \subseteq \Omega_T$ *such that*

$$|\Omega_T \backslash \Omega_0| = 0 \quad \textit{and} \quad u \in C^{0;\alpha,\alpha/2}_{loc}(\Omega_0, \mathbb{R}^N).$$

Moreover, for each $\alpha \in (0,1)$ the singular set $\Sigma := \Omega_T \backslash \Omega_0$ satisfies $\Sigma \subseteq \Sigma_1 \cup \Sigma_2$, where

$$\Sigma_1 := \left\{ z_0 \in \Omega_T : \liminf_{\rho \to 0^+} \fint_{Q_\rho(z_0)} |Du - (Du)_{z_0;\rho}|^2 \, dz > 0 \right\},$$
$$\Sigma_2 := \left\{ z_0 \in \Omega_T : \limsup_{\rho \to 0^+} |(Du)_{z_0;\rho}| = \infty \right\}.$$

By a weak solution to (1.1), we mean the following:

Defintion 3.1.1. *We say that $u \in C^0(-T,0;L^2(\Omega,\mathbb{R}^N)) \bigcap L^p(-T,0;W^{1,p}(\Omega,\mathbb{R}^N))$ is a **weak solution** to (3.1) if u satisfies*

$$\int_{\Omega_T} u \cdot \varphi_t - a(z,u,Du) \cdot D\varphi \, dz = 0, \qquad \textit{for all } \varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N).$$

We wish to conclude the Introduction of this chapter by providing more insight into the strategies utilized to establish the partial continuity result and the challenges presented in the problem we consider. The cornerstone of the argument is the A-caloric approximation lemma, found in Section 3.5. Here A is a bilinear form on $\mathbb{R}^{Nn} \times \mathbb{R}^{Nn}$ with constant coefficients satisfying certain growth and ellipticity assumptions that will be stated later. If A satisfies such conditions, then solutions, f, to $\int_{\Omega_T} f_t - \langle ADf, D\varphi \rangle \, dz = 0$ are A-caloric and have nice decay properties which are stated in Lemma 3.7.1. The A-caloric approximation lemma allows one to translate these decay estimates on f into the preservation of a smallness property for a certain excess functional (see (3.6)). This eventually allows one to obtain the desired partial continuity. When applying the A-caloric approximation method, it is necessary to use cylinders contained in Ω_T, which we represent by $Q_\rho(z_0) := B_\rho(x_0) \times (t_0 - \rho^2, t_0)$. If u is a solution of (3.1), let $\ell_\rho : B_\rho(x_0) \to \mathbb{R}^N$ be the unique time independent affine map minimizing $\ell \mapsto \fint_{Q_\rho(z_0)} |u - \ell|^2 \, dz$. We would like show $g := (u - \ell_\rho)$ approximately solves

$$\fint_{Q_{\rho/2}(z_0)} (g \cdot \varphi_t - \langle \partial_w a(z_0, \ell_\rho(x_0), D\ell_\rho) Dg, D\varphi \rangle)\, dz = 0$$

for all $\varphi \in C_0^\infty(Q_{\rho,\tau}(z_0); \mathbb{R}^N)$ by using a linearization lemma (Lemma 3.4.1). In this setting, we could then apply the A-caloric approximation lemma to establish the smallness, as $\rho \to 0$, for the following first order excess functional:

$$E(z_0, \ell_\rho, \ell_\rho) = \fint_{Q_\rho(z_0)} \left| \frac{u - \ell_\rho}{\rho} \right|^2 + \left| \frac{u - \ell_\rho}{\rho} \right|^p dz. \tag{3.5}$$

This would allow us to measure the oscillation in u with respect to an affine mapping. There are, however, some scaling issues that prevent one from showing $(u - \ell_\rho)$ is approximately A-caloric. The major obstacle is the hypothesis that $|\partial_w a(z_0, \ell_\rho(x_0), D\ell_\rho)|$ grows like $(1 + |D\ell_\rho|)^{p-2}$. Since $p < 2$ and we cannot bound $|D\ell_\rho|$ as ρ goes to zero, the ellipticity of $\partial_w a(z_0, \ell_\rho(x_0), D\ell_\rho)$ degenerates as $|D\ell_\rho|$ becomes large. (Note that $D\ell_\rho \approx Du$ at Lebesgue points.) Hence, we cannot apply the approximation lemma directly to $(u - \ell_\rho)$.

In order to overcome the growth of $|D\ell_\rho|$ and avoid the decay in the ellipticity of $\partial_w a(z_0, \ell_\rho(x_0), D\ell_\rho)$, we scale our system by an intrinsic factor $\lambda \approx (1 + |D\ell_\rho^{(\lambda)}|)$, where $\ell_\rho^{(\lambda)}$ is the unique affine minimizer of $\ell \mapsto \fint_{Q_\rho^{(\lambda)}(z_0)} |u - \ell|^2\, dz$ and $Q_\rho^{(\lambda)}(z_0) = B_\rho(x_0) \times (t_0 - \lambda^{2-p}\rho^2, t_0)$. Such a scaling provides a bilinear form that satisfies the growth and ellipticity bounds needed to apply the A-caloric approximation lemma. The structure of this bilinear form is given by $\langle Aw, w \rangle := \langle \partial_w a(z_0, \ell_\rho^{(\lambda)}(x_0), D\ell_\rho^{(\lambda)}) \lambda^{2-p} w, w \rangle$. With this intrinsic scaling, we also repair the aforementioned scaling problem that prevented us from showing $(u - \ell_\rho)$ was an approximate solution to the unscaled system. These scalings utilize the ideas of DiBenedetto's intrinsic geometry, which is discussed in [21]. Using the intrinsic scaling and Lemma 3.4.1, we are able to show

$$v := \frac{u(x, t_0 + \lambda^{2-p}(t - t_0)) - \ell^{(\lambda)}_{z_0;\rho}(x)}{\widetilde{c}\gamma(1 + |D\ell^{(\lambda)}_{z_0;\rho}|)}$$

is an approximate solution to $\fint_{Q_{\rho/2}(z_0)} (v \cdot \varphi_t - \langle ADv, D\varphi\rangle)\, dz = 0$, where γ is an intrinsically defined parameter and $\widetilde{c} \geq 1$ is a constant .

Having identified the map v to which the A-caloric approximation lemma can be applied, we now describe the compatible functional that will measure the oscillations in the gradient of our solution u to (3.1). Roughly speaking, the functional

$$E_\lambda(z_0, \rho, \ell^{(\lambda)}_\rho) = \fint_{Q^{(\lambda)}_\rho(z_0)} \left|\frac{u - \ell^{(\lambda)}_\rho}{(1 + |D\ell^{(\lambda)}_\rho|)\rho}\right|^2 + \left|\frac{u - \ell^{(\lambda)}_\rho}{(1 + |D\ell^{(\lambda)}_\rho|)\rho}\right|^p \, dz \tag{3.6}$$

measures the oscillations in Dv. Modulo the scaling factor $1 + |D\ell^{(\lambda)}_\rho|$, it also provides information about the oscillations in Du. This makes (3.6) the natural functional out of which one expects to obtain estimates on the oscillations of Du. By using the A-caloric approximation lemma, as described before, in Section 3.7 we are able to show that if this excess functional is small enough for some $\rho > 0$, then it remains small as $\rho \to 0$.

Once such smallness conditions are obtained for the excess functional, one can show that with $r > 0$ sufficiently small we have

$$\int_{Q_r(z)} |u - (u)_r|^2 \, dz \leq cr^{n+2+2\alpha}$$

for all $z \in Q_R(z_0)$ and all $z_0 \in \Omega_T \backslash (\Sigma_1 \cup \Sigma_2)$, where R is a fixed radius determined within the proof. Hence, u belongs to a Campanato space, and the result then follows from a Campanato embedding theorem. While we have sketched the argument with the excess functional in (3.6), the actual excess functional used must take into account the continuity of the coefficients in the system, which leads to a couple of additional terms in the functional.

3.2 BACKGROUND

Throughout this chapter, we use $z = (x, t)$ to represent points in $\mathbb{R}^{n+1}$. For the spatial ball of radius ρ centered at x_0, we use $B_\rho(x_0)$; i.e. $B_\rho(x_0) := \{x \in \Omega : |x - x_0| < \rho\}$. We will use three types of cylinders: general, standard, and scaled. We denote the general cylinder with spatial radius ρ and time length τ centered at $z_0 = (x_0, t_0)$ by

$$Q_{\rho,\tau}(z_0) := B_\rho(x_0) \times (t_0 - \tau, t_0),$$

and we define the standard and scaled cylinders by

$$Q_\rho(z_0) := Q_{\rho,\rho^2}(z_0) = B_\rho(x_0) \times (t_0 - \rho^2, t_0)$$

and

$$Q_\rho^{(\lambda)}(z_0) := Q_{\rho,\lambda^{2-p}\rho^2}(z_0) = B_\rho(x_0) \times (t_0 - \lambda^{2-p}\rho^2, t_0),$$

respectively. We use $|A|$ for the measure of a set A. If $|A| > 0$, then the mean value of $f \in L^1$ over A is given by

$$(f)_A = \fint_A f\,dz = \frac{1}{|A|}\int_A f\,dz.$$

For convenience, the mean value of f over scaled cylinders $Q_\rho^{(\lambda)}(z_0)$ is denoted by $(f)_{z_0;\rho}^{(\lambda)}$. If $\lambda = 1$, we write $(f)_{z_0;\rho}$.

As mentioned in the introduction, we denote the unique affine minimizer of

$$\ell(x) \mapsto \fint_{Q_\rho^{(\lambda)}(z_0)} |u(x,t) - \ell(x)|^2 \, dz \tag{3.7}$$

by $\ell_{z_0;\rho}^{(\lambda)} : B_\rho(x_0) \to \mathbb{R}^n$. It is well-known, for instance see [6], that $\ell_{z_0;\rho}^{(\lambda)}(x) = \xi_{Q_\rho}^{(\lambda)} + P_{z_0;\rho}^{(\lambda)}(x - x_0)$, where

$$\xi_{z_0;\rho}^{(\lambda)} = (u)_{z_0;\rho}^{(\lambda)} \quad \text{and} \quad P_{z_0;\rho}^{(\lambda)} = \frac{n+2}{\rho^2} \fint_{Q_\rho^{(\lambda)}(z_0)} u \otimes (x - x_0) \, dz. \tag{3.8}$$

We also have

$$|P_{z_0;\rho}^{(\lambda)} - w|^2 \leq \frac{n(n+2)}{\rho^2} \fint_{Q_\rho^{(\lambda)}(z_0)} |u - \xi - w(x - x_0)|^2 \, dz, \tag{3.9}$$

for all $\xi \in \mathbb{R}^n$ and $w \in \mathbb{R}^{Nn}$.

We now introduce a few functionals that will be used to measure the oscillations of u. Let u be a solution to (3.1) on Ω_T and $z_0 \in \Omega_T$. Given an affine map and $Q_\rho^{(\lambda)}(z_0) \subseteq \Omega_T$, the first order excess is given by

$$E_\lambda(z_0, \rho, \ell) = \fint_{Q_\rho^{(\lambda)}(z_0)} \left| \frac{u - \ell}{(1 + |D\ell|)\rho} \right|^2 + \left| \frac{u - \ell}{(1 + |D\ell|)\rho} \right|^p \, dz.$$

Defining the zero order excess by

$$\Psi_\lambda(z_0, \rho, \ell) := \fint_{Q_\rho^{(\lambda)}(z_0)} |u - \ell|^2 \, dz,$$

the full excess functional is defined to be

$$\widetilde{E}_\lambda(z_0, \rho, \ell) := E_\lambda(z_0, \rho, \ell) + \omega(\Psi_\lambda(z_0, \rho, \ell)) + \omega(\lambda^{2-p}\rho^2).$$

In the subquadratic setting, it is necessary to work with the function $V : \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$V(\xi) = (1 + |\xi|^2)^{\frac{p-2}{4}} \xi \tag{3.10}$$

in order to accommodate the growth conditions in (3.2). The following lemma lists several properties of V that will be used throughout the paper. It was first shown to hold in [17].

Lemma 3.2.1. *Let* $1 < p < 2$ *and* $V : \mathbb{R}^k \to R^k$ *be the function defined in* (3.10)*, then for any* $\xi, \eta \in \mathbb{R}^k, t > 0$

(i) $2^{(p-2)/4} \min\{|\xi|, |\xi|^{p/2}\} \leq |V(\xi)| \leq \min\{|\xi|, |\xi|^{p/2}\}$,

(ii) $|V(t\xi)| \leq \max\{t, t^{p/2}\}|V(\xi)|$,

(iii) $|V(\xi + \eta)| \leq c(p) \, [|V(\xi)| + |V(\eta)|]$,

(iv) $\frac{p}{2}|\xi - \eta| \leq \frac{|V(\xi) - V(\eta)|}{(1+|\xi|^2+|\eta|^2)^{\frac{p-2}{4}}} \leq c(k,p)|\xi - \eta|$,

(v) $|V(\xi) - V(\eta)| \leq c(k,p)|V(\xi - \eta)|$,

(vi) $|V(\xi - \eta)| \leq c(p, M)|V(\xi) - V(\eta)|$ *if* $|\eta| \leq M$.

The following standard lemma will be used extensively throughout the paper as well. It can be found in [1] with proof.

Lemma 3.2.2. *Given* $\sigma > -1$*, there exists constants* $c = c(\sigma) > 0$ *such that for each* $a, b \in \mathbb{R}^k$

$$c^{-1} (1 + |a| + |b|)^\sigma \leq \int_0^1 (1 + |a + sb|)^\sigma \, ds \leq c \, (1 + |a| + |b|)^\sigma$$

In section 6, we will use of the following lemma in order to establish a Poincaré-type inequality. We refer the interested reader to Chapter 6 of [34] for the proof.

Lemma 3.2.3. *Given $r < R$, let $f : [r, R] \to [0, \infty)$ be a bounded function. Assume there are constants $A, B, \alpha \in [0, \infty)$ and $\vartheta \in (0, 1)$ such that for all $r < \sigma < \rho < R$,*

$$f(\sigma) \leq \vartheta f(\rho) + \frac{A}{(\rho - \sigma)^\alpha} + B.$$

Then

$$f(\sigma_0) \leq c(\alpha, \vartheta) \left(\frac{A}{(\rho_0 - \sigma_0)^\alpha} + B \right),$$

for all $r \leq \sigma_0 \leq \rho_0 \leq R$.

Finally, we wish to comment on the spaces used at the end of this chapter. A function $u : \Omega_T \to \mathbb{R}^N$ is said to be Hölder continuous with exponent $\alpha \in (0, 1)$ if

$$\|u\|_{C^{\alpha/2,\alpha}(\Omega_T;\mathbb{R}^N)} := \|u\|_{L^p(\Omega_T;\mathbb{R}^N)} + \sup_{\substack{x,y\in\Omega, t,s\in(-T,0) \\ x\neq y, t\neq s}} \left(\frac{u(x,t) - u(y,s)}{d^\alpha_{\text{par}}((x,t),(y,s))} \right) < \infty.$$

However, we will avoid showing that a solution to the quasilinear systems we consider satisfies the above inequality by showing that the solution belongs to a Campanato space, $\mathscr{C}^{p,\lambda}(\Omega_T; \mathbb{R}^N)$, and using a result due to da Prato that states $\mathscr{C}^{p,\lambda}(\Omega_T; \mathbb{R}^N)$ is isomorphic to $C^{\alpha/2,\alpha}(\Omega; \mathbb{R}^N)$ if $\lambda > 1$ and $\alpha = \frac{n}{p}(\lambda - 1)$. This result is found in Theorem 3.1 of [18]. A function $u : \Omega_T \to \mathbb{R}^N$ is said to belong to $\mathscr{C}^{p,\lambda}(\Omega_T; \mathbb{R}^N)$ if it satisfies

$$\|u\|_{\mathscr{C}^{p,\lambda}(\Omega_T;\mathbb{R}^N)} := \|u\|_{L^p(\Omega_T;\mathbb{R}^N)} + |u|_{\mathscr{C}^{p,\lambda}(\Omega_T;\mathbb{R}^N)} < \infty,$$

where

$$|u|_{\mathscr{C}^{p,\lambda}(\Omega_T;\mathbb{R}^N)} := \sup_{\substack{z\in\Omega_T\\ \rho>0}} \left(|\Omega_T \cap Q_\rho(z_0)|^{-n\lambda} \int_{\Omega_T\cap Q_\rho(z_0)} |u(z) - u(z_0)|^p \, dz \right).$$

3.3 CACCIOPPOLI

We begin the work by presenting two Caccioppoli inequalities, or reverse Poincaré-type inequalities. The first will be used throughout the paper to prove Lemma 3.4.1 and the A-caloric approximation lemma, while the second inequality will be used to prove the first Poincaré inequality in section 3.6.1. Note also that the first Caccioppoli inequality can be used for general cylinders, while the second inequality can only be applied on standard cylinders.

Theorem 3.3.1. *(Caccioppoli's Inequality for Parabolic Systems with General Cylinders) Let $u \in C^0(-T, 0; L^2(\Omega, \mathbb{R}^N)) \cap L^p(-T, 0; W^{1,p}(\Omega, \mathbb{R}^N))$ be a weak solution to* (3.1) *in Ω_T, where* (3.2) *and* (3.3) *hold, and $Q_{\rho,\tau}(z_0) \subseteq \Omega_T$ be a general parabolic cylinder with center $z_0 = (x_0, t_0)$. Also assume $0 < \rho \le 1$ and $\tau \ge \rho^2$. Then for any affine map $\ell : \mathbb{R}^n \to \mathbb{R}^N$ we have*

$$\begin{aligned}
&\sup_{-\tau/4<s<0} \left(\fint_{B_{\rho/2}(x_0)} \left| \frac{u(s,x) - \ell}{\sqrt{\tau}/2} \right|^2 dz \right) \\
&\qquad + \fint_{Q_{\rho/2,\tau/4}(z_0)} (1 + |D\ell| + |Du|)^{p-2} |Du - D\ell|^2 \, dz \\
&\qquad \le c_0 \fint_{Q_{\rho,\tau}(z_0)} \left| \frac{u-\ell}{\sqrt{\tau}} \right|^2 + \left| \frac{u-\ell}{\rho} \right|^p dz \\
&\qquad\quad + c_0 \left(1 + |D\ell|\right)^p \left[\omega \left(\fint_{Q_{\rho,\tau}(z_0)} |u - \ell(x_0)|^2 \, dz \right) + \omega\left(\tau\right) \right] =: \xi,
\end{aligned}$$

where $c_0 \ge 1$ depends only on $p, n, M/\nu$, and N.

