7-2007

Pythagorean Triples

Diane Swartzlander
University of Nebraska-Lincoln

Master of Arts in Teaching (MAT)
Masters Exam

Diane Swartzlander

In partial fulfillment of the requirements for the Master of Arts in Teaching with a Specialization in the Teaching of Middle Level Mathematics in the Department of Mathematics.
Jim Lewis, Advisor

July 2007

Pythagorean Triples

Diane Swartzlander

July 2007

Pythagorean Triples

Who was Pythagoras after which the Pythagorean Theorem is named? Pythagoras was born between 580-572 BC and died between 500-490 BC. Pythagoras and his students believed that everything was related to mathematics and that numbers were the ultimate reality. Very little is known about Pythagoras because none of his writings have survived. Many of his accomplishments may actually have been the work of his colleagues and students. Pythagoras established a secret cult called the Pythagoreans. His cult was open to both females and males and they lived a structured life consisting of religious teaching, common meals, exercise, reading and philosophical study. The Pythagorean Theorem for which Pythagoras is given credit states: In a right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the legs. There is no evidence that Pythagoras himself worked on or proved this theorem, rather the earliest mention of his name with the theorem occurred five centuries after his death. Other accomplishments attributed to Pythagoras ~~are~~ include a system of analyzing music based on proportional intervals of one through four, the number system based on ten, the identification of square numbers and square roots, knowledge that the earth is round, and that all planets have an orbit that travels around one central point.

The Pythagorean Theorem may have more known proofs than any other mathematical theorem. In the book *Pythagorean Proposition*, by Elisha Scott Loomis, 367 proofs are included. Euclid's proof, found in the book *Elements*, is a favorite.

One well-known proof of the Pythagorean Theorem can be given using the illustration below. A square ABCD has been constructed with dimensions of $a + b$. Inside the square are four right triangles each having a corner of the square as its right angle and one leg of length *a*, the other of length *b*. Notice that every triangle's hypotenuse has length *c*, since all of the triangles are congruent to one another by the Side-Angle-Side theorem of congruent triangles. The quadrilateral inscribed in square ABCD is also a square since the measure of each of its angles is equal to 90° . (We know that the sum of the measures of one interior angle of the inner square and the two acute angles of one of the right triangles adds to 180° because they form a straight line. We also know that the sum of the measures of the acute angles of a right triangle is 90^{o}.) To find the area of square ABCD we take the produce of the lengths of two adjacent sides to obtain: Area of ABCD = $(a + b) * (a + b)$. The four right triangles and the inscribed square make up the area of ABCD. The area for a triangle is base * height * ½. Since there are four triangles we multiply by this by four to obtain $4(ab/2)$. We then add the area of the square that has side lengths of *c*. Our equation now is $4(ab/2) + c^2$. Since these two areas are equal to one another we now have