Proof. For notational convenience, we write $Q_{\rho,\tau}$ and B_ρ instead of $Q_{\rho,\tau}(z_0)$ and $B_\rho(x_0)$. Let u be a weak solution to (3.1) in Ω_T. Assume $Q_{\rho,\tau} \subseteq \Omega_T$ with $\rho \leq 1$, and $\ell : \mathbb{R}^n \to \mathbb{R}^N$ is an affine map. Define $\varphi(x,t) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^N$ by $\varphi(x,t) = \zeta^2(t)\chi^p(x)(u(x,t) - \ell(x))$, where $\chi \in C_0^\infty(B_\rho)$ and $\zeta \in C(\mathbb{R})$ are cutoff functions. In particular, $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $B_{\frac{\rho}{2}}$, $\chi \equiv 0$ on $\Omega \backslash B_\rho$, and $|D\chi| \leq \frac{4}{\rho}$ on B_ρ. Moreover, $\zeta : \mathbb{R} \to [0,1]$ is defined by

$$\begin{cases} \zeta \equiv 0, & \text{on } \left(-\infty, -\tau\right), \\ \zeta_t = \frac{4}{3\tau}, & \text{on } \left(-\tau, \frac{-\tau}{4}\right), \\ \zeta \equiv 1, & \text{on } \left(\frac{-\tau}{4}, s\right), \\ \zeta_t = \frac{-1}{\epsilon}, & \text{on } \left(s, s+\epsilon\right), \\ \zeta \equiv 0, & \text{on } \left(s+\epsilon, \infty\right) \end{cases}$$

for $s \in (-\tau/4, 0)$ and $0 \leq \epsilon \leq |s|$. Substituting φ into the weak formulation of (3.1) gives

$$\begin{aligned} &\int_{Q_{\rho,\tau}} \zeta^2(t)\chi^p(x) a(z,u,Du) \cdot D(u-\ell)\,dz \\ &\qquad = -p\int_{Q_{\rho,\tau}} \zeta^2(t)\chi^{p-1} a(z,u,Du) \cdot \left(D\chi \otimes (u-\ell)\right)\,dz + \int_{Q_{\rho,\tau}} u \cdot \varphi_t\,dz. \end{aligned}$$

By the definition of φ, we have

$$\begin{aligned} -\int_{Q_{\rho,\tau}} &\zeta^2(t)\chi^p a(z,u,D\ell) \cdot D(u-\ell)\,dz \\ &= p\int_{Q_{\rho,\tau}} \zeta^2(t)\chi^{p-1} a(z,u,D\ell) \cdot \left(D\chi \otimes (u-\ell)\right)\,dz \\ &\quad - \int_{Q_{\rho,\tau}} a(z,u,D\ell) \cdot D\varphi\,dz. \end{aligned}$$

Noting that $\int_{Q_{\rho,\tau}} \ell \cdot \varphi_t \, dz = 0$ and $\int_{Q_{\rho,\tau}} a(z_0, \ell(x_0), D\ell) \cdot D\ell \, dz = 0$, we obtain

$$\begin{aligned} I :=& \int_{Q_{\rho,\tau}} \zeta^2(t)\chi^p \left(a(z,u,Du) - a(z,u,D\ell)\right) \cdot D(u-\ell)\, dz \\ =& - p \int_{Q_{\rho,\tau}} \zeta^2(t)\chi^{p-1} \left(a(z,u,Du) - a(z,u,D\ell)\right) \cdot \left(D\chi \otimes (u-\ell)\right)\, dz \\ & - \int_{Q_{\rho,\tau}} \left(a(z,u,D\ell) - a(z_0,\ell(x_0),D\ell)\right) \cdot D\varphi \, dz + \int_{Q_{\rho,\tau}} (u-\ell)\cdot \varphi_t \, dz \\ =:& II + III + IV. \end{aligned}$$

We will first establish a lower estimate for I by means of assumption $(3.2)_2$ and the triangle inequality:

$$\begin{aligned} & \int_{Q_{\rho,\tau}} \zeta^2 \chi^p \left(a(z,u,Du) - a(z,u,D\ell)\right) \cdot D(u-\ell)\, dz \\ & \qquad = \int_{Q_{\rho,\tau}} \zeta^2\chi^p \int_0^1 \left\langle \partial_w a(z,u,D\ell + s(Du-D\ell))D(u-\ell), D(u-\ell)\right\rangle \, ds\, dz \\ & \qquad \geq \nu \int_{Q_{\rho,\tau}} \zeta^2\chi^p \int_0^1 \left(1 + |D\ell + s(Du - D\ell)|\right)^{p-2} |D(u-\ell)|^2 \, ds \, dz \\ & \qquad \geq \nu \int_{Q_{\rho,\tau}} \zeta^2\chi^p \left(1 + |D\ell| + |Du|\right)^{p-2} |D(u-\ell)|^2 \, dz. \end{aligned}$$

Using assumption $(3.2)_3$, Lemma 3.2.2, and Young's inequality with $\delta > 0$ to be chosen later, we also find

$$\begin{aligned} |II| \leq& p \int_{Q_{\rho,\tau}} \zeta^2\chi^{p-1} |a(z,u,Du) - a(z,u,D\ell)||D\chi||u-\ell|\, dz \\ \leq& cM \int_{Q_{\rho,\tau}} \zeta^2\chi^{p-1} \left(1+|Du|+|D\ell|\right)^{p-2} |Du - D\ell||D\chi||u-\ell|\, dz \\ \leq& \delta c M \int_{Q_{\rho,\tau}} \zeta^2\chi^p \left(1+|Du|+|D\ell|\right)^{\frac{p(p-2)}{p-1}} |Du - D\ell|^{\frac{p}{p-1}}\, dz \\ & + \delta^{\frac{1}{1-p}} cM \int_{Q_{\rho,\tau}} |D\chi|^p |u-\ell|^p \, dz \end{aligned}$$

$$\begin{aligned}
\leq & \delta c M \int_{Q_{\rho,\tau}} \zeta^2 \chi^p \left(1+|Du|+|D\ell|\right)^{p-2} |Du - D\ell|^2 \, dz \\
& + \delta^{\frac{1}{1-p}} c M \int_{Q_{\rho,\tau}} \left| \frac{u-\ell}{\rho} \right|^p \, dz.
\end{aligned}$$

Set $G = \{x \in Q_{\rho,\tau} : \; |Du| \geq 4|D\ell| \text{ and } |Du - D\ell| \geq 1\}$. By the continuity hypothesis $(3.3)_1$ and the definition of φ, the following holds:

$$\begin{aligned}
|III| \leq & \int_{Q_{\rho,\tau}} |a(z,u,D\ell) - a(z_0, \ell(x_0), D\ell)||D\varphi| \, dz \\
\leq & M \int_G \zeta^2 \chi^p \omega\left(|u-\ell(x_0)|^2 + \tau\right) \left(1+|D\ell|\right)^{p-1} |Du - D\ell| \, dz \\
& + M \int_{Q_{\rho,\tau} \setminus G} \zeta^2 \chi^p \omega\left(|u-\ell(x_0)|^2 + \tau\right) \left(1+|D\ell|\right)^{p-1} |Du - D\ell| \, dz \\
& + pM \int_{Q_{\rho,\tau}} \zeta^2 \chi^{p-1} \omega\left(|u-\ell(x_0)|^2 + \tau\right) \left(1+|D\ell|\right)^{p-1} |u-\ell||D\chi| \, dz \\
=: & III_1 + III_2 + III_3.
\end{aligned}$$

Using Young's inequality, the bound $\omega \leq 1$, the concavity of ω, and that $\omega(0) = 0$, we see that

$$III_3 \leq cM \left(1+|D\ell|\right)^p \int_{Q_{\rho,\tau}} \left\{\omega\left(|u-\ell(x_0)|^2\right) + \omega(\tau)\right\} \, dz + cM \int_{Q_{\rho,\tau}} \left|\frac{u-\ell}{\rho}\right|^p \, dz.$$

Similarly

$$\begin{aligned}
III_1 \leq & \delta c M \int_G \zeta^2 \chi^p |Du - D\ell|^p \, dz \\
& + \delta^{\frac{1}{1-p}} c M \left(1+|D\ell|\right)^p \left[\int_{Q_{\rho,\tau}} \omega\left(|u-\ell(x_0)|^2\right) + \omega(\tau) \, dz \right].
\end{aligned}$$

Recall that on G, we have $|Du| \geq 4|D\ell|$ and $|Du - D\ell| \geq 1$. Thus,

$$|Du - D\ell| \geq \frac{1}{4}\left(1 + |Du| + |D\ell|\right). \tag{3.11}$$

It follows that

$$\begin{aligned} III_1 \leq & \delta cM \int_{Q_{\rho,\tau}} \zeta^2 \chi^p \left(1 + |Du| + |D\ell|\right)^{p-2} |Du - D\ell|^2 \, dz \\ & + \delta^{\frac{1}{1-p}} cM \left(1 + |D\ell|\right)^p \left[\int_{Q_{\rho,\tau}} \omega\left(|u - \ell(x_0)|^2\right) + \omega(\tau) \, dz \right]. \end{aligned}$$

Now on $Q_{\rho,\tau} \backslash G$, one of the following cases must hold:

(i) $|Du| < 4|D\ell|$,

(ii) $|Du| \geq 4|D\ell|$ and $|Du - D\ell| < 1$.

We note that in either case, $|Du - D\ell| \leq c(1 + |D\ell|)$. Hence

$$III_2 \leq cM \left(1 + |D\ell|\right)^p \left[\int_{Q_{\rho,\tau}} \omega\left(|u - \ell(x_0)|^2\right) + \omega(\tau) \, dz \right].$$

Combining the estimates for III_1, III_2 and III_3 gives

$$\begin{aligned} |III| \leq & \delta cM \int_{Q_{\rho,\tau}} \zeta^2 \chi^p \left(1 + |Du| + |D\ell|\right)^{p-2} |Du - D\ell|^2 \, dz \\ & + \delta^{\frac{1}{1-p}} cM (1 + |D\ell|)^p \left[\int_{Q_{\rho,\tau}} \omega\left(|u - \ell(x_0)|^2\right) + \omega(\tau) \, dz \right] \\ & + cM \int_{Q_{\rho,\tau}} \left| \frac{u - \ell}{\rho} \right|^p \, dz. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned}
IV &= \int_{Q_{\rho,\tau}} |u-\ell|^2\chi^p \left(\zeta^2\right)_t \, dz + \int_{Q_{\rho,\tau}} \zeta^2\chi^p u_t \cdot (u-\ell)\, dz \\
&= \int_{Q_{\rho,\tau}} |u-\ell|^2\chi^p \left(\zeta^2\right)_t \, dz + \frac{1}{2}\int_{Q_{\rho,\tau}} \zeta^2\chi^p \left(|u-\ell|^2\right)_t \, dz \\
&= \int_{Q_{\rho,\tau}} |u-\ell|^2\chi^p\zeta\zeta_t\, dz \\
&= \frac{4}{3\tau}\int_{-\tau}^{\frac{-\tau}{4}}\int_{B_\rho(x_0)} |u-\ell|^2\chi^p\, dz - \frac{1}{\epsilon}\int_s^{s+\epsilon}\int_{B_\rho(x_0)} |u-\ell|^2\chi^p\, dz \\
&\leq c\int_{Q_{\rho,\tau}} \left|\frac{u-\ell}{\sqrt{\tau}}\right|^2 dz - \frac{1}{\epsilon}\int_s^{s+\epsilon}\int_{B_{\rho/2}(x_0)} |u-\ell|^2\, dz.
\end{aligned}$$

Combining all of our estimates, yields

$$\begin{aligned}
&\frac{1}{\epsilon}\int_s^{s+\epsilon}\int_{B_{\rho/2}(x_0)} |u-\ell|^2\, dz \\
&\quad + (\nu - 3\delta c(p,M))\int_{Q_{\rho,\tau}} \zeta^2\chi^p\left(1+|D\ell|+|Du|\right)^{p-2}|D(u-\ell)|^2\, dz \\
&\qquad \leq c(p,M,\delta^{\frac{1}{1-p}})\int_{Q_{\rho,\tau}} \left|\frac{u-\ell}{\sqrt{\tau}}\right|^2 + \left|\frac{u-\ell}{\rho}\right|^p dz \\
&\qquad\quad + c(p,M,\delta^{\frac{1}{1-p}})\int_{Q_{\rho,\tau}} (1+|D\ell|)^p\left[\omega\left(|u-\ell(x_0)|^2\right)+\omega\left(\tau\right)\right] dz.
\end{aligned}$$

Choosing $\delta > 0$ small enough, and recalling the definition of ζ and χ, we may take the limit as $\epsilon \to 0$ to get

$$\begin{aligned}
&\int_{B_{\rho/2}(x_0)} |u(s,x)-\ell|^2\, dz \\
&\quad + \int_{\frac{-\tau}{4}}^{s}\int_{B_{\rho/2}(x_0)} (1+|Du|+|D\ell|)^{p-2}|D(u-\ell)|^2\, dz \\
&\qquad \leq c\int_{Q_{\rho,\tau}} \left|\frac{u-\ell}{\sqrt{\tau}}\right|^2 + \left|\frac{u-\ell}{\rho}\right|^p dz
\end{aligned}$$

$$+\, c\,(1+|D\ell|)^p \left[\fint_{Q_{\rho,\tau}} \omega\left(|u-\ell(x_0)|^2\right)+\omega\left(\tau\right)\, dz\right].$$

As $s\in(-\tau/4,0)$ is arbitrary, we may conclude that

$$\begin{aligned}
\sup_{-\tau/4<s<0} & \fint_{B_{\rho/2}(x_0)} \left|\frac{u(s,x)-\ell}{\sqrt{\tau/2}}\right|^2 dz \\
&+ \fint_{\frac{-\tau}{4}}^{0}\fint_{B_{\rho/2}(x_0)} (1+|Du|+|D\ell|)^{p-2}\,|D(u-\ell)|^2\, dz \\
&\le c_0 \fint_{Q_{\rho,\tau}} \left|\frac{u-\ell}{\sqrt{\tau}}\right|^2 + \left|\frac{u-\ell}{\rho}\right|^p\, dz \\
&\quad + c_0\,(1+|D\ell|)^p \left[\fint_{Q_{\rho,\tau}} \omega\left(|u-\ell(x_0)|^2\right)+\omega\left(\tau\right)\, dz\right],
\end{aligned}$$

where $c_0 = 1+2^{n+2}c$. By Jensen's inequality and the concavity of ω, we have

$$\fint_{Q_{\rho,\tau}} \omega\left(|u-\ell(x_0)|^2\right)\, dz \le \omega\left(\fint_{Q_{\rho,\tau}} |u-\ell(x_0)|^2\, dz\right),$$

which gives the result. □

The following result will be used to prove the Poincaré-type inequality for solutions to (3.1) in section 3.6.1. It is stated here only for its obvious relationship with the above result and the likeness of their proofs.

Theorem 3.3.2. *(Caccioppoli's Inequality for Parabolic Cylinders)*
Let $u\in L^p(-T,0;W^{1,p}(\Omega,\mathbb{R}^N))\cap C^0(-T,0;L^2(\Omega,\mathbb{R}^N))$ *be a solution to* (3.1) *in* Ω_T *that satisfies* (3.2) *and* (3.3). *Let* $Q_\rho(z_0)\subseteq\Omega_T$*, where* $\rho\in(0,1)$. *For* $\sigma\in[\rho/2,\rho]$ *and any affine function* $\ell:\mathbb{R}^n\to\mathbb{R}^N$*, we have*

$$\sup_{s\in(-\sigma^2,0)}\left(\fint_{B_\sigma(z_0)\times\{s\}}\left|\frac{u-\ell}{\sigma}\right|^2 dx\right)$$

$$+\fint_{Q_\sigma(z_0)}(1+|D\ell|+|Du|)^{p-2}|Du-D\ell|^2\,dz$$
$$\leq c_1\frac{\rho^2}{(\rho-\sigma)^2}\fint_{Q_\rho(z_0)}\left|\frac{u-\ell}{\rho}\right|^2+\left|\frac{u-\ell}{\rho}\right|^p\,dz$$
$$+c_1(1+|D\ell|)^p\left[\omega\left(\fint_{Q_\rho(z_0)}|u-\ell(x_0)|^2\,dz\right)+\omega\left(\rho^2\right)\right],$$

where $c_1>1$ *depends on* p,n,N,M, *and* ν.

Proof. The result is proved similarly to the last theorem. Here we take

$$\varphi(x,t)=\zeta^2(t)\chi^p(x)\left(u(x,t)-\ell(x)\right),$$

where $\chi\in C_0^\infty(B_\rho)$ is a cutoff function with $0\leq\chi\leq 1,\chi\equiv 1$ on $B_\sigma,\chi\equiv 0$ on $\Omega\backslash B_\rho$, and $|D\chi|\leq\frac{2}{\rho-\sigma}$ on B_ρ, while $\zeta\in C^\infty(\mathbb{R})$, and for any $s\in(-\sigma^2,0)$ and $\epsilon\in(0,\sigma^2+s)$, ζ is a Lipschitz cutoff function with

$$\begin{cases}\zeta\equiv 0, & \text{on } (-\infty,-\rho^2],\\ |\zeta'|\leq\frac{2}{(\rho-\sigma)^2}, & \text{on } (-\rho^2,\sigma^2],\\ \zeta\equiv 1, & \text{on } (-\sigma^2,s-\epsilon],\\ \zeta(t)=\frac{-1}{\epsilon}(t-s), & \text{on } (s-\epsilon,s],\\ \zeta\equiv 0, & \text{on } (s,\infty).\end{cases}$$

□

3.4 LINEARIZATION

We now prove a lemma that allows us to compare our system to a linear system with constant coefficients. Such systems have A-caloric solutions with nice decay properties that can be transferred to our solution enabling us to bound our excess functional as mentioned in the introduction. In order to achieve this, our system and solution must give rise to the following inequality.

Lemma 3.4.1. *Let u be a weak solution to* (3.1) *in Ω_T satisfying* (3.2) *and* (3.3)*. Further let $Q_{\rho,\tau}(z_0) \subseteq \Omega_T$ with $\rho \leq 1$ and $\rho^2 \leq \tau$. Then for any affine function $\ell : \mathbb{R}^n \to \mathbb{R}^N$, we have*

$$
\begin{aligned}
&\fint_{Q_{\rho/2,\tau/4}(z_0)} \left((u-\ell)\cdot\varphi_t - \langle \partial_w a(z_0,\ell(x_0),D\ell)(Du-D\ell), D\varphi\rangle\right)\, dz \\
&\qquad \leq c_2 \left(1+|D\ell|\right)^{\frac{p-2}{2}} \mu^{1/2} \left(\left(1+|D\ell|\right)^{\frac{-p}{2}} \xi^{1/2}\right) \xi^{1/2} \sup_{Q_{\rho/2,\tau/4}(z_0)} |D\varphi| \\
&\qquad\quad + c_2 \left(1+|D\ell|\right)^{-1} \xi \sup_{Q_{\rho/2,\tau/4}(z_0)} |D\varphi|,
\end{aligned}
$$

for all $\varphi \in C_0^\infty(Q_{\rho,\tau}(z_0),\mathbb{R}^N)$. Here $c_2 \geq 1$ depends on p, n, N, M and ν.

Proof. In the following proof we write $Q_{\rho,\tau}$ for $Q_{\rho,\tau}(z_0)$ and B_ρ for $B_\rho(x_0)$. Since the result is trivial if $D\varphi \equiv 0$, we assume without loss of generality that $\sup_{Q_{\rho/2,\tau/4}} |D\varphi| = 1$. Now let u be a weak solution to (3.1) in Ω_T,and $Q_{\rho,\tau} \subseteq \Omega_T$ with $\rho \leq 1$. We begin by noting the following:

$$
\begin{aligned}
&\fint_{Q_{\rho/2,\tau/4}} (u-\ell)\cdot\varphi_t - \langle \partial_w a(z_0,\ell(x_0),D\ell)(Du-D\ell), D\varphi\rangle\, dz \\
&\qquad = \fint_{Q_{\rho/2,\tau/4}} \left[(u-\ell)\cdot\varphi_t - \langle a(z,u,Du), D\varphi\rangle\right] dz \\
&\qquad\quad + \fint_{Q_{\rho/2,\tau/4}} \langle a(z,u,Du) - a(z_0,\ell(x_0),Du), D\varphi\rangle\, dz \\
&\qquad\quad + \fint_{Q_{\rho/2,\tau/4}} \langle a(z_0,\ell(x_0),Du) - a(z_0,\ell(x_0),D\ell), D\varphi\rangle \\
&\qquad\qquad - \langle \partial_w a(z_0,\ell(x_0),D\ell)(Du-D\ell), D\varphi\rangle\, dz \\
&\qquad =: I + II + III.
\end{aligned}
$$

Since u is a weak solution to (3.1) and $\fint_{Q_{\rho/2,\tau/4}} \ell \cdot \varphi_t \, dz = 0$, we see immediately that $I = 0$. From the continuity assumption $(3.3)_1$, Young's inequality, and the fact that $\omega \leq 1$ is sublinear, we obtain the following estimate for II:

$$
\begin{aligned}
|II| &\leq c\fint_{Q_{\rho/2,\tau/4}} \omega\left(|u-\ell(x_0)|^2+\tau\right)(1+|Du|)^{p-1}\,|D\varphi|\,dz \\
&\leq c\fint_{Q_{\rho/2,\tau/4}} \left[\omega\left(|u-\ell(x_0)|^2\right)+\omega(\tau)\right](1+|D\ell|)^{p-1}\,|D\varphi|\,dz \\
&\quad + c\fint_{Q_{\rho/2,\tau/4}} \left[\omega\left(|u-\ell(x_0)|^2\right)+\omega(\tau)\right]|Du-D\ell|^{p-1}|D\varphi|\,dz.
\end{aligned}
$$

Taking $G := \{x \in Q_{\rho/2,\tau/4} : |Du| \geq 4|D\ell| \text{ and } |Du - D\ell| \geq 1\}$ as before, we can rewrite the above inequality as

$$
\begin{aligned}
|II| &\leq c\fint_{Q_{\rho/2,\tau/4}} \left[\omega\left(|u-\ell(x_0)|^2\right)+\omega(\tau)\right](1+|D\ell|)^{p-1}\,|D\varphi|\,dz \\
&\quad + c\fint_{G} \left[\omega\left(|u-\ell(x_0)|^2\right)+\omega(\tau)\right]|Du-D\ell|^{p-1}|D\varphi|\,dz \\
&\quad + c\fint_{Q_{\rho/2,\tau/4}\setminus G} \left[\omega\left(|u-\ell(x_0)|^2\right)+\omega(\tau)\right]|Du-D\ell|^{p-1}|D\varphi|\,dz \\
&=: II_1 + II_2 + II_3.
\end{aligned}
$$

On the set G, (3.11) holds. Using a similar argument to that in Theorem 3.3.1, we obtain

$$
II_2 \leq c\,(1+|D\ell|)^{-1}\fint_{Q_{\rho/2,\tau/4}} |Du-D\ell|^p|D\varphi|\,dz.
$$

Now using the Caccioppoli inequality, Theorem 3.3.1, and Jensen's Inequality along with the fact that ω is concave, yields

$$
II_2 \leq c\,(1+|D\ell|)^{-1}\,\xi.
$$

In order to estimate II_3, recall that on $Q_{\rho/2,\tau/4}\backslash G$ we have the estimate

$$|Du - D\ell| \leq c(1 + |D\ell|).$$

Hence,

$$II_3 \;\; \leq \;\; c\,(1+|D\ell|)^{p-1} \fint_{Q_{\rho/2,\tau/4}} \left[\omega\left(|u-\ell(x_0)|^2\right) + \omega(\tau)\right] |D\varphi|\, dz.$$

By the concavity of ω, Jensen's inequality gives

$$II_1 + II_3 \;\; \leq \;\; c\,(1+|D\ell|)^{p-1} \left[\omega\left(\fint_{Q_{\rho,\tau}} |u-\ell(x_0)|^2\, dz\right) + \omega(\tau)\right].$$

Combining the estimates for II_1, II_2, and II_3 and using $c_0 \geq 1$, we deduce that there is a $c \geq 1$ such that

$$|II| \leq c\,(1+|D\ell|)^{-1}\,\xi.$$

From the continuity assumption $(3.3)_2$, we see

$$\begin{aligned}
|III| \leq & \fint_{Q_{\rho/2,\tau/4}} \int_0^1 |\partial_w a(z_0, \ell(x_0), D\ell + s(Du - D\ell)) - \partial_w a(z_0, \ell(x_0), D\ell)| \\
& \quad \times |Du - D\ell|\, ds\, dz \\
\leq & M \fint_{Q_{\rho/2,\tau/4}} \int_0^1 \mu\left(\frac{s|Du - D\ell|}{1 + |D\ell + s(Du - D\ell)| + |D\ell|}\right) \\
& \quad \times (1 + |D\ell + s(Du - D\ell)| + |D\ell|)^{p-2}\, |Du - D\ell|\, ds\, dz \\
\leq & M \fint_{Q_{\rho/2,\tau/4}} \int_0^1 \mu\left(\frac{s|Du - D\ell|}{1 + s|D\ell| + s|Du|}\right) \\
& \quad \times (1 + |D\ell + s(Du - D\ell)| + |D\ell|)^{p-2}\, |Du - D\ell|\, ds\, dz.
\end{aligned}$$

Using $s \leq 1$ and lemma 3.2.2, we obtain

$$
\begin{aligned}
|III| &\leq c\fint_{Q_{\rho/2,\tau/4}} \int_0^1 \mu\left(\frac{|Du - D\ell|}{1+|Du|+|D\ell|}\right)(1+|Du|+|D\ell|)^{p-2}\,|Du - D\ell|\,ds\,dz \\
&\leq c\fint_{Q_{\rho/2,\tau/4}} \mu\left((1+|D\ell|)^{\frac{-p}{2}}\,|Du - D\ell|\,(1+|Du|+|D\ell|)^{\frac{p-2}{2}}\right) \\
&\qquad\qquad \times (1+|D\ell|)^{\frac{p-2}{2}}\,(1+|Du|+|D\ell|)^{\frac{p-2}{2}}\,|Du - D\ell|\,dz.
\end{aligned}
$$

Using Hölder's inequality, Jensen's inequality, the fact that $\mu \leq 1$, and $s \mapsto s^{1/2}$ is concave, we have

$$
\begin{aligned}
|III| \leq{}& (1+|D\ell|)^{\frac{p-2}{2}}\,c\left(\fint_{Q_{\rho/2,\tau/4}} (1+|Du|+|D\ell|)^{p-2}\,|Du - D\ell|^2\,dz\right)^{1/2} \\
&\times \left(\fint_{Q_{\rho/2,\tau/4}} \mu^2\left((1+|D\ell|)^{\frac{-p}{2}}\,|Du - D\ell|\,(1+|Du|+|D\ell|)^{\frac{p-2}{2}}\right)\,dz\right)^{1/2} \\
\leq{}& (1+|D\ell|)^{\frac{p-2}{2}}\,c\left(\fint_{Q_{\rho/2,\tau/4}} (1+|Du|+|D\ell|)^{p-2}\,|Du - D\ell|^2\,dz\right)^{1/2} \\
&\times \mu^{1/2}\left((1+|D\ell|)^{\frac{-p}{2}}\left(\fint_{Q_{\rho/2,\tau/4}} |Du - D\ell|^2\,(1+|Du|+|D\ell|)^{p-2}\,dz\right)^{1/2}\right).
\end{aligned}
$$

Now Theorem 3.3.1 implies

$$
|III| \;\leq\; (1+|D\ell|)^{\frac{p-2}{2}}\,c\mu^{1/2}\left((1+|D\ell|)^{\frac{-p}{2}}\,\xi^{1/2}\right)\xi^{1/2}.
$$

Combining all of the estimates for $|I|$, $|II|$, and $|III|$ gives the result. □

3.5 A-CALORIC APPROXIMATION

The cornerstone for proving Theorem 3.1.1 is the A-caloric approximation lemma. We point out that Scheven has recently produced an A-caloric approximation lemma for subquadratic problems [44]. Scheven's version, however, does not appear to be suitable for problems where only continuity of the coefficients is assumed. In this section, we prove a version that is compatible with the hypotheses for our problem. Before providing the argument for the lemma, we state the definition of an A-caloric function.

Defintion 3.5.1. *Let $A : \mathbb{R}^{nN} \times \mathbb{R}^{nN} \to \mathbb{R}^{nN}$ be a bilinear form with constant coefficients that satisfies*

$$\lambda|\tilde{w}|^2 \leq \langle A\tilde{w}, \tilde{w}\rangle, \quad \langle Aw, \tilde{w}\rangle \leq \Lambda|w||\tilde{w}|, \quad \textit{whenever } \omega, \tilde{\omega} \in \mathbb{R}^{Nn}, \tag{3.12}$$

where $\lambda, \Lambda > 0$. A map $f \in L^2(t_0 - \rho^2, t_0; W^{1,2}(B_\rho(x_0), \mathbb{R}^N))$ is called ***A-caloric in the cylinder*** *$Q_\rho(z_0)$ if it satisfies*

$$\fint_{Q_\rho(z_0)} f \cdot \varphi_t - \langle ADf, D\varphi\rangle \, dz = 0 \quad \textit{for all } \varphi \in C_0^\infty(Q_\rho(z_0), \mathbb{R}^N).$$

We now present the A-caloric approximation lemma. In the proof of the lemma we will exploit the convexity of the function $W : \mathbb{R}^k \to \mathbb{R}^k$ defined by

$$W(\xi) := (1 + |\xi|)^{\frac{p-2}{4}} \xi,$$

which satisfies the following estimate

$$2^{\frac{p-2}{4}} |V(\xi)| \leq |W(\xi)| \leq |V(\xi)| \quad \text{for all } \xi \in \mathbb{R}^k. \tag{3.13}$$

Lemma 3.5.1. *Given $\epsilon > 0$, $0 \leq \lambda \leq \Lambda$ and $p \in \left(\frac{2n}{n+2}, 2\right)$. There is a $\delta_0(n, p, \lambda, \Lambda, \epsilon) \leq 1$ with the following property: Whenever A is a bilinear form on R^{Nn} satisfying* (3.12) *and*

$\gamma \in (0,1]$*, and* w *is a map in*

$$C^0(t_0-\rho^2,t_0;L^2(B_\rho(x_0),\mathbb{R}^N))\bigcap L^p(t_0-\rho^2,t_0;W^{1,p}(B_\rho(x_0),\mathbb{R}^N))$$

with

$$\sup_{t_0-\rho^2\le t<t_0} \fint_{B_\rho(x_0)} \left|\frac{w(x,t)}{\rho}\right|^2 dx + \fint_{Q_\rho(z_0)} \left|\frac{w}{\rho}\right|^2 + \gamma^{p-2}\left|\frac{w}{\rho}\right|^p + |V(Dw)|^2\,dz \le 1$$

and

$$\left|\fint_{Q_\rho(z_0)} (w\cdot\varphi_t - \langle ADw, D\varphi\rangle)\,dz\right| \le \delta \sup_{Q_\rho(z_0)} |D\varphi|,$$

for all $\varphi \in C_0^\infty(Q_\rho(z_0);\mathbb{R}^N)$*, where* $\delta > 0$ *does not exceed the positive constant* δ_0*, then there exists a map*

$$f \in L^p(t_0-(\rho/4)^2,t_0;W^{1,p}(B_{\rho/4}(x_0),\mathbb{R}^N)) \cap L^2(t_0-(\rho/4)^2,t_0;W^{1,2}(B_{\rho/4}(x_0),\mathbb{R}^N))$$

which is A-caloric on $Q_{\rho/4}(z_0)$ *such that*

$$\fint_{Q_{\rho/4}(z_0)} \left|\frac{f}{\rho/4}\right|^2 + \gamma^{p-2}\left|\frac{f}{\rho/4}\right|^p + |V(Df)|^2\,dz \le 4^{n+8}$$

and

$$\fint_{Q_{\rho/4}} \left|\frac{w-f}{\rho/4}\right|^2 + \gamma^{p-2}\left|\frac{w-f}{\rho/4}\right|^p dz \le \epsilon.$$

Proof. Our strategy is along the same lines as the one used in [24].

Step 1: In this step we state the alternative to Lemma 3.5.1. For a detailed proof that we can make the reductions to the cylinder $Q_1 \equiv Q_1(0,0)$, instead of $Q_\rho(z_0)$, and consider only maps in $\varphi \in L^\infty(-1,0;W_0^{1,\infty}(B_1,\mathbb{R}^N))$, instead of $\varphi \in C_0^\infty(Q_1,\mathbb{R}^N)$, one should see [24]. We will proceed by contradiction. Suppose the lemma were not true, then we can find an $\epsilon > 0$, a sequence $\{w_k\}_{k=1}^\infty \subseteq C^0(-1,0;L^2(B_1(x_0),\mathbb{R}^N)\bigcap L^p(-1,0;W^{1,p}(B_1,\mathbb{R}^N))$, a sequence of bilinear forms $\{A_k\}$ satisfying our ellipticity and growth conditions, and $\gamma_k \in (0,1]$ such that

$$\sup_{-1\le t<0} ⨍_{B_1} |w_k(x,t)|^2 \, dx + ⨍_{Q_1} |w_k|^2 + \gamma_k^{p-2}|w_k|^p + |V(Dw_k)|^2 \, dz \le 1 \tag{3.14}$$

and

$$\left| \int_{Q_1} w_k \cdot \varphi_t - \langle A_k Dw_k, D\varphi \rangle \, dz \right| \le \frac{1}{k} \sup_{Q_1} |D\varphi| \tag{3.15}$$

for all $\varphi \in L^\infty(-1,0;W_0^{1,\infty}(B_1,\mathbb{R}^N))$ and $k \in \mathbb{N}$, but

$$⨍_{Q_{1/4}} 16|w_k - f|^2 + 4^p \gamma_k^{p-2}|w_k - f|^p \, dz > \epsilon \tag{3.16}$$

for all A_k-caloric maps f on $Q_{1/4}$ that satisfy

$$⨍_{Q_{1/4}} 16|f|^2 + \gamma_k^{p-2} 4^p |f|^p + |V(Df)|^2 \, dz \le 4^{n+8}.$$

Step 2: Here we obtain the weak convergence of $\{w_k\}_{k=1}^\infty$ in $L^2(Q_1,\mathbb{R}^N)$, $\{\widetilde{w}_k\}_{k=1}^\infty$ in $L^p(Q_1,\mathbb{R}^N)$ and $\{Dw_k\}_{k=1}^\infty$ in $L^p(Q_1,\mathbb{R}^{nN})$, where $\tilde{w}_k = \gamma_k^{\frac{p-2}{p}} w_k$. Note that $⨍_{Q_1} |\tilde{w}_k|^p \, dz$ ≤ 1 by (3.14). Now by part (i) of Lemma 3.2.1, Hölder's inequality, and (3.14), we have

$$\begin{aligned} ⨍_{Q_1} |Dw_k|^p \, dz &\le \frac{c(p)}{|Q_1|} \left[\int_{Q_1 \cap \{|Dw_k| \le 1\}} |V(Dw_k)|^p \, dz + \int_{Q_1 \setminus \{|Dw_k| \le 1\}} |V(Dw_k)|^2 \, dz \right] \\ &\le \frac{c(p)}{|Q_1|} \left[|Q_1| + \int_{Q_1} |V(Dw_k)|^2 \, dz \right] \le c(p). \end{aligned} \tag{3.17}$$