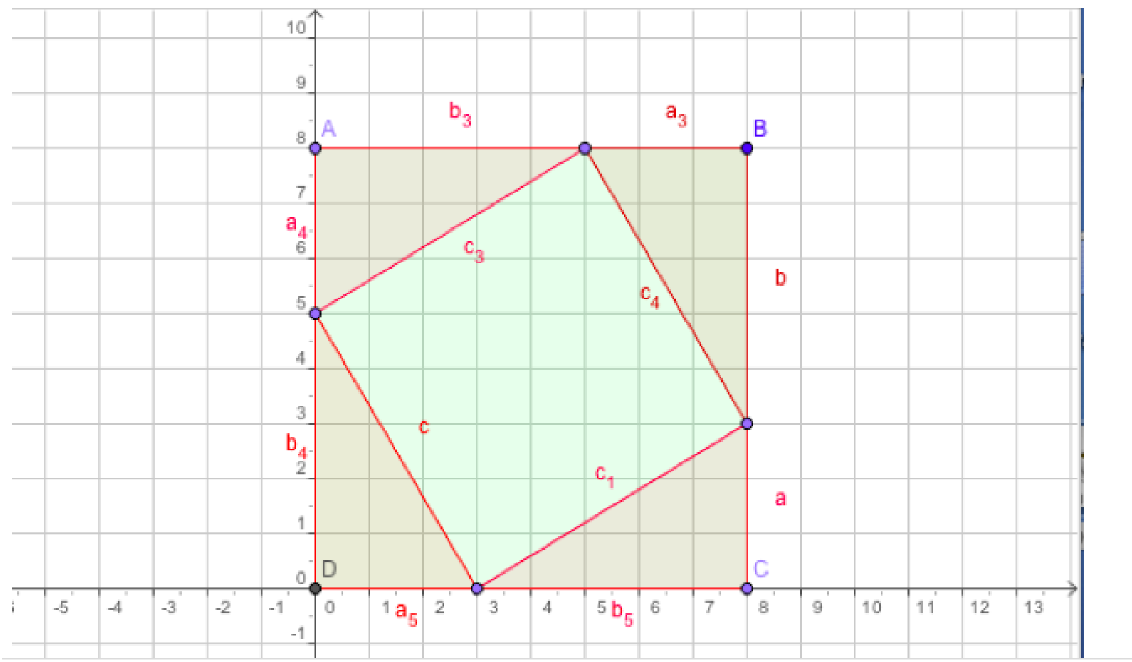
$$(a+b)^2 = 4(ab/2) + c^2 .$$

Multiplying out the left hand side and simplifying the right gives

$$a^2 + 2ab + b^2 = 2ab + c^2 .$$

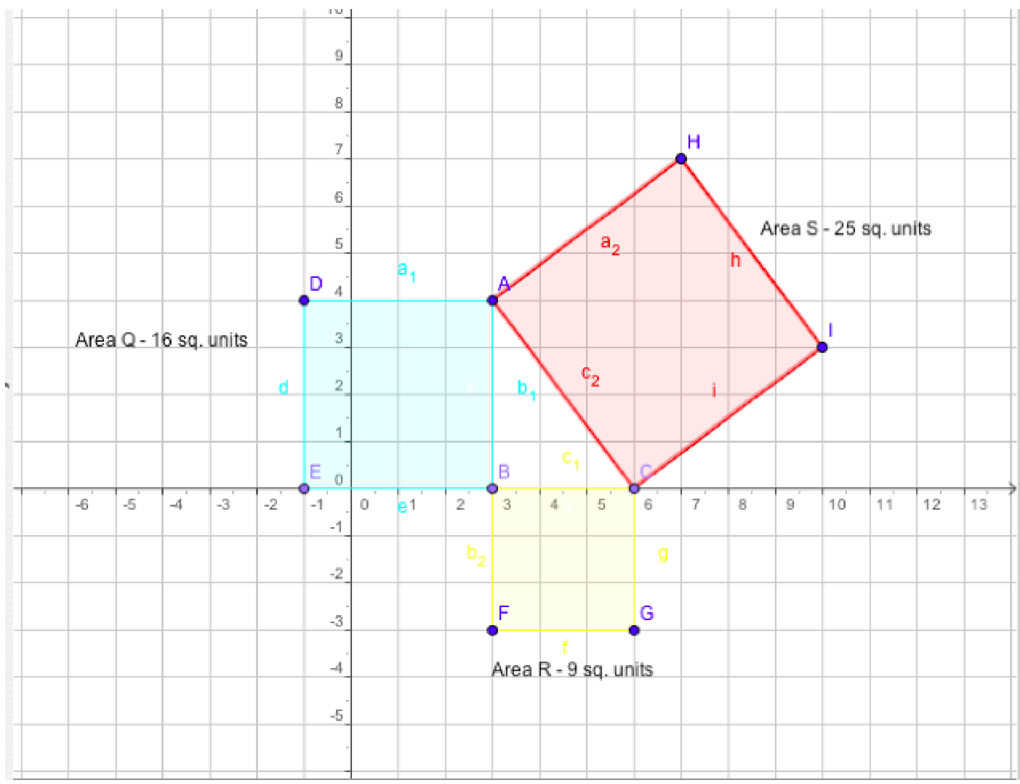
Subtract 2ab from both and you have

$$a^2 + b^2 = c^2 .$$



In this paper, we will also need the converse of the Pythagorean Theorem which is illustrated by the graph below. Specifically, if the square of the length of a side of ΔABC equals the sum of the squares of the lengths of the other two sides, then ΔABC is a right triangle with a right angle opposite the longest side. Papick's book *Algebra Connections* includes a proof of the converse to the Pythagorean Theorem.