So by (3.14) we can extract a subsequence such that $w \in L^2(-1,0;L^2(B_1;\mathbb{R}^N)), \tilde{w}, u \in L^p(-1,0;W^{1,p}(B_1;\mathbb{R}^N))$, and

$$\begin{cases} w_k \rightharpoonup w & \text{weakly in } L^2(Q_1,\mathbb{R}^N) \\ \tilde{w}_k \rightharpoonup \tilde{w} & \text{weakly in } L^p(Q_1,\mathbb{R}^N) \\ Dw_k \rightharpoonup u & \text{weakly in } L^p(Q_1,\mathbb{R}^{nN}) \\ A_k \to \widetilde{A} & \text{as bilinear forms on } \mathbb{R}^{nN} \\ \gamma_k \to \gamma & \text{in } [0,1] \end{cases} . \tag{3.18}$$

Note that if $\gamma = 0$, then by the definition of $\widetilde{w}_k$ and $(3.18)_2$ we have $w_k \to 0$. By this fact and the fact that $f \equiv 0$ is A_k-caloric for all $k \in N$, we arrive at a contradiction trivially. So we assume that $\gamma \in (0,1]$. Notice $(3.18)_1$ and $(3.18)_5$ imply

$$\tilde{w}_k = \gamma_k^{\frac{p-2}{p}} w_k \rightharpoonup \gamma^{\frac{p-2}{p}} w$$

weakly in $L^2(-1,0;L^2(B_1,\mathbb{R}^N))$. As $(3.18)_2$ gives $\tilde{w}_k \rightharpoonup \tilde{w}$ in $L^p(-1,0,L^p(B_1,\mathbb{R}^N))$, we see $\tilde{w} = \gamma^{\frac{p-2}{p}} w$. Hence we must have $u = Dw$ by uniqueness. Using the weak lower semicontinuity of $v \mapsto \int_{Q_1} |v|^2\,dz$ and the weak lower semicontinuity of $v \mapsto \int_{Q_1} |v|^p\,dz$, the estimate (3.13), the convexity of W, and (3.14), we have

$$\begin{aligned} &⨍_{Q_1} 4|w|^2 + 4\gamma^{p-2}|w|^p + |V(Dw)|^2\,dz \\ &\quad \le ⨍_{Q_1} 4|w|^2 + 4\gamma^{p-2}|w|^p + 4|W(Dw)|^2\,dz \\ &\quad \le 4\lim_{k\to\infty} ⨍_{Q_1} |w_k|^2 + \gamma^{p-2}|w_k|^p + |W(Dw_k)|^2\,dz \\ &\quad \le 4\lim_{k\to\infty} ⨍_{Q_1} |w_k|^2 + \gamma^{p-2}|w_k|^p + |V(Dw_k)|^2\,dz \le 4. \end{aligned} \tag{3.19}$$

Next, we need to show w is $\widetilde{A}$-caloric on Q_1. Let $\varphi \in C_0^\infty(Q_1, \mathbb{R}^N)$. Then,

$$\begin{aligned}
&\fint_{Q_1} \left(w \cdot \varphi_t - \left\langle \widetilde{A} Dw, D\varphi \right\rangle \right) dz \\
&= \fint_{Q_1} \left((w - w_k)\varphi_t - \left\langle \widetilde{A}(Dw - Dw_k), D\varphi \right\rangle \right) dz \\
&\quad + \fint_{Q_1} \left\langle (A_k - \widetilde{A}) Dw_k, D\varphi \right\rangle dz + \fint_{Q_1} \left(w_k \cdot \varphi_t - \langle A_k Dw_k, D\varphi \rangle \right) dz.
\end{aligned}$$

The first integral converges to zero by $(3.18)_1$ and $(3.18)_3$ as $\varphi \in C_0^\infty(Q_1, \mathbb{R}^N)$. The second integral goes to zero as (3.17) and $(3.18)_4$ hold. By assumption (3.15), the last integral tends to zero as k tends to infinity. Thus,

$$\fint_{Q_1} \left(w \cdot \varphi_t - \left\langle \widetilde{A} Dw, D\varphi \right\rangle \right) dz = 0, \qquad \text{for all } \varphi \in C_0^\infty(Q_1, \mathbb{R}^N). \tag{3.20}$$

Using the linearity of the above parabolic system, one can use a mollification argument to show $w \in C^\infty(B_1 \times (-1, 0], \mathbb{R}^N)$.

Step 3: In this step, we will obtain the strong convergence in $L^p(Q_1, \mathbb{R}^N)$ of $\{\widetilde{w}_k\}_{k=1}^\infty$. First note that if $w_k \to w$ strongly in L^p and $\gamma_k \to \gamma$ in $(0, 1]$, then we must have that $\widetilde{w}_k \to \widetilde{w}$ strongly in L^p. So we only need to obtain the strong convergence of $\{w_k\}_{k=1}^\infty$ to w in L^p. We first prove a useful inequality. Letting $\varphi \in L^\infty(-1, 0; W_0^{1,\infty}(B_1, \mathbb{R}^N))$, we see that (3.15) gives

$$\left| \int_{Q_1} w_k \cdot \varphi_t \, dz \right| \le \left| \int_{-1}^0 \int_{B_1} \langle A_k Dw_k, D\varphi \rangle \, dx dt \right| + \frac{1}{k} \sup_{-1 \le t \le 0} \| D\varphi(\cdot, t) \|_{L^\infty(B_1)} .$$

Using Hölder's inequality and (3.14),

$$\left| \int_{Q_1} w_k \cdot \varphi_t \, dz \right| \le \| A_k \| \left(\int_{-1}^0 \| D\varphi(\cdot, t) \|_{L^{\frac{p}{p-1}}(B_1)}^{\frac{p}{p-1}} \, dt \right)^{\frac{p-1}{p}}$$

$$+ \frac{1}{k} \sup_{-1 \le t \le 0} \|D\varphi(\cdot, t)\|_{L^\infty(B_1)}.$$

Let $-1 < s_1 < s_2 < 0$. Then choose $\beta > 0$ sufficiently small to define

$$\zeta_\nu := \begin{cases} 0, & -1 \le t \le s_1 - \beta, \\ \frac{1}{\beta}(t - s_1 + \beta), & s_1 - \beta \le t \le s_1, \\ 1, & s_1 \le t \le s_2, \\ -\frac{1}{\beta}(t - s_2 - \beta), & s_2 \le t \le s_2 + \beta, \\ 0, & s_2 + \beta \le t \le 1 \end{cases}.$$

Now let $\varphi(x, t) = \zeta_\beta(t)\Psi(x)$ with $\Psi \in C_0^\infty(B_1, \mathbb{R}^N)$. Substituting φ into our above inequality, we have

$$\left| \int_{B_1} \left(\frac{1}{\beta} \int_{s_1 - \beta}^{s_1} w_k(x,t)\, dt - \frac{1}{\beta} \int_{s_2}^{s_2 + \beta} w_k(x,t)\, dt \right) \cdot \Psi(x)\, dx \right|$$
$$\le \|A_k\| \left(\int_{-1}^{0} (\zeta_\beta(t))^{\frac{p}{p-1}}\, dt \right)^{\frac{p-1}{p}} \|D\Psi\|_{L^{\frac{p}{p-1}}(B_1)}$$
$$+ \frac{1}{k} \left[\sup_{-1 \le t \le 0} \zeta_\beta(t) \right] \|D\Psi\|_{L^\infty(B_1)}$$
$$\le \left[\|A_k\| \left(s_2 - s_1 + 2\beta \left(\frac{p-1}{2p+1} \right) \right)^{\frac{p-1}{p}} + \frac{1}{k} \right] \|D\Psi\|_{L^\infty(B_1)}.$$

The Sobolev embedding theorem gives $\|D\Psi\|_{L^\infty} \le c(n, r) \|\Psi\|_{W_0^{r, \frac{p}{p-1}}(B_1)}$ for $r > \frac{n(p-1)+p}{p}$. So we obtain

$$\left| \int_{B_1} \left(\frac{1}{\beta} \int_{s_1 - \beta}^{s_1} w_k(x,t)\, dt - \frac{1}{\beta} \int_{s_2}^{s_2 + \nu} w_k(x,t)\, dt \right) \cdot \Psi(x)\, dx \right|$$
$$\le c(n, r) \left(\|A_k\| \left(s_2 - s_1 + 2\beta \left(\frac{p-1}{2p+1} \right) \right)^{\frac{p-1}{p}} + \frac{1}{k} \right) \|\Psi\|_{W_0^{r, \frac{p}{p-1}}(B_1)}.$$

Now letting β tend to zero, yields

$$\left|\int_{B_1} (w_k(\cdot, s_1) - w_k(\cdot, s_2)) \cdot \Psi \, dx\right| \leq c(n,r)\left(\|A_k\| (s_2 - s_1)^{\frac{p-1}{p}} + \frac{1}{k}\right) \|\Psi\|_{W_0^{r,\frac{p}{p-1}}(B_1)}$$

for almost every s_1, s_2 such that $-1 < s_1 < s_2 < 0$ and for any $\Psi \in C_0^\infty(B_1, \mathbb{R}^N)$. By a density argument, the last inequality is valid for each $\Psi \in W_0^{r,\frac{p}{p-1}}(B_1, \mathbb{R}^N)$. Taking the supremum over all $\Psi \in W_0^{r,\frac{p}{p-1}}(B_1, \mathbb{R}^N)$ with $\|\Psi\|_{W_0^{r,\frac{p}{p-1}}(B_1)} \leq 1$, we find

$$\|w_k(\cdot, s_1) - w_k(\cdot, s_2)\|_{W_0^{-r,p}(B_1,\mathbb{R}^N)} \leq c(r,n)\left(\|A_k\| (s_2 - s_1)^{\frac{p-1}{p}} + \frac{1}{k}\right)$$

for almost every $-1 < s_1 < s_2 < 0$ when $r > \frac{n(p-1)+p}{p}$. In particular, for $0 < h < 1$,

$$\int_{-1}^{-h} \|w_k(\cdot, t) - w_k(\cdot, t+h)\|_{W^{-r,p}(B_1)}^p \, dt \leq c(n, M, r)(h^{\frac{p-1}{p}} + \frac{1}{k})^p.$$

As the left hand side in the above inequality tends to zero as h tends to 0 for each fixed $k \in \mathbb{N}$, the convergence above is uniform with respect to $k \in \mathbb{N}$. Furthermore we see that $\{w_k\}_{k=1}^\infty$ is uniformly bounded in $L^p(-1, 0; W^{1,p}(B_1, \mathbb{R}^N))$ by (3.14) as $\{\gamma_k\}_{k=1}^\infty \subset (0, 1]$. With the choice $(X, B, Y) = (W^{1,p}(B_1), L^p(B_1), W^{-\ell,p}(B_1))$, we have, possibly taking a subsequence, that $w_k \to w$ strongly in $L^p(Q_1)$ by Theorem 5 in [46]. Hence $\widetilde{w}_k \to \widetilde{w}$ strongly in L^p.

Step 4: Now we need to show $w_k \to w$ strongly in $L^2(Q_1)$. Note that $w_k \to w$ almost everywhere in Q_1 as $w_k \to w$ strongly in L^p. Hence $w_k \to w$ almost everywhere on $Q_{1/4}$. By Egoroff's theorem, given $\eta > 0$, there exists $M \subseteq Q_{1/4}$ such that $|Q_{1/4} \backslash M| < \eta$ and

$w_k \to w$ uniformly on M. Thus,

$$\lim_{k\to\infty} \int_{Q_{1/4}} |w_k - w|^2 \, dz = \lim_{k\to\infty} \int_{Q_{1/4}\setminus M} |w_k - w|^2 \, dz.$$

Choosing $\beta = \frac{p(n+2)}{n} > 2$ and using Hölder's inequality, we see

$$\int_{Q_{1/4}\setminus M} |w_k - w|^2 \, dz \quad < \quad \eta^{\frac{\beta-2}{\beta}} \left(\int_{Q_{1/4}\setminus M} |w_k - w|^\beta \, dz \right)^{2/\beta}.$$

By Proposition 3.1 in [21], there exists a constant c depending only on $N, p,$ and 2 such that

$$\begin{aligned} \int_{Q_{1/4}\setminus M} |w_k - w|^2 \, dz <& \eta^{\frac{\beta-2}{\beta}} c \left(\int_{Q_{1/4}\setminus M} |Dw_k - Dw|^p \, dz \right)^{\frac{2}{\beta}} \\ & \times \left(\sup_{-1/16<t<0} \int_{B_{1/4}} |w_k - w|^2 \, dx \right)^{2p/\beta n}. \end{aligned}$$

By (3.14) and the fact $\beta > 2$, we have

$$\lim_{k\to\infty} \int_{Q_{1/4}} |w_k - w|^2 \, dz = \lim_{k\to\infty} \int_{Q_{1/4}\setminus M} |w_k - w|^2 \, dz = 0.$$

So $w_k \to w$ strongly in $L^2(Q_{1/4})$.

Step 5: We represent the unique solution to

$$\begin{cases} \displaystyle\int_{Q_{1/4}} \left(v_k \cdot \partial_t \varphi - \langle A_k D v_k, D\varphi \rangle \right) dz = 0, & \text{for all } \varphi \in C_0^\infty(Q_{1/4}, \mathbb{R}^N) \\ v_k = w, & \text{on } \partial_{\text{par}} Q_{1/4} \end{cases} \tag{3.21}$$

by $\{v_k\}_{k=1}^\infty \subseteq C^0(-(1/4)^2, 0; L^2(B_{1/4}, \mathbb{R}^N)) \bigcap L^2(-(1/4)^2, 0; W^{1,2}(B_{1/4}, \mathbb{R}^N))$. We then set out to prove $v_k \to w$ and $V(Dv_k) \to V(Dw)$ in $L^2(Q_{1/4})$. This has been shown in

[44], but is included here for completeness. Since v_k and w are smooth and $v_k - w = 0$ on $\partial_{\text{par}} Q_{1/4}$, we can test (3.20) and (3.21) with $\varphi = v_k - w$ and via integration by parts obtain

$$\fint_{Q_{1/4}} \frac{\partial}{\partial t}|v_k - w|^2\,dz + \fint_{Q_{1/4}} \langle A_k(Dv_k - Dw), (Dv_k - Dw)\rangle\,dz = \fint_{Q_{1/4}} \langle (A - A_k)(Dw), (Dv_k - Dw)\rangle\,dz \tag{3.22}$$

Since

$$\int_{Q_{1/4}} \frac{\partial}{\partial t}|v_k - w|^2\,dz = \int_{-1/16}^{0} \frac{\partial}{\partial t}\int_{B_{1/4}} |v_k - w|^2\,dx\,dt = \int_{B_1} |v_k(\cdot,0) - w(\cdot,0)|^2\,dx \geq 0,$$

we can use (3.12) on the left side of (3.22) and Young's inequality on the right side of (3.22) to get

$$\frac{\lambda}{2}\fint_{Q_{1/4}} |Dv_k - Dw|^2\,dz \leq \frac{|A_k - A|}{2\lambda}\fint_{Q_{1/4}} |Dw|^2\,dz.$$

Since $Dw \in L^2(Q_{1/4})$, the right hand side tends to zero as k tends to infinity. Hence, $Dv_k \to Dw$ strongly in $L^2(Q_{1/4})$. The Sobolev embedding theorem on time slices and the Dominated Convergence Theorem give $v_k \to w$ strongly in $L^2(Q_{1/4})$. Thus

$$\lim_{k\to\infty} |v_k - w|^2 + |V(Dv_k) - V(Dw)|^2\,dx = 0 \tag{3.23}$$

by the convergence of Dv_k in $L^2(Q_{1/4})$ to Dw and (iv) of Lemma (3.2.1). By Hölder's inequality and (3.23), we also have $v_k \to w$ in $L^p(Q_{1/4})$.

Step 6: In this step, we obtain the contradiction. From the convergence discussed above, we have arrived at

$$\begin{aligned}
\fint_{Q_{1/4}} &|w_k - v_k|^2 + |V(Dw_k) - V(Dv_k)|^2 \, dz \\
&\leq 2 \fint_{Q_{1/4}} |w_k - w|^2 + |V(Dw_k) - V(Dw)|^2 \, dz \\
&\quad + 2 \fint_{Q_{1/4}} |w - v_k|^2 + |V(Dw) - V(Dv_k)|^2 \, dz \to 0.
\end{aligned} \tag{3.24}$$

Similarly,

$$\fint_{Q_{1/4}} |\widetilde{w}_k - \gamma_k^{\frac{p-2}{p}} v_k|^p \, dz \to 0$$

by the strong convergence of $\widetilde{w}_k \to \widetilde{w}$ in L^p and the convergence of $\gamma_k^{\frac{p-2}{p}} v_k \to \gamma^{\frac{p-2}{p}} w$ in L^p. Hence we have shown

$$\lim_{k\to\infty} \fint_{Q_{1/4}} 16|w_k - v_k|^2 + 4^p \gamma_k^{p-2} |w_k - v_k|^p \, dz = 0. \tag{3.25}$$

From (3.23), the strong convergence of v_k in L^p, the convergence of γ_k to γ in $(0, 1]$, and the bound (3.19), we see

$$\begin{aligned}
\lim_{k\to\infty} \fint_{Q_{1/4}} &16|v_k|^2 + \gamma_k^{p-2} 4^p |v_k|^p + |V(Dv_k)|^2 \, dz \\
&\leq 4 \fint_{Q_{1/4}} 16|w|^2 + \gamma^{p-2} 4^p |w|^p + |V(Dw)|^2 \, dz \\
&\leq 4^{n+6} \fint_{Q_1} |w|^2 + \gamma^{p-2} |w|^p + |V(Dw)|^2 \, dz \leq 4^{n+7}.
\end{aligned}$$

So for k large enough,

$$\fint_{Q_{1/4}} 16|v_k|^2 + \gamma_k^{p-2} 4^p |v_k|^p + |V(Dv_k)|^2 \, dz \leq 4 \cdot 4^{n+7}.$$

Since each v_k is A_k-caloric, there is a large enough k such that (3.25) contradicts (3.16). □

3.6 POINCARÉ INEQUALITY

Before setting out the prove the main result, we prove two useful Poincaré inequalities. It is important to note that the following inequalities can only be applied to solutions of (3.1). These results will be used in Section 3.7 to show the smallness assumptions in the excess decay estimate can be met for $z_0 \in \Sigma_1 \cup \Sigma_2$. We begin this section by proving a lemma that will enable us to prove the first Poincaré inequality. Both proofs are along the same lines as Lemma 9.1 and Lemma 9.2 in [44]. The argument for the second inequality is similar to the proof of (3.4) in Lemma 3.2 in [6].

Lemma 3.6.1. *Let $u \in L^p(-T, 0; W^{1,p}(\Omega, \mathbb{R}^N)) \cap C^0(-T, 0; L^2(\Omega, \mathbb{R}^N))$ be a solution to* (3.1) *in Ω_T, where the assumptions* (3.2) *and* (3.3) *hold for u. Assume $Q_\sigma(z_0) \subseteq \Omega_T$ is a parabolic cylinder with $\sigma \in (0, 1)$. Further let $\ell : \mathbb{R}^n \to \mathbb{R}^N$ be an affine map. Then for any $r, s \in (-\sigma^2, 0)$ and arbitrary test function $\chi \in C_0^\infty(B_\sigma, \mathbb{R}^N)$, there exists $c_3 = c_3(p, L)$ such that*

$$\begin{aligned}
&\left| \int_{B_\sigma} (u(x,r) - u(x,s)) \cdot \chi(x)\, dx \right| \\
&\qquad \le c_3 (r-s)^{\frac{1}{p}} \left\| D\chi \right\|_{L^p} |Q_\sigma|^{\frac{p-1}{p}} \left(\fint_{Q_\sigma} |Du - D\ell|^p \, dz \right)^{\frac{p-1}{p}} \\
&\qquad + c_3 (r-s)^{\frac{1}{p}} \left\| D\chi \right\|_{L^p} |Q_\sigma|^{\frac{p-1}{p}} (1 + |D\ell|)^{p-1} \omega \left(\fint_{Q_\sigma} |u - \ell(x_0)|^2 \, dz \right)^{\frac{p-1}{p}} \\
&\qquad + c_3 (r-s)^{\frac{1}{p}} \left\| D\chi \right\|_{L^p} |Q_\sigma|^{\frac{p-1}{p}} (1 + |D\ell|)^{p-1} \omega(\sigma^2)^{\frac{p-1}{p}}.
\end{aligned}$$

Proof. For notational convenience, we will eliminate the centers x_0 and z_0 from all balls and cylinders. Let $\varphi(x, t) = \zeta(t)\chi(x)$, where $\chi \in C_0^\infty(B_\sigma, \mathbb{R}^N)$ and ζ is a Lipschitz continuous cut-off function such that for $s, r \in (-\sigma^2, 0)$

$$\zeta(t) := \begin{cases} \frac{1}{h}(t-s), & \text{for } s < t \leq s+h, \\ 1, & \text{for } s+h < t \leq r-h, \\ \frac{-1}{h}(t-r), & \text{for } r-h < t \leq r, \\ 0, & \text{elsewhere.} \end{cases}$$

Substituting φ into the weak formulation of (3.1) gives

$$\int_{Q_\sigma} u \cdot \zeta_t \chi \, dz = \int_{Q_\sigma} a(z, u, Du) \cdot D\chi \zeta \, dz.$$

Letting h tend to zero, we see

$$\int_{B_\sigma} (u(x,r) - u(x,s)) \, \chi \, dx = \int_s^r \int_{B_\sigma} a(z, u, Du) \cdot D\chi \, dx \, dt. \tag{3.26}$$

We now need to establish an upper bound for the right hand side of the above equation. Note

$$\begin{aligned} \left| \int_{B_\sigma} a(z, u, Du) \cdot D\chi \, dx \right| \leq & \int_{B_\sigma} |a(z, u, Du) - a(z, u, D\ell)| \, |D\chi| \, dx \\ & + \int_{B_\sigma} |a(z, u, D\ell) - a(z_0, \ell(x_0), D\ell)| \, |D\chi| \, dx \\ =: & I + II. \end{aligned}$$

As in the proof of Theorem 3.3.1, we use $(3.3)_2$ and Lemma 3.2.2 to obtain

$$I \leq c \int_{B_\sigma} (1 + |Du| + |D\ell|)^{p-2} \, |Du - D\ell| |D\chi| \, dx.$$

Using Hölder's Inequality and the fact that $p < 2$, we have

$$
\begin{aligned}
I &\leq c \|D\chi\|_{L^p} \left(\int_{B_\sigma} (1 + |Du| + |D\ell|)^{\frac{p(p-2)}{p-1}} |Du - D\ell|^{\frac{p}{p-1}} \, dx \right)^{\frac{p-1}{p}} \\
&\leq c \|D\chi\|_{L^p} \left(\int_{B_\sigma} (1 + |Du| + |D\ell|)^{p-2} |Du - D\ell|^2 \, dx \right)^{\frac{p-1}{p}} \\
&\leq c \|D\chi\|_{L^p} \left(\int_{B_\sigma} |Du - D\ell|^p \, dx \right)^{\frac{p-1}{p}} .
\end{aligned}
$$

In order to obtain an upper bound for II, we use the continuity estimate $(3.3)_1$, Hölder's Inequality, $p < 2$, and the fact that $\omega \leq 1$ is sublinear as follows:

$$
\begin{aligned}
II &\leq c \int_{B_\sigma} \omega \left(|u - \ell(x_0)|^2 + \sigma^2 \right) (1 + |D\ell|)^{p-1} |D\chi| \, dx \\
&\leq c \, (1 + |D\ell|)^{p-1} \|D\chi\|_{L^p} \left(\int_{B_\sigma} \omega^{\frac{p}{p-1}} \left(|u - \ell(x_0)|^2 \right) \, dx + \omega^{\frac{p}{p-1}} (\sigma^2) \right)^{\frac{p-1}{p}} \\
&\leq c \, (1 + |D\ell|)^{p-1} \|D\chi\|_{L^p} \left(\int_{B_\sigma} \omega \left(|u - \ell(x_0)|^2 \right) \, dx + \omega(\sigma^2) \right)^{\frac{p-1}{p}}
\end{aligned}
$$

Combining these two estimates with (3.26) and using Hölder's Inequality gives

$$
\begin{aligned}
\left| \int_{B_\sigma} (u(x,r) - u(x,s)) \cdot \chi \, dx \right| &\leq c(r-s)^{\frac{1}{p}} \|D\chi\|_{L^p} \left(\int_{Q_\sigma} |Du - D\ell|^p \, dz \right)^{\frac{p-1}{p}} \\
&\quad + c(r-s)^{\frac{1}{p}} \|D\chi\|_{L^p} (1 + |D\ell|)^{p-1} \left(\int_{Q_\sigma} \omega \left(|u - \ell(x_0)|^2 \right) \, dz \right)^{\frac{p-1}{p}} \\
&\quad + c(r-s)^{\frac{1}{p}} \|D\chi\|_{L^p} (1 + |D\ell|)^{p-1} \left(\int_{Q_\sigma} \left[\omega(\sigma^2) \right] \, dz \right)^{\frac{p-1}{p}} .
\end{aligned}
$$

Noting $\omega \leq 1$ and concave, we can use Jensen's Inequality to arrive at the result. $\square$

Theorem 3.6.1. *(Poincaré's Inequality) Assume* $u \in L^p(-T, 0; W^{1,p}(\Omega, \mathbb{R}^N)) \cap$

$L^2(-T,0;L^2(\Omega,\mathbb{R}^N))$ *is a solution to* (3.1) *in* Ω_T *under the assumptions* (3.2) *and* (3.3), *and* $0<\rho<1$ *is a radius such that the parabolic cylinders* $Q_{2\rho}(z_0)\subseteq\Omega_T$. *Let* $A\in\mathbb{R}^{nN}$. *Then there exists* $c_4=c_4(p,n,N,M,\nu)$ *such that:*

$$\fint_{Q_\rho(z_0)}\left|\frac{u-(u)_{z_0,\rho}-A(x-x_0)}{\rho}\right|^2+\left|\frac{u-(u)_{z_0,\rho}-A(x-x_0)}{\rho}\right|^p dz$$
$$\leq c_4\left(\fint_{Q_{2\rho}(z_0)}|Du-A|^p\,dz\right)^{\frac{2(p-1)}{p}}+c_4\fint_{Q_{2\rho}(z_0)}|Du-A|^p\,dz$$
$$+(1+|A|)^p\,c_4\left[\omega^{\frac{2(p-1)}{p}}\left(\fint_{Q_{2\rho}(z_0)}|u-(u)_{2\rho}|^2\,dz\right)+\omega^{\frac{2(p-1)}{p}}(4\rho^2)\right].$$

Proof. For notational convenience, we will assume all cylinders are centered at $z_0=(0,0)$ and all balls are centered at $x_0=0$. Let σ and α be such that $\rho\leq\sigma<\alpha\leq 2\rho$. We choose a symmetric smoothing kernel $\psi\in C_0^\infty(B_1)$ with $\int_{B_1}\psi\,dx=1$ and $\|\psi\|_{L^\infty}+\|D\psi\|_{L^\infty}\leq 2(n+2)|B_1|^{-1}$. We rescale ψ to get $\psi_\rho:=\rho^{-n}\psi\left(\frac{x}{\rho}\right)$ which satisfy

$$\|D\psi_\rho\|_{L^p}\leq c(n)\rho^{-1-n\left(\frac{p-1}{p}\right)}\leq c(n)\sigma^{-1-n\left(\frac{p-1}{p}\right)}\tag{3.27}$$
$$\|\psi_\rho\|_{L^\infty}\leq c(n)\rho^{-n}\tag{3.28}$$

A variety of means will be applied throughout this proof. The means and ψ-means over a time slice for $t\in(-\rho^2,0)$ will be defined by

$$(\widetilde{u})_\rho(t):=\fint_{B_\rho}u(x,t)\,dx\quad\text{and}\quad(\widetilde{u})^\psi_\rho(t):=\int_{B_\rho}u(x,t)\psi_\rho(x)\,dx,$$

respectively. The means and ψ-means over cylinders, Q_ρ, will be defined by

$$(u)_\rho:=\fint_{Q_\rho}u(z)\,dz=\fint_{-\rho^2}^0(\widetilde{u})_\rho(t)\,dt\quad\text{and}\quad(u)^\psi_\rho:=\fint_{-\rho^2}^0(\widetilde{u})^\psi_\rho(t)\,dt.$$

Throughout the following proof, we will use several functions repeatedly. For notational brevity:

$$\Psi_q(r) = \fint_{Q_r} \left| \frac{u(z) - (u)_q - Ax}{r} \right|^2 + \left| \frac{u(z) - (u)_q - Ax}{r} \right|^p dz,$$
$$\Phi^q(r) = \left(\fint_{Q_r} |Du - A|^p \, dz \right)^{q(p-1)} + \left(\fint_{Q_r} |Du - A|^p \, dz \right)^q,$$
$$\Upsilon_q^s(r) = (1 + |A|)^{ps} \left[\omega^s \left(\fint_{Q_r} |u - (u)_q|^2 \, dz \right) + \omega^s(r^2) \right].$$

Also, throughout the proof we write $\Phi(r)$ and $\Upsilon_q(r)$ for $\Phi^1(r)$ and $\Upsilon_q^1(r)$, respectively. First note that by Hölder's inequality we have

$$\begin{aligned}
\Psi_\rho(\sigma) \leq & \fint_{-\sigma^2}^0 \left(\fint_{B_\sigma} \left| \frac{u(z) - (u)_\rho - Ax}{\sigma} \right|^2 dx \right)^{1-\frac{p}{2}} \\
& \times \left(\fint_{B_\sigma} \left| \frac{u(z) - (u)_\rho - Ax}{\sigma} \right|^2 dx \right)^{\frac{p}{2}} dt \\
& + \fint_{Q_\sigma} \left| \frac{u(z) - (u)_\rho - Ax}{\sigma} \right|^p dz \\
\leq & \sup_{s \in (-\sigma^2, 0)} \left(\fint_{B_\sigma} \left| \frac{u(x,s) - (u)_\rho - Ax}{\sigma} \right|^2 dx \right)^{1-\frac{p}{2}} \\
& \times \fint_{-\sigma^2}^0 \left(\fint_{B_\sigma} \left| \frac{u(z) - (u)_\rho - Ax}{\sigma} \right|^2 dx \right)^{\frac{p}{2}} dt \\
& + \fint_{Q_\sigma} \left| \frac{u(z) - (u)_\rho - Ax}{\sigma} \right|^p dz.
\end{aligned}$$

From Theorem 3.3.2, we know

$$\begin{aligned}
\Psi_\rho(\sigma) \leq & c \left[\left(\frac{\alpha}{\alpha - \sigma} \right)^2 \Psi_\rho(\alpha) + \Upsilon_\rho(\alpha) \right]^{1-\frac{p}{2}} \\
& \times \left(\fint_{-\sigma^2}^0 \left(\fint_{B_\sigma} \left| \frac{u(z) - (u)_\rho - Ax}{\sigma} \right|^2 dx \right)^{\frac{p}{2}} dt \right)
\end{aligned} \tag{3.29}$$

$$+ \fint_{Q_\sigma} \left| \frac{u(z) - (u)_\rho - Ax}{\sigma} \right|^p \, dz.$$

Next we want to estimate the term involving the power of $\frac{p}{2}$. Note that

$$\left(\fint_{-\sigma^2}^{0} \left(\fint_{B_\sigma} \left| \frac{u(z) - (u)_\rho - Ax}{\sigma} \right|^2 dx \right)^{\frac{p}{2}} dt \right)$$
$$\leq c \left(\fint_{-\sigma^2}^{0} \left(\fint_{B_\sigma} \left| \frac{u(z) - (u)_\sigma - Ax}{\sigma} \right|^2 dx \right)^{\frac{p}{2}} dt \right) \tag{3.30}$$
$$+ c\sigma^{-p} |(u)_\rho - (u)_\sigma|^p =: I + II.$$

Now note that

$$II \leq c\sigma^{-p} |(u)_\rho - (u)_\rho^\psi|^p + c\sigma^{-p} |(u)_\rho^\psi - (u)_\sigma^\psi|^p + c\sigma^{-p} |(u)_\sigma - (u)_\sigma^\psi|^p$$
$$=: II_1 + II_2 + II_3.$$

We begin by using Minkowski's inequality:

$$II_1 \leq c\sigma^{-p} \fint_{-\rho^2}^{0} \fint_{B_\rho} \left| (\widetilde{u})_\rho^\psi(t) - (\widetilde{u})_\rho(t) \right|^p \, dx \, dt$$
$$\leq c\sigma^{-p} \fint_{Q_\rho} |u - (\widetilde{u})_\rho(t) - Ax|^p + \left| u - (\widetilde{u})_\rho^\psi(t) - Ax \right|^p \, dz$$

From Poincaré's inequality for functions with vanishing mean value and vanishing ψ-mean value and noting $\sigma \in (\rho, 2\rho)$, we obtain

$$II_1 \leq c\sigma^{-p} \rho^p \fint_{Q_\rho} |Du - A|^p \, dz \leq c \fint_{Q_{2\rho}} |Du - A|^p \, dz.$$

Similarly, we can show

$$II_3 \leq c \fint_{Q_{2\rho}} |Du - A|^p \, dz.$$

Finally note that by the symmetry of ψ_ρ

$$\begin{aligned} II_2 &\leq c\sigma^{-p} \fint_{-\sigma^2}^{0} \left| (\widetilde{u})_\sigma^\psi(t) - (\widetilde{u})_\rho^\psi(t) \right|^p \, dt \\ &\leq \fint_{-\sigma^2}^{0} \left(\int_{B_\rho} \left| \frac{\left(u - (\widetilde{u})_\sigma^\psi(t) - Ax\right) \psi_\rho}{\sigma} \right| \, dx \right)^p \, dt. \end{aligned}$$

Employing Hölder's inequality and referring to (3.28), we see

$$\begin{aligned} II_2 &\leq \fint_{-\sigma^2}^{0} \left(\left(\int_{B_\rho} \left| \frac{u - (\widetilde{u})_\sigma^\psi(t) - Ax}{\sigma} \right|^2 \, dx \right)^{\frac{1}{2}} \sigma^{\frac{-n}{2}} |B_\rho|^{\frac{1}{2}} \right)^p \, dt \\ &\leq c \fint_{-\sigma^2}^{0} \left(\fint_{B_\rho} \left| \frac{u - (\widetilde{u})_\sigma^\psi(t) - Ax}{\sigma} \right|^2 \, dx \right)^{\frac{p}{2}} \, dt. \end{aligned}$$

Combining the estimates of $II_1, II_2,$ and II_3, and substituting them into (3.29), we obtain

$$\begin{aligned} \Psi_\rho(\sigma) \leq & c \left[\left(\frac{\alpha}{\alpha - \sigma} \right)^2 \Psi_\rho(\alpha) + \Upsilon_\rho(\alpha) \right]^{1 - \frac{p}{2}} \\ & \times \left(\fint_{-\sigma^2}^{0} \left(\fint_{B_\sigma} \left| \frac{u(z) - (u)_\sigma - Ax}{\sigma} \right|^2 \, dx \right)^{\frac{p}{2}} \, dt \right) \\ & + c \left[\left(\frac{\alpha}{\alpha - \sigma} \right)^2 \Psi_\rho(\alpha) + \Upsilon_\rho(\alpha) \right]^{1 - \frac{p}{2}} \left[\Phi(2\rho) + \Upsilon_\rho^{p-1}(2\rho) \right] \\ & + c \left[\fint_{Q_\sigma} \left| \frac{u(z) - (u)_\sigma - Ax}{\sigma} \right|^p \, dz + \Phi(2\rho) + \Upsilon_\rho^{p-1}(2\rho) \right]. \end{aligned}$$