Note: The illustration below was created using GeoGebra. In the figure a^2 = Area Q = 16 sq. units, b^2 = Area R = 9 sq. units and c^2 = Area S = 25 sq. units. Then by substitution 16 + 9 = 25.



Note that the graph illustrates one of the best known right triangles, namely one with side lengths 3 and 4 and hypotenuse of length 5. If we were to draw this triangle we could begin by drawing a right angle using sides of length 3 and 4. Because of the Pythagorean Theorem, we know that when we draw the third side (i.e. the hypotenuse) it must have length 5. The fact that the hypotenuse has integer length is something worth noting. Contrast this with what would happen if our two sides had length 2 and 3. By the Pythagorean Theorem, the hypotenuse of this triangle would have length $\sqrt{13}$.

This idea leads to the study of what are known as Pythagorean triples and primitive Pythagorean triples. A **Pythagorean triple** consists of three positive integers, *a*, *b*, and *c*, such that $a^2 + b^2 = c^2$. As mentioned previously, a well-known example is 3, 4, and 5 because $3^2 + 4^2 = 9 + 16 = 25 = 5^2$. Other examples include (5, 12, 13); (8, 15, 17); and (20, 48, 52). It is easy to see that there are an infinite number of Pythagorean triples because one can always multiply all three positive integers by a positive integer *k* and get another Pythagorean Triple. In other words if *(a, b, c)* is a Pythagorean triple, then so is (*ka, kb, kc*). For example, since (3, 4, 5) is a Pythagorean Triple, so is (6, 8, 10) and (9, 12, 15).

An effective way to generate Pythagorean triples is based on Euclid's formula found in his book *Elements*. This formula states that if *m* and *n* are two positive integers with $m > n$, then

$a = m^2 - n^2$

$b = 2mn$

$c = m^2 + n^2$, is a Pythagorean triple.

As an example, let $m = 5$ and $n = 3$. Then $a = 5^2 - 3^2 = 16$; $b = 2(5*3) = 30$; and $c = 5^2 + 3^2 = 34$. Note that, $16^2 + 30^2 = 34^2$, since $256 + 900 = 1156$.

Other examples are:

m	n	$a = m^2 - n^2$	$b = 2mn$	$c = m^2 + n^2$	Pythagorean Triple
6	4	20	48	52	20, 48, 52
5	3	16	30	34	16, 30, 34
4	2	12	16	20	12, 16, 20

We can show that this construction always gives a Pythagorean triple as follows:

$$\begin{aligned} a^2 + b^2 &= (m^2 - n^2)^2 + (2mn)^2 \\ &= m^4 - 2m^2n^2 + n^4 + 4m^2n^2 \\ &= m^4 + 2m^2n^2 \\ &= (m^2 + n^2)^2 \\ &= c^2 \end{aligned}$$

This work shows that if *m* and *n* are chosen by Euclid's formula the result is a Pythagorean triple.

Pythagorean triples can be used to construct right angles. Since the numbers in a Pythagorean triple are integers, if these are used as the lengths of the sides of a triangle it will necessarily be a right triangle. By tying knots on a string at even intervals and using the string to enclose a triangle with dimensions corresponding to a selected Pythagorean triple, the largest angle will be a right angle. This is proven to be true by the converse of the Pythagorean Theorem: If the square of the length of a side in a triangle equals the sum of the squares of the lengths of the other two sides, it is a right triangle with a right angle opposite the longest side. This method was used to step masts at sea and by Egyptians in construction work. Today the Pythagorean theorem is used by architects, engineers, meteorologists, airplane pilots, photographers, construction workers, plumbers, and quilters to name just a few.

A Pythagorean triple is called **primitive** if *a*, *b*, and *c* are relatively prime. Integers are said to be **relatively prime** if their greatest common divisor is 1.

To generate primitive Pythagorean triples using Euclid's formula, one needs to include the following parameters:

1. $a = m^2 - n^2$, $b = 2mn$, $c = m^2 + n^2$
2. $m > n$
3. gcd $(m,n) = 1$; and
4. of *m* and *n*, one is even and the other is odd.