By applying Poincaré's inequality for vanishing mean value functions and recalling that

$\sigma \in (\rho, 2\rho)$

$$\fint_{-\sigma^2}^{0} \left(\fint_{B_\sigma} \left| \frac{u(z) - (u)_\sigma - Ax}{\sigma} \right|^2 dx \right)^{\frac{p}{2}} dt \leq c \fint_{Q_\sigma} |Du - A|^p \, dz$$
$$\leq c \fint_{Q_{2\rho}} |Du - A|^p \, dz.$$

Finally employing Sobolev's inequality, we see

$$\fint_{Q_\sigma} \left| \frac{u(z) - (u)_\sigma - Ax}{\sigma} \right|^p dz \leq c \fint_{Q_\sigma} |Du - A|^p \, dz$$
$$\leq c \fint_{Q_{2\rho}} |Du - A|^p \, dz.$$

Hence,

$$\Psi_\rho(\sigma) \leq c \left[\left(\frac{\alpha}{\alpha - \sigma} \right)^2 \Psi_\rho(\alpha) + \Upsilon_\rho(\alpha) \right]^{1 - \frac{p}{2}} \left[\Phi(2\rho) + \Upsilon_\rho^{p-1}(2\rho) \right]$$
$$+ c \left[\Phi(2\rho) + \Upsilon_\rho^{p-1}(2\rho) \right].$$

Next, Young's Inequality yields

$$\Psi_\rho(\sigma) \leq \frac{1}{2} \Psi_\rho(\alpha) + \Upsilon_\rho(\alpha) + \left(\frac{\alpha}{\alpha - \sigma} \right)^{\frac{2(2-p)}{p}} c \left[\Phi^{\frac{2}{p}}(2\rho) + \Upsilon_\rho^{\frac{2(p-1)}{p}}(2\rho) \right]$$
$$+ c \left[\Phi(2\rho) + \Upsilon_\rho^{p-1}(2\rho) \right].$$

Since $\alpha \in (\rho, 2\rho)$,

$$\fint_{Q_\alpha} |u - (u)_\rho|^2 \, dz \leq c \fint_{Q_{2\rho}} |u - (u)_\rho|^2 \, dz$$
$$\leq c \fint_{Q_{2\rho}} |u - (u)_{2\rho}|^2 \, dz + c|(u)_{2\rho} - (u)_\rho|^2 \, dz$$

$$\leq c \fint_{Q_{2\rho}} |u-(u)_{2\rho}|^2 \, dz.$$

Hence, $\Upsilon_\rho(\alpha) \leq \Upsilon_\rho(2\rho) \leq \Upsilon_{2\rho}(2\rho)$, and

$$\Psi_\rho(\sigma) \leq \frac{1}{2}\Psi_\rho(\alpha) + \left(\frac{\rho}{\alpha-\sigma}\right)^{\frac{2(2-p)}{p}} c\left[\Phi^{\frac{2}{p}}(2\rho) + \Upsilon_{2\rho}^{\frac{2(p-1)}{p}}(2\rho)\right]$$
$$+ c\left[\Phi(2\rho) + \Upsilon_{2\rho}^{p-1}(2\rho) + \Upsilon_{2\rho}(2\rho)\right].$$

Applying Lemma 3.2.3 with $\sigma_0 = \rho$ and $\alpha_0 = 2\rho$ and then simplifying exponents using the fact that $1 < p < 2$ and $0 < \omega \leq 1$, we obtain the result. □

Below we prove a second poincare inequality which will be employed to gain some control on how quickly the gradient of our affine maps blow-up as we shrink ρ. This argument will be carried out at the end of the paper.

Lemma 3.6.2. *There exists a constant $c = c(n, N, p, M)$ such that the following holds: Suppose that $u \in L^p(-T, 0; W^{1,p}(\Omega; \mathbb{R}^N)) \bigcap C^0(-T, 0; L^2(\Omega; \mathbb{R}^N))$ is a weak solution to* (3.1) *in Ω_T under the assumptions* (3.2) *and* (3.3). *Let $Q_{2\rho}(z_0) \subseteq \Omega_T$ be a parabolic cylinder with reference point $z_0 = (x_0, t_0)$ and radius $0 \leq \rho \leq 1$. Then*

$$\fint_{Q_\rho(z_0)} \left|\frac{u-(u)_{z_0;\rho}}{\rho}\right|^q dz \leq c_5 \left(\fint_{Q_{2\rho}(z_0)} (1+|Du|)^p \, dz\right)^{\frac{q}{p}}, \textit{ for all } q \in [1, p].$$

Proof. For notational convenience, we drop the centers of all balls and cylinders in the proof below. All are centered at x_0 and z_0. Let ψ and ψ_ρ be defined as in the above lemma. We will apply the Poincaré inequality slicewise on $B_\rho \times \{t\}$ for almost every $t \in (-\rho^2, 0)$.

First note that for $r \in (-\rho^2, 0)$, we have the following:

$$\begin{aligned}
\fint_{Q_\rho} \left|\frac{u-(u)_\rho}{\rho}\right|^q dz \leq& c\left[\fint_{Q_\rho} \left|\frac{u-(u)_{2\rho}}{\rho}\right|^q dz + \fint_{Q_\rho} \left|\frac{(u)_{2\rho}-(u)_\rho}{\rho}\right|^q dz\right] \\
\leq& c\fint_{Q_{2\rho}} \left|\frac{u-(u)_{2\rho}}{2\rho}\right|^q dz \\
\leq& c\fint_{Q_{2\rho}} \left|\frac{u-(\widetilde{u})^\psi_{2\rho}(t)}{2\rho}\right|^q dz \\
&+ c\fint_{Q_{2\rho}} \left|\frac{(\widetilde{u})^\psi_{2\rho}(t)-(\widetilde{u})^\psi_{2\rho}(r)}{2\rho}\right|^q dz \\
&+ c\fint_{Q_{2\rho}} \left|\frac{(\widetilde{u})^\psi_{2\rho}(r)-(u)_{2\rho}}{2\rho}\right|^q dz \\
=:& I + II + III.
\end{aligned}$$

By applying Poincare's inequality for functions with vanishing ψ-mean value slicewise,

$$I \leq c(q)\fint_{Q_{2\rho}} |Du|^q\, dz.$$

Also,

$$III = (2\rho)^{-q}\left|(\widetilde{u})^\psi_{2\rho}(r) - \fint_{Q_{2\rho}} u\, dz\right|^q \leq I \leq c(q)\fint_{Q_{2\rho}} |Du|^q\, dz.$$

Lastly, we have

$$II \leq 2\rho^{-q} \sup_{(-4\rho^2,0)} \left|(\widetilde{u})^\psi_{2\rho}(t) - (\widetilde{u})^\psi_{2\rho}(r)\right|^q, \tag{3.31}$$

which leads us to consider bounding the term $\left|(\widetilde{u})^\psi_{2\rho}(t) - (\widetilde{u})^\psi_{2\rho}(r)\right|^q$. Without loss of generality, assume $t > r$. For $0 < \theta < \frac{t-r}{2}$, we define $\zeta_\theta \in W_0^{1,\infty}((r,t))$ as follows:

$$\zeta_\theta = \begin{cases} \frac{s-r}{\theta}, & s \in [r, r+\theta) \\ 1, & s \in [r+\theta, t-\theta] \\ \frac{t-s}{\theta}, & s \in (t-\theta, t] \end{cases}$$

Then for $i \in \{1, ..., N\}$, we take $\varphi_\theta : \mathbb{R}^{n+1} \to \mathbb{R}^N$ with $(\varphi_\theta)_i = \psi_{2\rho}\zeta_\theta$ and $(\varphi_\theta)_j = 0$ for $j \neq i$ as a testing function in the weak formulation of (3.1). Hence, we arrive at

$$\int_r^t \fint_{B_{2\rho}} u_i \psi_{2\rho} \frac{d}{ds}(\zeta_\theta)\, dx\, ds = \int_r^t \fint_{B_{2\rho}} a_i(\cdot, u, Du) \cdot D\psi_{2\rho}\zeta_\theta\, dx\, ds. \tag{3.32}$$

We now note that the choice of ζ_θ gives

$$\begin{aligned} \int_r^t \fint_{B_{2\rho}} u_i \psi_{2\rho} \frac{d}{ds}(\zeta_\theta)\, dx\, ds &= (2\rho)^{-n} \int_r^t (\widetilde{u}_i)^\psi_{2\rho} \frac{d}{ds}(\zeta_\theta)\, ds \\ &= (2\rho)^{-n} \left(\frac{1}{\theta} \int_r^{r+\theta} (\widetilde{u}_i)^\psi_{2\rho}\, ds - \frac{1}{\theta} \int_{t-\theta}^t (\widetilde{u}_i)^\psi_{2\rho}\, ds \right) \\ &\to (2\rho)^{-n} \left((\widetilde{u}_i)^\psi_{2\rho}(t) - (\widetilde{u}_i)^\psi_{2\rho}(r) \right). \end{aligned}$$

Letting $\theta \to 0$ in (3.32), we obtain

$$(\widetilde{u}_i)^\psi_{2\rho}(t) - (\widetilde{u}_i)^\psi_{2\rho}(r) = (2\rho)^n \int_r^t \fint_{B_{2\rho}} a_i(\cdot, u, Du) \cdot D\psi_{2\rho}\, dx\, ds.$$

Thus,

$$\begin{aligned} \left| (\widetilde{u}_i)^\psi_{2\rho}(t) - (\widetilde{u}_i)^\psi_{2\rho}(r) \right| &\leq M \left\| D\psi_{2\rho} \right\|_\infty (2\rho)^{n+2} \fint_{Q_{2\rho}} (1 + |Du|)^{p-1}\, dz \\ &\leq c(n, M) 2\rho \fint_{Q_{2\rho}} (1 + |Du|)^{p-1}\, dz \end{aligned}$$

Using this bound in (3.31) and combining all of the estimates gives the result. □

3.7 THE MAIN RESULT

In this section, we establish the main result. We begin by proving a partial decay estimate for the excess functional. In particular, we show the first order excess decays. This enables us to show the full excess functional preserves a smallness property as mentioned in the introduction. We obtain such an estimate using the A-caloric approximation lemma. The decay argument is then completed by means of an iteration lemma. Once this is established, we argue that a Campanato-type estimate holds whenever the excess functional is sufficiently small. We then assemble the results at the end of this section in order to prove the main theorem via a Campanato embedding theorem.

3.7.1 PARTIAL DECAY ESTIMATE

We begin by proving the excess decay estimate. As mentioned throughout the paper, we obtain the result by transferring decay estimates of A-caloric functions to our solution via the following lemma.

Lemma 3.7.1. *Let $h \in L^2(t_0 - \rho^2, t_0; W^{1,2}(B_\rho(x_0), \mathbb{R}^N))$ be an A-caloric map in $Q_\rho(z_0)$ as in* (3.5.1) *with A satisfying* (3.12)*. Then h is smooth in $B_\rho(x_0) \times (t_0 - \rho^2, t_0]$ and for any $s \geq 1$ there exists a constant $c = c(n, N, M/\nu, s) \geq 1$ such that for any affine function $\ell : \mathbb{R}^n \to \mathbb{R}^N$ there holds*

$$\fint_{Q_{\vartheta\rho}(z_0)} \left| \frac{h - \ell}{\vartheta\rho} \right|^s dz \leq c\vartheta^s \fint_{Q_\rho(z_0)} \left| \frac{h - \ell}{\rho} \right|^s dz \quad \textit{for every } \vartheta \in (0, 1].$$

The above lemma can be found in [13], [24], and [44]. Below is the statement and proof of the Excess decay estimate.

Lemma 3.7.2. *Suppose* $L \geq 1$ *and* $\theta \in (0, 2^{\frac{p-6}{2}}]$. *Then there exists* $\epsilon_0 \in (0,1]$ *and* $c_6 = c(p,n,N,M,\nu)$ *so that the following holds:*
Whenever $u \in L^p(-T,0;W^{1,p}(\Omega,\mathbb{R}^N)) \cap L^2(-T,0;L^2(\Omega,\mathbb{R}^N))$ *is a weak solution to* (3.1) *in* Ω_T *under the main assumptions* (3.2) *and* (3.3)*, and* $Q_\rho^{(\lambda)}(z_0) \subseteq \Omega_T$ *is a parabolic cylinder with radius* ρ *such that* $0 < \rho \leq 1$ *and scaling factor* $\lambda \geq 1$ *on which the intrinsic coupling*

$$\lambda \leq 1 + |D\ell_{z_0;\rho}^{(\lambda)}| \leq L\lambda \tag{3.33}$$

holds, and also the smallness condition,

$$\widetilde{E}_\lambda\left(z_0, \rho, \ell_{z_0;\rho}^{(\lambda)}\right) \leq \epsilon_0,$$

holds, then there exists $\lambda_1 \in \left[\frac{\lambda}{2}, 2L\lambda\right]$ *such that*

$$1 + |D\ell_{z_0;\theta\rho}^{(\lambda_1)}| = \lambda_1 \tag{3.34}$$

and

$$E_{\lambda_1}\left(z_0, \theta\rho, \ell_{z_0;\theta\rho}^{(\lambda_1)}\right) \leq c_6 \theta^p \widetilde{E}_\lambda\left(z_0, \rho, \ell_{z_0;\rho}^{(\lambda)}\right). \tag{3.35}$$

Proof. Recall that $\ell_{z_0;\rho}^{(\lambda)}$ is the unique affine minimizer of $\fint_{Q_\rho^{(\lambda)}(z_0)} |u-\ell|^2\,dz$, and define

$$\gamma := \widetilde{E}_\lambda^{1/2}(\rho), \quad v := \frac{u(x, t_0 + \lambda^{2-p}(t-t_0)) - \ell_{z_0;\rho}^{(\lambda)}(x)}{\widetilde{c}\gamma(1 + |D\ell_{z_0;\rho}^{(\lambda)}|)},$$
$$\widetilde{\varphi}(x,t) := \varphi(x, t_0 + \lambda^{2-p}(t-t_0)) \quad \text{and} \quad \mathcal{A} := \frac{\partial_w a(\widetilde{z}, \ell_{z_0;\rho}^{(\lambda)}(x_0), D\ell_{z_0;\rho}^{(\lambda)})}{\lambda^{p-2}},$$

where $\widetilde{c}$ is to be selected later and $\widetilde{z} = (x, t_0 + \lambda^{2-p}(t-t_0))$. Let $\tau = \lambda^{2-p}\rho^2$ and use the change of variables $t = t_0 + \lambda^{p-2}(s - t_0)$ to obtain

$$\begin{aligned}
&\fint_{(t_0-\frac{\tau}{4},t_0)}\fint_{B_{\frac{\varrho}{2}}(x_0)} (u - \ell^{(\lambda)}_{z_0;\rho}) \cdot \varphi_s \, dx \, ds \\
&- \fint_{(t_0-\frac{\tau}{4},t_0)}\fint_{B_{\frac{\varrho}{2}}(x_0)} \left\langle \partial_w a(\widetilde{z}, \ell^{(\lambda)}_{z_0;\rho}(x_0), D\ell^{(\lambda)}_{z_0;\rho}(x_0))(Du - D\ell^{(\lambda)}_{z_0;\rho}), D\varphi \right\rangle \, dx \, ds \\
&\qquad = \lambda^{(p-2)} \left(1 + |D\ell^{(\lambda)}_{z_0;\rho}|\right) \gamma \fint_{Q_{\rho/2}(z_0)} v \cdot \widetilde{\varphi}_t - \langle \mathcal{A}Dv, D\widetilde{\varphi} \rangle \, dxdt.
\end{aligned}$$

By the linear approximation lemma and the intrinsic coupling (3.33),

$$\begin{aligned}
&\left| \fint_{Q_{\rho/2}(z_0)} v \cdot \widetilde{\varphi}_t - \langle \mathcal{A}Dv, D\widetilde{\varphi} \rangle \, dz \right| \\
&\qquad \leq \frac{c}{\widetilde{c}} \lambda^{2-p}(1 + |D\ell^{(\lambda)}_{z_0;\rho}|)^{-1} \lambda^{p-1} L^2 \mu^{1/2} \left(\widetilde{E}^{1/2}_\lambda(\rho) \right) \sup_{Q_{\rho/2}} |D\widetilde{\varphi}| \\
&\qquad\quad + \frac{c}{\widetilde{c}} \lambda^{2-p}(1 + |D\ell^{(\lambda)}_{z_0;\rho}|)^{-1} \lambda^{p-1} L^2 \widetilde{E}^{1/2}_\lambda(\rho) \sup_{Q_{\rho/2}} |D\widetilde{\varphi}| \\
&\qquad \leq \frac{L^2 c}{\widetilde{c}} \left[\mu^{1/2} \left(\widetilde{E}^{1/2}_\lambda(\rho) \right) + \widetilde{E}^{1/2}_\lambda(\rho) \right] \sup_{Q_{\rho/2}} |D\widetilde{\varphi}|.
\end{aligned}$$

Assume the following smallness condition:

$$\frac{L^2 c}{\widetilde{c}} \left[\mu^{1/2} (\widetilde{E}^{1/2}_\lambda(\rho)) + \widetilde{E}^{1/2}_\lambda(\rho) \right] < \delta, \tag{3.36}$$

where $\delta > 0$ is the one given in the A-caloric approximation lemma. Hence,

$$\left| \fint_{Q_{\rho/2}(z_0)} v \cdot \widetilde{\varphi}_t - \langle \mathcal{A}Dv, D\widetilde{\varphi} \rangle \, dz \right| \leq \delta \sup_{Q_{\rho/2}(z_0)} |D\widetilde{\varphi}|.$$

Now note that since $D\ell/(1 + D\ell) \leq 1, \widetilde{c} \geq 1$, and $\gamma \leq 1$, Lemma 3.2.1 (vi) and (iv) give

$$|V(Dv)|^2 = \left| V \left(\frac{Du - D\ell}{\widetilde{c}\gamma(1 + |D\ell|)} \right) \right|^2$$

$$
\begin{aligned}
&\leq \frac{1}{\widetilde{c}^p\gamma^2}\left|V\left(\frac{Du}{1+|D\ell|}\right)-V\left(\frac{D\ell}{1+|D\ell|}\right)\right|^2\\
&\leq \frac{1}{\widetilde{c}^p\gamma^2}\left(1+\left|\frac{Du}{1+D\ell}\right|^2+\left|\frac{D\ell}{1+D\ell}\right|^2\right)^{\frac{p-2}{2}}\left|\frac{Du-D\ell}{1+|D\ell|}\right|^2\\
&\leq \frac{1}{\widetilde{c}^p\gamma^2}(1+|D\ell|)^{-p}(1+|Du|+|D\ell|)^{p-2}|Du-D\ell|^2.
\end{aligned}
$$

Using the Caccioppoli inequality, we have

$$
\begin{aligned}
&\sup_{t\in(t_0-\rho^2/4,t_0)}\fint_{B_{\rho/2}(x_0)}\left|\frac{v(x,t)}{\rho}\right|^2 dx+\fint_{Q_{\rho/2(z_0)}}|V(Dv)|^2\,dz\\
&\qquad\leq \frac{1}{\widetilde{c}^p\gamma^2}c_0\fint_{Q_\rho^{(\lambda)}(z_0)}\left\{(1+|D\ell|)^{-p}\left|\frac{u-\ell}{\lambda^{\frac{2-p}{2}}\rho}\right|^2+(1+|D\ell|)^{-p}\left|\frac{u-\ell}{\rho}\right|^p\right\}dz\\
&\qquad\quad+\frac{1}{\widetilde{c}^p\gamma^2}c_0\left[\omega\left(\fint_{Q_\rho^{(\lambda)}(z_0)}|u-\ell(x_0)|^2\,dz\right)+\omega\left(\lambda^{\frac{2-p}{2}}\rho^2\right)\right]\\
&\qquad\leq \frac{L^{2-p}c_0}{\widetilde{c}^p}.
\end{aligned}
$$

Hence, we can choose $\tilde{c}\geq 1$ large enough so that

$$
\begin{aligned}
\sup_{t\in(t_0-\rho^2/4,t_0)}\fint_{B_{\rho/2}(x_0)}\left|\frac{v(x,t)}{\rho/2}\right|^2 dx+\fint_{Q_{\rho/2}(z_0)}\left|\frac{v}{\rho/2}\right|^2+\gamma^{p-2}\left|\frac{v}{\rho/2}\right|^p+|V(Dv)|^2\,dz&\\
\leq \frac{2\cdot 2^{n+4}+L^{2-p}c_0}{\widetilde{c}^p}\leq 1,&
\end{aligned}
\tag{3.37}
$$

where c_0 is the constant from the Caccioppoli inequality. Thus there exists $f\in L^p(t_0-(\rho/8)^2,t_0;W^{1,p}(B_{\rho/8}(x_0),\mathbb{R}^N))\cap L^2(t_0-(\rho/8)^2,t_0;W^{1,2}(B_{\rho/8}(x_0),\mathbb{R}^N))$ which is A-caloric on $Q_{\rho/8}(z_0)$ such that

$$
\fint_{Q_{\rho/8}(z_0)}\left|\frac{f}{\rho/8}\right|^2+\gamma^{p-2}\left|\frac{f}{\rho/8}\right|^p+|V(Df)|^2\,dz\leq 4^{n+8}
$$

and

$$\fint_{Q_{\rho/8}(z_0)} \left|\frac{v-f}{\rho/8}\right|^2 + \gamma^{p-2}\left|\frac{v-f}{\rho/8}\right|^p \, dz \leq \epsilon$$

by the A-caloric approximation lemma. Hence f satisfies the assumptions in Lemma 3.7.1, and for $s=2$ and $s=p$ we have

$$\begin{aligned}
&\gamma^{s-2}\left(\frac{\vartheta\rho}{8}\right)^{-s}\fint_{Q_{\vartheta\rho/8}(z_0)} \left|f-(f)_{\vartheta\rho/8}-\gamma^{\frac{2-s}{s}}(Df)_{\vartheta\rho/8}(x-x_0)\right|^s dz \\
&\leq c\gamma^{s-2}\vartheta^s\left(\frac{\rho}{8}\right)^{-s}\fint_{Q_{\rho/8}(z_0)} \left|f-(f)_{\rho/8}-\gamma^{\frac{2-s}{s}}(Df)_{\rho/8}(x-x_0)\right|^s dz \\
&\leq c\gamma^{s-2}\vartheta^s\left[\left(\frac{\rho}{8}\right)^{-s}\left(\fint_{Q_{\rho/8}(z_0)} |f|^s\, dz + \left|(f)_{\rho/8}\right|^s\right) + \gamma^{2-s}\left|(Df)_{\rho/8}\right|^s\right] \\
&\leq c\vartheta^s\left[\fint_{Q_{\rho/8}(z_0)} \left\{\gamma^{s-2}\left|\frac{f}{\rho/8}\right|^s + |Df|^s\right\} dz\right],
\end{aligned}$$

as γ is a constant. Using Hölder's inequality and (i) in Lemma 3.2.1 on the second term in the integrand on the right-hand side, the following estimate holds:

$$\begin{aligned}
&\gamma^{s-2}\left(\frac{\vartheta\rho}{8}\right)^{-s}\fint_{Q_{\vartheta\rho/8}(z_0)} \left|f-(f)_{\vartheta\rho/8}-\gamma^{\frac{2-s}{s}}(Df)_{\vartheta\rho/8}(x-x_0)\right|^s dz \\
&\qquad\qquad \leq c\vartheta^s\left[\fint_{Q_{\rho/8}(z_0)} \gamma^{s-2}\left|\frac{f}{\rho/8}\right|^s + |V(Df)|^2\, dz + 1\right] \leq c\vartheta^s.
\end{aligned} \tag{3.38}$$

Hence for $s=2$ and $s=p$, we see

$$\begin{aligned}
&\gamma^{s-2}\left(\frac{\vartheta\rho}{8}\right)^{-s}\fint_{Q_{\vartheta\rho/8}(z_0)} \left|v-(f)_{\vartheta\rho/8}-\gamma^{\frac{2-s}{s}}(Df)_{\vartheta\rho/8}(x-x_0)\right|^s dz \\
&\leq c\left(\frac{\vartheta\rho}{8}\right)^{-s}\fint_{Q_{\vartheta\rho/8}(z_0)} \gamma^{s-2}\,|v-f|^s\, dz
\end{aligned}$$

$$
\begin{aligned}
&+ c\left(\frac{\vartheta\rho}{8}\right)^{-s}\gamma^{s-2}\fint_{Q_{\vartheta\rho/8}(z_0)}\left|f-(f)_{\vartheta\rho/8}-\gamma^{\frac{2-s}{s}}(Df)_{\vartheta\rho/8}(x-x_0)\right|^s dz\\
&\le c\left[\vartheta^{-n-2-s}\fint_{Q_{\rho/8}(z_0)}\gamma^{s-2}\left|\frac{v-f}{\rho/8}\right|^s dz + c(c_0,p)\vartheta^s\right]\\
&\le c\left[\vartheta^{-n-2-s}\epsilon+\vartheta^s\right]. \qquad (3.39)
\end{aligned}
$$

Now choose $\epsilon = \vartheta^{n+4+p}$. Remember this also determines δ. Scaling back to u on $Q^{(\lambda)}_{\rho/8}$ from v on $Q_{\rho/8}$ gives

$$
\begin{aligned}
\left(\frac{\vartheta\rho}{8}\right)^{-s}\fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} &|u-\ell^{(\lambda)}_{z_0;\rho}-\gamma\left(1+|D\ell^{(\lambda)}_{z_0;\rho}|\right)\\
&\times\left[(f)_{\vartheta\rho/8}+\gamma^{\frac{2-s}{s}}(Df)_{\vartheta\rho/8}(x-x_0)\right]|^s dz\\
&\le c\gamma^2\left(1+|D\ell^{(\lambda)}_{z_0;\rho}|\right)^s\vartheta^s \le c\widetilde{E}_\lambda(\rho)L^s\lambda^s\vartheta^s. \qquad (3.40)
\end{aligned}
$$

Using the fact that $\ell^{(\lambda)}_{\vartheta\rho/8}$ is the unique minimizer of the integral on the left hand side above,

$$
\fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)}\left|\frac{u-\ell^{(\lambda)}_{\vartheta\rho/8}}{\vartheta\rho/8}\right|^2 dz \le c\widetilde{E}_\lambda(\rho)L^2\lambda^2\vartheta^2. \qquad (3.41)
$$

Let $\ell^{(\lambda,p)}_{\vartheta\rho/8}$ be the unique minimizer of $\ell\mapsto\int_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)}|u-\ell|^p\,dz$, we also obtain

$$
\fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)}\left|\frac{u-\ell^{(\lambda,p)}_{\vartheta\rho/8}}{\vartheta\rho/8}\right|^p dz \le c\widetilde{E}_\lambda(\rho)L^p\lambda^p\vartheta^p.
$$

The next step is to replace $\ell^{(\lambda,p)}_{\vartheta\rho/8}=\xi^{(\lambda,p)}_{\vartheta\rho/8}+P^{(\lambda,p)}_{\vartheta\rho/8}(x-x_0)$ with $\ell^{(\lambda)}_{\vartheta\rho/8}=\xi^{(\lambda)}_{\vartheta\rho/8}+P^{(\lambda)}_{\vartheta\rho/8}(x-x_0)$. We use (3.8) and the identities

$$
\fint_{Q_{\vartheta\rho/8}(z_0)}\xi^{(\lambda,p)}_{\vartheta\rho/8}\otimes(x-x_0)\,dz=0
$$

and

$$P_{\vartheta\rho/8}^{(\lambda,p)} = \frac{n+2}{(\vartheta\rho/8)^2} \fint_{Q_{\vartheta\rho/8}(z_0)} P_{\vartheta\rho/8}^{(\lambda,p)}(x-x_0) \otimes (x-x_0)\, dz$$

to get

$$\begin{aligned}
&\left|P_{\vartheta\rho/8}^{(\lambda)} - P_{\vartheta\rho/8}^{(\lambda,p)}\right|^p \\
&\quad = \left|\frac{n+2}{(\vartheta\rho/8)^2} \fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} \left(u - \xi_{\vartheta\rho/8}^{(\lambda,p)} - P_{\vartheta\rho/8}^{(\lambda,p)}(x-x_0)\right) \otimes (x-x_0)\, dz\right|^p \\
&\quad \le \left|\frac{n+2}{(\vartheta\rho/8)} \fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} \left|u - \ell_{\vartheta\rho/8}^{(\lambda,p)}\right|\, dz\right|^p \\
&\quad \le c\left(\frac{\vartheta\rho}{8}\right)^{-p} \fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} \left|u - \ell_{\vartheta\rho/8}^{(\lambda,p)}\right|^p\, dz.
\end{aligned}$$

We also have

$$\begin{aligned}
\left|\xi_{\vartheta\rho/8}^{(\lambda)} - \xi_{\vartheta\rho/8}^{(\lambda,p)}\right|^p &= \left|\fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} u - \xi_{\vartheta\rho/8}^{(\lambda,p)} - P_{\vartheta\rho/8}^{(\lambda,p)}(x-x_0)\, dz\right|^p \\
&\le \fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} \left|u - \ell_{\vartheta\rho/8}^{(\lambda,p)}\right|^p\, dz.
\end{aligned}$$

Using the two estimates above, we finally obtain

$$\begin{aligned}
\fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} \left|\frac{u - \ell_{\vartheta\rho/8}^{(\lambda)}}{\vartheta\rho/8}\right|^p dz &\le \fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} \left|\frac{u - \ell_{\vartheta\rho/8}^{(\lambda,p)}}{\vartheta\rho/8}\right|^p dz \\
&\qquad + \fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} \left|\frac{\left(P_{\vartheta\rho/8}^{(\lambda)} - P_{\vartheta\rho/8}^{(\lambda,p)}\right)(x-x_0)}{\vartheta\rho/8}\right|^p dz \\
&\qquad + \fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} \left|\frac{\xi_{\vartheta\rho/8}^{(\lambda)} - \xi_{\vartheta\rho/8}^{(\lambda,p)}}{\vartheta\rho/8}\right|^p dz \\
&\le c\left(\frac{\vartheta\rho}{8}\right)^{-p} \fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} \left|u - \ell_{\vartheta\rho/8}^{(\lambda,p)}\right|^p dz
\end{aligned}$$

$$\leq c\widetilde{E}_\lambda(\rho)L^p\lambda^p\vartheta^p.$$

For $s = 2$ and $s = p$,

$$\fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} \left|\frac{u - \ell^{(\lambda)}_{\vartheta\rho/8}}{\vartheta\rho/8}\right|^s dz \leq c\widetilde{E}_\lambda(\rho)L^s\lambda^s\vartheta^s.$$

We now want to find the new scaling factor $\lambda_0 \in \left[\frac{\lambda}{2}, 2L\lambda\right]$ such that (3.34) holds. The following argument is given in [6]. It is included here for completeness with a few minor changes since $\frac{2n}{n+2} < p < 2$. Define $\theta := 2^{\frac{p-2}{2}}\vartheta$. Then $\theta \in (0, 2^{\frac{p-2}{2}}]$ since $\vartheta \in (0,1]$. Note for our choice of θ and $\mu \in \left[\frac{\lambda}{2}, 2L\lambda\right]$, $Q^{(\mu)}_{\theta\rho/8} \subseteq Q^{(\lambda)}_{\vartheta\rho/8}$. Recalling (3.9), we see

$$\begin{aligned}
&\left|D\ell^{(\mu)}_{\theta\rho/8} - D\ell^{(\lambda)}_{\vartheta\rho/8}\right|^2 \\
&\qquad \leq \frac{n(n+2)}{(\theta\rho/8)^2}\fint_{Q^{(\lambda)}_{\theta\rho/8}(z_0)} \left|u - \xi^{(\lambda)}_{\vartheta\rho/8} - D\ell^{(\lambda)}_{\vartheta\rho/8}(x - x_0)\right|^2 dz \\
&\qquad = c\fint_{Q^{(\mu)}_{\theta\rho/8}(z_0)} \left|\frac{u - \ell^{(\lambda)}_{\vartheta\rho/8}}{\theta\rho/8}\right|^2 dz \\
&\qquad \leq c\left(\frac{\vartheta}{\theta}\right)^{n+4}\left(\frac{\lambda}{\mu}\right)^{2-p}\fint_{Q^{(\lambda)}_{\vartheta\rho/8}(z_0)} \left|\frac{u - \ell^{(\lambda)}_{\vartheta\rho/8}}{\vartheta\rho/8}\right|^2 dz \\
&\qquad \leq c2^{\frac{(n+6)(2-p)}{2}}\vartheta^2\lambda^2\widetilde{E}_\lambda(\rho) = c\lambda^2\widetilde{E}_\lambda(\rho).
\end{aligned}$$

So provided we assume

$$c\widetilde{E}_\lambda(\rho) \leq 1/4, \tag{3.42}$$

we can obtain $\left|D\ell^{(\mu)}_{\theta\rho} - D\ell^{(\lambda)}_{\vartheta\rho}\right| \leq \frac{\lambda}{2}$. Hence we see

$$1 + \left|D\ell^{(\mu)}_{\theta\rho}\right| \leq 1 + \left|D\ell^{(\lambda)}_{\theta\rho}\right| + \left|D\ell^{(\mu)}_{\theta\rho} - D\ell^{(\lambda)}_{\theta\rho}\right| \leq L\lambda + \frac{\lambda}{2} \leq 2L\lambda,$$

and

$$1+\left|D\ell_{\theta\rho}^{(\mu)}\right| \geq 1+\left|D\ell_{\theta\rho}^{(\lambda)}\right| - \left|D\ell_{\theta\rho}^{(\mu)} - D\ell_{\theta\rho}^{(\lambda)}\right| \geq \lambda - \frac{\lambda}{2} = \frac{\lambda}{2}.$$

Define $f(\beta) := \beta - (1+|D\ell_{\theta\rho}^{(\beta)}|)$ for $\beta \in \left[\frac{\lambda}{2}, 2L\lambda\right]$. By the Intermediate Value Theorem, there exists $\lambda_1 \in \left[\frac{\lambda}{2}, 2L\lambda\right]$ such that $\lambda_1 - (1+|D\ell_{\theta\rho}^{(\lambda_1)}|) = 0$ as the function f is continuous. In order to see that f is continuous, rewrite $D\ell_{\theta\rho}^{(\lambda)}$ using (3.8).