Looking back at the table made with Euclid's formula, we can see that the triples are not primitive triples since they are all divisible by 2. We can also see that the numbers chosen for *m* and *n* are both even or odd, which generated even numbers for (a, b, c). To generate primitive Pythagorean triples, the last two conditions are necessary. This is because any prime that factors both *m* and *n* would have to factor *a, b.* and *c*, and, if both *m* and *n* were even or odd, then *a, b,* and *c* would have to be even.

But why do these last two conditions assure us that the resulting Pythagorean triple will be primitive? To argue this, suppose that *p* is a prime that divides *a, b* and *c*. Since $p|a$ and $p|c$, we know that $p|\ (a + c) = 2m^2$. This tells us that either $p = 2$ or $p|m$. But if $p = 2$, then *a, b* and *c* are all even, which only happens if condition 4 is violated. If $p|m$, since $p|a$ we know $p|m^2 - a = n^2$, which means that $p|n$. By condition 3 this can only be true if $p = 1$. Thus, the only common factor of *a*, *b* and *c* is 1, making this a primitive Pythagorean triple.

How can primitive Pythagorean triples help find other Pythagorean triples? Primitive Pythagorean triples generate a list of non-reducible triples. There are 16 primitive Pythagorean triples with $c \leq 100$. These are:

(3, 4, 5)	(5, 12, 13)	(7, 24, 25)	(8, 15, 17)
(9, 40, 41)	(11, 60, 61)	(12, 35, 37)	(13, 84, 85)
(16, 63, 65)	(20, 21, 29)	(28, 45, 53)	(33, 56, 65)
(36, 77, 85)	(39, 80, 89)	(48, 55, 73)	(65, 72, 97)

Using the previously stated definition of a Pythagorean triple, if (*a*, *b*, *c*) is a Pythagorean triple, then so is (*ka*, *kb*, *kc*). By multiplying a primitive Pythagorean triple by the same positive integer *k*, an infinite number of Pythagorean triples can be found.

Every primitive Pythagorean triple can be generated by using Euclid's formula. Start with any primitive Pythagorean triple *a*, *b*, and *c*. Our goal is to show that there exist positive integers *m*, *n* (with $m > n$) and such that using Euclid's formula with *m*, *n*, will produce the primitive Pythagorean triple *a*, *b*, and *c*. We know that gcd(*a*, *b*, *c*) = 1. If any two of *a*, *b*, and *c* shared a common divisor *d*, say $d|a$ and $d|b$, then we would also have $d^2|a^2$ and $d^2|b^2$. Since $a^2 + b^2 = c^2$ this tells us that d^2 would have to divide c^2, so that *d* would also divide *c*. This would contradict the assumption that our triple was primitive. Thus no two of *a*, *b*, and *c* have a common divisor greater than 1.

Not all of *a*, *b*, and *c* can be odd, because the sum of two odd integers is even. Neither can all of *a*, *b*, and *c* be even, because 2 would be a common factor, and the Pythagorean triple is primitive. It is not possible for exactly two of *a*, *b*, and *c* to be even, because by the argument above (letting $d = 2$), if 2 is a factor of *a* and *b*, then it is also a factor of *c*. So we are left to consider the case where exactly one of *a*, *b*, and *c* is even and the other two odd.

To examine this case, we first try letting the hypotenuse *c* be even, and *a* and *b* be odd. Then we can write $a = 2x + 1$, $b = 2y + 1$, and $c = 2z$, where *x*, *y* and *z* are positive integers. Using the Pythagorean Theorem we obtain the following:

$$(2x + 1)^2 + (2y + 1)^2 = (2z)^2$$
$$4x^2 + 4x + 1 + 4y^2 + 4y + 1 = 4z^2$$
$$4(x^2 + x + y^2 + y - z^2 + 1) = -2$$

This implies that 4 is a divisor of 2, which is false. So if we have a primitive Pythagorean triple, c (the hypotenuse) cannot be even. Thus either a or b must be even, and the other two odd.