Now we prove the last assertion of the theorem. For $s = 2$ and $s = p$, we have from our work above

$$\begin{aligned}
\fint_{Q_{\theta\rho/8}^{(\lambda_1)}(z_0)} \left|\frac{u-\ell_{\theta\rho/8}^{(\lambda_1)}}{\theta\rho/8}\right|^s dz &\leq \fint_{Q_{\theta\rho/8}^{(\lambda_1)}(z_0)} \left|\frac{u-\ell_{\vartheta\rho/8}^{(\lambda)}}{\theta\rho/8}\right|^s dz \\
&\leq c\left(\frac{\vartheta}{\theta}\right)^{n+s+2}\left(\frac{\lambda}{\lambda_1}\right)^{2-p} \fint_{Q_{\vartheta\rho/8}^{(\lambda)}(z_0)} \left|\frac{u-\ell_{\vartheta\rho/8}^{(\lambda)}}{\vartheta\rho/8}\right|^s dz \\
&\leq c(c_2)\left(\frac{\vartheta}{\theta}\right)^{n+s+2}\left(\frac{\lambda}{\lambda_1}\right)^{2-p} \vartheta^s\lambda^2\widetilde{E}_\lambda(\rho) \\
&\leq c(c_2)\left(\frac{\vartheta}{\theta}\right)^{n+2s+2}\left(\frac{\lambda}{\lambda_1}\right)^{2-p+s} \theta^s\lambda_1^2\widetilde{E}_\lambda(\rho) \\
&\leq c(c_2,n,p)\theta^s\lambda_1^s\widetilde{E}_\lambda(\rho) \\
&\leq c(c_2,n,p)\theta^s\left(1+|D\ell_{\theta\rho/8}^{(\lambda_1)}|\right)^s\widetilde{E}_\lambda(\rho).
\end{aligned}$$

Note that $\epsilon_0 = \epsilon_0(n,p,L,\nu,N,M,\theta,\mu(\cdot))$ in the statement of the problem must be chosen small enough to satisfy $\widetilde{E}_\lambda^{1/2}(\rho) \leq 1$ and (3.42). Thus the last argument gives the final claim provided $\widetilde{c} \geq 1$ is chosen large enough so that $(L^2c)/\widetilde{c} \leq 1$ in (3.36) and (3.37) holds. □

3.7.2 CHOICE OF THE CONSTANTS

For any given $\alpha \in (0,1)$, define

$$\vartheta := \min\left\{\left(\frac{1}{2}\right)^{\frac{6-p}{2}}, \left(\frac{1}{3c_6}\right)^{1/p}, \left(\frac{1}{2L}\right)^{\frac{2(n+4)}{1-\alpha}}\right\}. \tag{3.43}$$

Now set $\epsilon_1 = \frac{\epsilon_0}{3}$, where ϵ_0 is the epsilon from the excess decay theorem, so that ϵ_1 depends on $n, N, p, M, \nu, L, \mu(\cdot)$, and ϑ. Now select $\rho_0 \in (0,1]$ such that

$$\omega\left((2L\rho_0)^2\right) \le \epsilon_1, \tag{3.44}$$

whereρ_0 depends on $n, N, p, \nu, L, M, \omega(\cdot), \mu(\cdot), \vartheta$, and ϵ_1.

3.7.3 ITERATION ARGUMENT

Lemma 3.7.3. *Suppose for some $z_0 \in \Omega_T$ and some radius $0 < \rho \le \rho_0$ such that*

$$1 + |D\ell_{z_0;\rho}| \le L, \tag{3.45}$$

and

$$E_1(z_0, \rho, \ell_{z_0;\rho}) \le \epsilon_1 \tag{3.46}$$

hold on $Q_\rho(z_0)$. Then there exists $\{\lambda_j\}_{j=0}^\infty$ such that that the following hold:

$$\begin{cases} 1 \le \lambda_j \le (2L)^j, \\ \\ \lambda_j \le 1 + |D\ell_{z_0,\vartheta^j\rho}^{(\lambda_j)}| \le L\lambda_j, \\ \\ E_{\lambda_j}\left(z_0, \vartheta^j\rho, \ell_{z_0,\vartheta^j\rho}^{(\lambda_j)}\right) \le \epsilon_1, \end{cases} \tag{3.47}$$

and

$$\Psi_{\lambda_j}(z_0, \vartheta^j\rho, \ell_{z_0,\vartheta^j\rho}^{(\lambda_j)}) := \fint_{Q_{\vartheta^j\rho}^{(\lambda_j)}} \left| u - \ell_{z_0,\vartheta^j\rho}^{(\lambda_j)}(x_0) \right|^2 \, dz \le (\vartheta^j\rho)^2(2L)^{2j}(2L)^2. \tag{3.48}$$

Proof. We first show that (3.48) follows immediately from (3.47) if it holds. Assume (3.47) holds. In the following, we suppress notation by using $\ell_{z_0,\vartheta^j\rho}^{(\lambda_j)} = \ell_j$ and $D\ell_{z_0,\vartheta^j\rho}^{(\lambda_j)} = D\ell_j$. As (3.47) holds,

$$\begin{aligned} \Psi_{\lambda_j}(\vartheta^j\rho) &\le 2(\vartheta^j\rho)^2(1+|D\ell_j|)^2 \fint_{Q_{\vartheta^j\rho}^{(\lambda_j)}} \left| \frac{u-\ell_j}{\vartheta^j\rho(1+|D\ell_j|)} \right|^2 \, dz \\ &\quad + 2(\vartheta^j\rho)^2(1+|D\ell_j|)^2 \\ &\le 4(\vartheta^j\rho)^2(1+|D\ell_j|)^2 E_{\lambda_j} + 2(\vartheta^j\rho)^2(1+|D\ell_j|)^2 \\ &\le 4(\vartheta^j\rho)^2(1+|D\ell_j|)^2 \\ &\le 4(\vartheta^j\rho)^2(L\lambda_j)^2 \le (\vartheta^j\rho)^2(2L)^{2j}(2L)^2. \end{aligned}$$

We now use induction to prove (3.47). Let $\lambda_0 = 1$. Then (3.47) holds by our assumptions. Assume (3.47) holds for some $j \in \mathbb{N} \cup \{0\}$. We need to show (3.47) holds for $j+1$ using Theorem 3.7.2. Note that $\lambda_j \le 1 + |D\ell_j| \le L\lambda_j$. By (3.47), $E_{\lambda_j}(\vartheta^j\rho) \le \epsilon_1$. By (3.48) and our choice of the constant ϑ,

$$\Psi_{\lambda_j}(\vartheta^j\rho) \le (\vartheta^j\rho)^2(2L)^{2j}(2L)^2 \le (2L\vartheta)^{2j}(2L\rho)^2 \le (2L\rho)^2.$$

Thus by our assumption on ω,

$$\omega(\Psi_{\lambda_j}(\vartheta^j\rho)) \le \omega((2L\rho)^2) \le \omega((2L\rho_0)^2) \le \epsilon_1.$$

Also, $\omega((\vartheta^j\lambda_j^{\frac{2-p}{2}}\rho)^2) \le \omega(\rho^2) \le \omega(\rho_0^2) \le \epsilon_1$. Hence, $\widetilde{E}_{\lambda_j} \le \epsilon_0$. So by Lemma 3.7.2, there exists $\lambda_{j+1} \in \left[\frac{\lambda_j}{2}, 2L\lambda_j\right]$ such that $\lambda_{j+1} = 1 + |D\ell_{j+1}|$ and

$$E_{\lambda_{j+1}}(\vartheta^{j+1}\rho) \le c_0\vartheta^p\widetilde{E}_{\lambda_j}(\vartheta^j\rho) \le 3c_0\vartheta^p\epsilon_1 \le \epsilon_1.$$

We also see that $1 + |D\ell_{j+1}| \ge 1$ and $\lambda_{j+1} \le 2L\lambda_j \le (2L)^{j+1}$. Thus, (3.47) holds for $j+1$. Thus we have shown the result by induction. □

3.7.4 CAMPANATO-TYPE ESTIMATE

Let $z_0 \in \Omega_T$ and $0 < \rho \le \rho_0$. Further we assume that the smallness conditions in the iteration (3.45) and (3.46) hold. Then for any $j \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}\int_{Q_{\vartheta^j\rho}^{(\lambda_j)}} \left|u - (u)_{z_0,\vartheta^j\rho}^{(\lambda_j)}\right|^2 dz &\le |B_1(x_0)|\lambda_j^{2-p}(2L)^{2(j+1)}(\vartheta^j\rho)^{n+4} \\ &\le |B_1(x_0)|(2L)^2(2L)^{(4-p)j}(\vartheta^j\rho)^{n+4},\end{aligned}$$

by (3.48). Now set $\theta = (2L)^{\frac{p-2}{2}}\vartheta$. Note that $Q_{\theta^j\rho}(z_0) \subseteq Q^{(\lambda_j)}_{\vartheta^j\rho}$ by our choice of θ and the fact $\lambda_j \leq (2L)^j$. Hence,

$$\begin{aligned}
\int_{Q_{\theta^j\rho}} \left|u - (u)_{z_0,\theta^j\rho}\right|^2 dz &\leq \int_{Q_{\theta^j\rho}} \left|u - (u)^{(\lambda_j)}_{z_0,\vartheta^j\rho}\right|^2 dz \\
&\leq \int_{Q^{(\lambda_j)}_{\vartheta^j\rho}} \left|u - (u)^{(\lambda_j)}_{z_0,\vartheta^j\rho}\right|^2 dz \\
&= |B_1(x_0)|(2L)^2\rho^{n+4}\left((2L)^{4-p}\vartheta^{n+4}\right)^j \\
&\leq |B_1(x_0)|(2L)^2\rho^{n+4}\theta^{j(n+2+2\alpha)}\left((2L)^{\frac{(4-p)(2-p)(n+2+2\alpha)}{2}}\vartheta^{2-2\alpha}\right)^j \\
&\leq |B_1(x_0)|(2L)^2\rho^{n+4}\theta^{j(n+2+2\alpha)}\left((2L)^{4(n+4)}\vartheta^{2-2\alpha}\right)^j \\
&\leq |B_1(x_0)|(2L)^2\rho^{n+4}\theta^{j(n+2+2\alpha)},
\end{aligned}$$

where the last inequality uses the choice of ϑ in (3.43).

Now let $0 < r \leq \rho$. Then there exists $j \in \mathbb{N} \cup \{0\}$ such that $\theta^{j+1}\rho < r \leq \theta^j\rho$. So

$$\begin{aligned}
\int_{Q_r} |u - (u)_r|^2\, dz \leq \int_{Q_r} \left|u - (u)_{\theta^j\rho}\right|^2 dz &\leq \int_{Q_{\theta^j\rho}} \left|u - (u)_{\theta^j\rho}\right|^2 dz \\
&\leq |B_1(x_0)|(2L)^2\rho^{n+4}\theta^{j(n+2+2\alpha)} \\
&\leq |B_1(x_0)|(2L)^2\rho^{n+4}\theta^{-(n+2+2\alpha)}\left(\frac{r}{\rho}\right)^{n+2+2\alpha} \\
&\leq |B_1(x_0)|(2L)^2\theta^{-(n+2+2\alpha)}r^{n+2+2\alpha} \\
&\leq c(n, N, p, \nu, L, M, \alpha)r^{n+2+2\alpha}.
\end{aligned}$$

So for every $0 < r \leq \rho$, we have

$$\int_{Q_r} |u - (u)_r|^2\, dz \leq c(n, N, p, \nu, L, M, \alpha)r^{n+2+2\alpha}. \tag{3.49}$$

3.7.5 CONCLUSION OF THEOREM 3.1.1

Let $\alpha \in (0,1)$ and $L \geq 1$ be given. Also let $\epsilon_1(L)$ and $\rho_0(L)$ correspond to the ϵ_1 and ρ_0 given in the iteration argument. Let $z_0 \in \Omega_T \backslash (\Sigma_1 \cup \Sigma_2)$. Then there exists $0 < \rho_1 < 1$ and $L_0 \geq 1$ such that

$$|(Du)_{z_0;2\rho}| < L_0, \tag{3.50}$$

for all $0 < 2\rho < \rho_1$ with $Q_{4\rho}(z_0) \subseteq \Omega_T$, since $z_0 \notin \Sigma_2$. Now choose $0 < \rho_2 < \rho_1$ such that $Q_{4\rho_2}(z_0) \subseteq \Omega_T$,

$$\fint_{Q_{2\rho_2}(z_0)} |Du - (Du)_{z_0;2\rho_2}|^p \, dz \leq \epsilon_2, \quad \text{and} \quad \omega^{\frac{2(p-1)}{p}}((2\rho_2)^2) \leq \epsilon_2 \tag{3.51}$$

where ϵ_2 and L_0 satisfy

$$0 < \epsilon_2 < 1 < L_0, \quad 1 + cL_0 < L,$$

and

$$c\left[\epsilon_2^{\frac{2(p-1)}{p}} + \epsilon_2 + (1+L_0)^p \epsilon_2\right] < \epsilon_1(L).$$

In the above inequality $c = \max\{c_1, c_2\}$, where c_1 and c_2 are discussed below.

By Lemma 3.6.2, (3.50), and (3.51), we see

$$\begin{aligned} |D\ell_{z_0;\rho_2}| &= \frac{n+2}{\rho_2^2} \left| \fint_{Q_{\rho_2}(z_0)} (u - (u)_{z_0;\rho_2}) \otimes (x - x_0) \, dz \right| \\ &\leq (n+2) \fint_{Q_{\rho_2}(z_0)} \left| \frac{u - (u)_{z_0;\rho_2}}{\rho_2} \right| \, dz \end{aligned}$$

$$\leq c(n+2)\left(\fint_{Q_{2\rho_2}(z_0)}(1+|Du|)^p\,dz\right)^{1/p}$$
$$\leq c\left[(1+|(Du)_{z_0;2\rho_2}|)^p+\fint_{Q_{2\rho_2}(z_0)}|Du-(Du)_{z_0,2\rho_2}|^p\,dz\right]^{1/p}$$
$$\leq c\left[(1+L_0)^p+\epsilon_2\right]^{1/p}\leq c_1L_0,$$

where $c_1=c_1(n,N,p,M)$. So for all $0<\rho<1$, we have $1+|D\ell_{z_0;\rho_2}|\leq 1+c_1L_0<L$.

In the end we will obtain an estimate for $E_1(z_0,\rho_2,\ell_{z_0,\rho_2})$ as well, but in order to achieve this end, we must first estimate $\fint_{Q_{2\rho_2}(z_0)}|u-(u)_{z_0;2\rho_2}|^2\,dz$. Taking $A=0$ in Theorem 3.6.1 and recalling that $\omega\leq 1$, we find

$$\begin{aligned}\fint_{Q_{2\rho}(z_0)}|u-(u)_{z_0;2\rho}|^2\,dz \leq& c(2\rho)^2\left(\fint_{Q_{4\rho}(z_0)}|Du|^p\,dz\right)^{\frac{2(p-1)}{p}}\\ &+c(2\rho)^2\left[\fint_{Q_{4\rho}(z_0)}|Du|^p\,dz+1\right]\\ \leq& c(2\rho)^2 \end{aligned}\tag{3.52}$$

for all $0<2\rho<1$ such that $Q_{4\rho}(z_0)\subseteq\Omega_T$, since $u\in L^p(-T,0;W^{1,p}(\Omega,\mathbb{R}^N))$. Here $c=c(n,N,p,L,M)$. Now taking $A=(Du)_{z_0;2\rho_2}$ in Theorem 3.6.1 and using the fact that $\ell_{z_0;\rho_2}$ is the unique minimizer discussed above and (3.52), we have

$$\begin{aligned}E_1(z_0,\rho_2,\ell_{z_0,\rho_2})\leq& E_1(z_0,\rho_2,(u)_{z_0;\rho_2}+(Du)_{z_0;\rho_2}(x-x_0))\\ \leq& c\left(\fint_{Q_{2\rho_2}(z_0)}|Du-(Du)_{z_0,2\rho_2}|^p\,dz\right)^{\frac{2(p-1)}{p}}\\ &+c\fint_{Q_{2\rho_2}(z_0)}|Du-(Du)_{z_0;\rho_2}|^p\,dz\\ &+c\,(1+|(Du)_{z_0,2\rho_2}|)^p\,\omega^{\frac{2(p-1)}{p}}((2\rho_2)^2)\end{aligned}$$

$$\leq c_2 \left[\epsilon_2^{\frac{2(p-1)}{p}} + \epsilon_2 + (1+L_0)^p \epsilon_2 \right],$$

where $c_2 = c_2(n, N, p, M)$. Hence we have shown that we may select $L \geq 1$ and $0 < \rho_2 < \rho_0(L)$ such that $Q_{2\rho_2}(z_0) \subseteq \Omega_T$, and

$$1 + |(Du)_{z_0;\rho_2}| < L \quad \text{and} \quad E_1(z_0, \rho_2, \ell_{z_0;\rho_2}) < \epsilon_1(L)$$

By the continuity of the mappings $z \mapsto |D\ell_{z;\rho_2}|$ and $z \mapsto E_1(z, \rho_2, \ell_{z,\rho_2})$, there exists $0 < R < \rho_2/2$ such that

$$1 + |D\ell_{z;\rho_2}| < L \quad \text{and} \quad E_1(z, \rho_2, \ell_{z;\rho_2}) < \epsilon_1(L), \text{ for all } z \in Q_R(z_0).$$

Hence the assumptions for obtaining the Campanato estimate (3.49) hold uniformly for $z \in Q_R(z_0)$. Also note $Q_{\rho_2}(z) \subseteq Q_{2\rho}(z_0) \subseteq \Omega_T$. Thus we have shown

$$\int_{Q_r(z)} |u - (u)_r|^2 \, dz \leq c(n, N, p, \nu, L, M, \alpha) r^{n+2+2\alpha}$$

for all $r \in (0, \rho_2]$, $z \in Q_R(z_0)$, where $R > 0$ was fixed in a way that depended on $z \in \Omega_T$. Hence $u \in \mathscr{C}^{2,1+\frac{2\alpha}{n+2}}(Q_R(z_0), \mathbb{R}^N)$. By the Campanato-Da Prato integral characterization, Theorem 3.1 in [18], we have

$$u \in C^{0;\alpha,\alpha/2}(Q_R(z_0), \mathbb{R}^N) \text{ for all } z_0 \in \Omega_T \backslash (\Sigma_1 \cup \Sigma_2),$$

i.e. we have shown $u \in C^{0;\alpha,\alpha/2}$ for a small neighborhood around any $z_0 \in \Omega_T \backslash (\Sigma_1 \cup \Sigma_2)$. The union of all these neighborhoods gives an open set Ω_0. Since Σ_1 and Σ_2 are both of measure zero, we know Ω_0 has full measure.

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