Without loss of generality, let b be even, with a and c odd. Now note that $c^2 - a^2 = (c - a)*(c + a)$ and rewrite the equation $b^2 = c^2 - a^2$ in the form

$$(b/2)^2 = ([c - a]/2) * ([c + a]/2).$$

Notice that since b is even and a and c are odd, $b/2$, $(c - a)/2$, and $(c + a)/2$ are all positive integers. We claim that the greatest common divisor of $(c - a)/2$ and $(c + a)/2$ is 1. If d is any common divisor of these, then d would divide their sum, c, and their difference, a. We know, however, that the greatest common divisor of a and c is 1, so d must divide 1, so $d = 1$. This gives us the product of two whole numbers, $(c - a)/2$ and $(c + a)/2$, whose greatest common divisor is 1, and whose product is the square $(b/2)^2$. The only way that can happen is if each of them is a square itself. This means that there are positive whole numbers m and n such that:

$$m^2 = (c + a)/2,$$
$$n^2 = (c - a)/2,$$
$$mn = b/2.$$

Thus we let $m = \sqrt{(c+a)/2}$ and $n = \sqrt{(c-a)/2}$.

Then $m > n$, because $m^2 = n^2 + a > n^2$, and $n > 0$, because $c > a$. In addition, since m^2 and n^2 have greatest common divisor 1, likewise m and n have greatest common divisor 1. Lastly, m and n can't both be odd, or else $m^2 - n^2 = a$ would be even, which it is not since they cannot both be even as 2 cannot be a common divisor. This means that one of m and n is even; hence $m - n$ is odd. Now solving the above three equations for a, b, and c, we find that, for generating all primitive Pythagorean triples we can use Euclid's formula.

$a = m^2 - n^2$
$b = 2mn$
$c = m^2 + n^2$
$m > n > 0$ are whole numbers
$m - n$ is odd, (i.e. one is odd and one even) and the gcd of m and n = 1.

Example: Start with the primitive Pythagorean triple a = 7, b = 24, c = 25, and let

$m = \sqrt{(25+7)/2} = \sqrt{16} = 4$, and $n = \sqrt{(25-7)/2} = \sqrt{9} = 3$. Then $m > n > 0$, and they are relatively prime. In turn, using m = 4 and n = 3, Euclid's formula gives us 7, 24, and 25.

Did Pythagoras find all the primitive Pythagorean triples? The Pythagoreans stated that a primitive Pythagorean Triple could be found using the following:

$a = 2n + 1$, $b = 2n^2 + 2n$, and $c = 2n^2 + 2n + 1$, where $n > 0$ is a real number.

Example: If n=2, then $a = 2*2+1 = 5$, $b = 2(2^2)+2(2) = 12$, and $c = 2(2^2)+2(2)+1 = 13$.

Note that this formula produces values for b and c that are consecutive integers. Compare this with Euclid's formula:

1. $a = m^2 - n^2$, $b = 2mn$, *and* $c = m^2 + n^2$
2. $m > n$
3. gcd (m,n) = 1; and
4. of m and n, one is even and the other is odd.

By using Euclid's formula, we can see from the following table that there are primitive Pythagorean triples where b and c are not consecutive integers. The last column shows two examples where a primitive Pythagorean Triple does not fit Pythagoras's formula.

m	n	$m^2 - n^2$	$2mn$	$m^2 + n^2$	Pythagoras' Triple
2	1	3	4	5	yes
3	2	5	12	13	yes

4	1	15	8	17	no
4	3	7	24	25	yes
5	2	21	20	29	no

(Adapted from Algebra Connections)

During Pythagoras time, it was believed that he had found all the primitive Pythagorean Triples, but Euclid's formula found that there were more.

Many of Pythagoras's works influenced Plato, Aristotle, Euclid, and Copernicus. In addition to mathematics, the work of Pythagoras also impacted studies of religion, science, music, and literacy. Pythagoras was one of the first to propose that thought processes and the soul were located in the brain and not the heart. Where would our world be today, had it not been for Pythagoras?

References:

Books:

Saunders, Hal. (1988) *When Are We Ever Gonna Have to Use This?* Dale Seymour Publications.

Papick, Ira J. (2007) *Algebra Connections Mathematics for Middle School Teachers*. Pearson Prentice Hall

Internet Resources:

http://en.wikipedia.org/wiki/Pythagoras
http://en.wikipedia.org/wiki/Pythagorean_theorem
http://en.wikipedia.org/wiki/Pythagorean_triple
http://www.math.clemson.edu/~simms/neat/math/pyth/
http://www.mathopenref.com/pythagorastheorem.html
http://en.wikipedia.org/wiki/Pythgaoras_tree